Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(12): 31092–31103.
DOI: 10.3934/math.20231591
Received: 23 September 2023
Revised: 04 November 2023
Accepted: 15 November 2023
Published: 21 November 2023

Research article

$W^{1,\infty}$-seminorm superconvergence of the block finite element method for the five-dimensional Poisson equation

Jinghong Liu*

School of Science, Guangdong University of Petrochemical Technology, Maoming 525000, China

* **Correspondence:** Email: jhliu1129@sina.com.

Abstract: This study focused on the superconvergence of the finite element method for the five-dimensional Poisson equation in the $W^{1,\infty}$-seminorm. Specifically, we investigated the block finite element method, which is a tensor-product finite element approach applied to regular rectangular partitions of the domain. First, we introduced the finite element scheme for the equation and discussed various functions related to it, along with their properties. Next, we proposed a weight function and established its important properties, which play a crucial role in the theoretical analysis. By utilizing the properties of the weight function and employing weighted-norm analysis techniques, we derived an optimal order estimate in the $W^{2,1}$-seminorm for the discrete derivative Green's function (DDGF). Furthermore, we provided an interpolation fundamental estimate of the second type, also known as the weak estimate of the second type, for the block finite element. This weak estimate is based on a five-dimensional interpolation operator of the projection type. Finally, by combining the derived $W^{2,1}$-seminorm estimate for the DDGF and the weak estimate for the block finite element, we obtained a superconvergence estimate for the block finite element approximation in the pointwise sense of the $W^{1,\infty}$-seminorm. The correctness of the theoretical results was demonstrated through a numerical example.

Keywords: discrete derivative Green's function (DDGF); block finite element; superconvergence; interpolation operator of the projection type
Mathematics Subject Classification: 65N30

1. Introduction

Superconvergence is a crucial topic in the research field of Galerkin finite element methods. With advancements in research technologies, numerous superconvergence results have been obtained and a theoretical framework on superconvergence has been established [1–4]. Notably, superconvergence research in the finite element method is closely associated with the dimensions of the problems.

For one- and two-dimensional problems, superconvergence research has been nearly perfect, and significant progress has also been made for the three-dimensional setting [5–17]. However, for dimensions four and higher, superconvergence results are relatively scarce. One of the primary reasons is the difficulty in estimating the discrete Green's function (DGF) and discrete derivative Green's function (DDGF), which play critical roles in superconvergence research, particularly in pointwise superconvergence [10,12–14,18–20]. Unfortunately, the high-dimensional DGF and DDGF of dimensions four and higher cannot be straightforwardly extrapolated from those of low-dimensional cases. Consequently, establishing bounds for the high-dimensional DGF and DDGF becomes a vital aspect of researching high-dimensional pointwise superconvergence. Recently, we have obtained some results in this regard [12,14,19]. In this paper, we will illustrate a superconvergence analytic technique by considering a five-dimensional problem as an example, which may be applied to other high-dimensional superconvergence issues. First, by utilizing a weighted-function analysis technique, we will derive an optimal estimate for the $W^{2,1}$-seminorm of the five-dimensional DDGF. Second, in conjunction with a weak estimate for the finite element approximation, we can obtain a pointwise superconvergence estimate for the derivatives of the finite element approximation. Finally, we validate the theoretical results through a numerical example. It is important to note that although we provide the $W^{1,1}$-seminorm estimate for the five-dimensional DDGF, we do not discuss its applications to superconvergence [19]. Furthermore, the $W^{2,1}$-seminorm estimate for the DDGF in this paper is distinct from the $W^{1,1}$-seminorm estimate in [19]. Additionally, as high-dimensional partial differential equations (PDEs) in dimensions four and higher are encountered in various areas such as financial mathematics, particle physics, statistical physics and general relativity, research on high-dimensional finite element superconvergence holds significant importance.

In the following, we adhere to the standard notations for the Sobolev spaces and their norms. The symbol C represents a constant that is independent of the discretization parameter h.

Consider the following Poisson equation with the Dirichlet boundary condition:

$$\mathcal{L}u \equiv -\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1.1}$$

where $\Omega \subset \mathcal{R}^5$ is a bounded polytopic domain. The weak formulation of problem (1.1) is finding $u \in H_0^1(\Omega)$ such that

$$a(u,\, v) = (f,\, v) \;\forall v \in H_0^1(\Omega), \tag{1.2}$$

where

$$a(u,\, v) \equiv \int_\Omega \nabla u \cdot \nabla v \, dx_1 dx_2 dx_3 dx_4 dx_5, \quad (f,\, v) \equiv \int_\Omega f v \, dx_1 dx_2 dx_3 dx_4 dx_5.$$

Let $\{\mathcal{T}^h\}$ be a regular family of rectangular partitions of $\bar{\Omega}$. Denote by $S^h(\Omega)$ the space of continuous piecewise tensor-product m-order polynomials with respect to this kind of partition and let $S_0^h = S^h(\Omega) \cap H_0^1(\Omega)$. Discretizing problem (1.2) using S_0^h as approximating space means finding $u_h \in S_0^h$ such that $a(u_h\,,\, v) = (f\,,\, v)$ for all $v \in S_0^h$. Here, we have the following Galerkin orthogonality relation:

$$a(u - u_h\,,\, v) = 0 \;\; \forall\, v \in S_0^h. \tag{1.3}$$

For every $Z \in \Omega$, we define the discrete derivative δ function $\partial_{Z,\ell}\delta_Z^h \in S_0^h$ and the L^2-projection $P_h u \in S_0^h$ such that [19]

$$(v,\, \partial_{Z,\ell}\delta_Z^h) = \partial_\ell v(Z) \;\; \forall\, v \in S_0^h,$$

$$(u - P_h u,\ v) = 0\ \ \forall\, v \in S_0^h,$$

where $\ell \in \mathcal{R}^5$ and $|\ell| = 1$. $\partial_\ell v(Z)$ stands for the one-sided directional derivative

$$\partial_\ell v(Z) = \lim_{|\Delta Z| \to 0} \frac{v(Z + \Delta Z) - v(Z)}{|\Delta Z|},\ \ \Delta Z = |\Delta Z|\ell.$$

Remark 1.1. Since $\Delta Z = |\Delta Z|\ell$, that is, ΔZ is of the same direction as ℓ, provided that the direction ℓ is given for, the above limit exists, no matter what direction is given, $\partial_\ell v(Z)$ is well defined.

