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Research article

Existence theorems for the dbar equation and Sobolev estimates on q-convex domains

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Abstract: In this paper, we study a sufficient condition for subelliptic estimates in the weak Z(k) domain with C^3 boundary in an n-dimentionsl Stein manifold X. Consequently, the compactness of the $\overline{\partial}$-Neumann operator N on M is obtained and the closedness ranges of $\overline{\partial}$ and $\overline{\partial}^*$ are presented. The L^2-setting and the Sobolev estimates of N on M are proved. We study the $\overline{\partial}$ problem with support conditions in M for Ξ-valued (p, k) forms, where Ξ is the m-times tensor product of holomorphic line bundle $\Xi^{\otimes \mathrm{m}}$ for positive integer m. Moreover, the compactness of the weighted $\overline{\partial}$-Neumann operator is studied on an annular domain in a Stein manifold $M = M_1 \backslash \overline{M}_2$, between two smooth bounded domains M_1 and M_2 satisfy $\overline{M}_2 \Subset M_1$, M_1 is weak $Z(k)$, M_2 is weak $Z(n-1-k)$, $1 \leqslant k \leqslant n-2$ with $n \geqslant 3$. In all cases, the closedness of $\overline{\partial}$ and $\overline{\partial}^*$, global boundary regularity for $\overline{\partial}$ and $\overline{\partial}_b$ are studied.

Keywords: $\overline{\partial}$ and $\overline{\partial}$-Neumann operator; weakly q-convex domains
Mathematics Subject Classification: 32F10, 32W05, 32W10, 35J20, 35J60

1. Introduction

Several complex variables involve the $\overline{\partial}$ problem, and Kohn solved this problem in 1963 for strongly pseudoconvex domains. It is useful to use Sobolev estimates in various areas of mathematics, such as complex geometry and partial differential equations on pseudoconvex manifolds. Introducing a sequence of subelliptic multiplier ideals, he gave a sufficient condition for subellipticity in pseudoconvex domains with real analytical boundaries. Catlin proved the most general result regarding subelliptic estimates for the $\overline{\partial}$-Neumann problem. In [1], he showed that subelliptic estimates hold for k-forms at z_0 within a smooth and bounded pseudoconvex domain. Herbig [2] extended Catlin's result to a weak condition for boundedness in the sense of weight functions. Hörmander [3] and Folland-Kohn [4] proved that subelliptic $\frac{1}{2}$ estimate can be estimated on non-pseudoconvex domains. For more details, we refer the readers to [5–12].

We are motivated to give subelliptic estimates for the $\overline{\partial}$-Neumann problem on smooth bounded, weak Z(k) domains on a Stein manifold for (p, k)-forms, with $k \geqslant 1$ with values in holomorphic vector bundles. Sobolev estimates of N on M for all $\overline{\partial}$-closed (p, k)-forms. We also deduce some standard compactness consequences.

Further, if Ξ is the m-times tensor product of holomorphic line bundle $\Xi^{\otimes \mathrm{m}}$ for integer $m > 0$, we study the $\overline{\partial}$ problem with support conditions in M for Ξ-valued (p, k)-forms with values in $\Xi^{\otimes \mathrm{m}}$. This problem had already been discussed on domains like: Strongly q-convex (or concave) [13], pseudoconvex with C^1 boundary [14] and local Stein of the complex projective space [15]. We also refer the readers to [13, 16–23].

Finally, we assume that $M = M_1 \backslash \overline{M}_2$ is an annular domain in a Stein manifold, between two smooth bounded domains M_1 and M_2 satisfy $\overline{M}_2 \Subset M_1$, M_1 is weak $Z(k)$, M_2 is weak $Z(n-1-k)$, $1 \leqslant k \leqslant n-2$ with $n \geqslant 3$. We prove a basic prior estimate for the weighted $\overline{\partial}$-Neumann problem on M. This estimate is validated for vector bundle forms. Moreover, we also study the global boundary of $\overline{\partial}$ within such domains. Cho [24] says global boundary regularity is obtained when M_1 and M_2 are pseudoconvex manifolds. The boundary regularity and the closed range property of $\overline{\partial}$ were established in [14, 25, 26] for $0 < k < n-1$ and $n \geq 3$. There are also pseudoconvex and non-pseudoconvex domains in [15, 27, 28], as well as the author's results [29–39]. Similar results can be found in [40, 41].

The novelty of this study is the investigation of a sufficient condition for subelliptic estimates on the weak Z(k) domain. Moreover, we demonstrate that $\overline{\partial}$-Neumann operators are compact. In addition, we examine a weighted $\overline{\partial}$ Neumann operator over an annular domain between two smooth-bounded domains. Despite this, all results are obtained on weak Z(k) domains, which contrasts to previous works that were based on strong pseudoconvex domains and non-pseudoconvex domains.

2. Notations and definitions

Let $\mathrm{p}, \mathrm{k} \geq 0$, $\mathrm{n} \geqslant 1$ be an integer and let X be a complex manifold of dimension n. Let $M \Subset \mathrm{X}$ be a subset of X, and let ρ be its defining function. Let $T^{1,0}(bM)$ be the complex tangent bundle to the boundary bM, with $T^{0,1}(bM) = \overline{T^{1,0}(bM)}$. Suppose that Ξ^* is the dual of a holomorphic line bundle Ξ over X. In local coordinates $(z_j^1, z_j^2, \ldots, z_j^n)$ on open covering $\{\mathrm{V_j}\}_{j\in J}$ of X, $\Xi|_{V_j}$ is trivial. $\{f_{ab}\}$ is a transition function system of Ξ in sense of $\{\mathrm{V_j}\}_{j\in J}$. A (p, k) forms $\sigma = \{\sigma_j\}$ on X is given by:

$$\sigma_j = \sum_{C_\mathrm{p}, D_\mathrm{k}}{}' \sigma_{jC_\mathrm{p}\overline{D}_\mathrm{k}} dz_j^{C_\mathrm{p}} \wedge dz_j^{\overline{D}_\mathrm{k}},$$

where $C_\mathrm{p} = (c_1, \ldots, c_\mathrm{p})$ and $D_\mathrm{k} = (d_1, \ldots, d_\mathrm{k})$ are multiindices. A hermitian metric on X is

$$\mathrm{G} = \sum_{\sigma,\beta=1}^{n} \mathrm{g}_{j,\sigma\overline{\beta}}(z)\, dz_j^{\sigma}\, d\overline{z}_j^{\beta}.$$

Associate G with the differential form $\omega = \frac{\sqrt{-1}}{2}\sum_{\sigma,\beta=1}^{n} g_{j,\sigma\overline{\beta}}(z)\, dz_j^{\sigma} \wedge d\overline{z}_j^{\beta}$ of type $(1,1)$. $h = \{h_a\}$ is a hermitian metric of $\Xi = \{f_{ab}\}$ in sense of $\{V_a\}_{a\in J}$, so that $h_a = |f_{ab}|^2 h_b$ on $V_a \cap V_b$. $C^{\infty}_{p,k}(M,\Xi)$ is the complex vector space of C^{∞} Ξ-valued (p,k)-differential forms on M. $C^{\infty}_{p,k}(\overline{M},\Xi) = \{u|_{\overline{M}}; u \in C^{\infty}_{p,k}(X,\Xi)\}$. The space of Ξ-valued (p,k)-differential forms with compact support in M is denoted by $D_{p,k}(M,\Xi)$. $\divideontimes_{\Xi} : C^{\infty}_{p,k}(X,\Xi) \longrightarrow C^{\infty}_{k,p}(X,\Xi^*)$ is defined by $\divideontimes_{\Xi}\sigma = h\overline{\sigma}$, which commutes with the Hodge star operator $\star : C^{\infty}_{p,k}(X,\Xi) \longrightarrow C^{\infty}_{n-k,n-p}(X,\Xi)$. $\divideontimes_{\Xi^*} : C^{\infty}_{p,k}(X,\Xi^*) \longrightarrow C^{\infty}_{k,p}(X,\Xi)$ satisfies

$$\divideontimes_{\Xi^*}\sigma = \overline{(h)^*}\overline{\sigma} = \overline{{}^{t}(h)^{-1}}\overline{\sigma} = h\overline{\sigma} = \divideontimes_{\Xi}^{-1}\sigma,$$

with $\divideontimes_{\Xi^*}\sigma = \divideontimes_{\Xi}^{-1}\sigma$. $\mathscr{B}_{p,k}(\overline{M},\Xi) = \{\sigma \in C^{\infty}_{p,k}(\overline{M},\Xi);\ \star\ \divideontimes_{\Xi}\ \sigma|_{bM} = 0\}$. The volume element related to G is dV. $\overline{\partial} : C^{\infty}_{p,k-1}(M,\Xi) \longrightarrow C^{\infty}_{p,k}(M,\Xi)$ is the Cauchy-Riemann operator and ϑ its formal adjoint. $C^{\infty}_{p,k}(bM,\Xi) = C^{\infty}_{p,k}(\overline{M},\Xi)/\, D_{p,k}(M,\Xi)$. For $\sigma, \varrho \in C^{\infty}_{p,k}(X,\Xi)$,

$$(\sigma,\varrho)\, dV = \sigma_j \wedge \star\, h\overline{\varrho}_j = \sigma_j \wedge \star\ \divideontimes_{\Xi}\ \varrho_j,$$

is the inner product. For $\sigma, \varrho \in C^{\infty}_{p,k}(X,\Xi)$,

$$< \sigma, \varrho >_{C^{\infty}_{p,k}(\overline{M},\Xi)} = \int_{\overline{M}} \sigma \wedge \star\ \divideontimes_{\Xi}\ \varrho,$$
$$\|\sigma\|^2_{C^{\infty}_{p,k}(\overline{M},\Xi)} = < \sigma, \sigma >_{C^{\infty}_{p,k}(\overline{M},\Xi)},$$

are the global inner product and the norm, respectively. For $\sigma \in C^{\infty}_{p,k}(M,\Xi)$ and $\varrho \in D_{p,k-1}(M,\Xi)$, one obtains

$$\begin{aligned} < \vartheta\sigma, \varrho >_{C^{\infty}_{p,k}(\overline{M},\Xi)} &= < \sigma, \overline{\partial}\varrho >_{C^{\infty}_{p,k}(\overline{M},\Xi)}, \\ \vartheta &= -\ \divideontimes_{\Xi} \star \overline{\partial} \star\ \divideontimes_{\Xi}. \end{aligned} \tag{2.1}$$

