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Research article

Properties of generalized (p, q)-elliptic integrals and generalized (p, q)-Hersch-Pfluger distortion function

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Abstract: In this paper, we focus on investigating various properties of generalized (p, q)-elliptic integrals and the generalized (p, q)-Hersch-Pfluger distortion function. We establish the complete monotonicity, logarithmic, geometric concavity, and convexity of certain functions involving these generalized integrals and arcsine functions. Additionally, we derive several precise inequalities for the generalized (p, q)-Hersch-Pfluger distortion function, which enhance and extend previous results.

Keywords: generalized (p, q)-elliptic integrals; generalized (p, q)-Hersch-Pfluger distortion function; convexity; concavity; monotonicity; logarithmic
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1. Introduction

In this paper, we primarily focus on investigating different properties of generalized (p, q)-elliptic integrals and the generalized (p, q)-Hersch-Pfluger distortion function. In recent years, mathematicians have made significant progress in studying inequalities and various properties related to complete elliptic integrals, especially Legendre elliptic integrals and generalized complete elliptic integrals of the first and second types [3, 4, 5, 10, 11, 23, 26].

We first introduce some necessary notation. For complex numbers a, b, c with $c \neq 0, -1, -2, \ldots$, and $x \in (-1, 1)$, the Gaussian hypergeometric function [9] is defined as follows:

$$F(a, b; c; x) = {}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)} \frac{x^n}{n!}, \tag{1.1}$$

where $(a, n) \equiv a(a+1)\cdots(a+n-1)$ is the shifted factorial function for $n \in \mathbb{N}^{+}$, and $(a, 0) = 1$ for $a \neq 0$. As we all know, ${}_2F_1(a, b; c; x)$ has many important applications in the theory of geometric functions and several other contexts [19]. Many special functions in mathematical physics are special or limit cases of this function [1]. Bhayo studied a new form of the generalized (p, q)-complete elliptic integrals as an application of generalized (p, q)-trigonometric functions [6]. In recent years, the generalization of classical trigonometric functions has attracted significant interest [20, 8]. For this, we need the generalized arcsine function $\arcsin_{p,q}(x)$ and the generalized $\pi_{p,q}$. For $p, q \in (1, \infty)$, set

$$\arcsin_{p,q}(x) \equiv \int_0^x \frac{\mathrm{d}t}{(1-t^q)^{1/p}}, \quad x \in [0,1],$$

and the generalized $\pi_{p,q}$ is the number defined by

$$\pi_{p,q} = 2\arcsin_{p,q}(1) \equiv 2\int_0^1 \frac{\mathrm{d}t}{(1-t^q)^{1/p}} = \frac{2}{q}B\left(1-\frac{1}{p}, \frac{1}{q}\right),$$

where B is the beta function. For $\mathrm{Re}x > 0$ and $\mathrm{Re}y > 0$, the classical gamma function $\Gamma(x)$ and beta function $B(x, y)$ are respectively defined as

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\mathrm{d}t, \quad B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Clearly, $\arcsin_{p,q}(x)$ is an increasing homeomorphism from $[0, 1]$ onto $[0, \pi_{p,q}/2]$, and its inverse function is the generalized (p, q)-sine function $\sin_{p,q}$ is defined on the interval $[0, \pi_{p,q}/2]$. Moreover, the function $\sin_{p,q}$ can be extended to the interval $[0, \pi_{p,q}]$ by

$$\sin_{p,q}(x) = \sin_{p,q}(\pi_{p,q} - x), \quad x \in [\pi_{p,q}/2, \pi].$$

$\sin_{p,q}$ can be also extended to the whole $\mathbb{R}$, and the generalized (p, q)-sine function reduces to the classical sine function for $p = q = 2$.

Applying the definitions of $\sin_{p,q}(x)$ and $\pi_{p,q}$, we can define the generalized (p, q)-elliptic integrals of the first kind $\mathscr{K}_{p,q}$ and of the second kind $\mathscr{E}_{p,q}$ by

$$\mathscr{K}_{p,q}(r) = \int_0^{\pi_{p,q}/2} \frac{\mathrm{d}t}{\left(1-r^q\sin_{p,q}^q(t)\right)^{1-1/p}} = \int_0^1 \frac{\mathrm{d}t}{(1-t^q)^{1/p}(1-r^qt^q)^{1-1/p}}$$

and

$$\mathscr{E}_{p,q}(r) = \int_0^{\pi_{p,q}/2} \left(1-r^q\sin_{p,q}^q(t)\right)^{1/p}\mathrm{d}t = \int_0^1 \left(\frac{1-r^qt^q}{1-t^q}\right)^{1/p}\mathrm{d}t$$

respectively, for $p, q \in (1, \infty)$, $r \in (0, 1)$.

As a special case of the Gaussian hypergeometric function, these generalized (p, q)-elliptic integrals can be represented by Gaussian hypergeometric functions [15] as

$$\begin{cases} \mathscr{K}_{p,q} = \mathscr{K}_{p,q}(r) = \dfrac{\pi_{p,q}}{2}{}_2F_1\left(1-\dfrac{1}{p}, \dfrac{1}{q}; 1-\dfrac{1}{p}+\dfrac{1}{q}; r^q\right) \\ \mathscr{K}'_{p,q} = \mathscr{K}'_{p,q}(r) = \mathscr{K}_{p,q}(r') \\ \mathscr{K}_{p,q}(0) = \dfrac{\pi_{p,q}}{2}, \quad \mathscr{K}_{p,q}(1) = \infty \end{cases} \tag{1.2}$$

and

$$\begin{cases} \mathscr{E}_{p,q} = \mathscr{E}_{p,q}(r) = \dfrac{\pi_{p,q}}{2} {}_2F_1\left(-\dfrac{1}{p}, \dfrac{1}{q}; 1 - \dfrac{1}{p} + \dfrac{1}{q}; r^q\right) \\ \mathscr{E}'_{p,q} = \mathscr{E}'_{p,q}(r) = \mathscr{E}_{p,q}(r') \\ \mathscr{E}_{p,q}(0) = \dfrac{\pi_{p,q}}{2}, \quad \mathscr{E}_{p,q}(1) = 1, \end{cases} \tag{1.3}$$

where $p, q \in (1, \infty)$, $r \in (0, 1)$, $r' = (1 - r^q)^{1/q}$. If $p = q = 2$, we can derive the classical complete elliptic integrals $\mathscr{K}$ and $\mathscr{E}$, which are well-known complete elliptic integrals of the first kind and second kind, respectively. These complete elliptic integrals play an important role in many branches of quasiconformal mapping, complex analysis, and physics.

In 2017, Yang et al. [24] showed that the ratio $\mathscr{K}(r)/\ln(c/r')$ is strictly concave if and only if $c = e^{4/3}$ on $(0, 1)$, and $\mathscr{K}(r)/\ln(1 + 4/r')$ is strictly convex on $(0, 1)$.

In 2019, Wang et al. [22] presented the convexity of the function $(\mathscr{E}'_a - r^2\mathscr{K}'_a)/r'^2$ and some properties of the function $\alpha\mathscr{K}_a(r^\alpha)$ with respect to the parameter α.

In 2020, Huang et al. [13] established monotonicity properties for certain functions involving the complete p-elliptic integrals of the first and second kinds. They also presented the inequality $\pi/2 - \log 2 + \log(1 + 1/r') + \alpha(1 - r') < \mathscr{K}_p(r) < \pi/2 - \log 2 + \log(1 + 1/r') + \beta(1 - r')$ which holds for all $r \in (0, 1)$ with the best possible constants α and β. Moreover, these generalized elliptic integrals have significant applications in the theory of geometric functions and in the theory of mean values. More properties and applications of these integrals are given in [7, 13, 17, 21, 22, 24, 25].