Let $\partial_{Z,\ell} G_Z^* \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of the elliptic problem $\mathcal{L}\partial_{Z,\ell} G_Z^* = \partial_{Z,\ell}\delta_Z^h$. We may call $\partial_{Z,\ell} G_Z^*$ the regularized derivative Green's function (RDGF), and denote by $\partial_{Z,\ell} G_Z^h$ (the so-called DDGF) the finite element approximation to $\partial_{Z,\ell} G_Z^*$. Thus,

$$a(\partial_{Z,\ell} G_Z^* - \partial_{Z,\ell} G_Z^h, v) = 0\ \ \forall\, v \in S_0^h.$$

One of the main tasks of this article is how to obtain the optimal estimate for the $\partial_{Z,\ell} G_Z^h$.

As for $\partial_{Z,\ell}\delta_Z^h$ and P_h, we have [19]

$$\|\partial_{Z,\ell}\delta_Z^h\|_{\phi^{-\alpha}} \le C h^{\frac{5\alpha - 7}{2}},\ \ \alpha > 0, \tag{1.4}$$

$$\|P_h w\|_{1,q} \le C\|w\|_{1,q},\ \ 5 < q \le \infty. \tag{1.5}$$

The rest of the paper is arranged as follows. In section two, we will bound the terms $|\partial_{Z,\ell} G_Z^*|_{2,1}$ and $|\partial_{Z,\ell} G_Z^h|_{2,1}$. Section three is devoted to the weak estimate of the second type for the finite element method, and section four is about $W^{1,\infty}$-seminorm superconvergence of the finite element approximation. A numerical example is given in section five. Finally, we simply summarize the paper in section six.

2. The $W^{2,1}$-seminorm estimate for the DDGF

To derive the estimate for the DDGF, we introduce the weight function defined by

$$\phi \equiv \phi(X) = (|X - \bar{X}|^2 + \theta^2)^{-\frac{5}{2}}\ \ \forall\, X \in \bar{\Omega},$$

where $\bar{X} \in \bar{\Omega}$ is a fixed point, $\theta = \gamma h$ and $\gamma \in [5, +\infty)$ is a suitable real number.

For every $\alpha \in \mathcal{R}$, we give the following notations:

$$|\nabla^n v|^2 = \sum_{|\beta| = n} |D^\beta v|^2,\ \ |\nabla^n v|_{\phi^\alpha,\Omega} = \Big(\int_\Omega \phi^\alpha |\nabla^n v|^2 dX\Big)^{\frac{1}{2}},\ \ \|v\|_{m,\phi^\alpha,\Omega}^2 = \sum_{n=0}^{m} |\nabla^n v|_{\phi^\alpha,\Omega}^2,$$

where $\beta = (\beta_1, \beta_2, \beta_3, \beta_4, \beta_5)$, $|\beta| = \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5$ and $\beta_i \ge 0$, $i = 1, 2, 3, 4, 5$ are integers. In particular, for the case of $m = 0$, we write

$$\|v\|_{\phi^\alpha,\Omega} = \Big(\int_\Omega \phi^\alpha |v|^2\, dX\Big)^{\frac{1}{2}}.$$

As for the weight function ϕ, we have the properties [19]

$$|\nabla^n \phi^\alpha| \le C(\alpha, n)\phi^{\alpha + \frac{n}{5}},\ \ \alpha \in \mathcal{R},\ n = 1, 2, \tag{2.1}$$

$$\int_{\Omega} \phi^{\alpha}\, dX \leq C(\alpha-1)^{-1}\theta^{-5(\alpha-1)}\ \forall\, \alpha > 1, \tag{2.2}$$

$$\int_{\Omega} \phi\, dX \leq C(\tau)|\ln\theta|,\ \ \theta \leq \tau < 1, \tag{2.3}$$

$$\int_{\Omega} \phi^{\alpha}\, dX \leq C(1-\alpha)^{-1}\ \forall\ 0 < \alpha < 1. \tag{2.4}$$

In addition, we need to assume that there exists a q_0 ($5 < q_0 \leq \infty$) such that

$$\mathcal{L}:\ W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \longrightarrow L^q(\Omega)\ (1 < q < q_0)$$

is a homeomorphism [4]. It means that for all $v \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$, we have the so-called a priori estimate:

$$\|v\|_{2,q,\Omega} \leq C(q)\|\mathcal{L}v\|_{0,q,\Omega}, \tag{2.5}$$

where $C(q)$ denotes a positive constant only depending on q. Next, we give some lemmas used in the proofs of our main results.

