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Research article

Some properties of n-quasi-(m,q)-isometric operators on a Banach space

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Abstract: In this paper, we introduced the class of n-quasi-(m,q)-isometric operators on a Banach space. Such a class seems to be a natural generalization of m-isometric operators on Banach spaces and of n-quasi-m-isometric operators on Hilbert spaces. We started by giving some of their elementary properties and studying the products and the power of such operators. Next, we focused on the dynamic of a n-quasi-m-isometry. More precisely, we proved a result by characterizing the supercyclicity of such a class.

Keywords: m-isometry; (m,q)-isometry; n-quasi-m-isometry; C_0-semigroup
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1. Introduction

Let $\mathcal{H}$ be a complex separable Hilbert space. We denote by $\mathcal{B}(\mathcal{H})$ the Banach algebra of all bounded linear operators on $\mathcal{H}$.

The class of m-isometric operators on $\mathcal{H}$ has been introduced by J. Agler [1] and later has been intensively studied by J. Agler and M. Stankus (see [2–4]). In fact, an operator $T \in \mathcal{B}(\mathcal{H})$ is said to be a m-isometry if, and only if,

$$\beta_m(T) := \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} T^{*k} T^k = 0, \tag{1.1}$$

where T^* denotes the adjoint operator of T. Equivalently,

$$\Delta_m(T,x) := \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \|T^k x\|^2 = 0, \tag{1.2}$$

for all $x \in \mathcal{H}$. If $m = 1$, the operator T is said to be an isometry.

Recently, the study of the family of m-isometric operators has been developed by many researchers (see [5–8]). In quest of generality, the interest is devoted to introduce a new class of operators that generalizes the m-isometries, namely, the n-quasi-m-isometric operators. Such a class of bounded linear operators was first introduced by J. Shen and F. Zuo in [9]. Later, many authors investigated in details the study of the n-quasi-m-isometric operators. For example, we quote the readers to the works of S. Mecheri and T. Prasad [10] and O. A. Mahmoud Sid Ahmed, A. Saddi and K. Gherairi [11]. This class of bounded operators generalizes that of m-isometric operators on a Hilbert space, an operator $T \in \mathcal{B}(\mathcal{H})$ is said to be a n-quasi-m-isometry if, and only if,

$$\beta_{m,n}(T) := T^{*n}\left(\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}T^{*k}T^{k}\right)T^{n} = 0, \tag{1.3}$$

which is equivalent to

$$\Delta_{m,n}(T,x) := \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\|T^{k+n}x\|^{2} = 0, \quad \forall x \in \mathcal{H}. \tag{1.4}$$

In [12], O. A. M. Sid Ahmed extended the study of m-isometric operators to a Banach space structure. Let X be a Banach space. An operator $T \in \mathcal{B}(X)$ is called an m-isometry if

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\|T^{k}x\|^{2} = 0, \quad \forall x \in X. \tag{1.5}$$

In [13], F. Bayart has noted that the exponent two can be replaced by any real number $q \geq 1$ and has introduced the (m,q)-isometry by the following definition.

Definition 1.1. [13] Let $T \in \mathcal{B}(X)$, $m \geq 1$ and $q \in [1,+\infty)$. The operator T is said to be a (m,q)-isometry if, for any $x \in X$,

$$\Delta_{m}^{q}(T,x) := \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\|T^{k}x\|^{q} = 0. \tag{1.6}$$

It is called a m-isometry if it is an (m,p)-isometry for some $p \geq 1$.

We note that if X is an Hilbert space and $q = 2$ the $(m,2)$-isometry corresponds to Agler's definition of an m-isometry.

In this paper, we aim to generalize this notion by introducing and studying a new class called n-quasi-(m,q)-isometric operators on a Banach space. Throughout this paper $\mathbb{N}$ denotes the set of positive integers, X a Banach space and $I = I_X$ the identity operator. For every $T \in \mathcal{B}(X)$, we denote by $\mathcal{R}(T)$ the range of T. We notice that $\mathcal{R}(T)$ is a T-invariant subspace.

The paper is organized as follows. Section 2 is devoted to present some definitions and basic properties. The power and the product of n-quasi-(m,q)-isometric operators are discussed in Section 3. In the closing section, we study the dynamic of such a class. More precisely, we prove that a n-quasi-(m,q)-isometry on a Banach can never be supercyclic.

2. Basic properties of n-quasi-(m, q)-isometries

In this section, we give some properties of n-quasi-(m, q)-isometries and we prove results that generalize the existing ones corresponding to (m, q)-isometries on Banach spaces.

Let us begin with the following definition in which we generalize the (m, q)-isometry notion by defining the class of n-quasi-(m, q)-isometries.

