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Research article

Some generalizations for the Schwarz-Pick lemma and boundary Schwarz lemma

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Abstract: In this paper, we first obtain a Schwarz-Pick type lemma for the holomorphic self-mapping of the unit disk with respect to the q-distance. Second, we establish the general Schwarz-Pick lemma for the self-mapping of the unit disk satisfying the Poisson differential inequality. As an application, it is proven that this mapping is Lipschitz continuous with respect to the q-distance under certain conditions. Moreover, the corresponding explicit Lipschitz constant is given. Third, it is proved that there exists a self-mapping of the unit disk satisfying the Poisson differential inequality, which does not meet conditions of the boundary Schwarz lemma. Finally, with some additional conditions, a boundary Schwarz lemma for the self-mapping of the unit disk satisfying the Poisson differential inequality is established.

Keywords: Schwarz-Pick lemma; Poisson differential inequality; q-distance; boundary Schwarz lemma; explicit constant
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1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$ centered at the origin and denote $\mathbb{T} = \partial\mathbb{D}$. The symbol $C^n(\Omega)$ stands for the class of all complex-valued n-times continuously differentiable functions from the domain Ω into $\mathbb{C}$. In particular, we let $C(\Omega) = C^0(\Omega)$. A real-valued function $u \in C^2(\Omega)$ is called real harmonic if it satisfies the following Laplace's equation:

$$\Delta u(z) = \frac{\partial^2 u}{\partial x^2}(z) + \frac{\partial^2 u}{\partial y^2}(z) = 0, \quad z \in \Omega.$$

A complex-valued function $\omega = u + iv$ is harmonic if both u and v are real harmonic. We refer the readers to [12] for an excellent discussion on harmonic mappings in the plane.

Suppose that $g \in C(\mathbb{D})$. Then, it is well known that the solution of the Poisson's equation $\Delta\omega = g$ in $\mathbb{D}$ satisfying the boundary condition $\omega|_{\mathbb{T}} = f \in L^1(\mathbb{T})$ is given by

$$\omega(z) = P[f](z) - G[g](z), \quad z \in \mathbb{D}, \tag{1.1}$$

where

$$P[f](z) = \frac{1}{2\pi}\int_0^{2\pi} P(z, e^{i\varphi}) f(e^{i\varphi}) d\varphi, G[g](z) = \int_{\mathbb{D}} G(z, w) g(w) dm(w), \tag{1.2}$$

$$P(z, e^{i\theta}) = \frac{1 - |z|^2}{|z - e^{i\theta}|^2}, \quad z \in \mathbb{D}, \theta \in \mathbb{R}, G(z, w) = \frac{1}{2\pi} \log\left|\frac{1 - z\overline{w}}{z - w}\right|, \quad z, w \in \mathbb{D}, z \neq w,$$

and m denotes the Lebesgue measure in the plane.

The hyperbolic metric on $\mathbb{D}$, the Gaussian curvature of which is equal to -4, is given by $|dz|/(1-|z|^2)$. Then, the hyperbolic distance between two points $z_1, z_2 \in \mathbb{D}$ is defined by

$$d_{h_{\mathbb{D}}}(z_1, z_2) := \inf_{\gamma}\Big\{\int_{\gamma} \frac{|dz|}{1 - |z|^2}\Big\}, \tag{1.3}$$

where the infimum is taken over all the rectifiable curves γ which connect z_1 and z_2 with $\gamma \in \mathbb{D}$. It is well known that

$$d_{h_{\mathbb{D}}}(z_1, z_2) = \log \frac{|1 - z_1\overline{z_2}| + |z_1 - z_2|}{|1 - z_1\overline{z_2}| - |z_1 - z_2|}.$$

Also, we can consider the quasihyperbolic metric $|dz|/(1 - |z|)$ on $\mathbb{D}$. Then, it gives rise to the quasihyperbolic distance between two points $z_1, z_2 \in \mathbb{D}$, which is defined by

$$d_{qh_{\mathbb{D}}}(z_1, z_2) := \inf_{\gamma}\Big\{\int_{\gamma} \frac{|dz|}{1 - |z|}\Big\}, \tag{1.4}$$

where the infimum is taken over all rectifiable curves γ which connect z_1 and z_2 with $\gamma \in \mathbb{D}$. Actually, we can consider the general q-metric $|dz|/(1 - |z|^q)$ on $\mathbb{D}, q \in [1, 2]$, which leads to the following q-pseudo distance

$$d_q(z_1, z_2) := \inf_{\gamma}\Big\{\int_{\gamma} \frac{|dz|}{1 - |z|^q}\Big\}, q \in [1, 2], \tag{1.5}$$

where the infimum is taken over all rectifiable curves γ which connect z_1 and z_2 with $\gamma \in \mathbb{D}$. For completeness, we will prove that this q-pseudo distance is a distance in Lemma 7.