The generalized (p, q)-elliptic integrals of the first kind $\mathscr{K}_{p,q}$ and of the second kind $\mathscr{E}_{p,q}$ satisfy the following Legendre relation:

$$\mathscr{K}_{p,q}(r)\mathscr{E}'_{p,q}(r) + \mathscr{K}'_{p,q}(r)\mathscr{E}_{p,q}(r) - \mathscr{K}_{p,q}(r)\mathscr{K}'_{p,q}(r) = \frac{\pi_{p,q}}{2}. \tag{1.4}$$

This relation has important applications in many areas of mathematics and physics, including celestial mechanics, quantum mechanics, and statistical mechanics. when $p = q = 2$, the equation reduces to the classical Legendre relation.

Inspired by the papers [22], [24] and [13], we can consider extending the results of $\mathscr{K}(r)$, $\mathscr{K}_a(r)$ and $\mathscr{K}_p(r)$ to the generalized (p, q)-elliptic integrals of the first kind $\mathscr{K}_{p,q}$.

A generalized (p, q)-modular equation of degree $h > 0$ is

$$\frac{{}_2F_1(a, b; c; 1 - s^q)}{{}_2F_1(a, b; c; s^q)} = h\frac{{}_2F_1(a, b; c; 1 - r^q)}{{}_2F_1(a, b; c; r^q)}, \quad r \in (0, 1), \tag{1.5}$$

which $a, b, c > 0$ with $a + b \geq c$. Using the decreasing homeomorphism $\mu_{p,q} : (0, 1) \to (0, \infty)$ defined by

$$\mu_{p,q}(r) = \frac{\pi_{p,q}}{2}\frac{\mathscr{K}'_{p,q}(r)}{\mathscr{K}_{p,q}(r)},$$

for $p, q \in (1, \infty)$. The function $\mu_{p,q}$ is called the generalized (p, q)-Grӧtzsch ring function, we can rewrite (1.5) as

$$\mu_{p,q}(s) = h\mu_{p,q}(r), \quad r \in (0, 1). \tag{1.6}$$

The solution of (1.6) is given by

$$s = \varphi_K^{p,q}(r) = \mu_{p,q}^{-1}(\mu_{p,q}(r)/K). \tag{1.7}$$

For $p, q \in (1, \infty)$, $r \in (0, 1)$, $K \in (0, \infty)$, we have

$$\varphi_K^{p,q}(r)^q + \varphi_{1/K}^{p,q}(r')^q = 1. \tag{1.8}$$

The function $\varphi_K^{p,q}(r)$ is referred to as the generalized (p, q)-Hersch-Pfluger distortion function with degree $K = 1/h$. For $p = q = 2$, the functions $\mu_{p,q}(r)$ and $\varphi_K^{p,q}(r)$ reduces to well-known special cases that Grötzsch ring function $\mu(r)$ and Hersch-Pfluger distortion function $\varphi_K(r)$, respectively, which play important role in the theory of plane quasiconformal mappings.

In 2015, Alzer et al. [2] studied the monotonicity, convexity, and concavity properties of the function $\mu(r^\alpha)/\alpha$, and established various Grötzsch ring functional inequalities based on these properties.

In [3], the authors focused on studying the properties of the generalized Grötzsch ring function $\mu_a(r)$ and the generalized Hersch-Pfluger distortion function $\varphi_K^a(r)$. They derived various inequalities involving $\mu_a(r)$ and $r^{-1/K}\varphi_K^a(r)$ by utilizing the monotonicity, concavity, and convexity of the functions $\mu_a(r)$, $r^{-1/K}\varphi_K^a(r)$, $\log\left(1/\varphi_K^a(e^{-x})\right)$, and $\varphi_K^a(rx)/\varphi_K^a(x)$.

In 2022, Lin et al. [16] explored the monotonicity and convexity properties of the function $\mu_{p,q}(r)$ and obtained sharp functional inequalities that sharpen and extend some existing results on the modulus of $\mu(r)$.

Inspired by the papers [3], [12], and [16], our motivation is to extend the existing results of the functions $\varphi_K(r)$ and $\varphi_K^a(r)$ to the generalized (p, q)-Hersch-Pfluger distortion function. Our goal is to gain the properties of the generalized (p, q)-Hersch-Pfluger distortion function and derive sharp functional inequalities for this function.

Our main objective of this paper is to investigate various properties of the generalized (p, q)-elliptic integrals and the generalized (p, q)-Hersch-Pfluger distortion function. We specifically focus on establishing complete monotonicity, logarithmic, geometric concavity, and convexity properties of certain functions involving these generalized integrals and arcsine functions. Additionally, they derive several sharp functional inequalities for the generalized (p, q)-Hersch-Pfluger distortion function, which improve upon and generalize existing results. Apart from the introduction, this paper consists of three additional sections. Section 2 contains some preliminaries as well as several formulas and lemmas. In Section 3, we present some of the major results regarding the generalized (p, q)-elliptic integrals and provide their proof. In Section 4, we study the generalized (p, q)-Hersch-Pfluger distortion function, and present some of the major results and provide their proof.

2. Preliminaries

In this section, we present several formulas and lemmas that have been extensively utilized in the paper. These formulas and lemmas play a crucial role in the analysis and proofs of the major results. Throughout this paper, we denote $p, q \in (1, \infty)$, $r \in (0, 1)$ and $r' = (1 - r^q)^{1/q}$.

Lemma 2.1. [16] *Derivative formulas:*

$$(1)\ \frac{\mathrm{d}\mathscr{K}_{p,q}}{\mathrm{d}r}=\frac{\mathscr{E}_{p,q}-r'^{q}\mathscr{K}_{p,q}}{rr'^{q}},$$
$$(2)\ \frac{\mathrm{d}\mathscr{E}_{p,q}}{\mathrm{d}r}=\frac{q(\mathscr{E}_{p,q}-\mathscr{K}_{p,q})}{pr},$$
$$(3)\ \frac{\mathrm{d}\mu_{p,q}(r)}{\mathrm{d}r}=-\frac{\pi_{p,q}^{2}}{4rr'^{q}\mathscr{K}_{p,q}^{2}},$$
$$(4)\ \frac{\mathrm{d}\left(\mathscr{E}_{p,q}-r'^{q}\mathscr{K}_{p,q}\right)}{\mathrm{d}r}=\frac{(p-q)\left(\mathscr{K}_{p,q}-\mathscr{E}_{p,q}\right)+p(q-1)r^{q}\mathscr{K}_{p,q}}{pr},$$
$$(5)\ \frac{\mathrm{d}\left(\mathscr{K}_{p,q}-\mathscr{E}_{p,q}\right)}{\mathrm{d}r}=\frac{(p-qr'^{q})\mathscr{E}_{p,q}+(q-p)r'^{q}\mathscr{K}_{p,q}}{prr'^{q}}.$$

Based on the derivative formula Lemma 2.1, we derive the derivative formulas of the function $\varphi_K^{p,q}(r)$ in the following lemma.

Lemma 2.2. *Let* $p,q\in(1,\infty)$*, then*

$$\frac{\partial\varphi_K^{p,q}(r)}{\partial r}=\frac{ss'^{q}\mathscr{K}_{p,q}(s)\mathscr{K}'_{p,q}(s)}{rr'^{q}\mathscr{K}_{p,q}(r)\mathscr{K}'_{p,q}(r)}=\frac{1}{K}\frac{ss'^{q}\mathscr{K}_{p,q}(s)^{2}}{rr'^{q}\mathscr{K}_{p,q}(r)^{2}}=K\frac{ss'^{q}\mathscr{K}'_{p,q}(s)^{2}}{rr'^{q}\mathscr{K}'_{p,q}(r)^{2}}$$

for $r\in(0,1)$.