Definition 2.1. Let $T \in \mathcal{B}(X)$, $m, n \in \mathbb{N}$ and $q \in [1, +\infty)$. The operator T is said to be an n-quasi-(m, q)-isometry if

$$\Delta^q_{m,n}(T, x) := \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\|T^{n+k}x\|^q = 0, \quad \forall\, x \in X. \tag{2.1}$$

Remark 2.1. If X is a Hilbert space and $q = 2$, then the notion of n-quasi-$(m, 2)$-isometry corresponds to n-quasi-m-isometry on a Hilbert space; that is $\Delta^2_{m,n}(T) = \Delta_{m,n}(T)$.

The following proposition gives a characterization of the n-quasi-(m, q)-isometric operators. It will be used to prove results with some interest in this paper.

Proposition 2.1. *Let $T \in \mathcal{B}(X)$, then T is a n-quasi-(m, q)-isometry if, and only if, T is a (m, q)-isometry on $\overline{\mathcal{R}(T^n)}$.*

Proof. Let $T \in \mathcal{B}(X)$, T be a n-quasi-(m, q)-isometry on X if, and only if, for all $x \in X$, we have

$$\begin{aligned} 0 &= \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\|T^{n+k}x\|^q \\ &= \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\|T^k(\underbrace{T^n x}_{=y})\|^q \\ &= \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\|T^k(y)\|^q, \quad \forall y \in \mathcal{R}(T^n). \end{aligned}$$

Now, Let $z \in \overline{\mathcal{R}(T^n)}$, then there exists $(y_p)_p \subset \mathcal{R}(T^n)$ such that $z = \lim\limits_{p\to\infty} y_p$. Thus, we have

$$\begin{aligned} &\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\|T^k z\|^q \\ &= \lim_{p\to\infty}\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\|T^k y_p\|^q \\ &= 0, \end{aligned}$$

then $\sum\limits_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\|T^k z\|^q = 0, \quad \forall z \in \overline{\mathcal{R}(T^n)}.$

This ends the proof of Proposition 2.1. □

Remark 2.2. Regarding Proposition 2.1, if $\mathcal{R}(T^n)$ is dense in X, then T is a n-quasi-(m, q)-isometry on X if, and only if, T is a (m, q)-isometry on X. For this reason, we will assume throughout this paper that $\mathcal{R}(T^n)$ is not dense on X.

Proposition 2.2. *Let $T \in \mathcal{B}(X)$ be a n-quasi-(m,q)-isometry, then T is a n_1-quasi-(m,q)-isometry for all $n_1 \geq n$.*

Proof. Assume that T is a n-quasi-(m,q)-isometry on X. Referring to Proposition 2.1, we get that T is a (m,q)-isometry on $\overline{\mathcal{R}(T^n)}$. On the other hand, let $n_1 \geq n$ and $y \in \overline{\mathcal{R}(T^{n_1})}$

$$\begin{aligned}
&\Rightarrow \quad \exists\, (y_k)_k \subset \mathcal{R}(T^{n_1}) \text{ such that } \lim_{k\to\infty} y_k = y \\
&\Rightarrow \quad \forall k \in \mathbb{N}\ \exists x_k \in X \text{ such that } y_k = T^{n_1}(x_k) \\
&\Rightarrow \quad \forall k \in \mathbb{N}\ \exists x_k \in X \text{ such that } y_k = T^{n}(T^{n_1-n}(x_k)) \\
&\Rightarrow \quad \forall k \in \mathbb{N},\ y_k \in \mathcal{R}(T^n) \text{ and } y \in \overline{\mathcal{R}(T^n)} \\
&\Rightarrow \quad \overline{\mathcal{R}(T^n)} \supset \overline{\mathcal{R}(T^{n_1})} \quad \text{for all } n_1 \geq n.
\end{aligned}$$

This implies that T is a (m,q)-isometry on $\overline{\mathcal{R}(T^{n_1})}$. Therefore, T is a n_1-quasi-(m,q)-isometry for all $n_1 \geq n$. □

Example 2.1. Let $T \in \mathcal{B}(l^q(\mathbb{N}))$, where $q \geq 1$, be the unilateral weighted forward shift operator defined on $l^q(\mathbb{N})$ as follows

$$T(\alpha_1, \alpha_2, \alpha_3, \alpha_4, ...) := (0, 2\alpha_1, 3\alpha_2, \alpha_3, \alpha_4, ...).$$