Lemma 2.1. *For $\partial_{Z,\ell}G_Z^*$ the RDGF, we have the weighted-norm estimate*

$$\|\nabla^2\partial_{Z,\ell}G_Z^*\|_{\phi^{-1}} \leq Ch^{-1}|\ln h|^{\frac{17}{20}}. \tag{2.6}$$

Proof. By the triangular inequality, the a priori estimate (2.5) and the definition of $\partial_{Z,\ell}G_Z^*$, we have

$$\begin{aligned}
&\|\nabla^2\partial_{Z,\ell}G_Z^*\|_{\phi^{-1}}^2\\
= {}& \int_{\Omega}(\phi^{-\frac{1}{2}}|\nabla^2\partial_{Z,\ell}G_Z^*|)^2 dX \leq C(\int_{\Omega}|\nabla^2(\phi^{-\frac{1}{2}}\partial_{Z,\ell}G_Z^*)|^2 dX\\
&+\int_{\Omega}|\nabla^2\phi^{-\frac{1}{2}}\partial_{Z,\ell}G_Z^*|^2 dX + \int_{\Omega}|\nabla\phi^{-\frac{1}{2}}|^2|\nabla\partial_{Z,\ell}G_Z^*|^2 dX)\\
\leq {}& C(\|\nabla^2(\phi^{-\frac{1}{2}}\partial_{Z,\ell}G_Z^*)\|_0^2 + \|\partial_{Z,\ell}G_Z^*\|_{\phi^{-\frac{1}{5}}}^2 + |\partial_{Z,\ell}G_Z^*|_{1,\phi^{-\frac{3}{5}}}^2)\\
\leq {}& C(\|\mathcal{L}(\phi^{-\frac{1}{2}}\partial_{Z,\ell}G_Z^*)\|_0^2 + \|\partial_{Z,\ell}G_Z^*\|_{\phi^{-\frac{1}{5}}}^2 + |\partial_{Z,\ell}G_Z^*|_{1,\phi^{-\frac{3}{5}}}^2)\\
\leq {}& C(\|\mathcal{L}\partial_{Z,\ell}G_Z^*\|_{\phi^{-1}}^2 + |\partial_{Z,\ell}G_Z^*|_{1,\phi^{-\frac{3}{5}}}^2 + \|\partial_{Z,\ell}G_Z^*\|_{\phi^{-\frac{1}{5}}}^2)\\
\leq {}& C\|\partial_{Z,\ell}\delta_Z^h\|_{\phi^{-1}}^2 + C|a(\partial_{Z,\ell}G_Z^*, \phi^{-\frac{3}{5}}\partial_{Z,\ell}G_Z^*)| + C\|\partial_{Z,\ell}G_Z^*\|_{\phi^{-\frac{1}{5}}}^2\\
\leq {}& C\|\partial_{Z,\ell}\delta_Z^h\|_{\phi^{-1}}^2 + C|(\partial_{Z,\ell}\delta_Z^h, \phi^{-\frac{3}{5}}\partial_{Z,\ell}G_Z^*)| + C\|\partial_{Z,\ell}G_Z^*\|_{\phi^{-\frac{1}{5}}}^2\\
\leq {}& C\|\partial_{Z,\ell}\delta_Z^h\|_{\phi^{-1}}^2 + C\|\partial_{Z,\ell}G_Z^*\|_{\phi^{-\frac{1}{5}}}^2,
\end{aligned}$$

namely,

$$\|\nabla^2\partial_{Z,\ell}G_Z^*\|_{\phi^{-1}}^2 \leq C\|\partial_{Z,\ell}\delta_Z^h\|_{\phi^{-1}}^2 + C\|\partial_{Z,\ell}G_Z^*\|_{\phi^{-\frac{1}{5}}}^2. \tag{2.7}$$

Further, from the inverse estimate, the stability estimate (1.5), the a priori estimate (2.5) and the Sobolev embedding theorem [21],

$$\begin{aligned} & \left\|\partial_{Z,\ell}G_Z^*\right\|^2_{\phi^{-\frac{1}{5}}} = (\phi^{-\frac{1}{5}}\partial_{Z,\ell}G_Z^*\,,\,\partial_{Z,\ell}G_Z^*) = a(w\,,\,\partial_{Z,\ell}G_Z^*) = (\partial_{Z,\ell}\delta_Z^h\,,\,w) \\ = \;& \partial_\ell P_h w(Z) \le |P_h w|_{1,\infty} \le Ch^{-\frac{5}{q}}\,|P_h w|_{1,q} \le Ch^{-\frac{5}{q}}\|w\|_{1,q} \le Ch^{-\frac{5}{q}}q^{\frac{4}{5}}\|w\|_{2,5} \\ \le \;& Ch^{-\frac{5}{q}}q^{\frac{4}{5}}\|\phi^{-\frac{1}{5}}\partial_{Z,\ell}G_Z^*\|_{0,5} \le Ch^{-\frac{5}{q}}q^{\frac{4}{5}}\|\phi^{-\frac{1}{5}}\partial_{Z,\ell}G_Z^*\|_{2,\frac{5}{3}} \\ \le \;& Ch^{-\frac{5}{q}}q^{\frac{4}{5}}\|\mathcal{L}(\phi^{-\frac{1}{5}}\partial_{Z,\ell}G_Z^*)\|_{0,\frac{5}{3}}, \end{aligned} \tag{2.8}$$

where $\mathcal{L}w = \phi^{-\frac{1}{5}}\partial_{Z,\ell}G_Z^*$ in Ω and $w|_{\partial\Omega} = 0$. Taking $q = |\ln h|$ in (2.8) yields

$$\|\partial_{Z,\ell}G_Z^*\|^2_{\phi^{-\frac{1}{5}}} \le C|\ln h|^{\frac{4}{5}}\|\mathcal{L}(\phi^{-\frac{1}{5}}\partial_{Z,\ell}G_Z^*)\|_{0,\frac{5}{3}}. \tag{2.9}$$

However, from (2.1), the definition of $\partial_{Z,\ell}G_Z^*$ and the triangular inequality,

$$\begin{aligned} \|\mathcal{L}(\phi^{-\frac{1}{5}}\partial_{Z,\ell}G_Z^*)\|_{0,\frac{5}{3}} &\le C(\|\phi^{-\frac{1}{5}}\mathcal{L}\partial_{Z,\ell}G_Z^*\|_{0,\frac{5}{3}} + \|\nabla\partial_{Z,\ell}G_Z^*\|_{0,\frac{5}{3}} + \|\phi^{\frac{1}{5}}\partial_{Z,\ell}G_Z^*\|_{0,\frac{5}{3}}) \\ &= C(\|\phi^{-\frac{1}{5}}\partial_{Z,\ell}\delta_Z^h\|_{0,\frac{5}{3}} + \|\nabla\partial_{Z,\ell}G_Z^*\|_{0,\frac{5}{3}} + \|\phi^{\frac{1}{5}}\partial_{Z,\ell}G_Z^*\|_{0,\frac{5}{3}}). \end{aligned} \tag{2.10}$$