A self-mapping f of $\mathbb{D}$ is said to be Lipschitz continuous with respect to the q-distance if there exists a constant $L > 0$, such that the inequality

$$d_q(f(z_1), f(z_2)) \leq L \cdot d_q(z_1, z_2)$$

holds for any $z_1, z_2 \in \mathbb{D}$.

The classic Schwarz-Pick lemma plays an important role in complex analysis. It is stated as follows: If f is a holomorphic self-mapping of $\mathbb{D}$, then we have

$$|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}, \; z \in \mathbb{D}, \tag{1.6}$$

or

$$d_{h_{\mathbb{D}}}(f(z_1), f(z_2)) \le d_{h_{\mathbb{D}}}(z_1, z_2),\ z \in \mathbb{D}. \tag{1.7}$$

The Schwarz-Pick lemmas (1.6) or (1.7) have many generalizations. See the references [1, 6, 13, 30–32, 34, 35, 39, 46] on this topic. The first work of this paper is to extend (1.7) to the case for the q-distance. It is presented as follows:

Theorem 1. *Suppose that $q \in [1, 2]$. If f is a holomorphic self-mapping of $\mathbb{D}$ satisfying $f(0) = 0$, then the inequality*

$$d_q(f(z_1), f(z_2)) \le d_q(z_1, z_2) \tag{1.8}$$

holds for any $z_1, z_2 \in \mathbb{D}$.

Suppose that f satisfies the following the Poisson differential inequality:

$$|\Delta f(z)| \le a \cdot |Df(z)|^2 + b, \tag{1.9}$$

where $|Df| = |f_z| + |f_{\bar{z}}|$. This kind of mapping contains many classical mappings. For example, it includes a holomorphic mapping, a harmonic mapping, the mapping satisfying Poisson's equation $\Delta f = g$ (where g is continuous on $\overline{\mathbb{D}}$), the harmonic mapping between Riemannian surfaces satisfying some inequalities and so on. In addition, those mappings that satisfy (1.9) are also closely related to the following partial differential equations

$$\Delta u = Q\left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, u, x_1, x_2\right); u = (u_1(x_1, x_2), u_2(x_1, x_2), \ldots, u_m(x_1, x_2)), \tag{1.10}$$

where $Q = (Q_1, Q_2, \ldots, Q_m)$, Q_j are quadratic polynomial on $\partial u_i/\partial x_k$, $i = 1, \ldots, m, k = 1, 2$. This equation has a deep connection with the average curvature problem of surfaces in differential geometry, the conjugate isothermal coordinates of surfaces, and the Monge-Ampère equation and so on. Based on this backgrounds, the study of mappings that satisfy the Poisson differential inequality (1.9) has attracted much attention from researchers. See the articles [4, 18, 20, 42, 45] and the references therein. Recently, Kalaj [21] extended the results of Bernstein [4] and Heinz [18] to the case for spaces and applied the relevant results to the theory of harmonic quasiconformal mappings. The study of mappings satisfying the Poisson differential inequality that are also quasiconformal or quasiregular have also attracted many authors' interest, see [3, 10, 22].

Let f be a quasiconformal self-mapping of the unit disk satisfying the following Poisson differential equation

$$|\Delta f(z)| \le B \cdot |Df(z)|^2. \tag{1.11}$$

In [23], Kalaj obtained the following result.

Lemma 1. [23] *Suppose that f is a K-quasiconformal self-mapping of $\mathbb{D}$ satisfying the Poisson differential equation* (1.11)*, and that $f(0) = 0$. Then there exists a constant $C(B, K)$, which depends only on B and K, such that*

$$\frac{1 - |z|^2}{1 - |f(z)|^2} \le C(B, K). \tag{1.12}$$

The inequality (1.12) can be seen as a kind of Schwarz-Pick type inequality for the K-quasiconformal self-mapping of $\mathbb{D}$ satisfying the Poisson differential inequality (1.11). Generally, the condition for quasiconformality in Lemma 1 can not be removed. Recently, Zhong et al. [47] showed the following result.