$L^2_{p,k}(M,\Xi)$ is the Hilbert space obtained by completing the space $C^{\infty}_{p,k}(\overline{M},\Xi)$ under the norm $\|\sigma\|_M$. The maximal closed extension of $\overline{\partial}$ is $\overline{\partial} : L^2_{p,k-1}(M,\Xi) \longrightarrow L^2_{p,k}(M,\Xi)$, and $\overline{\partial}^*$ its Hilbert space adjoint. $\square = \square_{p,k} = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial} : \mathfrak{Dom}(\square_{p,k},\Xi) \longrightarrow L^2_{p,k}(M,\Xi)$ is the Laplace operator defined for Ξ-valued forms. $\mathfrak{Dom}(\square_{p,k},\Xi) = \{\sigma \in L^2_{p,k}(M,\Xi) : \sigma \in \mathfrak{Dom}(\overline{\partial},\Xi) \cap \mathfrak{Dom}(\overline{\partial}^*,\Xi), \overline{\partial}\sigma \in \mathfrak{Dom}(\overline{\partial}^*,\Xi)$ and $\overline{\partial}^*\sigma \in \mathfrak{Dom}(\overline{\partial},\Xi)\}$. $\mathcal{H}_{p,k}(\Xi) = \{\sigma \in \mathfrak{Dom}(\square_{p,k},\Xi) : \overline{\partial}\sigma = \overline{\partial}^*\sigma = 0\}$. $N = N_{p,k} : L^2_{p,k}(M,\Xi) \longrightarrow L^2_{p,k}(M,\Xi)$ is the $\overline{\partial}$-Neumann operator and is given as

$$N_{p,k}\,\sigma = \begin{cases} 0 & \text{if}\, \sigma \in \mathcal{H}_{p,k}(\Xi), \\ v\ \text{if}\, \sigma = \square_{p,k} v, & \text{and}\ v \perp \mathcal{H}_{p,k}(\Xi). \end{cases}$$

For $s \in \mathbb{R}$, the Sobolev Ξ-valued of (p,k)-forms is given by $W^s_{p,k}(M,\Xi)$ with $W^s(M,\Xi)$-coefficients and $\|\sigma\|_{W^s(\Xi)}$ their norms. The curvature form $\sum_{\sigma,\beta=1}^{n}\left(-\frac{\partial^2 \log h_j}{\partial z_j^{\sigma}\partial \overline{z}_j^{\beta}}\right) dz^{\sigma} \wedge d\overline{z}^{\beta}$ of Ξ provides a K*ä*hler metric $d\sigma^2 = \sum_{\sigma,\beta=1}^{n}\left(-\frac{\partial^2 \log h_j}{\partial z_j^{\sigma}\partial \overline{z}_j^{\beta}}\right) dz^{\sigma}\, d\overline{z}^{\beta}$ on V.

Definition 1. *A ϵ-subelliptic estimate for the $\overline{\partial}$-Neumann problem is satisfied at $z_0 \in \overline{M}$ on k-forms, $\epsilon > 0$, if there exists a constant $c > 0$ and a neighborhood $V \ni z_0$ such that*

$$\|\sigma\|^2_{W^{\epsilon}} \leqslant c\,(\|\overline{\partial}\sigma\|^2_{W^0} + \|\overline{\partial}^*\sigma\|^2_{W^0} + \|\sigma\|^2_{W^0}).$$

Definition 2. *[9, 10] A boundary bM is said to has the* $(\mathrm{k}-\mathrm{P})$ *property in* V *if for every* $\mathrm{T}>0$, *denote by* $\lambda_1^{\phi^{\mathrm{T}}} \leqslant \lambda_2^{\phi^{\mathrm{T}}} \leqslant ..... \leqslant \lambda_{n-1}^{\phi^{\mathrm{T}}}$ *the eigenvalues of the Levi form* (ϕ_{ij}^{T}), *there is a function* $\phi^{\mathrm{T}} \in C^{\infty}(\overline{M} \cap V)$ *with* $|\phi^{\mathrm{T}}| \leqslant 1$ *on* M *and so that*

$$\sum_{j=1}^{\mathrm{k}} \lambda_j^{\phi^{\mathrm{T}}} - \sum_{j=1}^{\mathrm{k}_0} \phi_{jj}^{\mathrm{T}} \geq c\left(\delta^{-2\epsilon} + \sum_{j=1}^{\mathrm{k}_0} |\phi_j|^2\right) \quad \text{on } \overline{M} \cap V,$$
$$\sum_{j=1}^{\mathrm{k}} \lambda_j^{\phi^{\mathrm{T}}} - \sum_{j=1}^{\mathrm{k}_0} \phi_{jj}^{\mathrm{T}} \geq C \text{ on } bM \cap V,$$

where $\epsilon > 0$ *and* $\mathrm{C} > 0$ *does not depend on* δ *and* s.

Define the Levi form $\mathscr{L}$ as: $\forall\ p \in bM$, with $\frac{\partial \zeta}{\partial z_j}(p) = 0\ \forall\ 1 \leqslant j \leqslant n-1$.

$$\mathscr{L}(\sigma, \sigma)(p) = \sum_{J \in \mathcal{I}_{k-1}} \sum_{j,k=1}^{n-1} \frac{\partial^2 \zeta}{\partial z_j \partial \bar{z}_k} \sigma_{kJ} \overline{\sigma}_{jJ}.$$

Definition 3. *[42] For* $1 \leqslant k \leqslant n-1$, *$bM$ is said satisfies weak* $\mathrm{Z}(k)$ *if there exists a real* $\Gamma \in T^{1,1}(bM)$ *satisfying*

(1) $|\gamma|^2 \geqslant (i\gamma \wedge \overline{\gamma})(\Gamma) \geqslant 0\ \forall\ \gamma \in \varrho^{1,0}(bM)$.

(2) $\mu\sigma_1 + \cdots + \mu\sigma_k - \mathscr{L}(\Gamma) \geqslant 0$ *where* $\mu\sigma_1, \ldots, \mu\sigma_{n-1}$ *are the eigenvalues of* $\mathscr{L}$.

(3) $M(\Gamma) \neq k$.

Lemma 1. *[42] For* $1 \leqslant k \leqslant n-2$, *let* $M \subset \mathrm{X}$ *be a bounded domain and* $B \subset \mathrm{X}$ *be a bounded pseudoconvex domain satisfies* $\overline{M} \subset B$. *Then bM satisfies weak* $\mathrm{Z}(k)$ *if and only if* $b(B/\overline{M})$ *satisfies weak* $\mathrm{Z}(n-k-1)$.

If $\mu\sigma_1, \ldots, \mu\sigma_{n-1}$ are the eigenvalues of $\mathscr{L}$, then one obtains

$$\mathscr{L}(\sigma, \sigma) \geqslant (\mu\sigma_1 + \cdots + \mu\sigma_k)\,|\sigma|^2.$$

Definition 4. *A form* $\sigma \in \mathrm{L}^2_{\mathrm{p,k}}(M, \Xi)$ *is supported in* $\overline{M}$ *if* σ *vanishes on* bM.

3. Main results

3.1. Subelliptic estimates

Theorem 1. *With a smooth boundary, let* $M \Subset \mathrm{C}^n$ *be a weak* $\mathrm{Z}(k)$ *domain. Suppose that bM has the property* $(\mathrm{k}-\mathrm{P})$. *Then,* ϵ*-subelliptic estimates at* z_0 *hold for* (p, k)*-forms. That is, there exists* $c > 0$ *such that*

$$\|\sigma\|^2_{\mathrm{W}^{\epsilon}(M)} \leqslant \mathrm{C}\left(\|\sigma\|^2_{\mathrm{L}^2_{\mathrm{p,k}}(M)} + \|\bar{\partial}\sigma\|^2_{\mathrm{L}^2_{\mathrm{p,k}}(M)} + \|\bar{\partial}^*\sigma\|^2_{\mathrm{L}^2_{\mathrm{p,k}}(M)}\right), \tag{3.1}$$

for $\sigma \in \mathrm{D}_{\mathrm{p,k}}(M)$.

Proof. Let $\mathrm{B}_\delta = \{z \in M : -\delta < \rho(z) \leqslant 0\}$ be a strip, where $\delta > 0$ small enough. As in Khanh and Zampieri [10],

$$\|\sigma\|^2_{\mathrm{W}^{\epsilon}(\mathrm{B}_\delta \cap M)} \leqslant \mathrm{C}\left(\|\sigma\|^2_{\mathrm{L}^2_{\mathrm{p,k}}(\mathrm{B}_\delta \cap M)} + \|\bar{\partial}\sigma\|^2_{\mathrm{L}^2_{\mathrm{p,k}}(\mathrm{B}_\delta \cap M)} + \|\bar{\partial}^*\sigma\|^2_{\mathrm{L}^2_{\mathrm{p,k}}(\mathrm{B}_\delta \cap M)}\right), \tag{3.2}$$

for $\sigma \in \mathrm{D}_{\mathrm{p,k}}(\mathrm{B}_\delta \cap M)$ with $k \geqslant 1$. From the compactness of bM, using a finite covering $\{\Delta_\phi\}_{\nu=1}$ of bM by neighborhoods Δ_ϕ as in (3.2), we have

$$\|\sigma\|^2_{\mathrm{W}^\epsilon(\mathrm{B}_\delta)} \leqslant \mathrm{C}(\|\sigma\|^2_{\mathrm{L}^2_{\mathrm{p,k}}(\mathrm{B}_\delta)} + \|\overline{\partial}\sigma\|^2_{\mathrm{L}^2_{\mathrm{p,k}}(\mathrm{B}_\delta)} + \|\overline{\partial}^*\sigma\|^2_{\mathrm{L}^2_{\mathrm{p,k}}(\mathrm{B}_\delta)}\Big), \tag{3.3}$$

with u is supported in B_δ.