Proof. By the definitions of $s=\varphi_K^{p,q}(r)$, $\mu_{p,q}(s)=\mu_{p,q}(r)/K$, and the derivative formulas of $\mathscr{K}_{p,q}(r)$, we can derive the following equation

$$-\frac{\pi_{p,q}^{2}}{4ss'^{q}\mathscr{K}_{p,q}(s)^{2}}\cdot\frac{\partial s}{\partial r}=-\frac{1}{K}\frac{\pi_{p,q}^{2}}{4rr'^{q}\mathscr{K}_{p,q}(r)^{2}},$$

then the derivative formulas of $\varphi_K^{p,q}(r)$ is following. □

Lemma 2.3. [4] *For* $p\in[0,\infty)$*, let* $I=[0,p)$*, and suppose that* $f,g:I\to[0,\infty)$ *are functions such that* $f(x)/g(x)$ *is decreasing on* $I\setminus\{0\}$ *and* $g(0)=0$*,* $g(x)>0$ *for* $x>0$*. Then*

$$f(x+y)(g(x)+g(y))\le g(x+y)(f(x)+f(y)),$$

for $x,y,x+y\in I$*. Moreover, if the monotoneity of* $f(x)/g(x)$ *is strict, then the above inequality is also strict on* $I\setminus\{0\}$*.*

The following result is a monotone form of L'Hôpital's Rule [4] and will be useful in deriving monotoneity properties and obtaining inequalities.

Lemma 2.4. [4] *For* $-\infty<a<b<\infty$*, let* $f,g:[a,b]\to R$ *be continuous on* $[a,b]$*, and be differentiable on* (a,b)*. Let* $g'(x)\neq0$ *on* (a,b)*. If* $f'(x)/g'(x)$ *is increasing (decreasing) on* (a,b)*, then so are*

$$[f(x)-f(a)]/[g(x)-g(a)]\quad\textit{and}\quad[f(x)-f(b)]/[g(x)-g(b)].$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

The following lemma presents some known results of generalized (p,q)-elliptic integrals, which can be utilized to prove the main results of this paper.

Lemma 2.5. [14] *For $p,q \in (1,\infty)$, $r \in (0,1)$, $a = 1-1/p, b = a+1/q$, then the functions*

(1) $h_1(r) = \dfrac{\mathscr{E}_{p,q}(r) - r'^q \mathscr{K}_{p,q}(r)}{r^q}$ is strictly increasing and convex from $(0,1)$ onto $(a\pi_{p,q}/(2b), 1)$.

(2) $h_2(r) = \dfrac{\mathscr{E}_{p,q}(r) - r'^q \mathscr{K}_{p,q}(r)}{r^q \mathscr{K}_{p,q}(r)}$ is strictly decreasing from $(0,1)$ onto $(0, a/b)$.

(3) $h_3(r) = r'^c \mathscr{K}_{p,q}(r)$ is decreasing(increasing) on $(0,1)$ iff $c \geq a/b$ ($c \leq 0$ respectively) with $h_3((0,1)) = (0, \pi_{p,q}/2)$ if $c \geq a/b$.

(4) $h_4(r) = \dfrac{\mathscr{K}_{p,q}(r) - \mathscr{E}_{p,q}(r)}{r^q \mathscr{K}_{p,q}(r)}$ is increasing from $(0,1)$ onto $(1/(qb), 1)$.

(5) $h_5(r) = r'^q \mathscr{K}_{p,q}(r)/\mathscr{E}_{p,q}(r)$ is strictly decreasing from $(0,1)$ onto itself.

Lemma 2.6. *For $r \in (0,1)$, $K, p, q \in (1,\infty)$, let $s = \varphi_K^{p,q}(r), t = \varphi_{1/K}^{p,q}(r)$.*

(1) The function $f(r) = \mathscr{K}_{p,q}(s)/\mathscr{K}_{p,q}(r)$ is increasing from $(0,1)$ onto $(1, K)$.

(2) For $q > \frac{3p-4}{p-1}$, the function $g(r) = \dfrac{s'^{q/2} \mathscr{K}_{p,q}(s)^2}{r'^{q/2} \mathscr{K}_{p,q}(r)^2}$ is decreasing from $(0,1)$ onto $(0,1)$.

Proof. (1) According to the Lemma 2.1, we have

$$\mathscr{K}_{p,q}^2(r) f'(r) = \frac{\mathscr{K}_{p,q}(s)}{r r'^q \mathscr{K}'_{p,q}(r)} \left\{ \mathscr{K}'_{p,q}(s) \left[\mathscr{E}_{p,q}(s) - s'^q \mathscr{K}_{p,q}(s) \right] - \mathscr{K}'_{p,q}(r) \left[\mathscr{E}_{p,q}(r) - r'^q \mathscr{K}_{p,q}(r) \right] \right\}.$$

Denote

$$\begin{aligned} f_1(r) &= \mathscr{K}'_{p,q}(r) \left[\mathscr{E}_{p,q}(r) - r'^q \mathscr{K}_{p,q}(r) \right] \\ &= r^q \mathscr{K}'_{p,q}(r) \left[\frac{\mathscr{E}_{p,q}(r) - r'^q \mathscr{K}_{p,q}(r)}{r^q} \right]. \end{aligned}$$

By applying Lemma 2.5(1)(3) and considering $s > r$, we can conclude that $f_1(r)$ is increasing. Hence, we can determine that $f'(r)$ is positive. Therefore, we can deduce that $f(r)$ is an increasing function. For the limiting values, we have $\lim_{r \to 0^+} f(r) = 1$ and $\lim_{r \to 1^-} f(r) = K$.

(2) Let

$$\begin{aligned} g_1(r) &= \mathscr{K}'_{p,q}(r) \left[q r^q \mathscr{K}_{p,q}(r) - 4 \left(\mathscr{E}_{p,q}(r) - r'^q \mathscr{K}_{p,q}(r) \right) \right] \\ &= r^q \mathscr{K}'_{p,q}(r) \mathscr{K}_{p,q}(r) \left[q - 4 \frac{\mathscr{E}_{p,q}(r) - r'^q \mathscr{K}_{p,q}(r)}{r^q \mathscr{K}_{p,q}(r)} \right]. \end{aligned}$$

By differentiation, we have

$$[r'^{q/2} \mathscr{K}_{p,q}(r)^2]^2 g'(r) = -\frac{s'^{q/2} \mathscr{K}_{p,q}(s)^2 \mathscr{K}_{p,q}(r)}{2 r r'^{q/2} \mathscr{K}'_{p,q}(r)} \left(g_1(s) - g_1(r) \right).$$

If $q > \frac{3p-4}{p-1}$, $g_1(r)$ is increasing by Lemma 2.5(2) and (3), thus $g'(r)$ is negative for $s > r$. Hence $g(r)$ is decreasing, the limiting values follow from the definitions (1.2) and (1.7)

$$\lim_{r \to 0^+} g(r) = 1, \quad \lim_{r \to 1^-} g(r) = 0.$$

□

Lemma 2.7. *For $r \in (0,1)$ and $p,q \in (1,\infty)$, the inequality*

$$\frac{r^q K_{p,q}(r)^2}{N(r)} > \frac{1}{q}$$

holds.

