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Robustness analysis of stability of Takagi-Sugeno type fuzzy neural network

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Abstract: In this paper, inequality techniques, stochastic analysis and algebraic methods are used to analyze the robustness of the stability of recurrent neural networks containing Takagi-Sugeno fuzzy rules. By solving the transcendental equations, the upper bounds of time delay and noise intensity are given, and the dynamic relationship between the two disturbance factors is derived. Finally, numerical examples are given to verify the results of this paper.

Keywords: Takagi-Sugeno type fuzzy neural network; time delay; stochastic disturbances; robustness; exponential stability
Mathematics Subject Classification: 93B35, 93D23

1. Introduction

As one of the extensions of artificial neural networks (ANNs), the cellular neural network (CNN) was initially proposed by Chua and Yang in [1,2]. The connections of neurons in CNNs are localized, which means that neurons in CNNs only connect to other neurons in a certain range. This local connection method not only reduces the connection between neurons, but also improves the efficiency of parallel computing. Based on this, CNNs can be used in many fields, such as image encryption [3], parallel computing [2], and so on [4,5].

However, for some problems which have more complexity and vagueness, the models established by using general neural networks are often not accurate enough. Hence, the Takagi-Sugeno fuzzy model (TSFM) [6] as an effective tool to solve this problem has received widespread attention in recent years. The TSFM is a nonlinear system description approach that employs linear weighted inequalities and differential equations, and TSFMs can simulate any smooth dynamical system in any precision which are defined on a compact set by connecting several local linear models with fuzzy membership functions piecewise smoothly. Based on these properties, very recently, TSFMs have been used to describe CNNs, e.g., T-S fuzzy CNN (TSFCNN) is presented [7–10]. TSFCNNs combine the advantages of fuzzy theory and features of neural networks which improve the ability of general

CNNs to handle complex problems by using the IF-THEN rules. Therefore, TSFCNNs are widely used in more fields than CNNs.

For a designed neural network model, stability is one of the indicators that determines whether it has good performance. However, throughout the real simulation procedure, due to the inevitable existence of disturbances, such as times delays and stochastic disturbances, the stability of TSFCNNs may not be achieved. Time delays are usually caused by the limited signal transmission speed of electronic devices, and different ways of operation of electrical devices might result in varying delays., such as time-varying delays [11], state-dependent delays [12] or distributed delays [13] etc. In addition, stochastic disturbance is a type of complicated and irregular disruption, and it is distinct from traditional processes. The intensities of both of the above disturbances can to some extent make the designed TSFCNN model unable to achieve the expected performance. Hence, the stability analysis of TSFCNNs have received more and more attention in recent decades [14–18]. In [14], Hou et al. analyzes the global exponential stability of delayed TSFCNNs (DTSFCNNs) by constructing Lyapunov-Krasovskii functionals as well as using the linear matrix inequality (LMI) method. In [15, 18], Balasubramaniam et al. derived some criteria based on the LMI method to ensure the stability of stochastic TSFCNNs (STSFCNNs) and DTSFCNNs, respectively. In [16], Yang and Sheng discuss the robust stability of uncertain STSFCNNs in detail. Long and Xu further considered the impacts of impulses based on delayed STSFCNNs (DSTSFCNNs) in [17], and explore its exponential p-stability by using the property of M-matrix.

It is worth mentioning that the above literature is all about the stability analysis of DTSFCNNs or STSFCNNs rather than robustness of stability. Robustness is the ability of a system to maintain its dynamic properties under parameter changes. Once the parameters change beyond a certain fixed range, the original dynamic properties of the system will change. In the actual modeling process, we always want to design a model with good robustness. However, few researchers have described the range of parameter variations that the systems can withstand. For the problem of the robustness of stability (RoS), the classical methods will no longer be applicable. Hence, Shen et al. first proposed an inequality method based on the Gronwall-Bellman lemma to explore the robustness of stability in [19]. Based on this method, many results which worked on the problem of RoS of neural networks are presented, for example, Si et al. investigated the RoS of the dynamical systems with deviating arguments based on Gronwall-Bellman lemma in [20, 21]. Moreover, Fang et al. further explore the RoS of a class of CNNs based on bidirectional associative memory in [22].

However, it should be pointed out that the above studies did not take into account the fuzziness in the modeling process, and did not explore the robustness of neural networks containing T-S fuzzy rules. Therefore, in order to fill this gap, this paper mainly analyzes the RoS of perturbed TSFCNNs. Key works and contributions of this paper are listed below.

- Time delay and random disturbance are two kinds of system disturbances mainly considered in this paper, both of which are unavoidable in practical modeling. Sufficient conditions are given in [9, 12, 14] for the stability of systems containing perturbations, but few have explored the upper bound on the strength of these perturbations. When the values of the disturbances change, the stability of the system will also change, so it is meaningful to discuss the maximum value of the disturbance that the system can withstand. The innovation of this paper lies in the following aspects: 1. This paper establishes a recurrent neural network with T-S fuzzy rules, and gives the robustness analysis of the model based on the classical Gronwall inequality. 2. The

constraint relationship between time delay and random disturbance is established, which means that the relationship between time delay and random disturbance is dynamic. Besides, the results obtained in this paper extend the results in [19], which means that when there is only one fuzzy rule in DTSFCNNs or DSTSFCNNs, the results in [19] can be obtained.
- In comparison to [19–21], the subsystems considered in this paper are interconnected by fuzzy rules, and the parameters of each subsystem are not the same. With the increase of fuzzy rules, the number of subsystems increases, which greatly enhances the fuzziness of the networks considered in the literature in dealing with practical problems, and the existence of fuzzy rules also increases the difficulty of system analysis to a certain extent.
- The methods used in this paper mainly include inequality techniques, random analysis and algebraic knowledge, so the results obtained in this paper are easy to verify. The results derived in this paper are helpful for designing TSFCNNs with better performance.

Finally, the organization of this paper is listed below. We introduce the models we considered and assumptions we needed in Section 2. The problem of the RoS of DTSFCNNs is explored in Section 3. In Section 4, we discuss the RoS of DSTSFCNNs. And, in Section 5, several examples are provided to verify the results in this paper.

Notations: $\mathbb{Z}^+ = \{a|a = 1, 2, \dots, w\}$. $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = (0, +\infty)$, $\mathbb{R}^w$ and $\mathbb{R}^{w\times v}$ are the sets of w-dimensional vectors, and $w \times v$ dimensional matrices, respectively. $\|\cdot\|$ denotes the Euclidean norm of any vectors or matrices. Define $(\zeta, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geq 0}, P)$ as the complete filtered probability space containing all P-null sets, where right continuous filtration $\mathfrak{F}_{t\geq 0}$ meets the usual conditions. Scalar Brownian movement $\varsigma(t)$ is defined in $(\zeta, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geq 0}, P)$. $L^2_{\mathfrak{F}_0}([-\mu, 0]; \mathbb{R}^w)$ is the set which contains all $\mathfrak{F}_0$ measurable $C([-\mu, 0]; \mathbb{R}^w)$ valued stochastic variables $\hbar = \{\hbar(\iota) : -\mu \leq \iota \leq 0\}$ and $\sup_{-\mu\leq\iota\leq 0} E\|\hbar(\iota)\|^2 \leq \infty$. E is the expectation operator. $\mu = \max\{\mu_1, \mu_2, \dots, \mu_n\} \in \mathbb{R}^+$, where μ_j is a constant delay. A^T represents the transposition of matrix or vector A.

2. Primaries

We consider the delayed CNNs which is in the following form:

$$\begin{cases} \dot{z}_h(t) = -c_h z_h(t) + \sum_{u=1}^{n} a_{hu} y_u(z_u(t)) + \sum_{u=1}^{n} b_{hu} y_u(z_u(t-\mu_u)) + J_h, \\ z_h(t) = \phi_h(t), \quad t \in [-\mu, 0], \end{cases} \tag{2.1}$$

where c_h represents the rates of hth neuron resetting its potential to the isolated resting state, and $c_h > 0$. a_{hu} and b_{hu} are the strengths of uth neuron on the hth neuron at time t and $t - \mu_u$, where μ_u represents the transmission delay, which is a positive constant. J_h represents the external bias which impacts the hth neuron.