By a direct computation, it holds for $\alpha = (\alpha_1, \alpha_2, \alpha_3, ...) \in l^q(\mathbb{N})$ that

$$\begin{aligned}
& \|T^4(\alpha_1, \alpha_2, \alpha_3, ...)\|^q - 2\|T^3(\alpha_1, \alpha_2, \alpha_3, ...)\|^q + \|T^2(\alpha_1, \alpha_2, \alpha_3, ...)\|^q \\
= & \left(|6\alpha_1|^q + |3\alpha_2|^q + \sum_{n=3}^{+\infty} |\alpha_n|^q\right) - 2\left(|6\alpha_1|^q + |3\alpha_2|^q + \sum_{n=3}^{+\infty} |\alpha_n|^q\right) \\
& + \left(|6\alpha_1|^q + |3\alpha_2|^q + \sum_{n=3}^{+\infty} |\alpha_n|^q\right) = 0.
\end{aligned}$$

Taking $(\alpha_1, \alpha_2, \alpha_3, \cdots) \in l^q(\mathbb{N})$ with $\alpha_1 \neq 0$, we obtain

$$\begin{aligned}
& \|T^3(\alpha_1, \alpha_2, \alpha_3, ...)\|^q - 2\|T^2(\alpha_1, \alpha_2, \alpha_3, ...)\|^q + \|T(\alpha_1, \alpha_2, \alpha_3, ...)\|^q \\
= & \left(|6\alpha_1|^q + |3\alpha_2|^q + \sum_{n=3}^{+\infty} |\alpha_n|^q\right) - 2\left(|6\alpha_1|^q + |3\alpha_2|^q + \sum_{n=3}^{+\infty} |\alpha_n|^q\right) \\
& + \left(|2\alpha_1|^q + |3\alpha_2|^q + \sum_{n=3}^{+\infty} |\alpha_n|^q\right) \\
= & |2\alpha_1|^q - |6\alpha_1|^q \neq 0.
\end{aligned}$$

It follows that T is a 2-quasi-$(2,q)$-isometry, but is not a quasi-$(2,q)$-isometry.

In the previous example, we have shown that an n-quasi-(m,q)-isometric operator is not necessarily a $(n-1)$-quasi-(m,q)-isomerty. In this context, O. A. M. Sid Ahmed, A. Saddi and K. Gherairi proved in [11, Theorem 2.9] that for $q = 2$, if $T \in \mathcal{B}(\mathcal{H})$ is an n-quasi-m-isometry for $n \geq 2$ and $\mathcal{N}(T^{*p}) = \mathcal{N}(T^{*(p+1)})$ for some $1 \leq p \leq n-1$, then T is a p-quasi-m-isometry. In the following result, we will add a suitable condition to have a similar result.

Theorem 2.1. *Let $T \in \mathcal{B}(X)$ be an n-quasi-(m, q)-isometry. If $\overline{\mathcal{R}(T^p)} = \overline{\mathcal{R}(T^{(p+1)})}$, then T is a p-quasi-(m, q)-isometric operator on X for some $1 \leq p \leq n-1$.*

Proof. Under the assumption $\overline{\mathcal{R}(T^p)} = \overline{\mathcal{R}(T^{p+1})}$, we have $\overline{\mathcal{R}(T^p)} = \overline{\mathcal{R}(T^n)}$. Indeed, we start by proving that $\overline{\mathcal{R}(T^{p+1})} = \overline{\mathcal{R}(T^{p+2})}$. The first inclusion $\overline{\mathcal{R}(T^{p+2})} \subset \overline{\mathcal{R}(T^{p+1})}$ is trivial. Conversely, let $x \in \overline{\mathcal{R}(T^{p+1})}$, then there exists $(y_k)_k \subset X$ such that

$$x = \lim_{k\to\infty} T^{p+1}(y_k) = \lim_{k\to\infty} T(T^p(y_k)),$$

and we denote by $z_k = T^p(y_k) \in \mathcal{R}(T^p) \subset \overline{\mathcal{R}(T^p)} = \overline{\mathcal{R}(T^{p+1})}, \forall k \in \mathbb{N}$. We have $x = \lim\limits_{k\to\infty} T(z_k)$. Thus, we get the existence of $(u_{k,m}) \subset X$ such that

$$x = \lim_{m\to\infty}\lim_{k\to\infty} TT^{p+1}(u_{k,m}) = \lim_{m\to\infty}\lim_{k\to\infty} T^{p+2}(u_{k,m}).$$