Proof. Let $M(r) = r^q K_{p,q}(r)^2/N(r)$, $M_1(r) = N(r)/(r^q K_{p,q}(r))$, we have

$$\begin{aligned} M_1(r) &= \frac{1}{r^q K_{p,q}(r)} \left\{ \frac{q}{p} \mathscr{K}_{p,q}(r) \left(\mathscr{E}_{p,q}(r) - r'^q \mathscr{K}_{p,q}(r) \right) + \mathscr{E}_{p,q}(r) \left(\mathscr{K}_{p,q}(r) - \mathscr{E}_{p,q}(r) \right) \right. \\ &\quad \left. - \left(\frac{q}{p} - q + 1 \right) r^q \mathscr{K}_{p,q}(r) \mathscr{E}_{p,q}(r) \right\} \\ &= \frac{q}{p} \frac{\mathscr{E}_{p,q}(r) - r'^q \mathscr{K}_{p,q}(r)}{r^q} + \mathscr{E}_{p,q}(r) \frac{\mathscr{K}_{p,q}(r) - \mathscr{E}_{p,q}(r)}{r^q \mathscr{K}_{p,q}(r)} - \left(\frac{q}{p} - q + 1 \right) \mathscr{E}_{p,q}(r). \end{aligned}$$

According to Lemma 2.5, we have

$$M_1(r) < \frac{q}{p} + q\left(1 - \frac{1}{p}\right) \mathscr{E}_{p,q}(r) < \frac{q\pi_{p,q}}{2} \le q\mathscr{K}_{p,q}(r),$$

and

$$M(r) = K_{p,q}(r)/M_1(r) > 1/q.$$

□

Lemma 2.8. *For $r \in (0,1)$, $p,q \in (1,\infty)$, the function*

$$h(r) = \frac{1}{\log(1/r)} - \frac{\mathscr{E}_{p,q}(r) - r'^q \mathscr{K}_{p,q}(r)}{r'^q K_{p,q}(r)}$$

is strictly increasing from $(0,1)$ onto $(0,\infty)$.

Proof. By differentiation, we have

$$\begin{aligned} h'(r) &= \frac{1}{r\left(\log(1/r)\right)^2} - \frac{N(r)}{r\left(r'^q \mathscr{K}_{p,q}(r)\right)^2} \\ &= \frac{N(r)}{r^{q+1}\left(\log(1/r)\right)^2 \mathscr{K}_{p,q}(r)^2} \left[\frac{r^q K_{p,q}(r)^2}{N(r)} - \left(\frac{r^{q/2}}{r'^q} \log\left(\frac{1}{r}\right) \right)^2 \right], \end{aligned}$$

where $N(r) = \frac{q}{p} \mathscr{K}_{p,q}(r) \left(\mathscr{E}_{p,q}(r) - r'^q \mathscr{K}_{p,q}(r) \right) + \mathscr{E}_{p,q}(r) \left(\mathscr{K}_{p,q}(r) - \mathscr{E}_{p,q}(r) \right) - \left(\frac{q}{p} - q + 1 \right) r^q \mathscr{K}_{p,q}(r) \mathscr{E}_{p,q}(r)$.

Let $h_2(r) = r^{q/2}\log(1/r)$, $h_3(r) = r'^q$, then $h_1(r) = h_2(r)/h_3(r)$, and $h_2(1^-) = h_3(1^-) = 0$.

$$\frac{h_2'(r)}{h_3'(r)} = -\frac{1}{qr^{q/2}} \left[\frac{q}{2}\log(1/r) - 1 \right].$$

By Lemma 2.4, the function $h_1(r)$ is strictly increasing from $(0,1)$ onto $(0,1/q)$. According to Lemma 2.7, we conclude that $r^q K_{p,q}(r)^2/N(r) > 1/q$. Therefore, it is easy to check that $h(r)$ is increasing. For the limiting values, $\lim_{r\to 0^+} h(r) = 0$. Since

$$\lim_{r\to 1^-} r'^q \mathscr{K}_{p,q}(r) = 0, \quad \lim_{r\to 1^-} \frac{r'^q}{\log(1/r)} = q, \quad \lim_{r\to 1^-} \mathscr{K}_{p,q}(r) = \infty,$$

we have

$$\lim_{r\to 1^-} h(r) = \lim_{r\to 1^-} \frac{1}{r'^q \mathscr{K}_{p,q}(r)} \left\{ \frac{r'^q \mathscr{K}_{p,q}(r)}{\log(1/r)} - \left(\mathscr{E}_{p,q}(r) - r'^q \mathscr{K}_{p,q}(r)\right) \right\} = \infty.$$

□

3. Generalized (p,q)-elliptic integrals

In this section, we present some of the main results regarding the generalized (p,q)-elliptic integrals.

Theorem 3.1. *For $p,q \in (1,\infty)$, the function $F_{p,q}(r) = (\mathscr{E}'_{p,q}(r) - r^q \mathscr{K}'_{p,q}(r))/r'^q$ is concave on $(0, r_0^*)$ and convex on $(r_0^*, 1)$ for some point $r_0^* \in (0,1)$.*

Proof. Let $F(r) = (\mathscr{E}_{p,q}(r) - r'^q \mathscr{K}_{p,q}(r))/r^q$, by the definitions (1.2) and (1.3), which can be expressed as

$$\begin{aligned} F(r) &= \frac{\mathscr{E}_{p,q}(r) - r'^q \mathscr{K}_{p,q}(r)}{r^q} = \frac{\pi_{p,q}}{2r^q}\left[F\left(-\frac{1}{p}, \frac{1}{q}; 1 - \frac{1}{p} + \frac{1}{q}; r^q\right) - r'^q F\left(1 - \frac{1}{p}, \frac{1}{q}; 1 - \frac{1}{p} + \frac{1}{q}; r^q\right)\right] \\ &= \frac{\pi_{p,q}}{2r^q}\left[\sum_{n=0}^{\infty}\left(\frac{\left(-\frac{1}{p}, n\right)\left(\frac{1}{q}, n\right)}{\left(1 - \frac{1}{p} + \frac{1}{q}, n\right)n!} - \frac{\left(1-\frac{1}{p}, n\right)\left(\frac{1}{q}, n\right)}{\left(1 - \frac{1}{p} + \frac{1}{q}, n\right)n!}\right) r^{qn} + \sum_{n=0}^{\infty} \frac{\left(1-\frac{1}{p}, n\right)\left(\frac{1}{q}, n\right)}{\left(1 - \frac{1}{p} + \frac{1}{q}, n\right)n!} r^{q(n+1)}\right] \\ &= \frac{\pi_{p,q}}{2}\sum_{n=0}^{\infty} \frac{\left(1-\frac{1}{p}, n\right)\left(\frac{1}{q}, n\right)}{\left(1 - \frac{1}{p} + \frac{1}{q}, n+1\right)(n+1)!}\left(1 - \frac{1}{p}\right)(n+1) r^{qn} = \frac{a\pi_{p,q}}{2(a+b)} {}_2F_1(a, b; a+b+1; r^q), \end{aligned}$$

with $a = 1 - 1/p, b = 1/q$. This implies that

$$F_{p,q}(r) = \frac{\mathscr{E}'_{p,q}(r) - r^q \mathscr{K}'_{p,q}(r)}{r'^q} = \frac{a\pi_{p,q}}{2(a+b)} {}_2F_1(a, b; a+b+1; r'^q).$$

By differentiation, we have

$$F'_{p,q}(r) = -\frac{a^2 b q \pi_{p,q}}{2(a+b)(a+b+1)} r^{q-1} {}_2F_1(a+1, b+1; a+b+2; r'^q).$$

Hence,

$$\begin{aligned} &-\frac{2(a+b)(a+b+1)}{a^2 b q \pi_{p,q}} F''_{p,q}(r) \\ &= (q-1) r^{q-2} {}_2F_1(a+1, b+1; a+b+2; r'^q) - \frac{q(a+1)(b+1)}{a+b+2} r^{2q-2} {}_2F_1(a+2, b+2; a+b+3; r'^q). \end{aligned}$$

According to [4, Theorem 1.19(10)],

$$ {}_2F_1(a,b;c;x) = (1-x)^{c-a-b}{}_2F_1(c-a,c-b;c;x) \quad (a+b>c, a,b,c>0), $$

we have

$$ \begin{aligned} &-\frac{2(a+b)(a+b+1)}{a^2bq\pi_{p,q}}F''_{p,q}(r) \\ &= (q-1)r^{q-2}{}_2F_1(a+1,b+1;a+b+2;r'^q) - \frac{q(a+1)(b+1)}{a+b+2}r^{q-2}{}_2F_1(a+1,b+1;a+b+3;r'^q) \\ &= (q-1)r^{q-2}{}_2F_1(a+1,b+1;a+b+3;r'^q)\left[\frac{{}_2F_1(a+1,b+1;a+b+2;r'^q)}{{}_2F_1(a+1,b+1;a+b+3;r'^q)} - \frac{q}{q-1}\frac{(a+1)(b+1)}{a+b+2}\right]. \end{aligned} \tag{3.1} $$