Remark 2.1. *Time delays are often present in real-world control systems, and there may be delays in states, control inputs or measurements. Delay will introduce uncertainty and interference, which will degrade the robustness of the controller. Therefore, the effect of time delay must be considered when designing the control system.*

Let $C = \text{diag}\{c_1, c_2, \dots, c_n\}$, $A = (a_{hu})_{n\times n}$, $B = (b_{hu})_{n\times n}$, $y(\cdot) = \{y_1(\cdot), y_2(\cdot), \dots, y_n(\cdot)\}^T$, and $\nu =$

$\{\mu_1, \mu_2, \ldots, \mu_n\}^T$. Then, CNN (2.1) has the following representation:

$$\begin{cases} \dot{z}(t) = -Cz(t) + Ay(z(t)) + By(z(t-\nu)) + J, \\ z(t) = \phi(t), \quad t \in [-\mu, 0]. \end{cases} \tag{2.2}$$

Assume the equilibrium point is $z^* = \{z_1^*, z_2^*, \ldots, z_n^*\}$, and $\rho_i(t) = z_i(t) - z_i^*$. Then, CNN (2.2) is equivalent to the following model

$$\begin{cases} \dot{\rho}(t) = -C\rho(t) + Ag(\rho(t)) + Bg(\rho(t-\nu)), \\ \rho(t) = \psi(t), \quad t \in [-\mu, 0], \end{cases} \tag{2.3}$$

where $g(\rho(t)) = y(\rho(t) + z^*) - y(z^*)$, $g(\rho(t-\nu)) = y(\rho(t-\nu) + z^*) - y(z^*)$ and $\psi(t) = \phi(t) - z^*$.

Remark 2.2. *The circuit implementations of* (2.1)–(2.3) *can be found in [1, 23].*

The following condition is one we assume the function $g(\cdot)$ needs to satisfy:

Assumption 2.1. *There is an* $L > 0$ *such that*

$$\|g(u) - g(v)\| \le L\|u - v\| \tag{2.4}$$

holds, where u *and* v *are two states of the CNN* (2.3).

Remark 2.3. *Since* $g(\rho(t)) = y(\rho(t) + z^*) - y(z^*)$, *hence,* $g(0) = 0$. *Therefore, from Assumption 2.1,* $\|g(u)\| \le L\|u\|$ *holds, i.e., the linear growth condition for activation function* $g(\cdot)$ *is also satisfied.*

Remark 2.4. *There are many functions that satisfy the Assumption 2.1, such as* $g(x) = \tanh(x)$, $g(x) = (|x+1| - |x-1|)/2$ *and so on. The selection of activation functions is closely related to the stability of CNNs. For example, consider the following neural network model*

$$\dot{x}(t) = -2x(t) + 1.1f(x(t)) + f(x(t-\tau)), \tag{2.5}$$

where $\tau = 3$. *We take the activation functions as* $f(x) = \tanh(x)$ *and* $f(x) = (|x+1| - |x-1|)/2$, *respectively. It can be seen from Figure 1 that when the model has different activation functions, its convergence rate to reach the stable state is affected.*

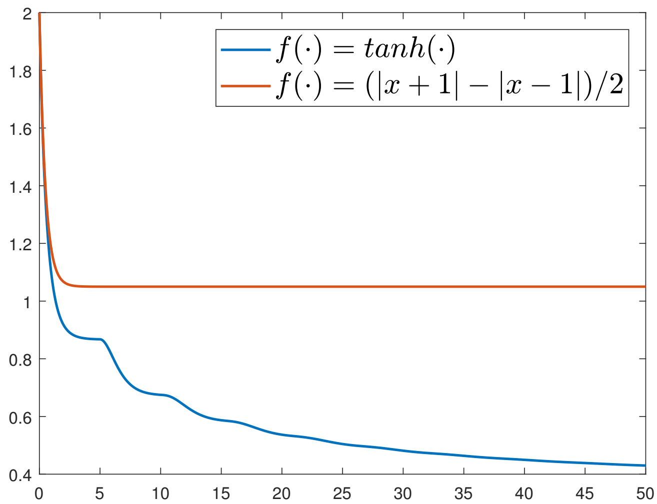

Figure 1. States of system (2.5) with different activation functions.

By applying fuzzy IF-THEN rules to the CNN (2.3), the following DTSFCNN model can be obtained:

Plant Rule k :
IF $\varpi_1(t)$ **is** υ_1^k **and** $\varpi_2(t)$ **is** υ_2^k **and** $\ldots$ **and** $\varpi_m(t)$ **is** υ_m^k
THEN

$$\begin{cases}\dot{\rho}(t) = -C_k\rho(t) + A_k g(\rho(t)) + B_k g(\rho(t-\nu)), \\ \rho(t) = \psi(t), \quad t \in [-\mu, 0], \quad k = 1, 2, \ldots, l,\end{cases} \tag{2.6}$$

where $\varpi_j(t)$ and υ_j^k are the premise variable and the fuzzy set respectively. l represents the amount of fuzzy **IF-THEN** rules. C_k, A_k, B_k are constant matrices.

Remark 2.5. *By combining T-S fuzzy logic with nonlinear differential equations, the described system simulates a smooth system with arbitrary precision by using fuzzy rules to connect several nonlinear models with fuzzy membership functions piecewise and smoothly on compact sets.*

Remark 2.6. *Compared with the existing literature [19–21], the system with T-S fuzzy logic considered in this paper is composed of multiple sets of nonlinear systems connected by T-S fuzzy logic, and the parameters of each nonlinear system are not fixed, which can more accurately describe the changes of system parameters brought by fuzziness during the operation of the actual dynamic system.*

The DTSFCNN (2.6) after being defuzzied is shown as follows:

$$\dot{\rho}(t) = \sum_{k=1}^{l} \Omega_k(\varpi(t))\{-C_k\rho(t) + A_k g(\rho(t)) + B_k g(\rho(t-\nu))\}, \tag{2.7}$$

where $\varpi(t) = \{\varpi_j(t)\}_{1\times m}$ and

$$\Omega_k(\varpi(t)) = \frac{\gamma_k(\varpi(t))}{\sum_{k=1}^{l}\gamma_k(\varpi(t))}, \quad \gamma_k(\varpi(t)) = \prod_{j=1}^{m} \upsilon_j^k(\varpi_j(t)). \tag{2.8}$$

Moreover, $\Omega_k(\varpi(t))$ represents the averaged weight of the rules. $\upsilon_j^k(\varpi_j(t))$ is the grade of the membership function of $\varpi_j(t)$ in υ_j^k. From fuzzy set theory, we can get

$$\gamma_k(\varpi(t)) \geq 0, \quad \sum_{k=1}^{l} \gamma_k(\varpi(t)) > 0, \tag{2.9}$$

and

$$\Omega_k(\varpi(t)) \geq 0, \quad \sum_{k=1}^{l} \Omega_k(\varpi(t)) = 1, \tag{2.10}$$

where $k = 1, 2, \ldots, l$.

When $\mu = 0$, DTSFCNN (2.6) degenerates into the following TSFCNN.

Plant Rule k :
IF $\varpi_1(t)$ **is** υ_1^k **and** $\varpi_2(t)$ **is** υ_2^k **and** $\ldots$ **and** $\varpi_m(t)$ **is** υ_m^k
THEN

$$\begin{cases} \dot{\kappa}(t) = -C_k\kappa(t) + A_k g(\kappa(t)) + B_k g(\kappa(t)), \\ \kappa(t_0) = \psi(t_0), \quad k = 1, 2, \ldots, l. \end{cases} \tag{2.11}$$

Then, we can get the following defuzzied output of the above TSFCNN.

$$\dot{\kappa}(t) = \sum_{k=1}^{l} \Omega_k(\varpi(t))\{-C_k\kappa(t) + A_k g(\kappa(t)) + B_k g(\kappa(t))\}. \tag{2.12}$$

Next, we give the definition of global exponential stability of system (2.7).

Definition 2.1. *A DTSFCNN* (2.7) *is globally exponentially stable (GES) if*

$$\|\rho(t)\| \leq \alpha \exp(-\beta(t - t_0)) \sup_{s\in[t_0-\mu,t_0]} \|\rho(s)\| \tag{2.13}$$

holds, where α, β are positive constants.

Remark 2.7. *When $\mu = 0$, system* (2.7) *will degenerate into the ideal delay-free system* (2.12)*, and Definition 2.1 can be rewritten as $\|\rho(t)\| \leq \alpha \exp(-\beta(t - t_0))\|\rho(t_0)\|$. In this paper, unless otherwise stated, TSFCNNs* (2.12) *are GES, i.e., $\|\kappa(t)\| \leq \alpha \exp(-\beta(t - t_0))\|\kappa(t_0)\|$ holds $\forall t > t_0$.*

Another assumption is needed to obtain our major results.

Assumption 2.2. *There are positive constants α, β and* ð *such that*

$$2m_2 \exp(2(m_1 + m_2)ð) + \alpha \exp(-\beta ð) < 1 \tag{2.14}$$

holds, where $m_1 = \sum_{k=1}^{l} \|C_k\| + L\sum_{k=1}^{l}$ and $m_2 = \sum_{k=1}^{l} \|B_k\|$.