Since $\overline{\mathcal{R}(T^{p+2})}$ is a closed set, we obtain that $x \in \overline{\mathcal{R}(T^{p+2})}$. Therefore, it holds that $\overline{\mathcal{R}(T^{p+1})} \subset \overline{\mathcal{R}(T^{p+2})}$ and, hence, we have $\overline{\mathcal{R}(T^{p+1})} = \overline{\mathcal{R}(T^{p+2})}$. By applying the same procedure $(n-p)$ times, we obtain that $\overline{\mathcal{R}(T^p)} = \overline{\mathcal{R}(T^n)}$. It results that $\overline{\mathcal{R}(T^p)} = \overline{\mathcal{R}(T^n)}$, which gives that T is an n-quasi-(m, q)-isometric operator on X, and so T is a (m, q)-isometry on $\overline{\mathcal{R}(T^n)} = \overline{\mathcal{R}(T^p)}$. Hence, T is a p-quasi-(m, q)-isometric operator on X for some $1 \leq p \leq n-1$. $\square$

Example 2.2. Let $T \in \mathcal{B}(l^q(\mathbb{N}))$, with $q \geq 1$, defined by

$$T(\alpha_1, \alpha_2, \alpha_3, \cdots) = (0, w_1\alpha_1, w_2\alpha_2, w_3\alpha_3, \cdots),$$

where the weights sequence $(w_n)_{n>0}$ is given by

$$w_n := \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{1}{\sqrt{2n-1}} & \text{if } n \text{ is odd.} \end{cases}$$

We can easily check that $\overline{\mathcal{R}(T)} \neq \overline{\mathcal{R}(T^2)}$ and T is a two quasi-$(2, 3)$-isometry but is not a quasi-$(2, 3)$-isometry.

On the other hand, a similar result can be found using the semigroup theory. Let's begin with the following elementary definition.

Definition 2.2. [14] A strongly continuous semigroup (or a C_0-semigroup) on a Banach space X is a mapping $T : \mathbb{R}_+ \longrightarrow \mathcal{B}(X)$, which satisfies:

(1) $T(0) = I$ (identity operator),
(2) $T(t+s) = T(t)T(s)$, for all $t, s \geq 0$ (semigroup property),
(3) $\lim\limits_{t\to 0^+} T(t)u = u$, for all $u \in X$ in the strong operator topology.

Definition 2.3. A C_0-semigroup $\{T(t)\}_{t\geq 0}$ is an n-quasi-(m, q)-isometry if $T(t)$ is an n-quasi-(m, q)-isometry operator for every $t \geq 0$.

Proposition 2.3. *Let $T := \{T(t)\}_{t\geq 0}$ be a C_0-semigroup, then T is an n-quasi-(m, q)-isometry if, and only if, T is a (m, q)-isometry.*

Proof. We know that every (m, q)-isometry is an n-quasi-(m, q)-isometry. Conversely, assume that T is an n-quasi-(m, q)-isometry, then

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\|T^{n+k}(t)x\|^q = 0, \quad \forall t \geq 0.$$

Since $\{T(t)\}_{t\geq 0}$ is a C_0-semigroup, we deduce that

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\|T^{k}(nt)x\|^q = 0, \quad \forall t \geq 0.$$

Thanks to a change of variable $t' = nt$, it holds that T is a (m, q)-isometry. □

Theorem 2.2. *Let $T \in \mathcal{B}(X)$ be an n-quasi-(m, q)-isometry, then T is an n-quasi-(l, q)-isometry for all $l \geq m$.*

Proof. It is enough to prove the result for $l = m + 1$. So, for all $x \in X$, we have

$$\begin{aligned}
\Delta^q_{m+1,n}(T, x) &= \sum_{k=0}^{m+1}(-1)^k\binom{m+1}{k}\|T^{n+m+1-k}x\|^q \\
&= \|T^{n+m+1}x\|^q + \sum_{k=1}^{m}(-1)^k\binom{m+1}{k}\|T^{n+m+1-k}x\|^q + (-1)^{m+1}\|T^n x\|^q \\
&= \|T^{n+m+1}x\|^q + \sum_{k=1}^{m}(-1)^k\left(\binom{m}{k} + \binom{m}{k-1}\right)\|T^{n+m+1-k}x\|^q \\
&\quad +(-1)^{m+1}\|T^n x\|^q \\
&= \|T^{n+m+1}x\|^q + \sum_{k=1}^{m}(-1)^k\binom{m}{k}\|T^{n+m+1-k}x\|^q \\
&\quad + \sum_{k=1}^{m}(-1)^k\binom{m}{k-1}\|T^{n+m+1-k}x\|^q + (-1)^{m+1}\|T^n x\|^q \\
&= \Delta^q_{m,n+1}(T, x) + \sum_{k=0}^{m-1}(-1)^{k+1}\binom{m}{k}\|T^{n+m-k}x\|^q + (-1)^{m+1}\|T^n x\|^q \\
&= \Delta^q_{m,n+1}(T, x) - \Delta^q_{m,n}(T, x).
\end{aligned}$$

Since T is an n-quasi-(m, q)-isometry, we have $\Delta^q_{m,n}(T, x) = 0$ for all $x \in X$. Referring to Proposition 2.2, we obtain that $\Delta^q_{m,n+1}(T, x) = 0$ for all $x \in X$.