If $\rho(z) \leqslant -\delta$ and $z \in M \setminus \mathrm{B}_\delta$, taking $\gamma_\delta \in \mathrm{D}(M)$ with $\gamma_\delta(z) = 1$. Using (3.3),

$$\begin{aligned}
\|\sigma\|^2_{\mathrm{W}^0(M)} &\leqslant \int_{\mathrm{B}_\delta} |\sigma|^2 dV + \|\gamma_\delta\sigma\|^2_{\mathrm{W}^0(\mathrm{B}_\delta)} \\
&\leqslant (\mathrm{C}_1 + 2\,\mathrm{C}_2\, st)(\|\overline{\partial}\sigma\|^2_{\mathrm{L}^2_{\mathrm{p,k}}(M)}) + \|\overline{\partial}^*\sigma\|^2_{\mathrm{L}^2_{\mathrm{p,k}}(M)}) + \|\sigma\|^2_{\mathrm{W}^0}) \\
&= (\mathrm{C}_1 + 2\,\mathrm{C}_2\, st)\Big(\|\overline{\partial}\sigma\|^2_{\mathrm{L}^2_{\mathrm{p,k}}(M)} + \|\overline{\partial}^*\sigma\|^2_{\mathrm{L}^2_{\mathrm{p,k}}(M)}\Big).
\end{aligned}$$

□

Theorem 2. *Let Let M, X be the same as in Theorem 1. Let Ξ be a holomorphic vector bundle, of rank r, on* X. *Suppose that bM has the property* (k − P). *Then, there exists $C > 0$ satisfies*

$$\|\sigma\|^2_{\mathrm{W}^\epsilon(\Xi)} \leqslant \mathrm{C}(\|\sigma\|^2_{\mathrm{L}^2(\Xi)} + \|\overline{\partial}\sigma\|^2_{\mathrm{L}^2(\Xi)} + \|\overline{\partial}^*\sigma\|^2_{\mathrm{L}^2(\Xi)}), \tag{3.4}$$

for $\sigma \in \mathrm{D}_{\mathrm{p,k}}(M, \Xi)$.

Proof. By a local patching, one assume that $\{U_j\}_{j=1}^N$ is a finite covering of bM. Extend the subelliptic estimate (3.1) to E-valued forms. An orthonormal basis could be $e_1, e_2, \ldots, e_r$ for $z \in U_j$; $j \in J$. Thus $\sigma(z) = \sum_{a=1}^r \sigma^a(z)\, e_a(z)$, where σ^a are the components of the restriction of σ on U_j. Let $\{\zeta_j\}_{j=0}^m$ be a partition unity. This partition of unity is $\zeta_0 \in \mathcal{D}_{\mathrm{p,k}}(M)$, $\zeta_j \in \mathcal{D}_{\mathrm{p,k}}(U_j)$, $j = 1, 2, ..., m$. $\sum_{j=0}^m \zeta_j^2 = 1$ on $\overline{M}$, where $\{U_j\}_{j=1,\ldots,m}$ is a covering of bM.

For a given $j_\nu \in I$, let U be a neighborhood of a given boundary point $\xi_0 \in bM$. Using $\sigma \in \mathcal{D}_{\mathrm{p,k}}(M, \Xi)$, $1 \leqslant k \leqslant n-2$, and $a = 1, ..., r$, we get a subelliptic estimate from (3.1), for $\sigma|_{M\cap U}$.

$$\|\zeta_0\sigma\|^2_{\mathrm{W}^\epsilon(M\cap U)} \lesssim C\, Q(\zeta_0\sigma, \zeta_0\sigma) \lesssim \epsilon Q(\sigma, \sigma).$$

Thus, subelliptic estimate for $\sigma|_{M\cap U_j}$ is

$$\|\zeta_j\sigma\|^2_{\mathrm{W}^\epsilon(M\cap U_j)} \lesssim C\, Q(\zeta_j\sigma, \zeta_j\sigma) \lesssim \epsilon Q(\sigma, \sigma).$$

Summing up over j, we get

$$\|\sigma\|^2_{\mathrm{W}^\epsilon(\Xi)} \leqslant c\, Q(\sigma, \sigma).$$

Thus (3.4) follows. □

3.2. Compactness estimates

Lemma 2. *[43] Let $M \Subset X$ be a weak* Z(k) *domain with* C^3 *boundary in* Stein manifold X.

(1) $\forall$ constant $\epsilon > 0$ there exists $t_\epsilon > 0$ and a $\mathrm{C}_\epsilon > 0$ satisfy $\forall\ t \geqslant t_\epsilon$ and $\sigma \in L^2_{\mathrm{p,k}}(M, e^{-\mathrm{t}\varrho}) \cap \mathfrak{Dom}(\overline{\partial}) \cap \mathfrak{Dom}\left(\overline{\partial}^*_{\mathrm{t}}\right)$ we have

$$\|\sigma\|^2_{L^2_{\mathrm{p,k}}(M, e^{-\mathrm{t}\varrho})} \leq \epsilon\left(\|\overline{\partial}\sigma\|^2_{L^2_{\mathrm{p,k}}(M, e^{-\mathrm{t}\varrho})} + \left\|\overline{\partial}^*_{\mathrm{t}}\sigma\right\|^2_{L^2_{\mathrm{p,k}}(M, e^{-\mathrm{t}\varrho})}\right) + \mathrm{C}_\epsilon\, \|\sigma\|^2_{\mathrm{t}, \mathrm{W}^{-1}(M)}. \tag{3.5}$$

(2) there exists constants $C > 0$ and $\tilde{t} > 0$ satisfy $\forall\ t \geqslant \tilde{t}$ and $\sigma \in L^2_{\mathrm{p,k}}(M, e^{-\mathrm{t}\varrho}) \cap \mathfrak{Dom}(\overline{\partial}) \cap \mathfrak{Dom}\left(\overline{\partial}^*_{\mathrm{t}}\right) \cap \left(\mathcal{H}^k_{\mathrm{t}}(M)\right)^{\perp}$ we have

$$\|\sigma\|^2_{L^2_{\mathrm{p,k}}(M,e^{-\mathrm{t}\varrho})} \leq C\left(\|\overline{\partial}\sigma\|^2_{L^2_{\mathrm{p,k}}(M,e^{-\mathrm{t}\varrho})} + \left\|\overline{\partial}^*_{\mathrm{t}}\sigma\right\|^2_{L^2_{\mathrm{p,k}}(M,e^{-\mathrm{t}\varrho})}\right).$$

(3) If bM is connected, $\forall$ constant $\epsilon > 0$ there exists $t_\epsilon > 0$ so that $\forall\ t \geqslant t_\epsilon$ and $\sigma \in L^2_{\mathrm{p,k}}(M, e^{-\mathrm{t}\varrho}) \cap \mathfrak{Dom}(\overline{\partial}) \cap \mathfrak{Dom}\left(\overline{\partial}^*_{\mathrm{t}}\right)$ we have

$$\|\sigma\|^2_{L^2_{\mathrm{p,k}}(M,e^{-\mathrm{t}\varrho})} \leq \epsilon\left(\|\overline{\partial}\sigma\|^2_{L^2_{\mathrm{p,k}}(M,e^{-\mathrm{t}\varrho})} + \left\|\overline{\partial}^*_{\mathrm{t}}\sigma\right\|^2_{L^2_{\mathrm{p,k}}(M,e^{-\mathrm{t}\varrho})}\right).$$

Theorem 3. *Let $M \Subset X$ be a weak $\mathrm{Z}(k)$ domain with C^3 boundary in* Stein manifold X. *Then the compactness estimate for a holomorphic vector bundle Ξ-valued $(\mathrm{p,k})$ form holds on M. Then, $\forall$ $c > 0$, there exists a $t > 0$ and $C_{c,t} > 0$ such that*

$$\|\sigma\|^2_{\mathrm{W}^0(\Xi)} \leq C\left(\|\overline{\partial}\sigma\|^2_{\mathrm{W}^0(\Xi)} + \|\overline{\partial}^*_{\mathrm{t}}\sigma\|^2_{\mathrm{W}^0(\Xi)} + \|\sigma\|^2_{\mathrm{W}^0(\Xi)}\right) + C_{\mathrm{t}}\,\|\sigma\|^2_{\mathrm{W}^{-1}(\Xi)}, \tag{3.6}$$

for $\sigma \in \mathscr{D}_{\mathrm{p,k}}(M, \Xi)$.