In addition, from the Hölder inequality,

$$\|\nabla\partial_{Z,\ell}G_Z^*\|_{0,\frac{5}{3}} \le \|\phi\|_{0,5}^{\frac{1}{2}}\|\nabla\partial_{Z,\ell}G_Z^*\|_{\phi^{-1}}. \tag{2.11}$$

Since [19]

$$\|\nabla\partial_{Z,\ell}G_Z^*\|_{\phi^{-1}} \le C|\ln h|^{\frac{9}{10}}, \tag{2.12}$$

from (2.2), (2.11) and (2.12),

$$\|\nabla\partial_{Z,\ell}G_Z^*\|_{0,\frac{5}{3}} \le Ch^{-2}|\ln h|^{\frac{9}{10}}. \tag{2.13}$$

Using the Hölder inequality again, we have

$$\|\phi^{-\frac{1}{5}}\partial_{Z,\ell}\delta_Z^h\|_{0,\frac{5}{3}} \le \|\phi\|_{0,3}^{\frac{3}{10}}\|\partial_{Z,\ell}\delta_Z^h\|_{\phi^{-1}}. \tag{2.14}$$

From (1.4), (2.2) and (2.14),

$$\|\phi^{-\frac{1}{5}}\partial_{Z,\ell}\delta_Z^h\|_{0,\frac{5}{3}} \le Ch^{-2}. \tag{2.15}$$

Similarly,

$$\|\phi^{\frac{1}{5}}\partial_{Z,\ell}G_Z^*\|_{0,\frac{5}{3}} \le \|\phi\|_{0,3}^{\frac{3}{10}}\|\partial_{Z,\ell}G_Z^*\|_{\phi^{-\frac{1}{5}}} \le Ch^{-1}\|\partial_{Z,\ell}G_Z^*\|_{\phi^{-\frac{1}{5}}}. \tag{2.16}$$

From (2.9), (2.10), (2.13), (2.15) and (2.16),

$$\begin{aligned} \left\|\partial_{Z,\ell}G_Z^*\right\|^2_{\phi^{-\frac{1}{5}}} &\le Ch^{-2}\,|\ln h|^{\frac{4}{5}} + Ch^{-2}\,|\ln h|^{\frac{17}{10}} + Ch^{-1}\,|\ln h|^{\frac{4}{5}}\,\|\partial_{Z,\ell}G_Z^*\|_{\phi^{-\frac{1}{5}}} \\ &\le Ch^{-2}\,|\ln h|^{\frac{4}{5}} + Ch^{-2}\,|\ln h|^{\frac{17}{10}} + C(\epsilon)h^{-2}\,|\ln h|^{\frac{8}{5}} + \epsilon\|\partial_{Z,\ell}G_Z^*\|^2_{\phi^{-\frac{1}{5}}}. \end{aligned} \tag{2.17}$$

Taking $\epsilon = \frac{1}{2}$ in (2.17) yields

$$\left\|\partial_{Z,\ell}G_Z^*\right\|^2_{\phi^{-\frac{1}{5}}} \le Ch^{-2}\,|\ln h|^{\frac{17}{10}}. \tag{2.18}$$

Combining (1.4), (2.7) and (2.18) immediately yields the result (2.6).

Lemma 2.2. *For $\partial_{Z,\ell}G_Z^*$ the RDGF, we have the $W^{2,1}$-seminorm estimate*

$$|\partial_{Z,\ell}G_Z^*|_{2,1} \le Ch^{-1}|\ln h|^{\frac{27}{20}}. \tag{2.19}$$

Proof. Obviously,

$$|\partial_{Z,\ell}G_Z^*|_{2,1} \le \left(\int_\Omega \phi\, dX\right)^{\frac{1}{2}} \cdot \|\nabla^2\partial_{Z,\ell}G_Z^*\|_{\phi^{-1}}. \tag{2.20}$$

By (2.3), (2.6) and (2.20), we immediately obtain the result (2.19).

In order to derive the estimate of the DDGF $\partial_{Z,\ell}G_Z^h$, we need the following result [19].

Lemma 2.3. *[19] For $\partial_{Z,\ell}G_Z^*$ and $\partial_{Z,\ell}G_Z^h$, the RDGF and the DDGF, respectively, we have*

$$\left|\partial_{Z,\ell}G_Z^* - \partial_{Z,\ell}G_Z^h\right|_{1,1} \le C|\ln h|^{\frac{13}{10}}. \tag{2.21}$$

Now, we can derive the following important estimate.

Theorem 2.1. *For $\partial_{Z,\ell}G_Z^h$ the DDGF, we have the $W^{2,1}$-seminorm estimate*

$$|\partial_{Z,\ell}G_Z^h|_{2,1}^h \le Ch^{-1}|\ln h|^{\frac{27}{20}}, \tag{2.22}$$

where $|\partial_{Z,\ell}G_Z^h|_{2,1}^h = \sum\limits_{e\in\mathcal{T}^h} |\partial_{Z,\ell}G_Z^h|_{2,1,e}$.