Let

$$ F_1(x) = \frac{{}_2F_1(a+1,b+1;a+b+2;x)}{{}_2F_1(a+1,b+1;a+b+3;x)} = \frac{\sum_{n=0}^{\infty} a_n x^n}{\sum_{n=0}^{\infty} b_n x^n}, $$

it is easy to obtain that $F_1(x)$ is strictly increasing from $(0,1)$ onto $(1,\infty)$, since

$$ \frac{a_n}{b_n} = \frac{a+b+2+n}{a+b+2} > 1. $$

Similarly, we can deduce that

$$ r \mapsto \frac{{}_2F_1(a+1,b+1;a+b+2;r'^q)}{{}_2F_1(a+1,b+1;a+b+3;r'^q)} - \frac{q}{q-1}\frac{(a+1)(b+1)}{a+b+2} $$

is strictly decreasing from $(0,1)$ onto $\left(\frac{1-b(2a+b+2)}{(1-b)(a+b+2)},\infty\right)$. Therefore, the sign of $F''_{p,q}(r)$ changes from negative to positive on $(0,1)$ by (3.1), we know that there exists $r_0^* \in (0,1)$ such that $F_{p,q}(r)$ is concave on $(0,r_0^*)$ and convex on $(r_0^*,1)$. $\square$

Theorem 3.2. *For $p,q \in (1,\infty)$, $r \in (0,1)$, $\alpha > 0$, Let $H_{p,q}(\alpha) = \alpha\mathscr{K}_{p,q}(r^\alpha)$, $G_{p,q}(\alpha) = \mathscr{K}_{p,q}(r^\alpha)/\alpha$. Then*

(1) The function $\alpha \mapsto H_{p,q}(\alpha)$ is strictly increasing and log-concave on $(0,\infty)$;

(2) The function $\alpha \mapsto 1/H_{p,q}(\alpha)$ is strictly convex on $(0,\infty)$;

(3) The function $\alpha \mapsto G_{p,q}(\alpha)$ is strictly decreasing and log-convex on $(0,\infty)$.

Proof. (1) Let $t = r^\alpha$, then $\mathrm{d}t/\mathrm{d}\alpha = t\log r < 0$, and

$$ \begin{aligned} \frac{\mathrm{d}H_{p,q}(\alpha)}{\mathrm{d}\alpha} &= \mathscr{K}_{p,q}(t) + \alpha\frac{\mathscr{E}_{p,q}(t) - t'^q\mathscr{K}_{p,q}(t)}{t'^q}\log r \\ &= \mathscr{K}_{p,q}(t) - \frac{\mathscr{E}_{p,q}(t) - t'^q\mathscr{K}_{p,q}(t)}{t'^q}\log\left(\frac{1}{t}\right) \\ &= \mathscr{K}_{p,q}(t)\log\left(\frac{1}{t}\right)\left[\frac{1}{\log(1/t)} - \frac{\mathscr{E}_{p,q}(t) - t'^q\mathscr{K}_{p,q}(t)}{t'^q K_{p,q}(t)}\right]. \end{aligned} $$

Hence, the monotonicity of $H_{p,q}(\alpha)$ follows from Lemma 2.8.

By logarithmic differentiation,

$$\frac{\mathrm{d}\left(\log H_{p,q}(\alpha)\right)}{\mathrm{d}\alpha} = \frac{1}{\alpha} - \frac{\mathscr{E}_{p,q}(t) - t'^q \mathscr{K}_{p,q}(t)}{t'^q K_{p,q}(t)} \log\left(\frac{1}{r}\right)$$
$$= \log\left(\frac{1}{r}\right)\left[\frac{1}{\log(1/t)} - \frac{\mathscr{E}_{p,q}(t) - t'^q \mathscr{K}_{p,q}(t)}{t'^q K_{p,q}(t)}\right].$$

It is not difficult to verify that $\mathrm{d}(\log H_{p,q}(\alpha))/\mathrm{d}\alpha$ is strictly increasing with respect to t by Lemma 2.8, and is strictly decreasing with respect to α. Thus the function $\alpha \mapsto H_{p,q}(\alpha)$ is strictly increasing and log-concave on $(0, \infty)$.

(2) Since $t = r^\alpha$, then $\alpha = \log(1/t)/\log(1/r)$. Differentiating $1/H_{p,q}(\alpha)$ yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\frac{1}{H_{p,q}(\alpha)}\right) &= -\frac{1}{H_{p,q}(\alpha)^2}\mathscr{K}_{p,q}(t)\log\left(\frac{1}{t}\right)\left[\frac{1}{\log(1/t)} - \frac{\mathscr{E}_{p,q}(t) - t'^q \mathscr{K}_{p,q}(t)}{t'^q K_{p,q}(t)}\right] \\ &= -\frac{1}{\alpha^2 \mathscr{K}_{p,q}(t)^2} K_{p,q}(t)\log\left(\frac{1}{t}\right)h(t) \qquad (3.2) \\ &= -\left(\log\frac{1}{r}\right)^2 \frac{h(t)}{\log(1/t)K_{p,q}(t)},\end{aligned}$$

where $h(t)$ is defined in Lemma 2.8. Let $f(r) = \log(1/r)K_{p,q}(r)$, $f_1(r) = \log(1/r)$, $f_2(r) = 1/K_{p,q}(r)$, We clearly see that $f_1(1) = f_2(1) = 0$, then

$$\frac{f_1'(r)}{f_2'(r)} = \frac{r'^q K_{p,q}(r)^2}{\mathscr{E}_{p,q}(r) - r'^q \mathscr{K}_{p,q}(r)} = \frac{r^q}{\mathscr{E}_{p,q}(r) - r'^q \mathscr{K}_{p,q}(r)} \cdot \frac{r'^q K_{p,q}(r)^2}{r^q},$$

which is decreasing follows from Lemma 2.5(1),(3). Hence the function $f(r)$ is decreasing from $(0, 1)$ onto $(0, \infty)$ by Lemma 2.4. Therefore, it follows from Lemma 2.8 and (3.2) that the function $\alpha \mapsto 1/H_{p,q}(\alpha)$ is strictly convex on $(0, \infty)$.

(3) Since $t = r^\alpha$, and $\mathrm{d}t/\mathrm{d}\alpha = t\log r < 0$, simple computations yields

$$\begin{aligned}\frac{\mathrm{d}G_{p,q}(\alpha)}{\mathrm{d}\alpha} &= \frac{1}{\alpha}\left[\frac{\mathscr{E}_{p,q}(t) - t'^q \mathscr{K}_{p,q}(t)}{t'^q}\log r - \frac{\mathscr{K}_{p,q}(t)}{\alpha}\right] \\ &= \frac{1}{\alpha}\left[\frac{\mathscr{E}_{p,q}(t) - t'^q \mathscr{K}_{p,q}(t)}{t'^q}\log r - \frac{\mathscr{K}_{p,q}(t)\log r}{\log t}\right] \\ &= \frac{\mathscr{K}_{p,q}(t)\log r}{\alpha}\left[\frac{\mathscr{E}_{p,q}(t) - t'^q \mathscr{K}_{p,q}(t)}{t'^q \mathscr{K}_{p,q}(t)} + \frac{1}{\log(1/t)}\right].\end{aligned}$$

Let $g(r) = \dfrac{\mathscr{E}_{p,q}(r) - r'^q \mathscr{K}_{p,q}(r)}{r'^q K_{p,q}(r)} + \dfrac{1}{\log(1/r)}$, we have

$$g'(r) = \frac{N(r)}{r^{q+1}\left(\log(1/r)\right)^2 \mathscr{K}_{p,q}(r)^2}\left[\frac{r^q K_{p,q}(r)^2}{N(r)} + \left(\frac{r^{q/2}}{r'^q}\log\left(\frac{1}{r}\right)\right)^2\right],$$

is positive by Lemma 2.8, where the defition of $N(r)$ is in Lemma 2.8. Hence the function $g(r)$ is increasing from $(0, 1)$ onto $(0, \infty)$. Since $\log r < 0$, the monotonicity of the function $G_{p,q}(\alpha)$ follows immediately.