Theorem 2. [47, Theorem 1.4] *For a given $q \in \{1\} \cup [2, +\infty)$. Suppose that $f : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ is continuous on $\overline{\mathbb{D}}$, $f|_{\mathbb{D}} \in C^2$, $f|_{\mathbb{T}} \in C^2$ and $f(0) = 0$, and that it satisfies the following Poisson differential inequality*

$$|\Delta f(z)| \le a \cdot |Df(z)|^2 + b,$$

as well as satisfying that $\left|\frac{\partial^2 f(e^{i\varphi})}{\partial \varphi^2}\right| \le K$, where $0 < a < 1/2, 0 < b, K < \infty$. If

$$\frac{2^{q-1}q + \max\{a, b\} \cdot \left(L^2(a, b, K) + 1\right)}{4} < \frac{2}{\pi},$$

then, we have

$$\frac{1 - |z|^q}{1 - |f(z)|^q} \le \begin{cases} \dfrac{1}{\frac{2}{\pi} - \frac{3\max\{a,b\}\cdot\left(L^2(a,b,K)+1\right)}{4}} & \text{when } q = 1, \\ \dfrac{1}{\frac{2}{\pi} - \frac{(2^{q-1}q+1)\cdot\max\{a,b\}\cdot\left(L^2(a,b,K)+1\right)}{4}} & \text{when } q \ge 2. \end{cases}$$

In the next result of this paper, we will improve Theorem 2 by giving the explicit constant $L(a, b, K)$ and showing that the result is still valid when we consider all of the cases for $q > 0$. Our result is presented as follows.

Theorem 3. *For given $q > 0$, suppose that $f : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ is continuous in $\overline{\mathbb{D}}$, $f|_{\mathbb{D}} \in C^2, f|_{\mathbb{T}} \in C^2$ and $f(0) = 0$, and that it satisfies the Poisson differential inequality $|\Delta f(z)| \le a \cdot |Df(z)|^2 + b$ and $\left|\frac{\partial^2 f(e^{i\varphi})}{\partial \varphi^2}\right| \le K$, where $0 < a < 1/2, 0 < b, K < \infty$. If, in addition, $48ab + 640a < 9$ and $\max\{a, b\} \cdot \left(L^2(a, b, K) + 1\right) < 2/\pi$, then we have*

$$\frac{1 - |z|^q}{1 - |f(z)|^q} \le \frac{\max\{1, q\}}{\min\{1, q\} \cdot \left(\frac{2}{\pi} - \max\{a, b\} \cdot (L^2(a, b, K) + 1)\right)}, \tag{1.13}$$

where

$$\begin{aligned} L(a, b, K) = 2 \max &\left\{8 + \frac{64a}{3(1 - a)}, b + \frac{8ab}{3(1 - a)}\right\} \\ &\times \left[1 + \frac{1 - a}{1 - 2a} \cdot \frac{3K\pi}{\sqrt{3}} + \frac{3a\pi}{2(1 - 2a)} \cdot \frac{8K^2\pi^2 + 2K}{\sqrt{3}} + \frac{3b}{2(1 - 2a)}\right]. \end{aligned} \tag{1.14}$$

As an application of Theorem 3, we provide the following more general form of [47, Corollary 1.5].

Corollary 1. *For a given $q > 0$, suppose that $f : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ is continuous in $\overline{\mathbb{D}}$, $f|_{\mathbb{D}} \in C^2, f|_{\mathbb{T}} \in C^2$ and $f(0) = 0$, and that it satisfies the Poisson differential inequality $|\Delta f(z)| \le a \cdot |Df(z)|^2 + b$ and $\left|\frac{\partial^2 f(e^{i\varphi})}{\partial \varphi^2}\right| \le K$, where $0 < a < 1/2, 0 < b, K < \infty$. If, in addition, $48ab + 640a < 9$ and $\max\{a, b\} \cdot \left(L^2(a, b, K) + 1\right) < 2/\pi$, then we have*

$$d_q(f(z_1), f(z_2)) \le \frac{\max\{1, q\} \cdot L(a, b, K)}{\min\{1, q\} \cdot \left(\frac{2}{\pi} - \max\{a, b\} \cdot (L^2(a, b, K) + 1)\right)} \cdot d_q(z_1, fz_2). \tag{1.15}$$

Proof. According to Lemma 9, we see that there exists a constant $L(a, b, K)$ such that the inequality

$$|Df(z)| \leq L(a, b, K)$$

holds for any $z \in \mathbb{D}$. Combining this with Theorem 3, we get

$$\frac{|Df(z)|}{1-|f(z)|^q} \leq \frac{\max\{1, q\} \cdot L(a, b, K)}{\min\{1, q\} \cdot \left(\frac{2}{\pi} - \max\{a, b\} \cdot (L^2(a, b, K) + 1)\right)} \cdot \frac{1}{1-|z|^q}.$$

By the definition of q-distance, we derive that inequality (1.15) is valid. □

The classic boundary Schwarz lemma for the holomorphic mapping states the following:

Lemma 2. ([14]) *Let f be a holomorphic self-mapping of $\mathbb{D}$ satisfying that $f(0) = 0$. If f is differentiable at the point $z = 1$ and satisfies that $f(1) = 1$, then we have the follwoing:*
(i) $f'(1) \geq 1$;
(ii) $f'(1) = 1$ *if and only if* $f(z) = z$.