Proof. We denote by $\Pi\partial_{Z,\ell}G_Z^*$ the interpolant of projection type to $\partial_{Z,\ell}G_Z^*$. Thus, by the triangle inequality, the interpolation error estimate and the inverse property, we have

$$\begin{aligned}
\left|\partial_{Z,\ell}G_Z^h\right|_{2,1}^h &\le \left|\partial_{Z,\ell}G_Z^* - \partial_{Z,\ell}G_Z^h\right|_{2,1}^h + \left|\partial_{Z,\ell}G_Z^*\right|_{2,1} \\
&\le \left|\partial_{Z,\ell}G_Z^*\right|_{2,1} + \left|\partial_{Z,\ell}G_Z^* - \Pi\partial_{Z,\ell}G_Z^*\right|_{2,1}^h + \left|\Pi\partial_{Z,\ell}G_Z^* - \partial_{Z,\ell}G_Z^h\right|_{2,1}^h \\
&\le C\left|\partial_{Z,\ell}G_Z^*\right|_{2,1} + Ch^{-1}\left|\Pi\partial_{Z,\ell}G_Z^* - \partial_{Z,\ell}G_Z^h\right|_{1,1} \\
&\le C\left|\partial_{Z,\ell}G_Z^*\right|_{2,1} + Ch^{-1}\left|\partial_{Z,\ell}G_Z^* - \Pi\partial_{Z,\ell}G_Z^*\right|_{1,1} \\
&\quad + Ch^{-1}\left|\partial_{Z,\ell}G_Z^* - \partial_{Z,\ell}G_Z^h\right|_{1,1} \\
&\le C\left|\partial_{Z,\ell}G_Z^*\right|_{2,1} + Ch^{-1}\left|\partial_{Z,\ell}G_Z^* - \partial_{Z,\ell}G_Z^h\right|_{1,1}.
\end{aligned} \tag{2.23}$$

Combining (2.19), (2.21) and (2.23) yields the result (2.22).

3. Weak estimate of the second type for the finite element

In this section, we first introduce an interpolation operator of projection type, and then derive the weak estimate of the second type for the finite element by using the interpolation operator of projection type.

Let element

$$\begin{aligned}
e &= (x_{1,e} - h_{1,e}, x_{1,e} + h_{1,e}) \times (x_{2,e} - h_{2,e}, x_{2,e} + h_{2,e}) \times (x_{3,e} - h_{3,e}, x_{3,e} + h_{3,e}) \\
&\quad \times (x_{4,e} - h_{4,e}, x_{4,e} + h_{4,e}) \times (x_{5,e} - h_{5,e}, x_{5,e} + h_{5,e}) \\
&\equiv I_1 \times I_2 \times I_3 \times I_4 \times I_5,
\end{aligned}$$

and let $\{l_{1,j}(x_1)\}_{j=0}^{\infty}$, $\{l_{2,j}(x_2)\}_{j=0}^{\infty}$, $\{l_{3,j}(x_3)\}_{j=0}^{\infty}$, $\{l_{4,j}(x_4)\}_{j=0}^{\infty}$, $\{l_{5,j}(x_5)\}_{j=0}^{\infty}$ be the normalized orthogonal Legendre polynomial systems on $L^2(I_1)$, $L^2(I_2)$, $L^2(I_3)$, $L^2(I_4)$, $L^2(I_5)$, respectively. Now, let $\partial_{x_1}\partial_{x_2}\partial_{x_3}\partial_{x_4}\partial_{x_5}u \in L^2(e)$ then we have the following expansion:

$$\partial_{x_1}\partial_{x_2}\partial_{x_3}\partial_{x_4}\partial_{x_5}u = \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{i_3=0}^{\infty}\sum_{i_4=0}^{\infty}\sum_{i_5=0}^{\infty}\alpha_{i_1i_2i_3i_4i_5}\prod_{k=1}^{5} l_{k,i_k}(x_k),$$

where

$$\alpha_{i_1i_2i_3i_4i_5} = \int_e \partial_{x_1}\partial_{x_2}\partial_{x_3}\partial_{x_4}\partial_{x_5}u \prod_{k=1}^{5} l_{k,i_k}(x_k)\,dX.$$

Set

$$\omega_{k,0}(x_k) = 1, \omega_{k,j+1}(x_k) = \int_{x_{k,e}-h_{k,e}}^{x_k} l_{k,j}(\xi)\,d\xi, \ \ k = 1,\cdots,5,\ j \geq 0.$$

By the Parseval equality for $X = (x_1, x_2, x_3, x_4, x_5) \in e$,

$$u(X) = \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{i_3=0}^{\infty}\sum_{i_4=0}^{\infty}\sum_{i_5=0}^{\infty}\beta_{i_1i_2i_3i_4i_5}\prod_{k=1}^{5}\omega_{k,i_k}(x_k), \tag{3.1}$$

where

$$\beta_{00000} = u(x_{1,e} - h_{1,e}, x_{2,e} - h_{2,e}, \cdots, x_{5,e} - h_{5,e}),$$

$$\beta_{i_10000} = \int_{I_1} \partial_{x_1}u(x_1, x_{2,e} - h_{2,e}, \cdots, x_{5,e} - h_{5,e}) \cdot l_{1,i_1-1}(x_1)\,dx_1,$$

$$\beta_{i_1i_2000} = \int_{I_1\times I_2} \partial_{x_1}\partial_{x_2}u(x_1, x_2, x_{3,e} - h_{3,e}, \cdots, x_{5,e} - h_{5,e}) \cdot l_{1,i_1-1}(x_1)l_{2,i_2-1}(x_2)\,dx_1dx_2,$$

$$\beta_{i_1i_2i_3i_4i_5} = \int_e \partial_{x_1}\partial_{x_2}\partial_{x_3}\partial_{x_4}\partial_{x_5}u(X)\prod_{k=1}^{5} l_{k,i_k-1}(x_k)\,dX,$$

where $i_k \geq 1, k = 1,\cdots,5$. Similarly, the other coefficients can also be given.