Since $\log G_{p,q}(\alpha) = \log \mathscr{K}_{p,q}(t) - \log \alpha$, by differentiation, we obtain that

$$\begin{aligned}
\frac{\mathrm{d}\left(\log G_{p,q}(\alpha)\right)}{\mathrm{d}\alpha} &= \frac{\mathscr{E}_{p,q}(t) - t'^q \mathscr{K}_{p,q}(t)}{t'^q K_{p,q}(t)} \log r - \frac{1}{\alpha} \\
&= \frac{\mathscr{E}_{p,q}(t) - t'^q \mathscr{K}_{p,q}(t)}{t'^q K_{p,q}(t)} \log r - \frac{\log r}{\log t} \\
&= \log r \left[\frac{\mathscr{E}_{p,q}(t)}{t'^q K_{p,q}(t)} - 1 + \frac{1}{\log(1/t)} \right]. \qquad (3.3)
\end{aligned}$$

It follows from (3.3) and Lemma 2.5(5) that $\mathrm{d}\left(\log G_{p,q}(\alpha)\right)/\mathrm{d}\alpha$ is strictly decreasing with respect to t. Therefore, $G_{p,q}(\alpha)$ is log-convex on $(0, \infty)$ with respect to α. □

Next, we apply Theorem 3.2 to obtain the inequality involving the generalized (p, q)-elliptic integrals $\mathscr{K}_{p,q}$.

Corollary 3.1. *For $p, q \in (1, \infty)$.*

(1) Let α,β be positive numbers with $\alpha > \beta > 0$. The double inequality

$$1 < \frac{\mathscr{K}_{p,q}(r^\beta)}{\mathscr{K}_{p,q}(r^\alpha)} < \frac{\alpha}{\beta}$$

holds for all $r \in (0, 1)$.

(2) Inequality

$$\mathscr{K}_{p,q}(\sqrt{xy}) \geq 2\frac{\sqrt{\log(1/x)\log(1/y)}}{\log[1/(xy)]} \sqrt{\mathscr{K}_{p,q}(x)} \sqrt{\mathscr{K}_{p,q}(y)}$$

holds with equality if and only if $x = y$ for all $x, y \in (0, 1)$.

(3) Inequality

$$\frac{4}{\log[1/(xy)]\mathscr{K}_{p,q}\left(\sqrt{xy}\right)} \leq \frac{1}{\log(1/x)\mathscr{K}_{p,q}(x)} + \frac{1}{\log(1/y)\mathscr{K}_{p,q}(y)}$$

holds with equality if and only if $x = y$ for all $x, y \in (0, 1)$.

(4) Let α,β be positive numbers with $\alpha > \beta > 0$. The double inequality

$$\frac{\beta}{\alpha} < \frac{\mathscr{K}_{p,q}(r^\beta)}{\mathscr{K}_{p,q}(r^\alpha)}$$

holds for all $r \in (0, 1)$.

(5) Inequality

$$\mathscr{K}_{p,q}(\sqrt{xy}) \leq \frac{1}{2} \log\left(\frac{1}{xy}\right) \sqrt{\frac{\mathscr{K}_{p,q}(x)\mathscr{K}_{p,q}(y)}{\log(1/x)\log(1/y)}}$$

holds with equality if and only if $x = y$ for all $x, y \in (0, 1)$.

Proof. (1) By utilizing the monotonicity of the function $H_{p,q}(\alpha)$ stated in Theorem 3.2, along with the monotonicity of the function $\mathscr{K}_{p,q}(r)$, we can establish that $\alpha\mathscr{K}_{p,q}(r^{\alpha}) > \beta\mathscr{K}_{p,q}(r^{\beta})$. Consequently, the double inequality holds.

(2) Since the function $\alpha \mapsto H_{p,q}(\alpha)$ is strictly log-concave on $(0, \infty)$, we can deduce that

$$\log H_{p,q}\left(\frac{\alpha+\beta}{2}\right) \geq \frac{1}{2}\left(\log H_{p,q}(\alpha) + \log H_{p,q}(\beta)\right) \Rightarrow H_{p,q}\left(\frac{\alpha+\beta}{2}\right) \geq \sqrt{H_{p,q}(\alpha)H_{p,q}(\beta)}$$

with equality if and only if $\alpha = \beta$ for $\alpha, \beta > 0$. For $x, y \in (0, 1)$ and set

$$\alpha = \frac{\log(1/x)}{\log(1/r)}, \quad \beta = \frac{\log(1/y)}{\log(1/r)}.$$

Simple computations yields

$$H_{p,q}(\alpha) = \alpha\mathscr{K}_{p,q}(r^{\alpha}) = \frac{\log(1/x)}{\log(1/r)}\mathscr{K}_{p,q}(x), \quad H_{p,q}(\beta) = \beta\mathscr{K}_{p,q}(r^{\beta}) = \frac{\log(1/y)}{\log(1/r)}\mathscr{K}_{p,q}(y), \tag{3.4}$$

$$H_{p,q}\left(\frac{\alpha+\beta}{2}\right) = \frac{1}{2}\frac{\log[1/(xy)]}{\log(1/r)}\mathscr{K}_{p,q}(\sqrt{xy}). \tag{3.5}$$

Hence, the inequality

$$\mathscr{K}_{p,q}(\sqrt{xy}) \geq 2\frac{\sqrt{\log(1/x)\log(1/y)}}{\log[1/(xy)]}\sqrt{\mathscr{K}_{p,q}(x)}\sqrt{\mathscr{K}_{p,q}(y)}$$

hold with equality if and only if $x = y$.

(3) Since the function $\alpha \mapsto 1/H_{p,q}(\alpha)$ is strictly convex on $(0, \infty)$, we get

$$\frac{1}{H_{p,q}\left(\frac{\alpha+\beta}{2}\right)} \leq \frac{1}{2}\left(\frac{1}{H_{p,q}(\alpha)} + \frac{1}{H_{p,q}(\beta)}\right) \tag{3.6}$$

with equality if and only if $\alpha = \beta$ for $\alpha, \beta > 0$. Set

$$\alpha = \frac{\log(1/x)}{\log(1/r)}, \quad \beta = \frac{\log(1/y)}{\log(1/r)}.$$

From (3.4)–(3.6), we conclude that the inequality hold with equality if and only if $x = y$.