The boundary Schwarz lemma is a basic and important result in complex analysis. Its generalizations and applications have received widespread attention; see [5, 7, 9, 15–17, 25–29, 40, 43, 44]. In [25], Lemma 2 was generalized to the following form:

Lemma 3. ([25]) *Let f be a holomorphic self-mapping of $\mathbb{D}$ satisfying that $f(0) = 0$. If f is differentiable at the point $z = \alpha \in \mathbb{T}$ and satisfies that $f(\alpha) = \beta \in \mathbb{T}$. Then we have the following*
(i′) $\overline{\beta} f'(\alpha)\alpha \geq 1$;
(ii′) $\overline{\beta} f'(\alpha)\alpha = 1$ *if and only if* $f(z) = e^{i\theta} z$, *where* $e^{i\theta} = \beta\alpha^{-1}$ *and* $\theta \in \mathbb{R}$.

Recently, the extension of the boundary Schwarz lemma to harmonic mappings has also attracted scholars' attention. In [50], Zhu and Wang established the following harmonic version of the boundary Schwarz lemma.

Lemma 4. ([50, Theorem 1.2]) *Suppose that $g \in C(\overline{\mathbb{D}})$. If $f \in C^2(\mathbb{D}) \bigcap C(\mathbb{T})$ is a self-mapping of $\mathbb{D}$ satisfying the Poisson equation $\Delta f = g$ and $f(0) = 0$, and if f is differentiable at the point $z = 1$ and satisfies that $f(1) = 1$, then we have*

$$\mathrm{Re}[f_x(1)] \geq \frac{2}{\pi} - \frac{3}{4}\|g\|_\infty, \tag{1.16}$$

where $\|g\|_\infty := \sup_{z\in\mathbb{D}} |g(z)|$.

This result was extended by Mohapatra et al. [37] and Mohapatra [38]. See the high dimensional version of the boundary Schwarz lemma for harmonic mappings in [11, 24, 33, 36, 49]. As a generalization of planar harmonic maps, those self-mappings that satisfy the Poisson differential inequality (1.11) can be naturally considered as analogous to the boundary Schwarz lemma. Unfortunately, those types of mappings generally are not associated with the boundary Schwarz lemma of types (1.16). This is because we have established the following results:

Theorem 4. *There exists a self-mapping f of $\mathbb{D}$, which is differentiable at $z = 1$ and satisfies that $f(0) = 0$ and $f(1) = 1$, in addition to the Poisson differential inequality $|\Delta f(z)| \leq a \cdot |Df(z)|^2$. But it satisfies the equation* $\mathrm{Re}(f_x(1)) = 0$.

According to Lemma 4, we see that there exists a self-mapping of $\mathbb{D}$ that satisfies the Poisson differential inequality (1.9), which is associated with a boundary Schwarz lemma of type (1.16). Therefore, we naturally ask the following question: which subset of the family

$$\mathcal{F} = \{f : |\Delta f(z)| \leq a \cdot |Df(z)|^2 + b, f(0) = 0, f \in C^2(\mathbb{D})\}$$

has boundary Schwarz lemma of type (1.16)? Next, we establish a result in this regard.

Theorem 5. *Suppose that $f : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ is continuous in $\overline{\mathbb{D}}$, $f|_{\mathbb{D}} \in C^2, f|_{\mathbb{T}} \in C^2$ and $f(0) = 0$, and that it satisfies the Poisson differential inequality $|\Delta f(z)| \leq a \cdot |Df(z)|^2 + b$ and $\left|\frac{\partial^2 f(e^{i\varphi})}{\partial \varphi^2}\right| \leq K$, where $0 < a < 1/2, 0 < b, K < \infty$. If, in addition, $48ab + 640a < 9$ and $\max\{a, b\} \cdot \left(L^2(a, b, K) + 1\right) < 8/(3\pi)$, then we have*

$$\mathrm{Re}(f_x(1)) > \frac{2}{\pi} - \frac{3 \max\{a, b\} \cdot \left(L^2(a, b, K) + 1\right)}{4}, \tag{1.17}$$

where

$$\begin{aligned} L(a, b, K) = 2 \max\left\{8 + \frac{64a}{3(1-a)}, b + \frac{8ab}{3(1-a)}\right\} \\ \times \left[1 + \frac{1-a}{1-2a} \cdot \frac{3K\pi}{\sqrt{3}} + \frac{3a\pi}{2(1-2a)} \cdot \frac{8K^2\pi^2 + 2K}{\sqrt{3}} + \frac{3b}{2(1-2a)}\right]. \end{aligned} \tag{1.18}$$

2. Some preparations

L' Hospital's rule for monotonicity is important to derive some inequalities and it will be used to prove some results.