We introduce standard tensor-product polynomial spaces of degree $m \geq 1$ denoted by T_m, i.e.,

$$q(X) = \sum_{(i_1,i_2,i_3,i_4,i_5)\in I} a_{i_1i_2i_3i_4i_5}x_1^{i_1}x_2^{i_2}x_3^{i_3}x_4^{i_4}x_5^{i_5},\ \ q \in T_m,$$

where the indexing set I is as follows:

$$I = \{(i_1, i_2, i_3, i_4, i_5)|0 \leq i_k \leq m, k = 1,\cdots,5\}.$$

Define the tensor-product interpolation operator of projection type by Π_m^e: $H^5(e) \to T_m(e)$ such that

$$\Pi_m^e u(X) = \sum_{(i_1,i_2,i_3,i_4,i_5)\in I}\beta_{i_1i_2i_3i_4i_5}\prod_{k=1}^{5}\omega_{k,i_k}(x_k). \tag{3.2}$$

By the definitions of the finite element space $S_0^h(\Omega)$ and Π_m^e, we have the tensor-product interpolation operator of project type

$$\Pi_m : H^5(\Omega) \cap H_0^1(\Omega) \to S_0^h(\Omega),$$

where $(\Pi_m u)|_e = \Pi_m^e u$.

For simplicity, we write

$$\lambda_{i_1 i_2 i_3 i_4 i_5} = \beta_{i_1 i_2 i_3 i_4 i_5} \prod_{k=1}^{5} \omega_{k,i_k}(x_k).$$

Thus, from (3.1) and (3.2),

$$\begin{aligned} & u - \Pi_m^e u \\ = \; & (\textstyle\sum_{i_1=0}^{m} \sum_{i_2=0}^{m} \sum_{i_3=0}^{m} \sum_{i_4=0}^{m} \sum_{i_5=m+1}^{\infty} + \sum_{i_1=0}^{m} \sum_{i_2=0}^{m} \sum_{i_3=0}^{m} \sum_{i_4=m+1}^{\infty} \sum_{i_5=0}^{\infty} \\ & + \textstyle\sum_{i_1=0}^{m} \sum_{i_2=0}^{m} \sum_{i_3=m+1}^{\infty} \sum_{i_4=0}^{\infty} \sum_{i_5=0}^{\infty} + \sum_{i_1=0}^{m} \sum_{i_2=m+1}^{\infty} \sum_{i_3=0}^{\infty} \sum_{i_4=0}^{\infty} \sum_{i_5=0}^{\infty} \\ & + \textstyle\sum_{i_1=m+1}^{\infty} \sum_{i_2=0}^{\infty} \sum_{i_3=0}^{\infty} \sum_{i_4=0}^{\infty} \sum_{i_5=0}^{\infty})\lambda_{i_1 i_2 i_3 i_4 i_5}. \end{aligned}$$

Next, we will derive the weak estimate of the second type for the finite element.

Theorem 3.1. *Let $\{\mathcal{T}^h\}$ be a regular family of rectangular partitions of $\bar{\Omega}$, $u \in W^{m+2,\infty}(\Omega) \cap H_0^1(\Omega)$ and $v \in S_0^h(\Omega)$ then the tensor-product m-degree interpolation operator of projection type Π_m satisfies the weak estimate of the second type*

$$|a(u - \Pi_m u, v)| \leq Ch^{m+2} \|u\|_{m+2,\infty,\Omega} |v|_{2,1,\Omega}^h, \quad m \geq 2, \tag{3.3}$$

where $|v|_{2,1,\Omega}^h = \sum_{e \in \mathcal{T}^h} |v|_{2,1,e}$.

Proof. By the properties of $\omega_{k,i}(x_k)$ as well as the orthogonality of the Legendre polynomial system, we have

$$\int_e \nabla(u - \Pi_m^e u) \cdot \nabla v \, dX = \int_e \nabla r \cdot \nabla v \, dX \equiv I_e \quad \forall e \in \mathcal{T}^h,$$

where

$$\begin{aligned} r \; = \; & (\textstyle\sum_{i_1=0}^{m} \sum_{i_2=0}^{m} \sum_{i_3=0}^{m} \sum_{i_4=0}^{m} \sum_{i_5=m+1}^{m+2} + \sum_{i_1=0}^{m} \sum_{i_2=0}^{m} \sum_{i_3=0}^{m} \sum_{i_4=m+1}^{m+2} \sum_{i_5=0}^{m+2} \\ & + \textstyle\sum_{i_1=0}^{m} \sum_{i_2=0}^{m} \sum_{i_3=m+1}^{m+2} \sum_{i_4=0}^{m+2} \sum_{i_5=0}^{m+2} + \sum_{i_1=0}^{m} \sum_{i_2=m+1}^{m+2} \sum_{i_3=0}^{m+2} \sum_{i_4=0}^{m+2} \sum_{i_5=0}^{m+2} \\ & + \textstyle\sum_{i_1=m+1}^{m+2} \sum_{i_2=0}^{m+2} \sum_{i_3=0}^{m+2} \sum_{i_4=0}^{m+2} \sum_{i_5=0}^{m+2})\lambda_{i_1 i_2 i_3 i_4 i_5}. \end{aligned} \tag{3.4}$$

Clearly, r only contains finite terms.

Among the indices i_k, $k = 1, \cdots, 5$, when some $i_k = m + 1$ or $m + 2$ and the others are zero, we have by the orthogonality of the Legendre polynomial system

$$\int_e \nabla \lambda_{i_1 i_2 i_3 i_4 i_5} \cdot \nabla v \, dX = 0. \tag{3.5}$$

For $m \geq 2$, without loss of generality, we assume $i_k \neq 0$, $k = 1, \cdots, j$ and $i_{j+1} = \cdots = i_5 = 0$.

$$I_{i_1 \cdots i_j 0 \cdots 0} \equiv \int_e \nabla \lambda_{i_1 \cdots i_j 0 \cdots 0} \cdot \nabla v \, dX = \sum_{s=1}^{j} \int_e \partial_{x_s} \lambda_{i_1 \cdots i_j 0 \cdots 0} \partial_{x_s} v \, dX = \sum_{s=1}^{j} I_s. \tag{3.6}$$