□

Usually, the reciprocal meaning is not verified, as shown in the following example.

Example 2.3. Consider the weighted shift operator $T \in \mathcal{B}(l^q(\mathbb{N}))$ given by $T(\alpha_1, \alpha_2, \alpha_3, \cdots) = (0, w_1\alpha_1, w_2\alpha_2, w_3\alpha_3, \cdots)$ with weights sequence $(w_n)_{n>0}$ given by $w_n := \left(\frac{n+1}{n}\right)^{\frac{2}{q}}$. By a direct calculation, we obtain that

$$\|T^4(\alpha_1, \alpha_2, \alpha_3, \cdots)\|^q - 3\|T^3(\alpha_1, \alpha_2, \alpha_3, \cdots)\|^q + 3\|T^2(\alpha_1, \alpha_2, \alpha_3, \cdots)\|^q$$

$$
\begin{aligned}
& -\|T(\alpha_1, \alpha_2, \alpha_3, \cdots)\|^q \\
= & \sum_{n=1}^{+\infty} \left(|w_n w_{n+1} w_{n+2} w_{n+3}|^q - 3|w_n w_{n+1} w_{n+2}|^q + 3|w_n w_{n+1}|^q - |w_n|^q \right) |\alpha_n|^q \\
= & \sum_{n=1}^{+\infty} \left(\left(\frac{n+4}{n}\right)^2 - 3\left(\frac{n+3}{n}\right)^2 + 3\left(\frac{n+2}{n}\right)^2 - \left(\frac{n+1}{n}\right)^2 \right) |\alpha_n|^q = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
& \|T^3(\alpha_1, \alpha_2, \alpha_3, \cdots)\|^q - 2\|T^2(\alpha_1, \alpha_2, \alpha_3, \cdots)\|^q + \|T(\alpha_1, \alpha_2, \alpha_3, \cdots)\|^q \\
= & \sum_{n=1}^{+\infty} \left(|w_n w_{n+1} w_{n+2}|^q - 2|w_n w_{n+1}|^q + |w_n|^q \right) |\alpha_n|^q \\
= & \sum_{n=1}^{+\infty} \left(\left(\frac{n+3}{n}\right)^2 - 2\left(\frac{n+2}{n}\right)^2 + \left(\frac{n+1}{n}\right)^2 \right) |\alpha_n|^q \\
= & 2\|(\alpha_1, \alpha_2, \alpha_3, \cdots)\|^q \neq 0.
\end{aligned}
$$

Hence, T is a quasi-$(3, q)$-isometry, which is not a quasi-$(2, q)$-isometry.

In the following result, we give some property of the approximate spectral of an n-quasi-(m, q)-isometric operator.

Proposition 2.4. *Let T be an n-quasi-(m, q)-isometry, then, a nonzero approximate eigenvalue of T lies in the unit circle.*

Proof. Let $\lambda \neq 0$ be an approximate eigenvalue of T, then, there exists $(x_j) \subset X$ with $\|x_j\| = 1$ and $(T - \lambda)x_j \longrightarrow 0$, so for all integer $k \geq 1$ we have $(T^{n+k} - \lambda^{n+k})x_j \longrightarrow 0$. Since T is an n-quasi-(m, q)-isometry, we get

$$
\begin{aligned}
0 &= \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \|T^{n+k} x_j\|^q \\
&= \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} |\lambda|^{q(n+k)} \\
&= |\lambda|^{qn} \left(|\lambda|^q - 1\right)^m .
\end{aligned}
$$

Since $\lambda \neq 0$, we obtain $|\lambda| = 1$. Hence, the desired claim follows from that.

□

3. Power and product of n-quasi-(m, q)-isometric operators

In this section we aim to study the stability of an n-quasi-(m, q)-isometry under products and powers.

Let's begin with the following result in which we will generalize [6, Theorem 3.1] and [11, Theorem 2.12]. More precisely, we will show that any power of an n-quasi-(m, q)-isometry is also an n-quasi-(m, q)-isometry.