3. Robustness analysis of the exponential stability of DTSFCNNs

Next, let us analyze definition of global exponential stability of DTSFCNNs (2.6) first.

Theorem 3.1. *Let Assumptions 2.1 and 2.2 hold, then a DTSFCNN* (2.6) *is GES when $\mu \leq \min\{ð/2, \hat{\mu}\}$, where $\hat{\mu}$ satisfies the following equation.*

$$\begin{aligned} m_2\Big\{\hat{\mu}[\alpha/\beta(m_1 + m_2) + m_2] + 2\Big\} \exp\Big\{2ð\Big[m_1 + m_2 \\ + m_2\hat{\mu}(m_1 + m_2)\Big]\Big\} + \alpha \exp(-\beta(ð - \hat{\mu})) = 1, \end{aligned} \tag{3.1}$$

where ð $> \ln \alpha/\beta$, $m_1 = \sum_{k=1}^{l} \|C_k\| + \sum_{k=1}^{l} L\|A_k\|$, and $m_2 = \sum_{k=1}^{l} L\|B_k\|$.

Proof. From (2.6) and (2.11),

$$\kappa(t)-\rho(t)=\int_{t_0}^{t}\sum_{k=1}^{l}\Omega_k(\varpi(\varsigma))\Big\{-C_k\Big(\kappa(\varsigma)-\rho(\varsigma)\Big) + A_k\Big(g(\kappa(\varsigma))-g(\rho(\varsigma))\Big)+B_k\Big(g(\kappa(\varsigma))-g(\rho(\varsigma-\nu))\Big)\Big\}d\varsigma. \tag{3.2}$$

According to (2.10), we can get $\Omega_k(\varpi(t))\leq 1$. Therefore,

$$\begin{aligned}\|\kappa(t)-\rho(t)\| &\leq \int_{t_0}^{t}\Big\{(\sum_{k=1}^{l}\|\Omega_k(\varpi(\varsigma))C_k\|+L\sum_{k=1}^{l}\|\Omega_k(\varpi(\varsigma))A_k\|)\\ &\quad\times\|\kappa(\varsigma)-\rho(\varsigma)\|+L\sum_{k=1}^{l}\|\Omega_k(\varpi(\varsigma))B_k\|\|\kappa(\varsigma)-\rho(\varsigma-\nu)\|\Big\}d\varsigma\\ &\leq\int_{t_0}^{t}m_1\|\kappa(\varsigma)-\rho(\varsigma)\|+m_2\|\kappa(\varsigma)-\rho(\varsigma-\nu)\|d\varsigma\\ &\leq\int_{t_0}^{t}(m_1+m_2)\|\kappa(\varsigma)-\rho(\varsigma)\|+m_2\|\rho(\varsigma)-\rho(\varsigma-\nu)\|d\varsigma,\end{aligned} \tag{3.3}$$

where $m_1=\sum_{k=1}^{l}\|C_k\|+\sum_{k=1}^{l}L\|A_k\|$, and $m_2=\sum_{k=1}^{l}L\|B_k\|$.

Since

$$\rho(\varsigma)-\rho(\varsigma-\nu)\leq\int_{\varsigma-\mu}^{\varsigma}\dot{\rho}(s)ds, \tag{3.4}$$

therefore,

$$\begin{aligned}\|\rho(\varsigma)-\rho(\varsigma-\nu)\| &\leq\int_{\varsigma-\mu}^{\varsigma}\|\dot{\rho}(s)\|ds\\ &\leq\int_{\varsigma-\mu}^{\varsigma}m_1\|\rho(s)\|+m_2\|\rho(s-\nu)\|ds.\end{aligned} \tag{3.5}$$

Thus, when $t>t_0+\mu$,

$$\begin{aligned}&\int_{t_0+\mu}^{t}\|\rho(\varsigma)-\rho(\varsigma-\nu)\|d\varsigma\\ \leq&\int_{t_0+\mu}^{t}\int_{\varsigma-\mu}^{\varsigma}m_1\|\rho(s)\|+m_2\|\rho(s-\nu)\|dsd\varsigma\\ \leq&\int_{t_0}^{t}\int_{\max\{t_0+\mu,s\}}^{\min\{s+\mu,t\}}m_1\|\rho(s)\|+m_2\|\rho(s-\nu)\|d\varsigma ds\\ \leq&\mu\int_{t_0}^{t}m_1\|\rho(s)\|+m_2\|\rho(s-\nu)\|ds\\ \leq&\mu\int_{t_0}^{t}m_1\|\rho(s)-\kappa(s)\|ds+\mu\int_{t_0}^{t}m_1\|\kappa(s)\|ds+\mu\int_{t_0}^{t}m_2\|\rho(s-\nu)\|ds\end{aligned}$$

$$\leq m_1\mu\int_{t_0}^{t}\|\rho(s)-\kappa(s)\|ds+m_1\mu\frac{\alpha}{\beta}\sup_1\|\kappa(s)\|+m_2\mu\sup_1\|\rho(s)\|+m_2\mu\int_{t_0}^{t}\|\rho(s)\|ds$$
$$\leq\mu(m_1+m_2)\int_{t_0}^{t}\|\rho(s)-\kappa(s)\|ds+\mu[\alpha/\beta(m_1+m_2)+m_2]\sup_1\|\rho(s)\|, \tag{3.6}$$

where $\sup_1=\sup_{s\in[t_0-\mu,t_0]}$.

On the other hand,

$$\int_{t_0}^{t_0+\mu}\|\rho(\varsigma)-\rho(\varsigma-\nu)\|d\varsigma\leq\sup_2\|\rho(s)\|+\sup_1\|\rho(s)\|, \tag{3.7}$$

where $\sup_2=\sup_{s\in[t_0,t_0+\mu]}$.

Thus, we have

$$\begin{aligned}&\int_{t_0}^{t}\|\rho(\varsigma)-\rho(\varsigma-\nu)\|d\varsigma\\ \leq&\mu(m_1+m_2)\int_{t_0}^{t}\|\rho(s)-\kappa(s)\|ds+\mu[\alpha/\beta(m_1+m_2)+m_2]\sup_1\|\rho(s)\|\\ &+\sup_2\|\rho(s)\|+\sup_1\|\rho(s)\|\\ \leq&\mu(m_1+m_2)\int_{t_0}^{t}\|\rho(s)-\kappa(s)\|ds+\Big\{\mu[\alpha/\beta(m_1+m_2)+m_2]+2\Big\}\sup_3\|\rho(s)\|,\end{aligned} \tag{3.8}$$

where $\sup_3=\sup_{s\in[t_0-\mu\leq t\leq t_0+\mu]}$.

Then, from (3.3),

$$\begin{aligned}&\|\kappa(t)-\rho(t)\|\\ \leq&\Big\{m_1+m_2+m_2\mu(m_1+m_2)\Big\}\int_{t_0}^{t}\|\rho(s)-\kappa(s)\|ds\\ &+m_2\Big\{\mu[\alpha/\beta(m_1+m_2)+m_2]+2\Big\}\sup_3\|\rho(s)\|\\ =:&m_3\int_{t_0}^{t}\|\rho(s)-\kappa(s)\|ds+m_4\sup_3\|\rho(s)\|.\end{aligned} \tag{3.9}$$

By applying the Gronwall-Bellman lemma, when $t\leq t_0+2ð$,

$$\|\kappa(t)-\rho(t)\|\leq m_4\exp(2ðm_3)\sup_3\|\rho(s)\|. \tag{3.10}$$

Therefore, for $t_0+\mu\leq t\leq t_0+2ð$,

$$\begin{aligned}\|\rho(t)\|\leq&\|\kappa(t)-\rho(t)\|+\|\kappa(t)\|\\ \leq&m_4\exp(2ðm_3)\sup_3\|\rho(s)\|+\alpha\exp(-\beta(t-t_0))\sup_1\|\rho(s)\|.\end{aligned} \tag{3.11}$$

Notice that $\mu\leq ð/2$, and thus, for $t_0-\mu+ð\leq t_0-\mu+2ð$, we have

$$\|\rho(t)\|\leq\Big\{m_4\exp(2ðm_3)+\alpha\exp(-\beta(ð-\mu))\Big\}\sup_4\|\rho(s)\|. \tag{3.12}$$

where $\sup_4 = \sup_{s\in[t_0-\mu\le t\le t_0-\mu+ð]}$.

Let $F(\mu) = m_4\exp(2ðm_3) + \alpha\exp(-\beta(ð-\mu))$. Then, we can easily get that F is strictly increasing with respect to μ, and $F(0) < 1$. Hence, there must be a $\hat{\mu}$ such that $F(\hat{\mu}) = 1$ holds, i.e., when $\mu \le \min\{ð/2, \hat{\mu}\}$, $F(\mu) < 1$ holds.