Lemma 5. ([2, Theorem 2]) *Let $-\infty < a < b < +\infty$ and let $f, g : [a, b] \to \mathbb{R}$ be continuous functions that are differentiable on (a, b) with $f(a) = g(a) = 0$ or $f(b) = g(b) = 0$. Suppose that $g'(x) \neq 0$ for each $x \in (a, b)$. If f'/g' is increasing (decreasing) on (a, b), then so is f/g.*

The following lemma can be obtained in [48] by letting $x = t^2$.

Lemma 6. *For any $q > 0$, the inequalities*

$$\min\{1, q\} \leq \frac{1 - x^q}{1 - x} \leq \max\{1, q\} \tag{2.1}$$

hold for any $x \in [0, 1)$.

Remark 1. *In the rest of this paper, we will always use the constant c_q, which is defined by*

$$c_q := \frac{\min\{1, q\}}{\max\{1, q\}}. \tag{2.2}$$

In Section 1, we defined the following general q-pseudo distance in $\mathbb{D}$:

$$d_q(z_1, z_2) := \inf_{\gamma} \left\{ \int_{\gamma} \frac{1}{1 - |z|^q} |dz| \right\}, \; q \in [1, 2], \tag{2.3}$$

where the infimum is taken over all rectifiable curves γ in $\mathbb{D}$ connected z_1 and z_2. Next, we will prove that it is a distance.

Lemma 7. *The function $d_q(\cdot,\cdot)$ defined by* (2.3) *is a distance.*

Proof. Obviously, for any $z_1, z_2 \in \mathbb{D}$, we have that $d_q(z_1,z_2) \geq 0$ and $d_q(z_1,z_2) = d_q(z_2,z_1)$. Hence, we only prove that if $d_q(z_1,z_2) = 0$, then $z_1 = z_2$. This is because, by taking $x = |z|^2$ in Lemma 6, we get

$$\frac{1}{1-|z|^2} \leq \max\{1, q/2\} \cdot \frac{1}{1-|z|^q}.$$

Hence, by the definitions of hyperbolic distance and q-pseudo distance, we have

$$d_{h_{\mathbb{D}}}(z_1,z_2) \leq \max\{1, q/2\} \cdot d_q(z_1,z_2).$$

Therefore, it must be that $d_{h_{\mathbb{D}}}(z_1,z_2) = 0$ if $d_q(z_1,z_2) = 0$. Since $d_{h_{\mathbb{D}}}(\cdot,\cdot)$ is a distance, we get that $z_1 = z_2$. This shows that the q-pseudo distance $d_q(\cdot,\cdot)$ defined in $\mathbb{D}$ is a distance. □

We also need the following general Schwarz lemma for harmonic self-mappings of the unit disk, which was obtained by Hethcote [19].

Lemma 8. ([19, Theorem 1]) *Suppose that f is a harmonic self-mapping of $\mathbb{D}$. Then we have*

$$\left| f(z) - \frac{1-|z|^2}{1+|z|^2} \cdot f(0) \right| \leq \frac{4}{\pi} \cdot \arctan|z|, z \in \mathbb{D}. \tag{2.4}$$

In [18], Heinz obtained a gradient estimate for self-mappings of $\mathbb{D}$ satisfying the Poisson differential inequality. The upper bound he obtained was abstract. In what follows, we give an explicitly upper bound based on the proof in [18]. The disadvantage is that the range of a, b becomes smaller.

Lemma 9. *Suppose that $f : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ is continuous in $\overline{\mathbb{D}}$, $f|_{\mathbb{D}} \in C^2$, $f|_{\mathbb{T}} \in C^2$ and $f(0) = 0$, and that it satisfies the Poisson differential inequality $|\Delta f(z)| \leq a \cdot |Df(z)|^2 + b$ and $\left|\frac{\partial^2 f(e^{i\varphi})}{\partial \varphi^2}\right| \leq K$, where $0 < a < 1/2, 0 < b, K < \infty$. If $48ab + 640a < 9$, then the inequality*

$$|Df(z)| \leq L(a,b,K) \tag{2.5}$$

holds for any $z \in \mathbb{D}$, where

$$\begin{aligned} L(a,b,K) = {} & 2\max\left\{8 + \frac{64a}{3(1-a)}, b + \frac{8ab}{3(1-a)}\right\} \\ & \times \left[1 + \frac{1-a}{1-2a} \cdot \frac{3K\pi}{\sqrt{3}} + \frac{3a\pi}{2(1-2a)} \cdot \frac{8K^2\pi^2 + 2K}{\sqrt{3}} + \frac{3b}{2(1-2a)}\right]. \end{aligned} \tag{2.6}$$