Theorem 3.1. *Let T be an n-quasi-(m, q)-isometric operator on X, then T^k is also an n-quasi-(m, q)-isometry, for all positive integer k.*

Proof. Let $T \in \mathcal{B}(X)$ be an n-quasi-(m,q)-isometry on X, then T is a (m,q)-isometry on $\overline{\mathcal{R}(T^n)}$. Referring to [6, Theorem 3.1], we get that T^k is a (m,q)-isometry on $\overline{\mathcal{R}(T^n)}$. We obtain

$$\overline{\mathcal{R}(T^n)} \supset \overline{\mathcal{R}(T^{nk})}.$$

This implies that T^k is a (m,q)-isometry on $\overline{\mathcal{R}((T^k)^n)}$. Hence, T^k is an n-quasi-(m,q)-isometry on X for all $k \geq 1$.

□

Example 3.1. Let T be the bounded linear operator defined as in Example 2.1. By a simple calculation we can show that T^2 is a quasi-$(2,q)$-isometry but T is not quasi-$(2,q)$-isometric.

Proposition 3.1. *Let $T \in \mathcal{B}(X)$ and n_1, n_2, r, s, m, l be positive integers. If T^r is an n_1-quasi-(m,q)-isometry and T^s is an n_2-quasi-(l,q)-isometry, then T^t is an n_0-quasi-(p,q)-isometry, where t is the greatest common divisor of r and s, $n_0 = \max\left(\frac{n_1 r}{t},\ \frac{n_2 s}{t}\right)$ and $p = \min(m,\ l)$.*

Proof. Since T^r is an n_1-quasi-(m,q)-isometry and T^s is an n_2-quasi-(l,q)-isometry on X, we deduce that T^r is a (m,q)-isometry on $\overline{\mathcal{R}(T^{rn_1})}$ and T^s is a (l,q)-isometry on $\overline{\mathcal{R}(T^{sn_2})}$. On the other hand, if we define t as the greatest common divisor of r and s, then

$$\overline{\mathcal{R}(T^{rn_1})} = \overline{\mathcal{R}((T^t)^{\frac{r}{t}n_1})} \quad \text{and} \quad \overline{\mathcal{R}(T^{sn_2})} = \overline{\mathcal{R}((T^t)^{\frac{s}{t}n_2})}.$$

Let $n_0 := \max\left(\frac{r}{t}n_1,\ \frac{s}{t}n_2\right)$, then $\overline{\mathcal{R}(T^{rn_1})} \supset \overline{\mathcal{R}(T^{tn_0})}$ and $\overline{\mathcal{R}(T^{sn_2})} \supset \overline{\mathcal{R}(T^{tn_0})}$. It follows that T^r is a (m,q)-isometry and T^s is a (l,q)-isometry on $\overline{\mathcal{R}(T^{tn_0})}$. By using [6, Theorem 3.6], we can easily show that T^t is a (p,q)-isometry on $\overline{\mathcal{R}(T^{tn_0})}$, where $p = \min(m,\ l)$. According to Proposition 2.1, we get that T^t is an n_0-quasi-(p,q)-isometry on X.

□

As an immediate consequence of Proposition 3.1, we have the following result.

Corollary 3.1. *Let $T \in \mathcal{B}(X)$ and r, s, m, n, l be positive integers, then the following properties hold.*

(1) If T is an n-quasi-(m,q)-isometry such that T^s is an n-quasi-$(1,q)$-isometry, then T is an ns-quasi-$(1,q)$-isometry.
(2) If T^r and T^{r+1} are n-quasi-(m,q)-isometries, then T is a $n(r+1)$-quasi-(m,q)-isometry.
(3) If T^r is an n-quasi-(m,q)-isometry and T^{r+1} is an n-quasi-(l,q)-isometry with $m < l$, then T is a $n(r+1)$-quasi-(m,q)-isometry.

T. Bermúdez, A. Martinón and J. A. Noda [7] have proved that if T is a (m,q)-isometry and S is a (l,q)-isometry with $TS = ST$, then ST is a $(m+l-1,q)$-isometry. In the following theorem, we will generalize this result for the class of n-quasi-(m,q) isometric operators.

Theorem 3.2. *Let T, $S \in \mathcal{B}(X)$ such that $TS = ST$. If T is an n_1-quasi-(m,q)-isometry and S is an n_2-quasi-(l,q)-isometry, then TS is an n-quasi-$(m+l-1,q)$-isometry, where $n = \max(n_1,n_2)$.*

Proof. For all $x \in X$, we have

$$\Delta^q_{m+l-1,n}(TS,x) \;=\; \sum_{k=0}^{m+l-1}(-1)^{m+l-1-k}\binom{m+l-1}{k}\|(TS)^{n+k}x\|^q.$$