(4) Since the function $\alpha \mapsto G_{p,q}(\alpha)$ is strictly decreasing, and the monotonicity of the function $\mathscr{K}_{p,q}(r)$, we have

$$\frac{\mathscr{K}_{p,q}(r^{\alpha})}{\alpha} < \frac{\mathscr{K}_{p,q}(r^{\beta})}{\beta}.$$

(5) Since the function $\alpha \mapsto G_{p,q}(\alpha)$ is log-convex on $(0, \infty)$,

$$\log G_{p,q}\left(\frac{\alpha+\beta}{2}\right) \leq \frac{1}{2}\left(\log G_{p,q}(\alpha) + \log G_{p,q}(\beta)\right) \Rightarrow G_{p,q}\left(\frac{\alpha+\beta}{2}\right) \leq \sqrt{G_{p,q}(\alpha)G_{p,q}(\beta)}$$

with equality if and only if $\alpha = \beta$ for $\alpha, \beta > 0$. Set

$$\alpha = \frac{\log(1/x)}{\log(1/r)}, \quad \beta = \frac{\log(1/y)}{\log(1/r)}.$$

Simple computations yields

$$G_{p,q}(\alpha) = \frac{\mathscr{K}_{p,q}(r^\alpha)}{\alpha} = \frac{\log(1/r)}{\log(1/x)}\mathscr{K}_{p,q}(x), \quad G_{p,q}(\beta) = \frac{\mathscr{K}_{p,q}(r^\beta)}{\beta} = \frac{\log(1/r)}{\log(1/y)}\mathscr{K}_{p,q}(y),$$

$$G_{p,q}\left(\frac{\alpha+\beta}{2}\right) = \frac{2\log(1/r)}{\log[1/(xy)]}\mathscr{K}_{p,q}(\sqrt{xy}).$$

Hence, the inequality

$$\mathscr{K}_{p,q}(\sqrt{xy}) \le \frac{1}{2}\log\left(\frac{1}{xy}\right)\sqrt{\frac{\mathscr{K}_{p,q}(x)\mathscr{K}_{p,q}(y)}{\log(1/x)\log(1/y)}}$$

hold with equality if and only if $x = y$. □

Remark 3.1. *In [22], Wang et al. provided the proof for the convexity of the function* $(\mathscr{E}_a' - r^2\mathscr{K}_a')/r'^2$ *and presented certain properties of the functions* $\alpha\mathscr{K}_a(r^\alpha)$ *and* $1/\alpha\mathscr{K}_a(r^\alpha)$ *with respect to the parameter* α*. It is worth noting that Theorem 3.1 and Theorem 3.2(1),(2) can be reduced to [22, Theorem 1.1, Theorem 1.3] if* $p = q = 1/a$*.*

4. Generalized (p, q)-Hersch-Pfluger distortion function

In this section, we study the complete monotonicity, logarithmic, geometric concavity and convexity of the generalized (p, q)-Hersch-Pfluger distortion function, and present some of the main results about $\varphi_K^{p,q}$.

Theorem 4.1. *For* $K, p, q \in (1, \infty)$*, and* $q > \frac{3p-4}{p-1}$*, let* $a = 1 - 1/p, b = 1/q$*,* f, g *be defined on* $(0, 1]$ *by*

$$f(r) = r^{-1/K}\varphi_K^{p,q}(r), g(r) = r^{-K}\varphi_{1/K}^{p,q}(r).$$

Then f *is decreaseing from* $(0, 1]$ *onto* $\left[1, e^{bR(a,b)(1-1/K)}\right)$*, and* g *is increasing from* $(0, 1]$ *onto* $\left(e^{bR(a,b)(1-K)}, 1\right]$*.*

Proof. Let $s = \varphi_K^{p,q}(r)$, then $f(r) = \dfrac{s}{r^{1/K}}$, we have

$$(r^{1/K})^2 f'(r) = \frac{s}{K}r^{1/K-1}\left[\left(\frac{s'^{q/2}\mathscr{K}_{p,q}(s)}{r'^{q/2}\mathscr{K}_{p,q}(r)}\right)^2 - 1\right].$$

Hence,

$$\frac{f'(r)}{f(r)} = \frac{1}{Kr}\left[\left(\frac{s'^{q/2}\mathscr{K}_{p,q}(s)}{r'^{q/2}\mathscr{K}_{p,q}(r)}\right)^2 - 1\right].$$

According to Lemma 2.6(2), $f'(r)$ is negative. Combine with $f(1^-) = 1$ and by [18, Theorem 2],

$$\begin{aligned}\lim_{r\to 0^+}\log\left(r^{-1/K}s\right) &= \lim_{r\to 0^+}\left[\left(\mu_{p,q}(s)+\log s\right)-\frac{1}{K}\left(\mu_{p,q}(r)+\log r\right)\right]\\ &= bR(a,b)\left(1-\frac{1}{K}\right).\end{aligned}$$

Let $t = \varphi_{1/K}^{p,q}(r)$, thus $r = \varphi_K^{p,q}(t)$ and

$$g(r) = \varphi_K^{p,q}(t)^{-K}\cdot t = \left(t^{-1/K}\varphi_K^{p,q}(t)\right)^{-K} = f(t)^{-K}.$$

According to the monotonicity of $f(r)$, $g(r)$ is increasing on $(0, 1]$. The limiting values can also be derived from [18, Theorem 2]. □

Next, we utilize Theorem 4.1 to derive the inequality concerning the generalized (p, q)-Hersch-Pfluger distortion function $\varphi_K^{p,q}$.

Corollary 4.1. *For $K, p, q \in (1,\infty)$, and $q > \frac{3p-4}{p-1}$, let $a = 1 - 1/p, b = 1/q$, then*

(1) The double inequality

$$\left|\varphi_K^{p,q}(r)-\varphi_K^{p,q}(s)\right| \le \varphi_K^{p,q}\left(|r-s|\right) \le e^{bR(a,b)(1-1/K)}|r-s|^{1/K} \tag{4.1}$$

hold with equality if and only if $r = s$.

(2) The double inequality

$$\left|\varphi_{1/K}^{p,q}(r)-\varphi_{1/K}^{p,q}(s)\right| \ge \varphi_{1/K}^{p,q}\left(|r-s|\right) \ge e^{bR(a,b)(1-K)}|r-s|^{K} \tag{4.2}$$

hold with equality if and only if $r = s$.

Proof. (1) According to Lemma 2.3 and the monotonicity of the function $f(r) = r^{-1/K}\varphi_K^{p,q}(r)$, we can conclude that

$$\varphi_K^{p,q}(x+y) \le \varphi_K^{p,q}(x)+\varphi_K^{p,q}(y)$$

for $x, y \in (0, 1)$. Set $r = x + y$ and $s = y$, we get

$$\left|\varphi_K^{p,q}(r)-\varphi_K^{p,q}(s)\right| \le \varphi_K^{p,q}\left(|r-s|\right).$$

According to $f(r)$ is decreaseing from $(0, 1]$ onto $\left[1, e^{bR(a,b)(1-1/K)}\right)$, we obtain that

$$\varphi_K^{p,q}\left(|r-s|\right) \le e^{bR(a,b)(1-1/K)}|r-s|^{1/K}$$

with equality if and only if $r = s$.

(2) By Lemma 2.3 and the monotonicity of $g(r)$, we have

$$\varphi_{1/K}^{p,q}(x)+\varphi_{1/K}^{p,q}(y) \le \varphi_{1/K}^{p,q}(x+y)$$

for $x, y \in (0,1)$. Set $r = x + y$ and $s = y$, we obtain that

$$\left|\varphi_{1/K}^{p,q}(r) - \varphi_{1/K}^{p,q}(s)\right| \geq \varphi_{1/K}^{p,q}(|r-s|)$$

with equality if and only if $r = s$.

According to $g(r)$ increasing from $(0,1]$ onto $\left(e^{bR(a,b)(1-K)}, 1\right]$, we get

$$\varphi_{1/K}^{p,q}(|r-s|) \geq e^{bR(a,b)(1-K)}|r-s|^K$$

with equality if and only if $r = s$. Therefore, the double inequality (4.2) hold. □

Theorem 4.2. *For $K, p, q \in (1,\infty)$, $q > \frac{3p-4}{p-1}$, the function $f(x) = \log\left(1/\varphi_K^{p,q}(e^{-x})\right)$ is increasing and convex on $(0,\infty)$, $g(x) = \log\left(1/\varphi_{1/K}^{p,q}(e^{-x})\right)$ is increasing and concave on $(0,\infty)$, and*

$$\varphi_K^{p,q}(r)\varphi_K^{p,q}(t) \leq \left(\varphi_K^{p,q}\left(\sqrt{rt}\right)\right)^2,$$
$$\varphi_{1/K}^{p,q}(r)\varphi_{1/K}^{p,q}(t) \geq \left(\varphi_{1/K}^{p,q}\left(\sqrt{rt}\right)\right)^2,$$

with equality if and only if $K = 1$ for each $r, t \in (0,1)$.