Proof. As Theorem 2, for $\sigma \in \mathscr{D}_{\mathrm{p,k}}(M, \Xi)$, $1 \leq k \leq n-2$, over each σ^a by applying (3.5) and adding for $a = 1, ..., r$, we get compactness estimate for $\sigma\ |_{M\cap U}$

$$\begin{aligned}\|\zeta_0\sigma\|^2_{\mathrm{t}} &\leq c\ \mathrm{Q}^{\mathrm{t}}(\zeta_0\sigma, \zeta_0\sigma) + C_{\mathrm{t}}\|\zeta_0\sigma\|^2_{\mathrm{W}^{-1}} \\ &\leq c\ \mathrm{Q}^{\mathrm{t}}(\sigma, \sigma) + C_{\mathrm{t}}\|\sigma\|_{\mathrm{W}^{-1}}.\end{aligned}$$

For $j = 1, ..., m$, for $u|_{M\cap\sigma_j}$, we have

$$\begin{aligned}\|\zeta_j\sigma\|^2_{\mathrm{t}} &\leq c\ \mathrm{Q}^{\mathrm{t}}(\zeta_j\sigma, \zeta_j\sigma) + C_{\mathrm{t}}\|\zeta_j u\|^2_{\mathrm{W}^{-1}} \\ &\leq c\ \mathrm{Q}^{\mathrm{t}}(\sigma, \sigma) + C_{\mathrm{t}}\|\sigma\|^2_{\mathrm{W}^{-1}}.\end{aligned}$$

Let's sum j up

$$\|\sigma\|^2_{\mathrm{t}} \leq c\ \mathrm{Q}^{\mathrm{t}}(\sigma, \sigma) + C_{\mathrm{t}}\,\|\sigma\|^2_{\mathrm{W}^{-1}(\Xi)}.$$

Thus (3.6) follows. □

Proposition 1. *Assuming the same assumptions as Theorem 3, let us assume the following:* $\ker(\Box, \Xi)$ *is* finite dimension*al and* $\mathrm{Ran}(\Box, \Xi)$ *is closed in* $\mathrm{L}^2_{\mathrm{p,k}}(M, \Xi)$ *and there exists a bounded linear operator* $\mathrm{N} : \mathrm{L}^2_{\mathrm{p,k}}(M, \Xi) \longrightarrow \mathrm{L}^2_{\mathrm{p,k}}(M, \Xi)$ *satisfies*

(i) $\mathrm{Ran}(\mathrm{N}, \Xi) \subset \mathfrak{Dom}(\Box, \Xi)$, *$N\Box = I - \mathbb{H}$ on $\mathfrak{Dom}(\Box, E)$,*

(ii) for $\sigma \in \mathrm{L}^2_{\mathrm{p,k}}(M, \Xi)$, $\sigma = \overline{\partial}\,\overline{\partial}^* N\sigma \oplus \overline{\partial}^*\overline{\partial}N\sigma \oplus \mathbb{H}\sigma$,

(iii) $N\overline{\partial} = \overline{\partial}N$, and $N\overline{\partial}^* = \overline{\partial}^* N$.

(iv) $\forall\ \sigma \in \mathrm{L}^2_{\mathrm{p,k}}(M, \Xi)$,

$$\begin{aligned}\|\,\mathrm{N}\,\sigma\|_{\mathrm{W}^0(\Xi)} &\leq C\|\sigma\|_{\mathrm{W}^0(\Xi)}, \\ \|\overline{\partial}\,\mathrm{N}\,\sigma\|_{\mathrm{W}^0(\Xi)} + \|\overline{\partial}^*\,\mathrm{N}\,\sigma\|_{\mathrm{W}^0(\Xi)} &\leq \sqrt{C}\|\sigma\|_{\mathrm{W}^0(\Xi)}.\end{aligned}$$

(v) If $\sigma \in \mathrm{L}^2_{\mathrm{p,k}}(M, \Xi)$, with $\overline{\partial}\sigma = 0$ (resp. $\overline{\partial}^*\sigma = 0$), then $\overline{\partial}^*\,\mathrm{N}\,\sigma$ (resp.$\overline{\partial}\,\mathrm{N}\,\sigma$) gives the solution of $\overline{\partial}u = \sigma$ (resp. $\overline{\partial}^* u = \sigma$) of minimal $\mathrm{L}^2_{\mathrm{p,k}-1}(M, \Xi)$(resp. $\mathrm{L}^2_{\mathrm{p,k}+1}(M, \Xi)$)-norm.

Proof. Applying (3.6) at $\epsilon = \frac{1}{2}$, for $\sigma \in \mathrm{L}^2_{\mathrm{p,k}}(M,\Xi) \cap \mathfrak{Dom}\,(\overline{\partial},\Xi) \cap \mathfrak{Dom}\,(\overline{\partial}^*,\Xi)$, one obtains

$$\|\sigma\|^2_{\mathrm{W}^{\frac{1}{2}}(\Xi)} \leqslant C\,(\|\sigma\|^2_{\mathrm{W}^0(\Xi)} + \|\overline{\partial}\sigma\|^2_{\mathrm{W}^0(\Xi)} + \|\overline{\partial}^*\sigma\|^2_{\mathrm{W}^0(\Xi)}). \tag{3.7}$$

Then, $\mathrm{N}_{\mathrm{p,k}} : \mathrm{L}^2_{\mathrm{p,k}}(M,\Xi) \longrightarrow \mathrm{W}^1_{\mathrm{p,k}}(M,\Xi)$. Following (3.7), every sequence $\{\sigma_\phi\}_{\phi=1}^\infty$ in $\mathfrak{Dom}\,(\overline{\partial},\Xi) \cap \mathfrak{Dom}\,(\overline{\partial}^*,\Xi)$ with $\|\sigma_\phi\|$ bounded, $\overline{\partial}\sigma_\phi \longrightarrow 0$ in $\mathrm{L}^2_{\mathrm{p,k+1}}(M,\Xi)$ and $\overline{\partial}^*\sigma_\phi \longrightarrow 0$ in $\mathrm{L}^2_{\mathrm{p,k-1}}(M,\Xi)$ as $\phi \longrightarrow \infty$, then (3.6) implies that $\left\|\sigma_\phi\right\|^2_{\mathrm{W}^{\frac{1}{2}}(\Xi)} \leqslant C$ for some constant C. Thus, the inclusion map $i_M : \mathrm{W}^{\frac{1}{2}}_{\mathrm{p,k}}(M,\Xi) \longrightarrow \mathrm{L}^2_{\mathrm{p,k}}(M,\Xi)$ is compact. By Rellich Lemma, a subsequence of the sequence σ_ϕ can be extracted which converges in the $\mathrm{L}^2_{\mathrm{p,k}}(M,\Xi)$-norm. Thus, following Theorem 1.1.3 in [3], for $\sigma \in \mathfrak{Dom}\,(\overline{\partial},\Xi) \cap \mathfrak{Dom}\,(\overline{\partial}^*,\Xi)$, $\sigma \perp \ker(\Box,\Xi)$, $\ker(\Box,\Xi)$ is finite dimensional and one obtains

$$\|\sigma\|^2_{\mathrm{W}^0(\Xi)} \leqslant C\,(\|\overline{\partial}\sigma\|^2_{\mathrm{W}^0(\Xi)} + \|\overline{\partial}^*\sigma\|^2_{\mathrm{W}^0(\Xi)}).$$

Then

$$\|\sigma\|^2_{\mathrm{W}^0(\Xi)} \leqslant C\,\|\Box\sigma\|^2_{\mathrm{W}^0(\Xi)}, \quad \text{for } \sigma \in \mathfrak{Dom}\,(\Box,\Xi), \sigma \perp \ker(\Box,\Xi). \tag{3.8}$$

Since $\Box$ is self-adjoint, thus following Theorem 1.1.1 in [3], one obtains

$$\mathrm{L}^2_{\mathrm{p,k}}(M,\Xi) = \mathrm{Ran}(\Box,\Xi) \oplus \ker(\Box,\Xi) = \overline{\partial}\,\overline{\partial}^*\,\mathfrak{Dom}\,(\Box,\Xi) \oplus \overline{\partial}^*\overline{\partial}\,\mathfrak{Dom}\,(\Box,\Xi) \oplus \ker(\Box,\Xi).$$

According to (3.8) there's a unique bounded operator N on $\mathrm{L}^2_{\mathrm{p,k}}(M,\Xi)$ that inverts $\Box$ on $\ker(\Box,\Xi)^\perp$. Extend N to the whole $\mathrm{L}^2_{\mathrm{p,k}}(M,\Xi)$ space by setting $N = 0$ on $\ker(\Box,\Xi)$. The rest of the proof follows Theorem 3.1.14 in [28]. □

Corollary 1. *Assuming the same assumptions as Theorem 3, we have the following:*

(i) the $\overline{\partial}$*-Neumann operator* N *exists and* $\mathrm{N} : \mathrm{L}^2_{\mathrm{p,k}}(M,\Xi) \longrightarrow \mathrm{W}^1_{\mathrm{p,k}}(M,\Xi)$.

(ii) For $\sigma \in \mathrm{W}^{\frac{1}{2}}_{\mathrm{p,k}}(M,\Xi)$, *there exists* $u \in \mathrm{W}^{\frac{1}{2}}_{\mathrm{p,k-1}}(M,\Xi)$ *with* $\overline{\partial}u = \sigma$.

(iii) $\mathrm{N} : \mathrm{L}^2_{\mathrm{p,k}}(M,\Xi) \longrightarrow \mathrm{L}^2_{\mathrm{p,k}}(M,\Xi)$ *is compact.*

Proof. (i) From (3.8), for $\sigma \in \mathrm{L}^2_{\mathrm{p,k}}(M,\Xi) \cap \mathfrak{Dom}\,(\overline{\partial},\Xi) \cap \mathfrak{Dom}\,(\overline{\partial}^*,\Xi)$,

$$\|\sigma\|^2_{\mathrm{W}^{\frac{1}{2}}(\Xi)} \leqslant C\Big(\|\overline{\partial}\sigma\|^2_{\mathrm{W}^0(\Xi)} + \|\overline{\partial}^*\sigma\|^2_{\mathrm{W}^0(\Xi)} + \|\sigma\|^2_{\mathrm{W}^0(\Xi)}\Big).$$

Thus, the existence of $\mathrm{N} : \mathrm{L}^2_{\mathrm{p,k}}(M,\Xi) \longrightarrow \mathrm{W}^1_{\mathrm{p,k}}(M,\Xi)$ follows.

(ii) From Eq (3.7), $\forall\,\sigma \in \mathrm{L}^2_{\mathrm{p,k}}(M,\Xi) \cap \ker(\overline{\partial},\Xi)$ and $\sigma \perp \ker(\Box,\Xi)$, there exists a $u \in \mathrm{W}^{\frac{1}{2}}_{\mathrm{p,k-1}}(M,\Xi)$ with $\overline{\partial}u = \sigma$.