We assume $i_1 \geq m + 1$; thus, $i_1 \geq m + 1 \geq 3$. By the orthogonality of the Legendre polynomial system,

$$I_1 = \beta_{i_1 \cdots i_j 0 \cdots 0} \times \int_e l_{1,i_1-1}(x_1) \omega_{2,i_2}(x_2) \cdots \omega_{j,i_j}(x_j) \partial_{x_1} v \, dX = 0. \tag{3.7}$$

In addition,

$$\begin{aligned} I_2 &= \beta_{i_1\cdots i_j0\cdots 0}\times\int_e \omega_{1,i_1}(x_1)l_{2,i_2-1}(x_2)\cdots\omega_{j,i_j}(x_j)\partial_{x_2}v\,dX \\ &= -\beta_{i_1\cdots i_j0\cdots 0}\times\int_e D^{-1}\omega_{1,i_1}(x_1)l_{2,i_2-1}(x_2)\cdots\omega_{j,i_j}(x_j)\cdot\partial_{x_1}\partial_{x_2}v\,dX, \end{aligned} \tag{3.8}$$

and it is easy to prove

$$|\beta_{i_1\cdots i_j0\cdots 0}|\le Ch^{m+2-\frac{j}{2}}\|u\|_{m+2,\,\infty,\,e}. \tag{3.9}$$

In fact,

$$D^{-1}\omega_{1,i_1}(x_1)l_{2,i_2-1}(x_2)\cdots\omega_{j,i_j}(x_j)=O(h^{\frac{j}{2}}). \tag{3.10}$$

Combining (3.8)–(3.10) yields

$$|I_2|\le Ch^{m+2}\|u\|_{m+2,\,\infty,\,e}|v|_{2,\,1,\,e}. \tag{3.11}$$

Similarly, we have

$$|I_k|\le Ch^{m+2}\|u\|_{m+2,\,\infty,\,e}|v|_{2,\,1,\,e},\ k=3,\cdots,j. \tag{3.12}$$

From (3.6), (3.7), (3.11) and (3.12),

$$|I_{i_1\cdots i_j0\cdots 0}|\le Ch^{m+2}\|u\|_{m+2,\,\infty,\,e}|v|_{2,\,1,\,e},\ k=3,\cdots,j. \tag{3.13}$$

When each $i_k\ne 0,\ k=1,\cdots,5$, similar to the above arguments, we easily get

$$|\int_e \nabla\lambda_{i_1i_2i_3i_4i_5}\cdot\nabla v\,dX|\le Ch^{m+2}\|u\|_{m+2,\,\infty,\,e}|v|_{2,\,1,\,e}. \tag{3.14}$$

From (3.4), (3.5), (3.13) and (3.14),

$$|I_e|\le Ch^{m+2}\|u\|_{m+2,\,\infty,\,e}|v|_{2,\,1,\,e}.$$

Summing over all elements proves the result (3.3).

4. $W^{1,\infty}$-seminorm superconvergence of the block finite element approximation

In this section, we will give the superconvergent estimate for the m-degree block finite element approximation by using the weak estimate of the second type and the $W^{2,1}$-seminorm estimate for the DDGF.

Let u_h be the m-degree block finite element approximation to u, the solution of problem (1.2) and $\Pi_m u$ the corresponding interpolant of projection type to u. Thus, we have the following theorem.

Theorem 4.1. *Let $\{\mathcal{T}^h\}$ be a regular family of rectangular partitions of $\bar\Omega$ and $u\in W^{m+2,\infty}(\Omega)\cap H_0^1(\Omega)$ then we have the superconvergent estimate*

$$|u_h-\Pi_m u|_{1,\,\infty,\,\Omega}\le Ch^{m+1}|\ln h|^{\frac{27}{20}}\|u\|_{m+2,\,\infty},\ \ m\ge 2. \tag{4.1}$$

Proof. For every $Z\in\Omega$, applying the definition of $\partial_{Z,\ell}G_Z^h$ and the Galerkin orthogonality relation (1.3), we derive

$$\partial_{Z,\ell}(u_h-\Pi_m u)(Z)=a(u_h-\Pi_m u\,,\,\partial_{Z,\ell}G_Z^h)=a(u-\Pi_m u\,,\,\partial_{Z,\ell}G_Z^h). \tag{4.2}$$

From (2.22), (3.3) and (4.2), we immediately obtain the result (4.1).

5. A numerical example

Example 5.1. Consider the following Poisson equation:

$$\begin{cases} -\Delta u = f & \text{in } \Omega = (0,1)\times(0,1)\times(0,1)\times(0,1)\times(0,1), \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$f = 5\pi^2 \sin(\pi x_1)\sin(\pi x_2)\sin(\pi x_3)\sin(\pi x_4)\sin(\pi x_5).$$

The true solution is

$$u = \sin(\pi x_1)\sin(\pi x_2)\sin(\pi x_3)\sin(\pi x_4)\sin(\pi x_5).$$

Let u_h be the tensor-product two-degree finite element approximation to u on uniform rectangular meshes and Πu the corresponding interpolant of the projection type. Set $X^* = (0.5, 0.5, 0.5, 0.5, 0.5)$, $Y^* = (0.25, 0.25, 0.25, 0.25, 0.25)$ and $Z^* = (0.125, 0.125, 0.125, 0.125, 0.125)$. We solve Example 5.1 and obtain the following numerical results (see Table 1):

Table 1. Numerical results at X^*, Y^* and Z^*.

h	$\lvert\bar{\nabla}(u_h - \Pi u)(X^*)\rvert$	$\lvert\bar{\nabla}(u_h - \Pi u)(Y^*)\rvert$	$\lvert\bar{\nabla}(u_h - \Pi u)(Z^*)\rvert$
0.5	8.0218e-003	6.7152e-003	4.9384e-003
0.25	1.9724e-003	5.3517e-004	5.1322e-004
0.125	3.1481e-004	5.8873e-005	6.0237e-005

Here, the operator $\bar{\nabla} = (\bar{\partial}_{x_1}, \bar{\partial}_{x_2}, \bar{\partial}_{x_3}, \bar{\partial}_{x_4}, \bar{\partial}_{x_5})$ and $\bar{\partial}_{x_i} v = \frac{1}{2}(|\partial^+_{x_i} v| + |\partial^-_{x_i} v|)$, $i = 1, \cdots, 5$. The numerical results demonstrate our theoretical results.