Proof. Let $r = e^{-x}$, $s = \varphi_K^{p,q}(r)$, according to the Lemma 2.6, we have

$$f'(x) = \frac{1}{K}\left(\frac{s'^{q/2}\mathscr{K}_{p,q}(s)}{r'^{q/2}\mathscr{K}_{p,q}(r)}\right)^2$$

is positive and increasing with respect to x. Thus f is increasing and convex. Therefore,

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}\left(f(x) + f(y)\right),$$

and putting $r = e^{-x}, t = e^{-y}$, we obtain

$$\varphi_K^{p,q}(r)\varphi_K^{p,q}(t) \leq \left(\varphi_K^{p,q}\left(\sqrt{rt}\right)\right)^2,$$

with equality if and only if $K = 1$ for each $r, t \in (0,1)$. The proof for $g(x)$ follows a similar approach. □

Theorem 4.3. *For $K, p, q \in (1,\infty)$, $q > \frac{3p-4}{p-1}$, $r \in (0,1)$, the function $f(x) = \varphi_K^{p,q}(rx)/\varphi_K^{p,q}(x)$ is increasing from $(0,1)$ onto $\left(r^{1/K}, \varphi_K^{p,q}(r)\right)$, while function $g(x) = \varphi_{1/K}^{p,q}(rx)/\varphi_{1/K}^{p,q}(x)$ is decreasing from $(0,1)$ onto $\left(\varphi_{1/K}^{p,q}(r), r^K\right)$. In particular,*

$$\varphi_K^{p,q}(rt) \leq \varphi_K^{p,q}(r)\varphi_K^{p,q}(t),$$
$$\varphi_{1/K}^{p,q}(rt) \geq \varphi_{1/K}^{p,q}(r)\varphi_{1/K}^{p,q}(t),$$

with equality if and only if $K = 1$ for each $r, t \in (0,1)$.

Proof. Let $t = rx, u = \varphi_K^{p,q}(t), s = \varphi_K^{p,q}(x)$,

$$f'(x) = \frac{u}{Ksx}\left[\left(\frac{u'^{q/2}\mathcal{K}_{p,q}(u)}{t'^{q/2}\mathcal{K}_{p,q}(t)}\right)^2 - \left(\frac{s'^{q/2}\mathcal{K}_{p,q}(s)}{x'^{q/2}\mathcal{K}_{p,q}(x)}\right)^2\right],$$

then

$$\frac{f'(x)}{f(x)} = \frac{1}{Kx}\left[\left(\frac{u'^{q/2}\mathcal{K}_{p,q}(u)}{t'^{q/2}\mathcal{K}_{p,q}(t)}\right)^2 - \left(\frac{s'^{q/2}\mathcal{K}_{p,q}(s)}{x'^{q/2}\mathcal{K}_{p,q}(x)}\right)^2\right].$$

Since $t < x$ and $\frac{s'^{q/2}\mathcal{K}_{p,q}(s)}{x'^{q/2}\mathcal{K}_{p,q}(x)}$ is decreasing with respect to r by Lemma 2.6, $f'(x)$ is positive on $(0, 1)$. For the limiting values, by using L'Hôpital's Rule, we get

$$\lim_{r\to 0^+} f(r) = r^{1/K}, \quad \lim_{r\to 1^-} h(r) = \varphi_K^{p,q}(r).$$

Since the monotonicity of the function $f(x)$, along with the definition of the function $\varphi_K^{p,q}(r)$, we obtain

$$\varphi_K^{p,q}(rt) \le \varphi_K^{p,q}(r)\varphi_K^{p,q}(t),$$

with equality if and only if $K = 1$ for each $r, t \in (0, 1)$. The proof for $g(x)$ follows a similar approach. As a result, we will omit the detailed proof. □

Theorem 4.4. *For $K, p \in (1, \infty)$, $r \in (0, 1)$, $q > 3$ and $p > \frac{q(q-3)}{q^2-2q-1}$, the function $f(r)$ defined by*

$$f(r) = \frac{\arcsin\left(\varphi_K^{p,q}(r)\right)}{\arcsin\left(r^{1/K}\right)}$$

is strictly decreasing from $(0, 1]$ into $\left[1, e^{(1-1/K)bR(a,b)}\right)$, the function $g(r)$ defined by

$$g(r) = \frac{\arcsin\left(\varphi_{1/K}^{p,q}(r)\right)}{\arcsin\left(r^{K}\right)}$$

is strictly increasing from $(0, 1]$ into $\left(e^{(1-K)bR(a,b)}, 1\right]$, where $a = 1 - 1/p, b = a + 1/q$.

Proof. Let $s = \varphi_K^{p,q}(r)$, $f_1(r) = \arcsin(s)$, $f_2(r) = \arcsin\left(r^{1/K}\right)$, $f(r) = f_1(r)/f_2(r)$. Since $f_1(0) = f_2(0) = 0$, we have

$$\frac{f_1'(r)}{f_2'(r)} = \frac{s}{r^{1/K}}\left(\frac{1 - r^{2/K}}{1 - r^2}\right)^{1/2} \frac{s'^{q-1}\mathcal{K}_{p,q}(s)^2}{r'^{q-1}\mathcal{K}_{p,q}(r)^2}.$$

Let $f_3(r) = \frac{s'^{q-1}\mathcal{K}_{p,q}(s)^2}{r'^{q-1}\mathcal{K}_{p,q}(r)^2}$, according to the Lemma 2.1 , we have

$$[r'^{q-1}\mathcal{K}_{p,q}(r)^2]^2 f_3'(r) = -\frac{s'^{q-1}\mathcal{K}_{p,q}(s)^2\mathcal{K}_{p,q}(r)}{rr'\mathcal{K}_{p,q}'(r)}\left(f_4(s) - f_4(r)\right),$$

where

$$f_4(r) = \mathscr{K}'_{p,q}(r)\left[(q-1)r^q\mathscr{K}_{p,q}(r) - 2\left(\mathscr{E}_{p,q}(r) - r'^q\mathscr{K}_{p,q}(r)\right)\right]$$
$$= r^q\mathscr{K}'_{p,q}(r)\mathscr{K}_{p,q}(r)\left[(q-1) - 2\frac{\mathscr{E}_{p,q}(r) - r'^q\mathscr{K}_{p,q}(r)}{r^q\mathscr{K}_{p,q}(r)}\right].$$

Using Lemma 2.5(2) and (3), we can observe that $f_4(r)$ is positive. Consequently, we can deduce that $f_3(r)$ is decreasing. As a result, we obtain that $f(r)$ is decreasing on the interval $(0, 1]$ by Theorem 4.1. Furthermore, it can be deduced that the monotonicity of $g(r)$ is similar to that of $f(r)$. □

Remark 4.1. *Theorems 4.1 to 4.4 can be seen as variations and extensions of the results presented in [3, Theorem 1.14, Theorem 1.15, Theorem 6.7, Theorem 6.13]. When $p = q = 1/a$, the results obtained in Theorems 4.1 to 4.4 can be reduced to those obtained in [3].*

5. Conclusions

In this paper, we investigate the properties of the generalized (p, q)-elliptic integrals and the generalized (p, q)-Hersch-Pfluger distortion function. Through our analysis, we have established complete monotonicity, logarithmic, geometric concavity, and convexity properties for certain functions involving these integrals and arcsine functions. These properties provide valuable insights into the behavior of these functions. Furthermore, we have derived several sharp functional inequalities for the generalized (p, q)-elliptic integrals and the generalized (p, q)-Hersch-Pfluger distortion function. These inequalities not only improve upon existing results but also generalize them.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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