(iii) To prove the compactness of N, since $N = 0$ on $\ker(\Box,\Xi)$, it suffices to show compactness on $\ker(\Box,\Xi)^\perp$. When $\sigma \in \ker(\Box,\Xi)^\perp$ and hence $N\sigma \in \ker(\Box,\Xi)^\perp$, the integration by parts, inequality (3.8) and the Cauchy-Schwarz inequality imply

$$\|\overline{\partial}\,\mathrm{N}\,\sigma\|^2_{\mathrm{W}^0(\Xi)} + \|\overline{\partial}^*N\sigma\|^2_{\mathrm{W}^0(\Xi)} = <\sigma,\mathrm{N}\,\sigma>_{\mathrm{W}^0(\Xi)} \leqslant \|\sigma\|_{\mathrm{W}^0(\Xi)}\|\,\mathrm{N}\,\sigma\|_{\mathrm{W}^0(\Xi)} \leqslant \|\sigma\|^2_{\mathrm{W}^0(\Xi)}. \tag{3.9}$$

Following (3.7)–(3.9), we get

$$\|\,\mathrm{N}\,\sigma\|^2_{\mathrm{W}^{\frac{1}{2}}(\Xi)} \leqslant C(\|\overline{\partial}\,\mathrm{N}\,\sigma\|^2_{\mathrm{W}^0(\Xi)} + \|\overline{\partial}^*N\sigma\|^2_{\mathrm{W}^0(\Xi)} + \|\,\mathrm{N}\,\sigma\|^2_{\mathrm{W}^0(\Xi)}) \leqslant K\|\sigma\|^2_{\mathrm{W}^0(\Xi)},$$

where K is a positive constant. Thus, by the Rellich Lemma, the compactness of N follows on $\mathrm{L}^2_{\mathrm{p,k}}(M,\Xi)$, that is, the embedding of $\mathrm{W}^{\frac{1}{2}}_{\mathrm{p,k}}(M,\Xi)$ into $\mathrm{L}^2_{\mathrm{p,k}}(M,\Xi)$ is compact. □

3.3. Global regularity and closed range for $\overline{\partial}$

Lemma 3. *Assuming the same assumptions as Theorem 3. Let* $1 \le \mathrm{k} \le n-2$*,* $n \ge 3$*, then there exists* $C > 0$ *satisfies for all* $\sigma \in \mathscr{D}^{\mathrm{p,k}}(M, \Xi)$ *with* $\sigma \perp \mathcal{H}_{\mathrm{t}}^{\mathrm{p,k}}(\Xi)$*, we have*

$$\|\sigma\|^2_{\mathrm{W}^0(\Xi)} \le C(\|\overline{\partial}\sigma\|^2_{\mathrm{W}^0(\Xi)} + \|\vartheta_{\mathrm{t}}\sigma\|^2_{\mathrm{W}^0(\Xi)}). \tag{3.10}$$

Proof. If for any $\nu \in \mathbb{N}$ there exists a $\sigma_\nu \perp \mathcal{H}_{\mathrm{t}}^{\mathrm{p,k}}(\Xi)$, with $\|\sigma_\nu\|_{\mathrm{t}} = 1$ so that

$$\|\sigma_\nu\|^2_{\mathrm{W}^0(\Xi)} \ge \nu\Big(\|\overline{\partial}\sigma_\nu\|^2_{\mathrm{W}^0(\Xi)} + \|\vartheta_{\mathrm{t}}\sigma_\nu\|^2_{\mathrm{W}^0(\Xi)}\Big).$$

Combining with (3.6), we have

$$\|\sigma_\nu\|^2_{\mathrm{W}^0(\Xi)} \le C\|\sigma_\nu\|^2_{\mathrm{W}^{-1}(\Xi)}.$$

Then $\sigma_\nu \to \sigma$ in L^2, where $\sigma \perp \mathcal{H}^{\mathrm{p,k}}_{\mathrm{W}^0(\Xi)}(\Xi)$. By (3.7) we have that $\sigma \in \mathcal{H}_{\mathrm{t}}^{\mathrm{p,k}}(\Xi)$, a contradiction. Thus (3.10) must hold $\forall\ \sigma \perp \mathcal{H}_{\mathrm{t}}^{\mathrm{p,k}}(\Xi)$. □

Using (3.10), as in [3, 28], we have

Lemma 4. *Assuming the same assumptions as Theorem 3. Let* $1 \le \mathrm{k} \le n-2$*,* $n \ge 3$*, then we have*
(1) $\mathcal{H}_{\mathrm{t}}^{\mathrm{p,k}}(\Xi)$ *is* finite dimension*al.*
(2) $\Box^{\mathrm{t}}$ *has closed range in* $\mathrm{L}^2_{\mathrm{p,k}}(M, \Xi)$*.*
(3) $\overline{\partial}$ *(resp.* $\overline{\partial}^*_{\mathrm{t}}$*) has closed range in* $\mathrm{L}^2_{\mathrm{p,k}}(M, \Xi)$ *and* $\mathrm{L}^2_{\mathrm{p,k+1}}(M, \Xi)$ *(resp.* $\mathrm{L}^2_{\mathrm{p,k-1}}(M, \Xi)$*).*

Proof. Following (3.10), every sequence $\{\sigma_\nu\}_{\nu=1}^\infty$ in $L^2_{\mathrm{p,k}}(M, \Xi)$ with $\|\sigma_\nu\|_{\mathrm{t}}$ is bounded and $\overline{\partial}\sigma_\nu \longrightarrow 0$, $\overline{\partial}^*_{\mathrm{t}}\sigma_\nu \longrightarrow 0$, one can extract a subsequence which converges in $L^2_{\mathrm{p,k}}(M, \Xi)$. Since $L^2_{\mathrm{p,k}}(M, \Xi) \hookrightarrow \mathrm{W}^{-1}_{\mathrm{p,k}}(M, \Xi)$ is compact, (3.7) implies that such a subsequence is convergent in $L^2_{\mathrm{p,k}}(M, \Xi)$. Following Theorems 1.1.3 and 1.1.2 in [3], implies that $\mathcal{H}_{\mathrm{t}}^{\mathrm{p,k}}(\Xi)$ is finite dimensional. Thus, $\overline{\partial} : \mathrm{L}^2_{\mathrm{p,k}}(M, \Xi) \longrightarrow \mathrm{L}^2_{\mathrm{p,k+1}}(M, \Xi)$ and $\overline{\partial}^*_{\mathrm{t}} : \mathrm{L}^2_{\mathrm{p,k}}(M, \Xi) \longrightarrow \mathrm{L}^2_{\mathrm{p,k-1}}(M, \Xi)$ have closed range. □

Theorem 4. *Assuming the same assumptions as Theorem 3. Let* $1 \le \mathrm{k} \le n-2$*,* $n \ge 3$*, then for* $\sigma \in \mathrm{C}^\infty_{\mathrm{p,k}}(\overline{M}, \Xi)$*, satisfying* $\overline{\partial}\sigma = 0$ *in the distribution sense in* X*, there exists* $u \in \mathrm{C}^\infty_{\mathrm{p,k-1}}(\overline{M}, \Xi)$*, satisfies* $\overline{\partial}u = \sigma$ *in* X*.*

Proof. The proof follows as [29, 30]. □

4. Solution of the dbar problem with support conditions

Proposition 2. *[43] Let* $M \Subset X$ *be a weak* $\mathrm{Z}(k)$ *domain with smooth boundary in a complex manifold* X*. Assume that* $\mathrm{s} > 0$ *is an integer,* $\overline{\nabla}$ *is the covariant differentiation of type* $(0, 1)$ *associated with the metric* G*, and* $\Xi^{\otimes \mathrm{s}}$ *is the* s*-times tensor product of a holomorphic line bundle* Ξ*. Suppose that there* *exists a strongly plurisubharmonic function on a neighborhood U^* of bM. We have*

$$\|\overline{\partial}\sigma\|^2_{L^2_{p,k}(M,\Xi^{\otimes s})} + \|\overline{\partial}^*_s\sigma\|^2_{L^2_{p,k}(M,\Xi^{\otimes s})} = \|\overline{\nabla}\sigma\|^2_{L^2_{p,k}(M,\Xi^{\otimes s})} + \int_{bM} h^s_j |\nabla \rho|^{-1} \sum_{\beta,\gamma=1}^{n} \frac{\partial^2\rho}{\partial z^\beta \partial z^{\overline{\gamma}}} \sigma^\beta_{jC_p\overline{B}_{k-1}} \overline{\sigma_j^{\overline{C}_p\gamma B_{k-1}}}\, dS + \int_s h^s_j \sum_{\beta,\gamma=1}^{n} \Big(\delta^\sigma_\epsilon [s\,\gamma^{\overline{\beta}}_{\overline{\sigma}} + R^{\overline{\beta}}_{\overline{\sigma}}] - R^{\sigma\overline{\beta}}_{\epsilon\overline{\sigma}}\Big) \times \sigma^\beta_{jC_p\overline{B}_{k-1}} \overline{\sigma_j^{\overline{C}_p\gamma B_{k-1}}}\, dV,$$

for $\sigma \in \mathcal{B}_{p,k}(\overline{M}, \Xi^{\otimes s})$, so that σ is supported in U^*, and $k \geqslant 1$, where

$$\|\overline{\nabla}\sigma\|^2_{L^2_{p,k}(M,\Xi^{\otimes s})} = \int_M \sum_{\sigma,\beta=1}^{n} g_j^{\overline{\beta}\sigma} \overline{\nabla}_\beta \sigma_{jC_p\overline{D}_s} \overline{\overline{\nabla}_\sigma \sigma_j^{\overline{C}_p D_k}}\, dV,$$

and

$$R^\sigma_{\beta\overline{\nu}\gamma} = -\frac{\partial}{\partial \overline{z}^\nu_j}\left(\sum g_j^{\overline{\sigma}\sigma} \frac{\partial}{\partial z^\gamma_j} g_{j\beta\overline{\sigma}}\right),$$

is the Riemann curvature tensor,

$$R_{\sigma\overline{\nu}} = -\frac{\partial^2}{\partial z^\sigma_j \partial \overline{z}^\nu_j}(log\ det\ g_{j\sigma\overline{\beta}}),$$

is the Ricci curvature tensor, and the curvature tensor of Ξ is given by

$$\gamma_{\sigma\overline{\nu}} = -\frac{\partial^2}{\partial z^\sigma_j \partial \overline{z}^\nu_j}(log\ h),$$

where δ^σ_ϵ is the Kronecker's delta.