Since $TS = ST$, we obtain

$$\begin{aligned}\Delta^q_{m+l-1,n}(TS,x) &= \sum_{k=0}^{m+l-1}(-1)^{m+l-1-k}\binom{m+l-1}{k}\|(TS)^k(\underbrace{(TS)^n(x)}_{=y})\|^q \quad (\forall x\in X)\\ &= \sum_{k=0}^{m+l-1}(-1)^{m+l-1-k}\binom{m+l-1}{k}\|(TS)^k(y)\|^q \quad (\forall y\in\mathcal{R}((TS)^n)).\end{aligned}$$

Likewise to the proof of Proposition 2.1, we deduce that

$$\sum_{k=0}^{m+l-1}(-1)^{m+l-1-k}\binom{m+l-1}{k}\|(TS)^k(y)\|^q \quad (\forall y\in\overline{\mathcal{R}((TS)^n)}).$$

Thanks to [7, Theorem 3.3], we have

$$\sum_{k=0}^{m+l-1}(-1)^{m+l-1-k}\binom{m+l-1}{k}\|T^kS^kx\|^q = 0 \text{ for all } x\in X,$$

in particular for all $x \in \overline{\mathcal{R}((TS)^n)}$. This implies that TS is a $(m+l-1,q)$-isometry on $\overline{\mathcal{R}((TS)^n)}$. Referring to Proposition 2.1, we get that TS is an n-quasi-$(m+l-1,q)$-isometry, with $n=\max(n_1,n_2)$.

□

The following example shows that Theorem 3.2 is not necessarily true if $TS \neq ST$.

Example 3.2. Let $q\geq 1$ and $T,\ S\in\mathcal{B}(l^q(\mathbb{N}))$ be the weighted shift operators defined by

$$T(\alpha_1,\alpha_2,\alpha_3,\cdots) = (0,w_1\alpha_1,w_2\alpha_2,w_3\alpha_3,\cdots),\ S(\alpha_1,\alpha_2,\alpha_3,\cdots) = (0,\gamma_1\alpha_1,\gamma_2\alpha_2,\gamma_3\alpha_3,\cdots),$$

with $w_k := \left(\frac{3k+4}{3k+1}\right)^{\frac{1}{q}}$ and $\gamma_k := \left(\frac{k+2}{k+1}\right)^{\frac{1}{q}}$. It is immediate to verify that $TS\neq ST$. The operators T and S are quasi-$(2,q)$-isometries. Indeed,

$$\begin{aligned}&\|T^3(\alpha_1,\alpha_2,\alpha_3,\cdots)\|^q - 2\|T^2(\alpha_1,\alpha_2,\alpha_3,\cdots)\|^q + \|T(\alpha_1,\alpha_2,\alpha_3,\cdots)\|^q\\ =& \sum_{n=1}^{+\infty}|w_nw_{n+1}w_{n+2}|^q|\ \alpha_n|^q - 2\sum_{n=1}^{+\infty}|w_nw_{n+1}|^q|\alpha_n|^q + \sum_{n=1}^{+\infty}|w_n|^q|\alpha_n|^q\\ =& \sum_{n=1}^{+\infty}\left(\frac{3n+10}{3n+1} - 2\frac{3n+7}{3n+1} + \frac{3n+4}{3n+1}\right)|\alpha_n|^q = 0,\end{aligned}$$

and

$$\begin{aligned}&\|S^3(\alpha_1,\alpha_2,\alpha_3,\cdots)\|^q - 2\|S^2(\alpha_1,\alpha_2,\alpha_3,\cdots)\|^q + \|S(\alpha_1,\alpha_2,\alpha_3,\cdots)\|^q\\ =& \sum_{n=1}^{+\infty}|\gamma_n\gamma_{n+1}\gamma_{n+2}|^q|\ \alpha_n|^q - 2\sum_{n=1}^{+\infty}|\gamma_n\gamma_{n+1}|^q|\alpha_n|^q + \sum_{n=1}^{+\infty}|\gamma_n|^q|\alpha_n|^q\\ =& \sum_{n=1}^{+\infty}\left(\frac{n+4}{n+1} - 2\frac{n+3}{n+1} + \frac{n+2}{n+1}\right)|\alpha_n|^q = 0.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
& \|(TS)^4(\alpha_1,\alpha_2,\alpha_3,\cdots)\|^q - 3\|(TS)^3(\alpha_1,\alpha_2,\alpha_3,\cdots)\|^q + 3\|(TS)^2(\alpha_1,\alpha_2,\alpha_3,\cdots)\|^q \\
& -\|TS(\alpha_1,\alpha_2,\alpha_3,\cdots)\|^q \\
= & \sum_{n=1}^{+\infty} |\gamma_n\gamma_{n+2}\gamma_{n+4}\gamma_{n+6}w_{n+1}w_{n+3}w_{n+5}w_{n+7}|^q|\,\alpha_n|^q \\
& -3\sum_{n=1}^{+\infty} |\gamma_n\gamma_{n+2}\gamma_{n+4}w_{n+1}w_{n+3}w_{n+5}|^q|\,\alpha_n|^q \\
& +3\sum_{n=1}^{+\infty} |\gamma_n\gamma_{n+2}w_{n+1}w_{n+3}|^q|\,\alpha_n|^q - \sum_{n=1}^{+\infty} |\gamma_n w_{n+1}|^q|\,\alpha_n|^q \\
= & \sum_{n=1}^{+\infty} \frac{-48(9n^3+90n^2+263n+238)}{(n+1)(n+3)(n+5)(n+7)(3n+4)(3n+10)(3n+16)(3n+22)}|\,\alpha_n|^q \\
\neq & \; 0,
\end{aligned}$$