Proof. First, we calculate $c'_{10}(a,b,K)$ in [18, P. 239]. According to [18, Lemma 11] and given

$$|\widehat{F}''(\varphi)| \leq 8K^2\pi^2 + 2K$$

in [18, P. 239] and [18, (2.2.45), P. 238], we get

$$|X(w) - X(w_0)| \leq \left[\frac{1-a}{1-2a} \cdot \frac{3K\pi}{\sqrt{3}} + \frac{3a\pi}{2(1-2a)} \cdot \frac{8K^2\pi^2 + 2K}{\sqrt{3}} + \frac{3b}{2(1-2a)}\right](1-r).$$

Hence, by [18, (2.2.47), P. 239], we can take

$$c'_{10}(a,b,K) = \frac{1-a}{1-2a}\cdot\frac{3K\pi}{\sqrt{3}} + \frac{3a\pi}{2(1-2a)}\cdot\frac{8K^2\pi^2+2K}{\sqrt{3}} + \frac{3b}{2(1-2a)}. \tag{2.7}$$

Next, we will estimate the constant $c_5(a,b)$ in [18]. According to the proof of [18, Theorem 2], we can take $\alpha = 2$, $\beta = 2(1-a)$ and $\gamma = 2b$ in [18, (2.1.27)]. Therefore, the formula given by [18, (2.1.27)] can be rewritten as

$$4AB \le \frac{16a\lambda}{\theta(1-\lambda)^2} + \frac{4ab\lambda}{(1-\lambda)^2} + \frac{a^2(b+16)}{(1-a)(1-\lambda)^2\ln\frac{1}{\theta}}, \tag{2.8}$$

where λ and θ are constants satisfying that $0 < \lambda < \theta < 1$. Now, we take $\theta = 2\lambda$ and assume that $0 < \lambda < 1/4$. Since $\ln\frac{1}{\theta} \ge 1-\theta$ and $\frac{a^2}{1-a} \le a$ (as $a < 1/2$), we have

$$4AB \le \frac{8a}{(1-\lambda)^2} + \frac{4ab\lambda}{(1-\lambda)^2} + \frac{a(b+16)}{(1-\lambda)^2(1-2\lambda)}. \tag{2.9}$$

As the functions $\varphi_1(\lambda) = 8a+4ab\lambda+\frac{a(b+16)}{1-2\lambda}$, $\varphi_2(\lambda) = -(\lambda-1)^2$ are monotonically increasing on $(0,1/4)$, we see that the inequality

$$4AB \le \frac{8a}{(1-\lambda)^2} + \frac{4ab\lambda}{(1-\lambda)^2} + \frac{a(b+16)}{(1-\lambda)^2(1-2\lambda)} < 1 \tag{2.10}$$

holds for any $0 < \lambda < 1/4$ when $48ab + 640a < 9$. Especially, inequality (2.10) holds for any $0 < \lambda \le 1/8$. Now, we take $\lambda_0 = 1/8$ and $\theta_0 = 1/4$. Then by [18, (2.1.31), P. 227], we have

$$M \le 2B(\lambda_0) \le 2\max\left\{8 + \frac{64a}{3(1-a)}, b + \frac{8ab}{3(1-a)}\right\}\cdot(K+R_0).$$

Hence, by the inequality given by [18, (2.1.31)], we get

$$c_5(a,b) = 2\max\left\{8 + \frac{64a}{3(1-a)}, b + \frac{8ab}{3(1-a)}\right\}. \tag{2.11}$$

Combining [18, P. 239] with (2.7) and (2.11), we have

$$\begin{aligned} L(a,b,K) =& c_{10}(a,b,K) = 2\max\left\{8 + \frac{64a}{3(1-a)}, b + \frac{8ab}{3(1-a)}\right\} \\ &\times\left[1 + \frac{1-a}{1-2a}\cdot\frac{3K\pi}{\sqrt{3}} + \frac{3a\pi}{2(1-2a)}\cdot\frac{8K^2\pi^2+2K}{\sqrt{3}} + \frac{3b}{2(1-2a)}\right]. \end{aligned} \tag{2.12}$$

This completes the proof. □

The proof of following estimate can be found in [8].

Lemma 10. ([8]) *Suppose that* $g \in C(\mathbb{D})$. *Then,* $|G[g](z)| \le \frac{(1-|z|^2)\cdot\|g\|_\infty}{4}$, *where* $\|g\|_\infty := \sup_{z\in\mathbb{D}}|g(z)|$.