Proposition 3. *[43] With the same assumptions as in Proposition 2, let us assume the following: There exists a constant $C > 0$ not depending on s and an integer $s_0 > 0$ so that for all $s \geqslant s_0$, $k \geqslant 1$, we have*

$$\|\overline{\nabla}\sigma\|^2_{L^2_{p,k}(M\setminus K,\Xi^{\otimes s})} + (s - s_0)\|\sigma\|^2_{L^2_{p,k}(M\setminus K,\Xi^{\otimes s})} \leqslant C\Big(\|\overline{\partial}\sigma\|^2_{L^2_{p,k}(M,\Xi^{\otimes s})} + \|\overline{\partial}^*_s\sigma\|^2_{L^2_{p,k}(M,\Xi^{\otimes s})}\Big) + \|\sigma\|^2_{L^2_{p,k}(K,\Xi^{\otimes s})}, \tag{4.1}$$

where $K = M\backslash(M \cap V)$ is the compact subset of M.

Proposition 4. *With the same assumptions as in Proposition 2, let us assume the following: There exists a constant $s_* > 0$ satisfies $\forall\, s \geqslant s_*$, the harmonic space $\mathcal{H}^s_{p,k}(\Xi^{\otimes s})$ has* finite dimension *and there exists a constant $C_s > 0$ depending on s such that*

$$\|\sigma\|^2_{L^2_{p,k}(M,\Xi^{\otimes s})} \leqslant C_s\Big(\|\overline{\partial}\sigma\|^2_{L^2_{p,k}(M,\Xi^{\otimes s})} + \|\overline{\partial}^*_s\sigma\|^2_{L^2_{p,k}(M,\Xi^{\otimes s})}\Big), \tag{4.2}$$

for $\sigma \in L^2_{p,k}(M, \Xi^{\otimes s}) \cap \mathfrak{Dom}\,(\overline{\partial}, \Xi^{\otimes s}) \cap \mathfrak{Dom}\,(\overline{\partial}^*_s, \Xi^{\otimes s})$ with $k \geqslant 1$.

Proof. Using (4.1), the proof follows as in Saber [29, 30]. □

Proposition 5. *With the same assumptions as in Proposition 2. Assume that there exists a positive integer m^* satisfies, for $s \geqslant s^*$, $k \geqslant 1$, then there exists a bounded linear operator $N^s : L^2_{p,k}(M, \Xi^{\otimes s}) \longrightarrow L^2_{p,k}(M, \Xi^{\otimes s})$ such that*

(i) $\mathfrak{Ran}(N^s, \Xi^{\otimes s}) \subset \mathfrak{Dom}(\Box^s, \Xi^{\otimes s})$, $N^s \Box^s = I - \Pi^s$ *on* $\mathfrak{Dom}(\Box^s, \Xi^{\otimes s})$,

(ii) for $\sigma \in L^2_{p,k}(M, \Xi^{\otimes s})$, *we have* $\sigma = \overline{\partial}\,\overline{\partial}^*_s N^s \sigma \oplus \overline{\partial}^*_s \overline{\partial} N^s \sigma \oplus \Pi^s \sigma$,

(iii) $N^s \overline{\partial} = \overline{\partial} N^s$ *and* $N^s \overline{\partial}^*_s = \overline{\partial}^*_s N^s$,

(iv) N^s, $\overline{\partial} N^s$, $\overline{\partial}^*_s N^s$ *are bounded operators on* $L^2_{p,k}(M, \Xi^{\otimes s})$.

Proof. The proof follows as [3, 28]. □

Theorem 5. *With the same assumptions as in Proposition 2. For $\alpha \in L^2_{p,k}(X, \Xi^{\otimes s})$, α is supported in $\overline{M}$, with $k \geqslant 1$, satisfying $\overline{\partial}\alpha = 0$ in X, there exists $w \in L^2_{p,k-1}(X, \Xi^{\otimes s})$, w is supported in $\overline{M}$ such that $\overline{\partial} w = \alpha$ in X.*

Proof. Let $\alpha \in L^2_{p,k}(X, \Xi^{\otimes s})$, α is supported in $\overline{M}$, then $\alpha \in L^2_{p,k}(M, \Xi^{\otimes s})$. Following Theorem 2, $N^s_{n-p,n-k}$ exists for $n - k \geqslant 1$. Define

$$w = - \star \divideontimes_{\Xi^{\otimes s}} \overline{\partial}\, N^s_{n-p,n-k} \divideontimes_{\Xi^{\otimes s}} \star\ \alpha, \tag{4.3}$$

for $w \in L^2_{p,k-1}(M, \Xi^{\otimes s})$. Set $w = 0$ in $X \setminus \overline{M}$.

To solve $\overline{\partial} w = \alpha$ in X, first solve $\overline{\partial} w = \alpha$ in M.

$$< \overline{\partial}\varrho, \divideontimes_{\Xi^{\otimes s}} \star \alpha >_{L^2_{p,k}(M,\Xi^{\otimes s})} = (-1)^{p+k} < \alpha, \divideontimes_{\Xi^{*\otimes s}} \star \overline{\partial}\varrho >_{L^2_{p,k}(M,\Xi^{\otimes s})},$$

if $\varrho \in \text{dom}(\overline{\partial}, \Xi^{*\otimes s})$. From the fact that $\vartheta^s = \overline{\partial}^*_s$ on $\mathcal{B}_{p,k}(\overline{M}, \Xi^{\otimes s})$ and the density of $\mathcal{B}_{p,k}(\overline{M}, \Xi^{\otimes s})$ in $\mathfrak{Dom}(\overline{\partial}, \Xi^{\otimes s}) \cap \mathfrak{Dom}(\overline{\partial}^*, \Xi^{\otimes s})$, and from (4.3), we obtain

$$< \overline{\partial}\varrho, \divideontimes_{\Xi^{\otimes s}} \star \alpha >_{L^2_{p,k}(M,\Xi^{\otimes s})} = < \alpha, \overline{\partial}^*_s \divideontimes_{\Xi^{*\otimes s}} \star\varrho >_{L^2_{p,k}(M,\Xi^{\otimes s})} .$$

Thus, α is supported in $\overline{M}$, implies $\overline{\partial}^*_s(\divideontimes_{\Xi^{\otimes s}} \star \alpha) = 0$ on M. Proposition 5(iii) implies

$$\overline{\partial}^*_s N^s_{n-p,n-k}(\divideontimes_{\Xi^{\otimes s}} \star \alpha) = N^s_{n-p,n-k-1} \overline{\partial}^*_s(\divideontimes_{\Xi^{\otimes s}} \star \alpha) = 0. \tag{4.4}$$

Thus, from (1.1), (4.3) and (4.4), one obtains

$$\begin{aligned} \overline{\partial} w &= -\overline{\partial} \star \divideontimes_{\Xi^{*\otimes s}} \overline{\partial} N^s_{n-p,n-k} \divideontimes_{\Xi^{\otimes s}} \star \alpha \\ &= (-1)^{p+k} \star \divideontimes_{\Xi^{*\otimes s}} \overline{\partial}^*_s \overline{\partial} N^s_{n-p,n-k} \divideontimes_{\Xi^{\otimes s}} \star \alpha \\ &= (-1)^{p+k} \star \divideontimes_{\Xi^{*\otimes s}} \divideontimes_{\Xi^{\otimes s}} \star\alpha \\ &= \alpha. \end{aligned} \tag{4.5}$$

Because $w = 0$ in $X \setminus M$, for $\varrho \in L^2_{p,k}(X, \Xi^{\otimes s})$, one obtains

$$< w, \overline{\partial}^*_s\varrho >_{L^2_{p,k}(X,\Xi^{\otimes s})} = < w, \overline{\partial}^*_s\varrho >_{L^2_{p,k}(M,\Xi^{\otimes s})} = < \divideontimes_{\Xi^{\otimes s}} \star \overline{\partial}^*_s\varrho, \divideontimes_{\Xi^{\otimes s}} \star w >_{L^2_{p,k}(M,\Xi^{\otimes s})} .$$

Since

$$\divideontimes_{\Xi^{\otimes s}} \star \mathrm{w} = (-1)^{\mathrm{p}+\mathrm{k}+1}\,\overline{\partial}\,\mathrm{N}^{\mathrm{s}}_{n-\mathrm{p},n-\mathrm{k}} \divideontimes_{\Xi^{\otimes s}} \star\, \alpha.$$

Equation (4.5) gives

$$<\mathrm{w}, \overline{\partial}^{*}_{\mathrm{s}}\varrho>_{\mathrm{L}^2_{\mathrm{p,k}}(\mathrm{X},\Xi^{\otimes s})} = (-1)^{\mathrm{p}+\mathrm{k}} < \overline{\partial} \divideontimes_{\Xi^{\otimes s}} \star\varrho, \divideontimes_{\Xi^{\otimes s}} \star \mathrm{w} >_{\mathrm{L}^2_{\mathrm{p,k}}(M,\Xi^{\otimes s})} =< \overline{\partial}\,\mathrm{w}, \varrho >_{\mathrm{L}^2_{\mathrm{p,k}}(M,\Xi^{\otimes s})}.$$

Thus

$$<\mathrm{w}, \overline{\partial}^{*}_{\mathrm{s}}\varrho>_{\mathrm{L}^2_{\mathrm{p,k}}(\mathrm{X},\Xi^{\otimes s})} =< \alpha, \varrho >_{\mathrm{L}^2_{\mathrm{p,k}}(M,\Xi^{\otimes s})} =< \alpha, \varrho >_{\mathrm{L}^2_{\mathrm{p,k}}(\mathrm{X},\Xi^{\otimes s})}.$$

Thus $\overline{\partial}\,\mathrm{w} = \alpha$ in X. □

5. Annular domains

5.1. Compactness estimates

Theorem 6. *Assume that $M = M_1\backslash\overline{M}_2$ is* an annulus between two smooth bounded domains M_1 *and* M_2 *in a* Stein manifold X *of dimension n satisfy $\overline{M}_2 \Subset M_1$, M_1 is weak $Z(k)$, M_2 is weak $Z(n-1-k)$ and $1 \leqslant k \leqslant n-2$ with $n \geqslant 3$. Let ϱ be a smooth function on $\overline{M}$ satisfy $\varrho = \mu$ in a neighborhood of bM_1 and $\varrho = -\mu$ in a neighborhood of bM_2. Then, there exists $c, T > 0$ satisfy for every $t \geq T$ with $C_{\mathtt{t}} > 0$, one obtains*

$$t\|\sigma\|^2_{L^2_{\mathrm{p,k}}(M,e^{-\mathrm{t}\varrho})} \leqslant c\,\|\overline{\partial}\sigma\|^2_{L^2_{\mathrm{p,k}}(M,e^{-\mathrm{t}\varrho})} + \|\overline{\partial}^{*}_{\mathrm{t}}\sigma\|^2_{L^2_{\mathrm{p,k}}(M,e^{-\mathrm{t}\varrho})} + \mathrm{C_t}\,\|\sigma\|^2_{\mathrm{W}^{-1}(M)}, \tag{5.1}$$

for $\sigma \in \mathscr{D}_{\mathrm{p,k}}(M)$.