which yields that TS is not a quasi-$(3,q)$-isometry.

In [11, Theorem 2.18] and for $q=2$, it was proven that if $S,\ T \in \mathcal{B}(\mathcal{H})$ are doubly commuting, T is a n_1-quasi-m-isometric and S is a n_2-quasi-l-isometric, then TS is a $\max\{n_1,n_2\}$-quasi-$(m+l-1)$-isometry. By using Theorem 3.2, we can show this result by assuming only that T and S are commuting.

Corollary 3.2. *Let $T,\ S \in \mathcal{B}(X)$ be commuting operators. If T is an n_1-quasi-(m,q)-isometry and S is an n_2-quasi-(l,q)-isometry, then T^tS^r is a $\max\{n_1,\ n_2\}$-quasi-$((m+l-1),q)$-isometry for all positive integers $t,\ r$.*

Proof. Since T is an n_1-quasi-(m,q)-isometry and S is an n_2-quasi-(l,q)-isometry, it follows from Theorem 3.1 that T^t is an n_1-quasi-(m,q)-isometry and S^r is an n_2-quasi-(l,q)-isometry for all positive integers $t,\ r$. Moreover, since $TS=ST$, we deduce that $T^tS^r=S^rT^t$. Referring to Theorem 3.2, it holds that T^tS^r is a $\max\{n_1,\ n_2\}$-quasi-$((m+l-1),q)$-isometry.

□

4. Dynamic of a n-quasi-(m,q)-isometric operator

In this section, we aim to study the supercyclicity of a n-quasi-(m,q)-isometry on a complex Banach space X.

Definition 4.1. Let X be a separable Banach space.

(1) The orbit of $E \subset X$ under T is defined by:

$$Orb(T,E) = \cup_{k=0}^{\infty} T^k(E).$$

(2) An operator $T \in \mathcal{B}(X)$ is said to be supercyclic, if $E=\mathrm{span}\langle x\rangle$ with supercyclic vector x such that

$$\overline{\mathbb{C}\ Orb(T,x)} := \overline{\{\lambda T^n x, \lambda \in \mathbb{C}, n \geq 0\}} = X.$$

(3) An operator $T \in \mathcal{B}(X)$ is said to be N-supercyclic with $N \geq 1$ if there exists a subspace $E \subset X$ with $dim(E) = N$ such that
$$\overline{Orb(T, E)} = X.$$

It's clear that a supercyclic operator is a one supercyclic, then we get the relation between the properties as following

$$\begin{array}{ccc} \text{supercyclic} & \Rightarrow & N - \text{supercyclic} \\ \Downarrow & & \\ \text{cyclic} & & \end{array}$$

In the following theorem we will investigate the supercycilcity of n-quasi-(m, q)-isometry operators.

Theorem 4.1. *On an infinite-dimensional Banach space X, an n-quasi-(m, q)-isometry is never supercyclic.*

Proof. We know for any operator $T \in \mathcal{B}(X)$ that $T\left(\overline{\mathcal{R}(T)}\right) \subset \overline{\mathcal{R}(T)}$. Let $T \in \mathcal{B}(T)$ be an n-quasi-(m, q)-isometry. We will discuss two cases:

- If $\mathcal{R}(T)$ is dense, then by the Proposition 2.1 we get that T is a (m, q)-isometry. According to [13, Thoerem 3.3], T is not an N-supercyclic operator for any $N \geq 1$, then T is not a supercyclic operator.
- If $\mathcal{R}(T)$ is not dense, then $\overline{\mathcal{R}(T)}$ is a nontrivial closed T-invariant subspace. By [15], T is not supercyclic operator.

□

5. Conclusions

In this article, we introduce and study n-quasi-(m, q)-isometric operators in Banach space settings. The supercyclicity and some fundamental properties, including the power and the product, of such operators are explored. As a future work, we can generalize our study on a metric space.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

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