Take $\Gamma = -\ln F/ð$. Then, from (3.12), we can obtain

$$\sup_5 \|\rho(s)\| \le \exp(-\Gamma ð)\sup_4 \|\rho(s)\|, \tag{3.13}$$

where $\sup_5 = \sup_{s\in[t_0-\mu+ð, t_0-\mu+2ð]}$.

Then, there is a $\xi \in \mathbb{Z}^+$ such that

$$\begin{aligned}\lambda &= \sup_{s\in[t_0+\xi ð-\mu, t_0+(\xi+1)ð-\mu]} \|\rho(s)\| \\ &\le \exp(-\Gamma ð)\sup_{s\in[t_0+(\xi-1)ð-\mu, t_0+\xi ð-\mu]} \|\rho(s)\| \\ &\quad \dots \\ &\le \exp(-\Gamma\xi ð)\sup_5 \|\rho(s)\|.\end{aligned} \tag{3.14}$$

Clearly, for $\forall t > t_0 + ð - \mu$,

$$\|\rho(s)\| \le \exp(\Gamma ð)\exp(-\Gamma(t-t_0))\lambda. \tag{3.15}$$

We can verify this is also true for $t_0 \le t \le t_0 + ð - \mu$. Thus, the DTSFCNN (2.6) is GES when $\mu < \min\{ð/2, \hat{\mu}\}$. □

Remark 3.1. *Compared with the existing results [16, 19–21], this paper combines T-S fuzzy rules with nonlinear differential equations, which makes the system similar to a switching system, where each fuzzy rule corresponds to a definite subsystem. On this basis, the addition of fuzzy rules allows nonlinear systems to be fitted using combinations of linear systems.*

4. Robustness analysis of the stability of stochastic DTSFCNNs

This section will mainly consider the RoS of TSFCNN under the combined effects of time delays and stochastic disturbances.

We consider the following DSTSFCNNs under the same fuzzy rules of DTSFCNNs (2.6).

Plant Rule k :
IF $\varpi_1(t)$ **is** υ_1^k **and** $\varpi_2(t)$ **is** υ_2^k **and** $\dots$ **and** $\varpi_m(t)$ **is** υ_m^k
THEN

$$\begin{cases} d\wp(t) = \Big[-C_k\wp(t) + A_k g(\wp(t)) + B_k g(\wp(t-\nu))\Big]dt + \ell H_k\wp(t)d\varsigma(t), \\ \wp(t) = \psi(t), \quad t\in[-\mu, 0], \quad k = 1, 2, \dots, l, \end{cases} \tag{4.1}$$

where ℓ represents the intensity of stochastic disturbances. H_k is a constant matrix.

Remark 4.1. *Since the errors are generated by rounding when doing numerical calculations, although these errors are determined, they are actually impossible to calculate. The way to overcome this phenomenon is to model these errors as small random perturbations of the system, for example, large-scale deterministic systems used to model climate evolution may exhibit "noise-like" characteristics in some subsystems on short time scales. By using noise to model these subsystems, calculations can be greatly accelerated. Hence, under normal circumstances, random factors must be considered in the system to be more realistic.*

Similarly, the defuzzied model is as follows.

$$d\wp(t) = \sum_{k=1}^{l} \Omega_k(\varpi(t))\Big\{\Big[-C_k\wp(t) + A_k g(\wp(t)) + B_k g(\wp(t-\nu))\Big]dt + \ell H_k \wp(t) d\varsigma(t)\Big\}. \tag{4.2}$$

Next, we introduce the definition exponential stability of DSTSFCNN (4.1) in mean square.

Definition 4.1. *DSTSFCNN* (4.1) *is mean square exponentially stable (MSES) if there exist* $\omega > 0$, $\varrho > 0$, *such that*

$$E\|\wp(t)\|^2 \le \omega^2 \sup_{s\in[t_0-\mu,t_0]} E\|\wp(s)\|^2 \exp(-2\varrho(t-t_0)) \tag{4.3}$$

holds.

Theorem 4.1. *Let Assumption 2.1 hold, DSTSFCNN* (4.1) *is MSES if* $|\ell| < \hat{\ell}/\sqrt{2}$ *and* $\mu < \min\{\eth/2, \hat{\mu}\}$, *where* $\hat{\ell}$ *and* $\hat{\mu}$ *satisfies two equations listed below, respectively.*

$$12\hat{\ell}^2 m_5^2 \exp\Big\{2\eth\Big[6\eth(m_1^2 + 2m_2^2) + 6\hat{\ell}^2 m_5^2\Big]\Big\} + 2\alpha^2 \exp(-2\beta\eth) = 1, \tag{4.4}$$

$$\begin{aligned} &2\Big\{12\eth m_2^2\Big[\hat{\mu}[3\hat{\mu}m_1^2 + 3m_5^2\hat{\ell}^2/2]\alpha^2/\beta + 3\hat{\mu}^2 m_2^2 + 4\hat{\mu}\Big] + 3\hat{\ell}^2 m_5^2\Big\} \\ &\times \exp\Big\{2\eth\Big[6\eth(m_1^2 + 2m_2^2) + 3\hat{\ell}^2 m_5^2 + 12\eth m_2^2(6\hat{\mu}(\hat{\mu}m_1^2 + m_5^2\hat{\ell}^2/2))\Big]\Big\} \\ &+2\alpha^2 \exp(-2\beta(\eth - \hat{\mu})) = 1, \end{aligned} \tag{4.5}$$

where m_1 *and* m_2 *are defined in Theorem 3.1, and* $m_5 = \sum_{k=1}^{l} \|H_k\|$.

Proof. From (4.2) and (2.7), we can obtain

$$\begin{aligned} \wp(t) - \kappa(t) = &\int_{t_0}^{t} \sum_{k=1}^{l} \Omega_k(\varpi(s))\Big\{-C_k(\wp(s) - \kappa(s)) + A_k(g(\wp(s)) - g(\kappa(s))) \\ &+ B_k(g(\wp(s-\nu)) - g(\kappa(s)))\Big\}ds + \int_{t_0}^{t} \sum_{k=1}^{l} \Omega_k(\varpi(t))\Big[\ell H_k \wp(s)\Big]d\varsigma(s). \end{aligned} \tag{4.6}$$

Thus,

$$\|\wp(t) - \kappa(t)\| \le \int_{t_0}^{t} m_1\|\wp(s) - \kappa(s)\| + m_2\|\wp(s-\nu) - \kappa(s)\|dt + \int_{t_0}^{t} m_5\ell\|\wp(s)\|d\varsigma(s), \tag{4.7}$$

where $m_5 = \sum_{k=1}^{l} \|H_k\|$.

By utilizing the *Itô* formula, when $t \le t_0 + 2ð$,

$$\begin{aligned}
&E\|\wp(t) - \kappa(t)\|^2 \\
\le& 6ð m_1^2 \int_{t_0}^{t} E\|\wp(s) - \kappa(s)\|^2 ds + 6ð m_2^2 \int_{t_0}^{t} E\|\wp(s-\nu) - \kappa(s)\|^2 ds \\
&+ 3\ell^2 m_5^2 \int_{t_0}^{t} E\|\wp(s)\|^2 ds \\
\le& 6ð(m_1^2 + 2m_2^2) \int_{t_0}^{t} E\|\wp(s) - \kappa(s)\|^2 ds + 12ð m_2^2 \int_{t_0}^{t} E\|\wp(s-\nu) - \wp(s)\|^2 ds \\
&+ 6\ell^2 m_5^2 \int_{t_0}^{t} E\|\wp(s) - \kappa(s)\|^2 + E\|\kappa(s)\|^2 ds \\
\le& \Big[6ð(m_1^2 + 2m_2^2) + 6\ell^2 m_5^2\Big] \int_{t_0}^{t} E\|\wp(s) - \kappa(s)\|^2 ds \\
&+ 12ð m_2^2 \int_{t_0}^{t} E\|\wp(s-\nu) - \wp(s)\|^2 ds + 6\ell^2 m_5^2 \int_{t_0}^{t} E\|\kappa(s)\|^2 ds. \qquad (4.8)
\end{aligned}$$