Proof. As in [27, 29] (resp. [30]). Let σ be supported in a small neighborhood V of bM_1. Let (ζ_{ij}) be a Levi matrix of a defining function ζ of M_1. If $\overline{U} \subset M$, one obtains

$$\|\sigma\|^2_{\mathrm{W}^1(\mathrm{U})} \leqslant c'\,(\|\overline{\partial}\sigma\|^2_{L^2_{\mathrm{p,k}}(\mathrm{U},e^{-\mathrm{t}\varrho})} + \|\overline{\partial}^{*}_{\mathrm{t}}\sigma\|^2_{L^2_{\mathrm{p,k}}(\mathrm{U},e^{-\mathrm{t}\varrho})}), \quad \text{for } \sigma \in \mathscr{D}_{\mathrm{p,k}}(\mathrm{U}). \tag{5.2}$$

Following [25], one obtains

$$\|\sigma\|^2_{L^2_{\mathrm{p,k}}(\mathrm{U},e^{-\mathrm{t}\varrho})} \leqslant t\|\sigma\|^2_{\mathrm{W}^1(\mathrm{U})} + \mathrm{C_t}\,\|\sigma\|^2_{\mathrm{W}^{-1}(\mathrm{U})}. \tag{5.3}$$

If $c = t^2c'$ and using (5.2), (5.3), inequality (5.1) follows for $\sigma \in \mathscr{D}_{\mathrm{p,k}}(U)$ when $\overline{U} \cap bM = \varnothing$.

Since bM_1 is weak $Z(k)$ as the boundary of M_1 and is weak $Z(n-1-k)$ as a part of bM. Thus, for $\sigma \in \mathscr{D}_{\mathrm{p,k}}(\mathrm{U}\cap M_1)$ with $1 \leqslant k \leqslant n-1$ and $\forall\, c > 0$, it follows that

$$t\int_{\mathrm{U}\cap M_1} |\sigma|^2 e^{-\mathrm{t}\varrho} dV \leqslant c\,(\|\overline{\partial}\sigma\|^2_{L^2_{\mathrm{p,k}}(\mathrm{U}\cap M_1,e^{-\mathrm{t}\varrho})} + \|\overline{\partial}^{*}_{\mathrm{t}}\sigma\|^2_{L^2_{\mathrm{p,k}}(\mathrm{U}\cap M_1,e^{-\mathrm{t}\varrho})}) + \mathrm{C_t}\,\|\sigma\|^2_{\mathrm{W}^{-1}(\mathrm{U}\cap M_1)}. \tag{5.4}$$

Let $\Delta_{\delta_1} = \{z \in \mathrm{X} : -\delta_1 < \zeta(z) \leqslant 0\}$, where $\delta_1 > 0$ is a number (depend on t) small enough. From the compactness of bM_1, by using a finite covering $\{V_\nu\}^{\mathrm{s}}_{\nu=1}$ of bM_1 by neighborhoods V_ν as in (5.4), one obtains

$$t\int_{\Delta_{\delta_1}} |\sigma|^2 e^{-\mathrm{t}\varrho} dV \leqslant c\,(\|\overline{\partial}\sigma\|^2_{L^2_{\mathrm{p,k}}(\Delta_{\delta_1},e^{-\mathrm{t}\varrho})} + \|\overline{\partial}^{*}_{\mathrm{t}}\sigma\|^2_{L^2_{\mathrm{p,k}}(\Delta_{\delta_1},e^{-\mathrm{t}\varrho})}) + \mathrm{C_t}\,\|\sigma\|^2_{\mathrm{W}^{-1}(\Delta_{\delta_1})}, \tag{5.5}$$

when σ is supported in the strip Δ_{δ_1}.

Since bM_2 is weak $Z(n-1-k)$ as the boundary of M_2 and is weak $Z(k)$ as a part of bM. Following Lemma 3,

$$t\int_{\mathrm{U}\cap M_2}|\sigma|^2 e^{-\mathrm{t}\varrho}dV \leqslant C\,\|\overline{\partial}\sigma\|^2_{L^2_{\mathrm{p,k}}(\mathrm{U}\cap M_2,e^{-\mathrm{t}\varrho})}+\|\overline{\partial}^*_{\mathrm{t}}\sigma\|^2_{L^2_{\mathrm{p,k}}(\mathrm{U}\cap M_2,e^{-\mathrm{t}\varrho})}+\mathrm{C_t}\,\|\sigma\|^2_{\mathrm{W}^{-1}(\mathrm{U}\cap M_2)}, \tag{5.6}$$

for $\sigma\in\mathscr{D}_{\mathrm{p,k}}(U\cap M_2)$, $1\leqslant k\leqslant n-2$.

Let $\Delta_{\delta_2}=\{z\in X:0\leqslant\zeta(z)<\delta_2\}$, where $\delta_2>0$ small enough. From the compactness of bM_2, by a finite covering $\{V_\nu\}_{\nu=1}^{\mathrm{s}}$ of bM_2 by neighborhoods V_ν as in (5.5),

$$t\int_{\Delta_{\delta_2}}|\sigma|^2 e^{-\mathrm{t}\varrho}dV \leqslant c\,\|\overline{\partial}\sigma\|^2_{L^2_{\mathrm{p,k}}(\Delta_{\delta_2},e^{-\mathrm{t}\varrho})}+\|\overline{\partial}^*_{\mathrm{t}}\sigma\|^2_{L^2_{\mathrm{p,k}}(\Delta_{\delta_2},e^{-\mathrm{t}\varrho})}+\mathrm{C_t}\,\|\sigma\|^2_{\mathrm{W}^{-1}(\Delta_{\delta_2})}, \tag{5.7}$$

when σ is supported in the strip Δ_{δ_2}.

Let $\Delta_\delta=\Delta_{\delta_1}\cup\Delta_{\delta_2}$ with $\delta=\min\{\delta_1,\delta_2\}$. Thus, by (5.5) and (5.7), one obtains

$$t\int_{\Delta_\delta}|\sigma|^2\,e^{-\mathrm{t}\varrho}\,dV \leqslant c\,\|\overline{\partial}\sigma\|^2_{L^2_{\mathrm{p,k}}(\Delta_\delta,e^{-\mathrm{t}\varrho})}+\|\overline{\partial}^*_{\mathrm{t}}\sigma\|^2_{L^2_{\mathrm{p,k}}(\Delta_\delta,e^{-\mathrm{t}\varrho})}+\mathrm{C_t}\,\|\sigma\|^2_{\mathrm{W}^{-1}(\Delta_\delta)}. \tag{5.8}$$

The integral on $M\backslash\Delta_\delta$ can be estimated by choosing $\gamma_\delta\in\mathscr{D}(M)$ with $\gamma_\delta(z)=1$, $\zeta(z)\leqslant-\delta$ and $z\in M\backslash\Delta_\delta$ as

$$\|\gamma_\delta\sigma\|^2_{L^2_{\mathrm{p,k}}(M\backslash\Delta_\delta,e^{-\mathrm{t}\varrho})}\leqslant k\,\|\gamma_\delta\sigma\|^2_{\mathrm{W}^1(M\backslash\Delta_\delta)}+\frac{1}{k}\|\gamma_\delta\sigma\|^2_{\mathrm{W}^{-1}(M\backslash\Delta_\delta)}. \tag{5.9}$$

Because $\mathrm{Q^t}$ is elliptic, by Gårding's inequality [28],

$$\begin{aligned}\|\gamma_\delta\sigma\|^2_{\mathrm{W}^1(M)}&\leqslant\|\overline{\partial}(\gamma_\delta\sigma)\|^2_{L^2_{\mathrm{p,k}}(M,e^{-\mathrm{t}\varrho})}+\|\overline{\partial}^*_{\mathrm{t}}(\gamma_\delta\sigma)\|^2_{L^2_{\mathrm{p,k}}(M,e^{-\mathrm{t}\varrho})}\\&\leqslant(\|\gamma_\delta(\overline{\partial}\sigma)\|^2_{L^2_{\mathrm{p,k}}(M,e^{-\mathrm{t}\varrho})}+\|\gamma_\delta(\overline{\partial}^*_{\mathrm{t}}\sigma)\|^2_{L^2_{\mathrm{p,k}}(M,e^{-\mathrm{t}\varrho})}\\&+\|[\gamma_\delta,\overline{\partial}]\sigma\|^2_{L^2_{\mathrm{p,k}}(M,e^{-\mathrm{t}\varrho})}+\|[\gamma_\delta,\overline{\partial}^*_{\mathrm{t}}]\sigma\|^2_{L^2_{\mathrm{p,k}}(M,e^{-\mathrm{t}\varrho})}+\|\gamma_\delta\sigma\|^2_{L^2_{\mathrm{p,k}}(M,e^{-\mathrm{t}\varrho})})\\&\leqslant\|\overline{\partial}\sigma\|^2_{L^2_{\mathrm{p,k}}(M,e^{-\mathrm{t}\varrho})}+\|\overline{\partial}^*_{\mathrm{t}}\sigma\|^2_{L^2_{\mathrm{p,k}}(M,e^{-\mathrm{t}\varrho})}+\mathrm{C}_\delta\,\|\sigma\|^2_{L^2_{\mathrm{p,k}}(M,e^{-\mathrm{t}\varrho})}.\end{aligned} \tag{5.10}$$