Lemma 11. *For any* $t \in [0,1)$, *we have*

$$\frac{1-\frac{4}{\pi}\cdot\arctan t}{1-t} \ge \frac{2}{\pi}.$$

Proof. Let $\varphi_1(t) = 1 - \frac{4}{\pi} \cdot \arctan t$ and $\varphi_2(t) = 1 - t, t \in [0, 1)$. Then, $\varphi_1(1) = \varphi_2(1) = 0$ and the function

$$\frac{\varphi_1'(t)}{\varphi_2'(t)} = \frac{4}{\pi} \cdot \frac{1}{1 + t^2}$$

is monotonically decreasing on $t \in [0, 1)$. By L' Hospital's rule for monotonicity [2], we see that the function φ_1/φ_2 is monotonically decreasing on $t \in [0, 1)$. Hence, by L'Hospital's rule, we get

$$\frac{1 - \frac{4}{\pi} \cdot \arctan t}{1 - t} \geq \lim_{t \to 1^-} \frac{1 - \frac{4}{\pi} \cdot \arctan t}{1 - t} = \lim_{t \to 1^-} \frac{4}{\pi} \cdot \frac{1}{1 + t^2} = \frac{2}{\pi}.$$

This finishes the proof. □

The following result was proved by Ruscheweyh [41].

Lemma 12. ([41]) *Suppose that f is a holomorphic self-mapping of $\mathbb{D}$. Then,*

$$\frac{|f^{(n)}(z)|}{1 - |f(z)|^2} \leq \frac{n!(1 + |z|)^{n-1}}{(1 - |z|^2)^n}. \tag{2.13}$$

In what follows, we generalize Lemma 12 to the following form.

Proposition 1. *For any $q \in [1, 2]$, suppose that f is a holomorphic self-mapping from the unit disk $\mathbb{D}$ into itself; then, we have*

$$|f^n(z)| \leq \frac{n!}{(1 - |z|)^{n-1}} \cdot \frac{1 - |f(z)|^q}{(1 + |z| \cdot |f(0)|)^q - (|z| + |f(0)|)^q} \cdot \frac{1 - |f(0)|^2}{(1 + |z| \cdot |f(0)|)^{2-q}}, z \in \mathbb{D}, \tag{2.14}$$

where $n \in \{1, 2, \cdots\}$.

Proof. Let

$$\varphi(t) = \frac{1 - t^2}{1 - t^q}, t \in [0, 1).$$

Since $\frac{(1-t^2)'}{(1-t^q)'} = \frac{2}{q} \cdot t^{2-q}$ is monotonically increasing on $[0, 1)$, the function φ is also monotonically increasing on $[0, 1)$. By Lindelöf's inequality, we have that $|f(z)| \leq \frac{|z|+|f(0)|}{1+|z| \cdot |f(0)|}, z \in \mathbb{D}$. Therefore,

$$\frac{1 - |f(z)|^2}{1 - |f(z)|^q} \leq \frac{1 - \left(\frac{|z|+|f(0)|}{1+|z| \cdot |f(0)|}\right)^2}{1 - \left(\frac{|z|+|f(0)|}{1+|z| \cdot |f(0)|}\right)^q} = \frac{1 - |z|^2}{(1 + |z| \cdot |f(0)|)^q - (|z| + |f(0)|)^q} \cdot \frac{1 - |f(0)|^2}{(1 + |z| \cdot |f(0)|)^{2-q}}.$$

Combining this with Lemma 12, we get

$$\begin{aligned} |f^{(n)}(z)| &\leq \frac{n!}{(1 - |z|)^{n-1}} \cdot \frac{1 - |f(z)|^2}{1 - |z|^2} \\ &\leq \frac{n!}{(1 - |z|)^{n-1}} \cdot \frac{1 - |f(z)|^q}{(1 + |z| \cdot |f(0)|)^q - (|z| + |f(0)|)^q} \cdot \frac{1 - |f(0)|^2}{(1 + |z| \cdot |f(0)|)^{2-q}}. \end{aligned} \tag{2.15}$$

This has finished the proof. □

Remark 2. *If $q = 2$, then Proposition 1 becomes Lemma 12.*

3. Proof of Theorem 1

Proof. By using Proposition 1 for $n = 1$ and $f(0) = 0$, we get

$$|f'(z)| \leq \frac{1 - |f(z)|^q}{1 - |z|^q}. \tag{3.1}$$

Namely,

$$\frac{|f'(z)|}{1 - |f(z)|^q} \leq \frac{1}{1 - |z|^q}.$$

According to the definition of q-distance, we have that $d_q(f(z_1), f(z_2)) \leq d_q(z_1, z_2)$. This finishes the proof. □

Remark 3. *Considering the case for $n = 1$ and $f(0) = 0$ in Proposition 1, we get the following Schwarz-Pick lemma: Suppose that f is a holomorphic self-mapping from unit disk $\mathbb{D}$ into itself with $f(0) = 0$; then, we have*

$$|f'(z)| \leq \frac{1 - |f(z)|^q}{1 - |z|^q}, z \in \mathbb{D}. \tag{3.2}$$

From the proof of Proposition 1, we see that the inequality (3.2) is sharp. The equation holds if and only if $|f(z)| = |z|$, which is subject to the condition if and only if $f(z) = e^{i\theta} \cdot z, \theta \in \mathbb{R}$ is true.