When $t > t_0 + \mu$,

$$\begin{aligned}
&\int_{t_0+\mu}^{t} E\|\wp(s-\nu) - \wp(s)\|^2 ds \\
\le& \int_{t_0+\mu}^{t} \int_{s-\mu}^{s} \Big\{\Big[3\mu m_1^2 + 3\ell^2 m_5^2\Big] E\|\wp(s)\|^2 + 3\mu m_2^2 E\|\wp(s-\nu)\|^2\Big\} d\vartheta ds \\
\le& \int_{\max\{\vartheta, t_0+\mu\}}^{\min\{\vartheta+\mu, t\}} \int_{t_0}^{t} \Big\{\Big[3\mu m_1^2 + 3\ell^2 m_5^2\Big] E\|\wp(s)\|^2 + 3\mu m_2^2 E\|\wp(s-\nu)\|^2\Big\} ds d\vartheta \\
\le& \mu\Big[3\mu m_1^2 + 3\ell^2 m_5^2\Big] \int_{t_0}^{t} E\|\wp(s)\|^2 ds + 3\mu^2 m_2^2 \int_{t_0}^{t} E\|\wp(s-\nu)\|^2 ds. \qquad (4.9)
\end{aligned}$$

Let $m_6 = 6\mu\Big[\mu m_1^2 + \ell^2 m_5^2\Big]$, and $m_7 = 3\mu^2 m_2^2$. Thus, we can obtain

$$\begin{aligned}
&\int_{t_0+\mu}^{t} E\|\wp(s-\nu) - \wp(s)\|^2 ds \\
\le& m_6 \int_{t_0}^{t} E\|\wp(s) - \kappa(s)\|^2 ds + m_6 \int_{t_0}^{t} E\|\kappa(s)\|^2 ds + m_7 \int_{t_0-\mu}^{t_0} E\|\wp(s)\|^2 ds \\
\le& m_6 \int_{t_0}^{t} E\|\wp(s) - \kappa(s)\|^2 ds + \Big(m_6 \frac{\alpha^2}{2\beta} + m_7\Big) \sup_1 E\|\wp(s)\|^2, \qquad (4.10)
\end{aligned}$$

where $\sup_1 = \sup_{s\in[t_0-\mu, t_0]}$.

In addition,

$$\int_{t_0}^{t_0+\mu} E\|\wp(s-\nu) - \wp(s)\|^2 ds \le 2\mu \sup_1 E\|\wp(s)\|^2 + 2\mu \sup_2 E\|\wp(s)\|^2, \qquad (4.11)$$

where $\sup_2 = \sup_{s\in[t_0,t_0+\mu]}$.

Hence,

$$\begin{aligned}
&E\|\wp(t)-\kappa(t)\|^2\\
\le&\Big[6ð(m_1^2+2m_2^2)+6\ell^2m_5^2\Big]\int_{t_0}^{t}E\|\wp(s)-\kappa(s)\|^2ds\\
&+12ðm_2^2\int_{t_0}^{t}E\|\wp(s-\nu)-\wp(s)\|^2ds+6\ell^2m_5^2\int_{t_0}^{t}E\|\kappa(s)\|^2ds\\
\le&\Big[6ð(m_1^2+2m_2^2)+6\ell^2m_5^2\Big]\int_{t_0}^{t}E\|\wp(s)-\kappa(s)\|^2ds\\
&+12ðm_2^2\Big[m_6\int_{t_0}^{t}E\|\wp(s)-\kappa(s)\|^2ds+\Big(m_6\frac{\alpha^2}{2\beta}+m_7\Big)\sup_1 E\|\wp(s)\|^2\\
&+4\mu\sup_3 E\|\wp(s)\|^2\Big]+6\ell^2m_5^2\int_{t_0}^{t}E\|\kappa(s)\|^2ds\\
=:&m_8\int_{t_0}^{t}E\|\wp(s)-\kappa(s)\|^2ds+m_9\sup_3 E\|\wp(s)\|^2,
\end{aligned}\tag{4.12}$$

where

$$\begin{aligned}
&m_8=6ð(m_1^2+2m_2^2)+6\ell^2m_5^2+12ðm_2^2m_6,\\
&m_9=12ðm_2^2(m_6\alpha^2/2\beta+m_7+4\mu)+3\ell^2m_5^2\alpha^2/\beta,\\
&\sup_3=\sup_{s\in[t_0-\mu,t_0+\mu]}.
\end{aligned}$$

Therefore, when $t_0+\mu\le t\le t_0+2ð$, using the Gronwall-Bellman lemma, we can get

$$E\|\wp(t)-\kappa(t)\|^2\le m_9\exp(2m_8ð)\sup_3 E\|\wp(s)\|^2.\tag{4.13}$$

Then, for $t_0-\mu+ð\le t\le t_0-\mu+2ð$

$$\begin{aligned}
E\|\wp(t)\|^2\le&2E\|\wp(t)-\kappa(t)\|^2+2E\|\kappa(t)\|^2\\
\le&2\Big[m_9\exp(2m_8ð)+\alpha^2\exp(-2\beta(ð-\mu))\Big]\sup_4 E\|\wp(s)\|^2,
\end{aligned}\tag{4.14}$$

where $\sup_4=\sup_{s\in[t_0-\mu,t_0-\mu+ð]}$.

Select $\aleph(\mu,\ell)=m_9\exp(2m_8ð)+\alpha^2\exp(-2\beta(ð-\mu))$. Then, $\aleph(0,\ell)$ is strictly increasing with respect to ℓ. Hence, there exists an $\hat{\ell}>0$ such that $\aleph(0,\hat{\ell})=1$. Besides, when $\ell=\sqrt{2}$, $\aleph(\mu,\ell/\sqrt{2})$ is also strictly increasing for μ, and thus there exists a $\hat{\mu}$ such that $\aleph(\hat{\mu},\hat{\ell}/\sqrt{2})=1$. That is, $\aleph(\mu,\ell)<1$ when $\mu\le\min\{ð/2,\hat{\mu}\}$ and $\ell\le\hat{\ell}/\sqrt{2}$.

Furthermore, the remaining part of the proof is exactly the same with Theorem 3.1, and hence it is omitted here. Hence, a DSTSFCNN (4.1) is MSES when $\mu\le\min\{ð/2,\hat{\mu}\}$, and $\ell\le\hat{\ell}/\sqrt{2}$. □

Remark 4.2. *Figure 2 shows the detailed analysis steps of Theorems 3.1 and 4.1. In addition, in the derivation of Theorem 4.1, due to the existence of random perturbations, the number of terms of the* *system increases, which also makes the inequality used in Theorem 4.1 and Theorem 3.1 have some differences, so Theorem 4.1 is not a simple superposition of Theorem 3.1. Besides, if there is only one fuzzy rule in* (2.3) *and* (4.2)*, then* (2.3) *and* (4.2) *can be seen as certain neural networks, and we can obtain the Theorems 2 and 3 of [19] respectively.*