Thus, from (5.8)–(5.10), one obtains

$$\|\gamma_\delta\sigma\|^2_{L^2_{\mathrm{p,k}}(M,e^{-\mathrm{t}\varrho})}-\frac{1}{2}\|\sigma\|^2_{L^2_{\mathrm{p,k}}(M,e^{-\mathrm{t}\varrho})}\leqslant k(\|\overline{\partial}\sigma\|^2_{L^2_{\mathrm{p,k}}(M,e^{-\mathrm{t}\varrho})}+\|\overline{\partial}^*_{\mathrm{t}}\sigma\|^2_{L^2_{\mathrm{p,k}}(M,e^{-\mathrm{t}\varrho})})+\frac{1}{k}\|\gamma_\delta\sigma\|^2_{\mathrm{W}^{-1}(M)}. \tag{5.11}$$

Thus, from (5.10), (5.11), we get

$$\begin{aligned}\frac{t}{2}\|\sigma\|^2_{L^2_{\mathrm{p,k}}(M,e^{-\mathrm{t}\varrho})}&\leqslant t\int_{\Delta_\delta}|\sigma|^2\,e^{-\mathrm{t}\varrho}\,dV+t\|\gamma_\delta\sigma\|^2_{L^2_{\mathrm{p,k}}(M,e^{-\mathrm{t}\varrho})}-\frac{t}{2}\|\sigma\|^2_{L^2_{\mathrm{p,k}}(M,e^{-\mathrm{t}\varrho})}\\&\leqslant(c+kt)\,\|\overline{\partial}\sigma\|^2_{L^2_{\mathrm{p,k}}(M,e^{-\mathrm{t}\varrho})}+\|\overline{\partial}^*_{\mathrm{t}}\sigma\|^2_{L^2_{\mathrm{p,k}}(M,e^{-\mathrm{t}\varrho})}+(C_{\mathtt{t}}+\frac{t}{k})\|\sigma\|^2_{\mathrm{W}^{-1}(M)}.\end{aligned}$$

Thus (5.1) follows by choosing $c+\mathtt{kt}<\frac{C}{2}$ and $C'_{\mathtt{t}}+\frac{t}{k}<\frac{C_{\mathrm{t}}}{2}$. □

Theorem 7. *Let* X, Ξ, *M be as in Theorem 6. Then, the compactness estimate of the weighted* $\overline{\partial}$-*Neumann problem holds on M for a holomorphic vector bundle* Ξ*-valued* (p, k) *form. Then, for all* $c>0$, *there exists a* $\mathtt{t}>0$ *and* $C_{c,\mathtt{t}}>0$ *such that*

$$\mathtt{t}\,\|\sigma\|^2_{\mathrm{L}^2_{\mathrm{p,k}}(M,\Xi)}\leq c\Big(\|\overline{\partial}\sigma\|^2_{\mathrm{L}^2_{\mathrm{p,k}}(M,\Xi)}+\|\overline{\partial}^*_{\mathtt{t}}\sigma\|^2_{\mathrm{L}^2_{\mathrm{p,k}}(M,\Xi)}\Big)+C_{\mathtt{t}}\,\|\sigma\|^2_{\mathrm{W}^{-1}(M,\Xi)},$$

for $\sigma\in\mathscr{D}_{\mathrm{p,k}}(M,\Xi)$.

Proof. The proof follows as Theorem 2. □

5.2. Global regularity up to the boundary

Lemma 5. *With the same assumptions as in Theorem 7, let us assume the following: For* $1 \leq \mathrm{k} \leq n-2$, $n \geq 3$, *there exists* $C > 0$ *satisfies* $\forall\ \sigma \in \mathscr{D}^{\mathrm{p,k}}(M, \Xi)$ *with* $\sigma \perp \mathcal{H}_{\mathrm{t}}^{\mathrm{p,k}}(\Xi)$, *we have*

$$\|\sigma\|^2_{\mathrm{L}^2_{\mathrm{p,k}}(M,\Xi)} \leq C\Big(\|\overline{\partial}\sigma\|^2_{\mathrm{L}^2_{\mathrm{p,k}}(M,\Xi)} + \|\vartheta_{\mathrm{t}}\sigma\|^2_{\mathrm{L}^2_{\mathrm{p,k}}(M,\Xi)}\Big). \tag{5.12}$$

Proof. The proof follows as Lemma 3. □

By using (5.12), as Proposition 3.5 in [3], we prove the following theorem:

Lemma 6. *With the same assumptions as in Theorem 7, let us assume the following: for* $1 \leq \mathrm{k} \leq n-2$, $n \geq 3$, *we have*

(1) $\overline{\partial}$ *(resp.* $\overline{\partial}^*_{\mathrm{t}}$*) has closed range in* $\mathrm{L}^2_{\mathrm{p,k}}(M, \Xi)$ *and* $\mathrm{L}^2_{\mathrm{p,k}+1}(M, \Xi)$ *(resp.* $\mathrm{L}^2_{\mathrm{p,k}-1}(M, \Xi)$*)*,

(2) $\mathcal{H}_{\mathrm{t}}^{\mathrm{p,k}}(\Xi)$ *is* finite dimension*al,*

(3) $\Box^{\mathrm{t}}$ *has closed range in* $\mathrm{L}^2_{\mathrm{p,k}}(M, \Xi)$,

(4) $\mathfrak{Ran}(N^{\mathrm{t}}) \subset Dom\,\Box^{\mathrm{t}}$, $N^{\mathrm{t}}\Box^{\mathrm{t}} = I - \mathcal{H}_{\mathrm{t}}^{\mathrm{p,k}}(\Xi)$ *on* $Dom(\Box^{\mathrm{t}}, \Xi)$,

(5) for $\sigma \in \mathrm{L}^2_{\mathrm{p,k}}(M, \Xi)$, *we have* $\sigma = \overline{\partial}\,\overline{\partial}^{\star}_{\mathrm{t}} N\sigma \oplus \overline{\partial}^{\star}\overline{\partial} N^{\mathrm{t}}\sigma \oplus \mathbb{H}_{\mathrm{t}}\sigma$.

Proof. The proof follows as in [28]. □

By Lemma 6 (ii) and the density of $\mathrm{C}^{\infty}_{\mathrm{p,k}}(\overline{M}, \Xi)$ in $\mathrm{W}^{k}_{\mathrm{p,k}}(M, \Xi)$, the following result is immediate.

Lemma 7. *[44] With the same assumptions as in Theorem 7, let us assume the following: If* $f \in \mathrm{C}^{\infty}_{\mathrm{p,k}}(\overline{M}, \Xi)$ *with* $1 \leq \mathrm{k} \leq n-2$, $n \geq 3$ *and* $N^{\mathrm{t}} f \in \mathrm{C}^{\infty}_{\mathrm{p,k}}(\overline{M}, \Xi)$, *then for all* $s \geq 0$, *there exists constants* C_s *and* T_s *such that*

$$\|N^{\mathrm{t}} f\|_{\mathrm{W}^s(M,\Xi)} \leq C_s \|f\|_{\mathrm{W}^s(M,\Xi)}, \text{ for every } t > T_s.$$

One can prove the following theorem by using the elliptic regularization method used in [44]:

Lemma 8. *Assuming the same assumption as in Theorem 7, for every integer* $s \geq 0$ *and real* $t > T > 0$, N^{t} *is bounded from* $\mathrm{W}^s_{\mathrm{p,k}}(M, \Xi)$ *into itself for* $1 \leq \mathrm{k} \leq n-2$, $n \geq 3$.

By Lemma and the density of $\mathrm{C}^{\infty}_{\mathrm{p,k}}(\overline{M}, \Xi)$ in $\mathrm{W}^s_{\mathrm{p,k}}(M, \Xi)$, the following is immediate.

Corollary 2. *Let* M, Ξ *and* X *be the same as in Theorem 7. Then, if* $f \in \mathrm{W}^s_{\mathrm{p,k}}(M, \Xi)$, $s = 0, 1, 2, 3, ...$ *satisfies* $\overline{\partial} f = 0$, *where* $1 \leq \mathrm{k} \leq n-2$, $n \geq 3$, *there exits* $\sigma \in \mathrm{W}^s_{\mathrm{p,k}-1}(M, \Xi)$ *so that* $\overline{\partial}\sigma = f$ *on* M *and*

$$\|\sigma\|_{\mathrm{W}^s(M,\Xi)} \leq C_s \|f\|_{\mathrm{W}^s(M,\Xi)}.$$

Theorem 8. *With the same assumptions as in Theorem 7, let us assume the following: for* $\sigma \in \mathrm{C}^{\infty}_{\mathrm{p,k}}(\overline{M}, \Xi)$, $1 \leq \mathrm{k} \leq n-2$, $n \geq 3$, *satisfying* $\overline{\partial}\sigma = 0$, *there exists* $\mathrm{w} \in \mathrm{C}^{\infty}_{\mathrm{p,k}-1}(\overline{M}, \Xi)$, *satisfies* $\overline{\partial}\,\mathrm{w} = \sigma$ *in* X.

Proof. The proof follows as [30, 44]. □

6. Conclusions

In this paper, we are concerned with the Sobolev estimates of the $\overline{\partial}$-Neumann operator N and the resulting results (Compactness and Global regularity, etc.). Existence theorems and Sobolev estimates for the $\overline{\partial}$ and the $\overline{\partial}$-Neumann operator on the weak Z(k) domain with C^3 boundary in an n-dimensional Stein manifold X are fundamental results in complex analysis. In this way, we can gain a deeper understanding of holomorphic functions, and we can implement tools to solve the $\overline{\partial}$-equation more efficiently.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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