6. Conclusions

In this paper, we proposed two important analytic tools: The DDGF and the weak estimate of the second type. By combining these tools, we obtained a pointwise superconvergence estimate for the finite element approximation in the $W^{1,\infty}$-seminorm, which is a challenging issue in the field of superconvergence for the finite element method. The main difficulty of the paper lies in deriving the optimal order estimate for the DDGF in the $W^{2,1}$-seminorm. Notably, the estimates for the DDGF vary for different dimensions of elliptic equations, necessitating different approaches for them. Although the methods presented in the paper are specifically developed for the five-dimensional second-order elliptic equation, they can also be applied to other high-dimensional second-order elliptic equations.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by the Special Projects in Key Fields of Ordinary Colleges and Universities in Guangdong Province (2022ZDZX3016) and the Projects of Talents Recruitment of Guangdong University of Petrochemical Technology (2021rc003).

Conflict of interest

The author declares no conflict of interest.

References

1. C. M. Chen, *Construction theory of superconvergence of finite elements (in Chinese)*, Changsha: Hunan Science and Technology Press, 2001.
2. C. M. Chen, Y. Q. Huang, *High accuracy theory of finite element methods (in Chinese)*, Changsha: Hunan Science and Technology Press, 1995.
3. L. Wahlbin, *Superconvergence in Galerkin finite element methods*, Berlin: Springer Verlag, 1995. https://doi.org/10.1007/bfb0096835
4. Q. D. Zhu, Q. Lin, *Superconvergence theory of the finite element methods (in Chinese)*, Changsha: Hunan Science and Technology Press, 1989.
5. J. H. Brandts, M. Křížek, *History and future of superconvergence in three dimensional finite element methods*, In: Proceedings of the Conference on Finite Element Methods: Three-dimensional Problems, GAKUTO Int. Ser. Math. Sci. Appl., Gakkotosho, Tokyo, **15** (2001), 22–33.
6. J. H. Brandts, M. Křížek, Gradient superconvergence on uniform simplicial partitions of polytopes, *IMA J. Numer. Anal.*, **23** (2003), 489–505. https://doi.org/10.1093/imanum/23.3.489
7. J. H. Brandts, M. Křížek, Superconvergence of tetrahedral quadratic finite elements, *J. Comput. Math.*, **23** (2005), 27–36.
8. C. M. Chen, Optimal points of the stresses for the linear tetrahedral element (in Chinese), *Nat. Sci. J. Xiangtan Univ.*, **3** (1980), 16–24.
9. L. Chen, Superconvergence of tetrahedral linear finite elements, *Int. J. Numer. Anal. Mod.*, **3** (2006), 273–282. https://doi.org/10.1080/00207160601117354
10. G. Goodsell, Pointwise superconvergence of the gradient for the linear tetrahedral element, *Numer. Meth. Part. D. E.*, **10** (1994), 651–666. https://doi.org/10.1002/num.1690100511
11. V. Kantchev, R. D. Lazarov, *Superconvergence of the gradient of linear finite elements for 3D Poisson equation*, In: Proceedings of the Conference on Optimal Algorithms, Bulgarian Academy of Sciences, Sofia, 1986, 172–182.
12. J. H. Liu, B. Jia, Q. D. Zhu, An estimate for the three-dimensional discrete Green's function and applications, *J. Math. Anal. Appl.*, **370** (2010), 350–363. https://doi.org/10.1016/j.jmaa.2010.05.002

13. J. H. Liu, Q. D. Zhu, Maximum-norm superapproximation of the gradient for the trilinear block finite element, *Numer. Meth. Part. D. E.*, **23** (2007), 1501–1508. https://doi.org/10.1002/num.20237

14. J. H. Liu, Q. D. Zhu, Pointwise supercloseness of tensor-product block finite elements, *Numer. Meth. Part. D. E.*, **25** (2009), 990–1008. https://doi.org/10.1002/num.20384

15. A. Pehlivanov, *Superconvergence of the gradient for quadratic 3D simplex finite elements*, In: Proceedings of the Conference on Numerical Methods and Application, Bulgarian Academy of Sciences, Sofia, 1989, 362–366.

16. A. H. Schatz, I. H. Sloan, L. B. Wahlbin, Superconvergence in finite element methods and meshes that are locally symmetric with respect to a point, *SIAM J. Numer. Anal.*, **33** (1996), 505–521. https://doi.org/10.1137/0733027

17. Z. M. Zhang, R. C. Lin, Locating natural superconvergent points of finite element methods in 3D, *Int. J. Numer. Anal. Mod.*, **2** (2005), 19–30. https://doi.org/10.1080/00207160412331291107

18. G. Goodsell, J. R. Whiteman, Pointwise superconvergence of recovered gradients for piecewise linear finite element approximations to problems of planar linear elasticity, *Numer. Meth. Part. D. E.*, **6** (1990), 59–74. https://doi.org/10.1002/num.1690060105

19. J. H. Liu, Q. D. Zhu, The $W^{1,1}$-seminorm estimate for the discrete derivative Green's function for the 5D Poisson equation, *J. Comput. Anal. Appl.*, **13** (2011), 1143–1156.

20. G. H. Zhou, R. Rannacher, Pointwise superconvergence of the streamline diffusion finite-element method, *Numer. Meth. Part. D. E.*, **12** (1996), 123–145. https://doi.org/10.1002/(sici)1098-2426(199601)12:1<123::aid-num7>3.0.co;2-u

21. R. A. Adams, *Sobolev spaces*, New York: Academic Press, 1975.

AIMS Press