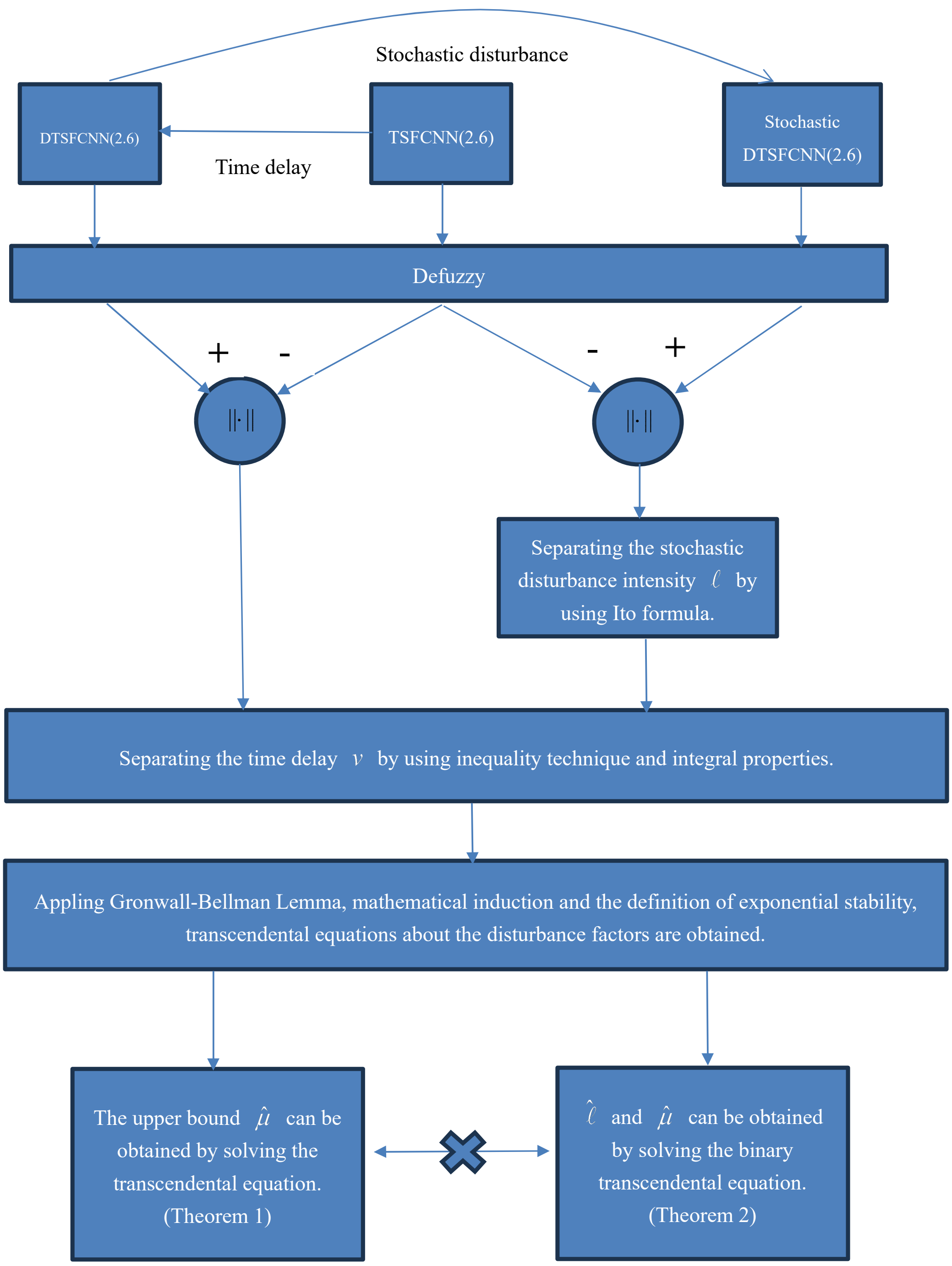

Figure 2. The steps of the analysis of Theorems 1 and 2.

Remark 4.3. *Table 1 provides a comparison of the existing literature with this paper. Elements to be compared are time delays (T-D), stochastic disturbances (S-D), RoS, T-S type fuzzy model (T-S fuzzy), global asymptotic stability (GAS), GES and MSES.*

Table 1. The differences between this paper and several existing literature.

	T-S fuzzy	T-D	S-D	RoS	GAS	GES	MSES
Singh et al. [9] (2021)	-	✔	-	-	-	-	-
Li et al. [12] (2022)	-	✔	-	-	✔	-	-
Almarri et al. [13] (2022)	-	✔	-	-	-	-	-
Si et al. [20] (2021)	-	-	✔	✔	-	✔	✔
Wenxiang et al. [22] (2023)	-	-	✔	✔	-	✔	✔
Zhang et al. [24] (2023)	-	-	✔	✔	-	✔	✔
This paper	✔	✔	✔	✔	-	✔	✔

5. Examples

Example 1. We take the following DTSFCNN model into account:

$$
\begin{aligned}
&\textbf{Plant Rule } 1:\\
&\textbf{IF } \varpi_1(t) \textbf{ is } \upsilon_1^1 \textbf{ and } \varpi_2(t) \textbf{ is } \upsilon_2^1\\
&\textbf{THEN}\\
&\begin{cases} \dot{\rho}(t) = -C_1\rho(t) + A_1 g(\rho(t)) + B_1 g(\rho(t-\nu)), \\ \rho(t) = \psi(t), \quad t \in [-\mu, 0], \end{cases}\\
&\textbf{Plant Rule } 2:\\
&\textbf{IF } \varpi_1(t) \textbf{ is } \upsilon_1^2 \textbf{ and } \varpi_2(t) \textbf{ is } \upsilon_2^2\\
&\textbf{THEN}\\
&\begin{cases} \dot{\rho}(t) = -C_2\rho(t) + A_2 g(\rho(t)) + B_2 g(\rho(t-\nu)), \\ \rho(t) = \psi(t), \quad t \in [-\mu, 0], \end{cases}
\end{aligned} \tag{5.1}
$$

where

$$
C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1.2 \end{bmatrix}, C_2 = \begin{bmatrix} 1.1 & 0 \\ 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} 0.01 & -0.01 \\ 0.01 & -0.01 \end{bmatrix},
$$
$$
A_2 = \begin{bmatrix} -0.01 & 0.01 \\ 0.01 & 0.01 \end{bmatrix}, B_1 = \begin{bmatrix} 0.01 & 0.01 \\ 0.01 & -0.01 \end{bmatrix}, B_2 = \begin{bmatrix} -0.01 & -0.01 \\ -0.01 & 0.01 \end{bmatrix}.
$$

Taking the follow functions as the fuzzy membership functions:

$$
\Omega_1(\varpi(t)) = \frac{\gamma_1(\varpi(t))}{\sum_{k=1}^{2} \gamma_k(\varpi(t))}, \quad \Omega_2(\varpi(t)) = \frac{\gamma_2(\varpi(t))}{\sum_{k=1}^{2} \gamma_k(\varpi(t))},
$$

where $\hbar_1 = \rho_1(t) + 3, \hbar_2 = \rho_2(t) + 3$, and

$$
\gamma_1(\varpi(t)) = \exp\Big[-\frac{\hbar_1^2}{1.4426}\Big]\exp\Big[-\frac{\hbar_2^2}{1.4426}\Big],
$$

$$\gamma_2(\varpi(t)) = \exp\left[-\frac{\hbar_1^2}{1.4426}\right]\exp\left[-\frac{(\hbar_2-6)^2}{1.4426}\right].$$

Let $g(\cdot) = \tanh(\cdot)$, $\alpha = 1, \beta = 0.9$, $\eth = 0.2$ and $L = 1$. Then, we can obtain that

$$0.0566\exp(0.9450) + \exp(-0.18) = 0.9808 < 1, \tag{5.2}$$

and from Theorem 3.1, we have

$$(0.075\hat{\mu} + 0.0566)\exp(0.4(2.3624 + 0.0668\hat{\mu})) + \exp(-0.9(0.2-\hat{\mu})) = 1. \tag{5.3}$$

Therefore, we can get that $\hat{\mu} = 0.0201$ by solving Eq (5.3). Thus, the DTSFCNN (5.1) is GES when $\mu < \min\{\eth/2, \hat{\mu}\}$.

Figure 3 shows the relationship between the states of the DTSFCNN (5.1) and the membership functions $\Omega_k(\varpi(t))$. We can find that, as the state values decrease, the values of both membership functions tend to approach 0.5. In Figure 4, we choose $\mu = 0.015 < 0.0201$, and then the DTSFCNN is GES.

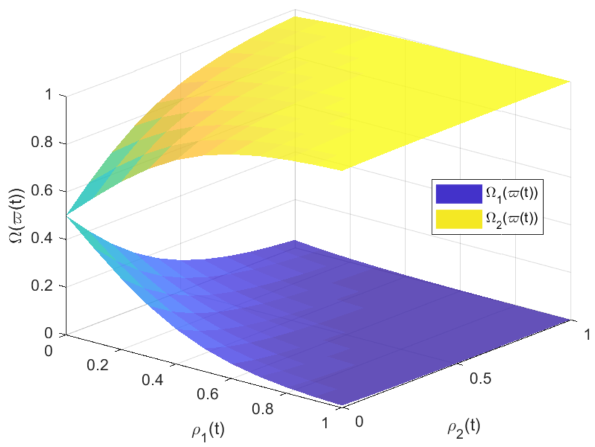

Figure 3. Values of membership functions $\Omega_k(\varpi(t))$.

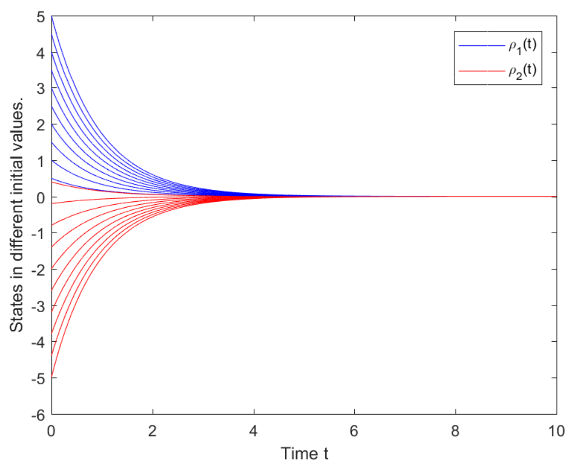

Figure 4. The states of the DTSFCNN (5.1) with $\mu = 0.015$ for different initial values.

Example 2. The DSTSFCNN model we consider is of the following form:

Plant Rule 1 :
IF $\varpi_1(t)$ **is** υ_1^1 **and** $\varpi_2(t)$ **is** υ_2^1
THEN

$$\begin{cases} d\wp(t) = \Big[-C_1\wp(t) + A_1 g(\wp(t)) + B_1 g(\wp(t-\nu))\Big]dt + \ell H_1\wp(t)d\varsigma(t), \\ \wp(t) = \psi(t), \quad t \in [-\mu, 0], \end{cases}$$

Plant Rule 2 :
IF $\varpi_1(t)$ **is** υ_1^2 **and** $\varpi_2(t)$ **is** υ_2^2
THEN

$$\begin{cases} d\wp(t) = \Big[-C_2\wp(t) + A_2 g(\wp(t)) + B_2 g(\wp(t-\nu))\Big]dt + \ell H_2\wp(t)d\varsigma(t), \\ \wp(t) = \psi(t), \quad t \in [-\mu, 0], \end{cases} \tag{5.4}$$

where membership function $\Omega_k(\varpi(t))$ is the same as that in Example 1.

Accordingly, the following is the defuzzied DSTSFCNN (5.4).

$$d\wp(t) = \sum_{k=1}^{l} \Omega_k(\varpi(t))\Big\{\Big[-C_k\wp(t) + A_k g(\wp(t)) + B_k g(\wp(t-\nu))\Big]dt + \ell H_k\wp(t)d\varsigma(t)\Big\}, \tag{5.5}$$

where

$$C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} -0.01 & 0.01 \\ -0.01 & -0.01 \end{bmatrix}, A_2 = \begin{bmatrix} -0.01 & 0.01 \\ 0.01 & 0.01 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 0.01 & -0.01 \\ 0.01 & -0.01 \end{bmatrix}, B_2 = \begin{bmatrix} -0.01 & 0.01 \\ 0.01 & 0.01 \end{bmatrix}, H_1 = \begin{bmatrix} 0.1 & -0.1 \\ 0.1 & 0.1 \end{bmatrix}, H_2 = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{bmatrix}.$$

Take $\alpha = 1$, $\beta = 0.9$, $\eth = 0.4$ and $L = 1$. Then, we can get the following two equations from Theorem 4.1:

$$1.92\hat{\ell}^2 \exp(7.9032 + 0.7680\hat{\ell}^2) + 0.9735 = 1, \tag{5.6}$$

and

$$\begin{aligned} &\Big\{0.0112\Big[1.1111\hat{\mu}[12.3418\hat{\mu} + 0.24\hat{\ell}^2] + 0.0035\hat{\mu}^2 + 4\hat{\mu}\Big] + 1.0667\hat{\ell}^2\Big\} \\ &\times \exp\Big\{7.9032 + 0.3840\hat{\ell}^2 + 0.0045(6\hat{\mu}(4.1189\hat{\mu} + 0.08\hat{\ell}^2))\Big\} \\ &+ 2\exp(-1.8(0.4 - \hat{\mu})) = 1. \end{aligned} \tag{5.7}$$

Take $\zeta(\mu, \ell) = \aleph(\mu, \ell) - 1$, $\hat{\zeta}(\ell) = \aleph(0, \ell) - 1$ and $\bar{\zeta}(\mu) = \aleph(\mu, \hat{\ell}/\sqrt{2}) - 1$. Then, the problem of solving Eqs (5.6) and (5.7) can be transformed into finding the zero point of functions $\hat{\zeta}(\ell)$ and $\bar{\zeta}(\mu)$. In addition, Figure 5 shows the values of functions $\zeta(\mu, \ell)$, $\hat{\zeta}(\ell)$ and $\bar{\zeta}(\mu)$, respectively.

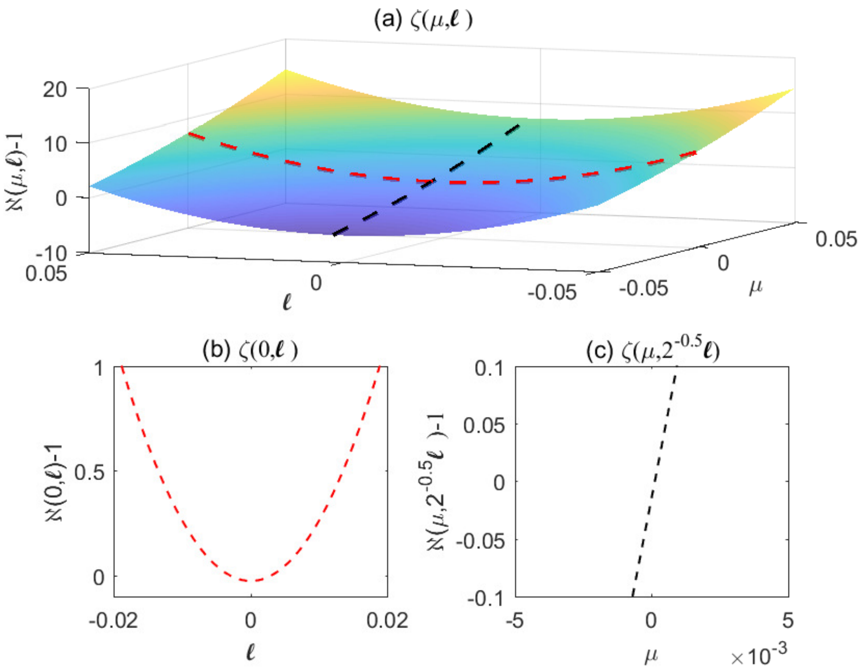

Figure 5. Values of functions $\zeta(\mu, \ell)$, $\hat{\zeta}(\ell)$ and $\bar{\zeta}(\mu)$.

From Figure 5b,c and Eqs (5.6) and (5.7), we can easily obtain that $\hat{\ell} = 0.0030$ and $\hat{\mu} = 1.0777 \times 10^{-4}$. This means that the system can remain globally exponentially stable when the perturbation intensity is less than the upper bounds derived in this paper.

In Figure 6, we take $\ell = 0.0001 < 0.003$ and $\mu = 0.0001 < 1.0777 \times 10^{-4}$, and hence, the states of (5.5) with different initial values are MSES. Figure 7 depicts the states of (5.5) in the sense of mean square. When time delay and stochastic noise intensity are all bigger than the results we derive, that is, when $\ell = 0.01$ and $\mu = 0.001$, (5.5) exhibits unstable behavior and cannot maintain global exponential stability. This can easily be seen in Figure 8.

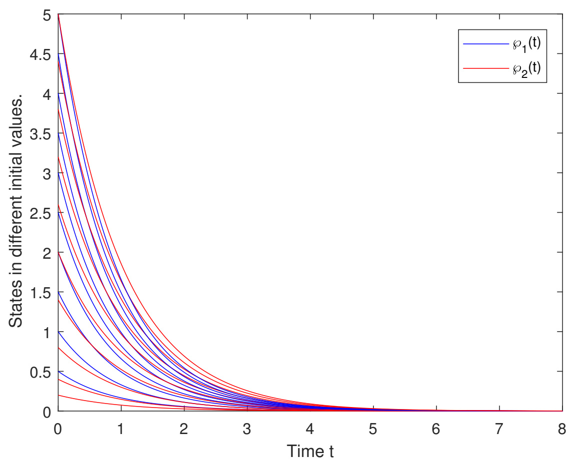

Figure 6. The states of (5.5) with $\ell = 0.0001$, $\mu = 0.0001$ for different initial values.

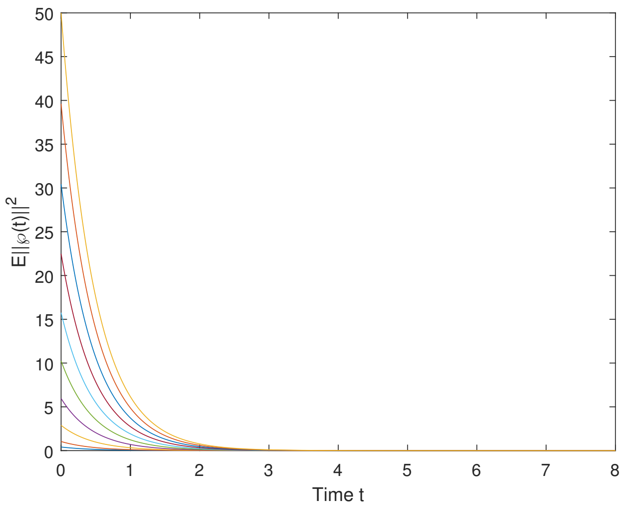

Figure 7. The values of (5.5) with $\ell = 0.0001$, $\mu = 0.0001$ in the sense of mean square.

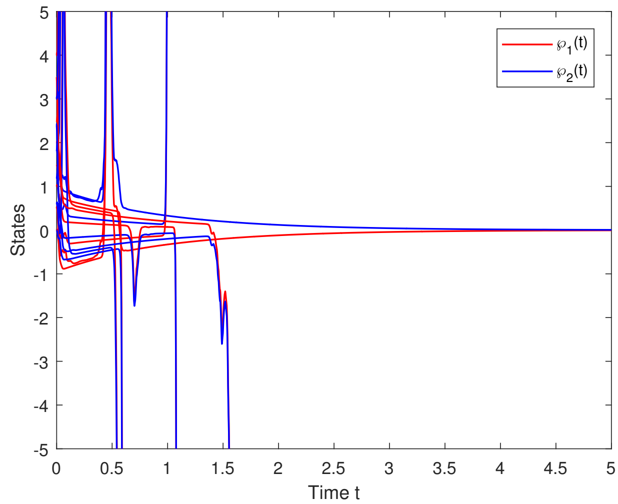

Figure 8. The states of (5.5) with $\hat{\mu} = 0.001$, $\hat{\ell} = 0.01$ for different initial values.

Figures 9 and 10 show that when the strength of one of the disturbance factors is greater than the upper bound derived by Theorem 4.1, the system still cannot maintain global exponential stability. Therefore, it can be seen that the system can maintain exponential stability only when the strength of both disturbances is smaller than the upper bound derived in this paper. This also fully proves the validity of the results obtained in this paper.

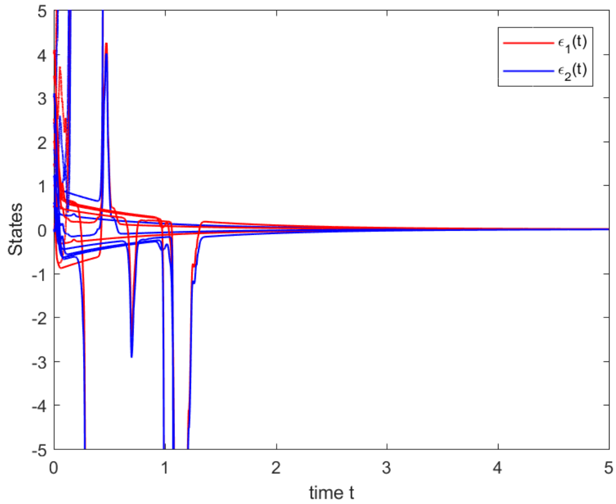

Figure 9. The states of (5.5) with $\hat{\mu} = 0.0001$, $\hat{\ell} = 0.01$ for different initial values.

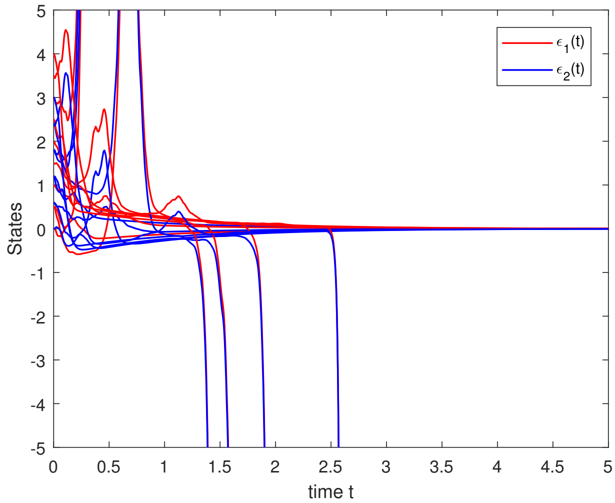

Figure 10. The states of (5.5) with $\hat{\mu} = 0.1$, $\hat{\ell} = 0.0001$ for different initial values.

Remark 5.1. *These examples show that if the requirements in Theorems 3.1 and 4.1 are met, the perturbed systems are globally exponentially stable. The transcendental equations satisfied by the maximum delay and maximum noise intensity are derived by inequality technology, and they can be easily solved by MATLAB, so the conditions in the theorem are easy to verify. In addition, Figure 11 shows the brief calculation process of Examples 1 and 2.*

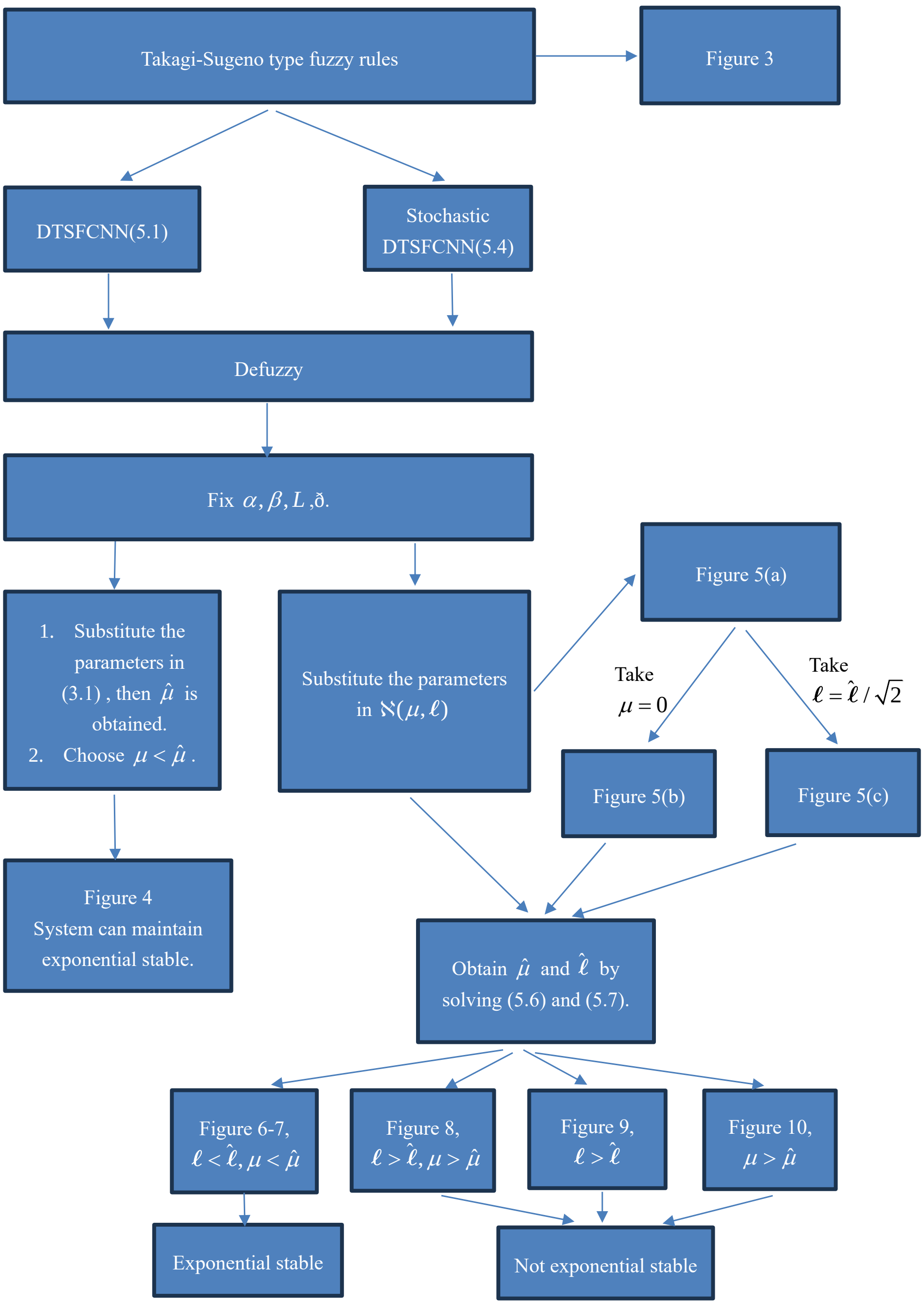

Figure 11. The calculation process of Examples 1 and 2.

Remark 5.2. *Since the inequality technique is mainly used in this paper, error is inevitable. Therefore, the upper bound of the perturbation obtained in this paper is only a sufficient condition for the system to maintain exponential stability. Furthermore, due to the monotonically increasing nature of the exponential function, this leads to the fact that Assumption 2.2 is not easily satisfied. Future research will focus on using the new inequalities to weaken the conservativeness of Assumption 2.2 and further derive more accurate upper bounds of perturbations.*

6. Conclusions

In short, we analyze the robustness of the stability of recurrent neural networks with T-S fuzzy rules. First, we decouple the proposed neural network models. Second, we discuss the maximum delay length and maximum noise intensity of DTSFCNNs and DSTSFCNNs by using the inequality technique, stochastic analysis and algebraic methods. Moreover, numerical examples show that the disturbed system can still maintain exponential stability when the intensity of the perturbations is lower than the results derived in this paper.

Future research may concentrate on merging well-known methodologies such as the LMI method, Lyapunov theory, and so on to lessen the conservativeness of this work. Besides, on the basis of this paper, we can continue to consider the uncertainties in the modeling, such as deviating parameters and Markov switching. We can also continue to consider more complex neural network models, such as higher-order neural networks, neutral neural networks, etc.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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