4. Proof of Theorem 3

Proof. According to Lemma 9, we see that there exists a constant $L(a, b, K)$, such that the inequality

$$|Df(z)| \leq L(a, b, K)$$

holds for any $z \in \mathbb{D}$, where

$$\begin{aligned} L(a, b, K) = 2 \max &\left\{8 + \frac{64a}{3(1-a)}, b + \frac{8ab}{3(1-a)}\right\} \\ &\times \left[1 + \frac{1-a}{1-2a} \cdot \frac{3K\pi}{\sqrt{3}} + \frac{3a\pi}{2(1-2a)} \cdot \frac{8K^2\pi^2 + 2K}{\sqrt{3}} + \frac{3b}{2(1-2a)}\right]. \end{aligned} \tag{4.1}$$

Now, let

$$\Delta f(z) = h(z), z \in \mathbb{D}, \tag{4.2}$$

where $h(z) = l(z)(|Df(z)|^2 + 1)$ and $\|l\|_\infty \leq \max\{a, b\}$. For example, we can define l as follows:

$$l(z) := \Delta f(z) \cdot (|Df(z)|^2 + 1)^{-1}, z \in \mathbb{D}.$$

Since $f \in C^2(\mathbb{D})$, we see that $h \in C(\mathbb{D})$. Hence, by Lemma 9, we get

$$\|h\|_\infty \leq \max\{a, b\} \cdot (L^2(a, b, K) + 1).$$

By virtue of the solution of Poisson's equation, we have

$$f(z) = P[k](z) - G[h](z), \tag{4.3}$$

where $f|_{\mathbb{T}} := k$. Since $P[k]$ is a harmonic self-mapping of $\mathbb{D}$, we get the following by Lemma 8

$$\left| P[k](z) - \frac{1-|z|^2}{1+|z|^2} \cdot P[k](0) \right| \le \frac{4}{\pi} \cdot \arctan |z|. \tag{4.4}$$

Combining Lemmas 6, 8, 10 and 11, we get

$$\begin{aligned}
&\frac{1-|f(z)|^q}{1-|z|^q} = \frac{1-|f(z)|^q}{1-|f(z)|} \cdot \frac{1-|z|}{1-|z|^q} \cdot \frac{1-|f(z)|}{1-|z|} \ge c_q \cdot \frac{1-|f(z)|}{1-|z|} \\
=&c_q \cdot \frac{1 - \left| P[k](z) - \frac{1-|z|^2}{1+|z|^2} \cdot P[k](0) + \frac{1-|z|^2}{1+|z|^2} \cdot P[k](0) - G[h](z) \right|}{1-|z|} \\
\ge&c_q \cdot \frac{1 - \left| P[k](z) - \frac{1-|z|^2}{1+|z|^2} \cdot P[k](0) \right| - \frac{1-|z|^2}{1+|z|^2} \cdot |P[k](0)| - |G[h](z)|}{1-|z|} \\
\ge&c_q \cdot \frac{1 - \frac{4}{\pi} \cdot \arctan |z| - \frac{1-|z|^2}{1+|z|^2} \cdot |P[k](0)| - \frac{(1-|z|^2)\|h\|_\infty}{4}}{1-|z|} \\
=&c_q \cdot \left(\frac{1 - \frac{4}{\pi} \cdot \arctan |z|}{1-|z|} - \frac{1+|z|}{1+|z|^2} \cdot |G[h](0)| - \frac{(1+|z|) \cdot \|h\|_\infty}{4} \right) \\
\ge&c_q \cdot \left(\frac{2}{\pi} - \frac{1}{2} \cdot \|h\|_\infty - \frac{1}{2} \cdot \|h\|_\infty \right) = c_q \cdot \left(\frac{2}{\pi} - \|h\|_\infty \right) \\
\ge&c_q \cdot \left(\frac{2}{\pi} - \max\{a, b\} \cdot (L^2(a, b, K) + 1) \right) > 0.
\end{aligned} \tag{4.5}$$

This completes the proof. □

5. Proof of Theorem 4

Proof. Consider the function

$$f(z) = \frac{(3-|z|^2)z}{2}, z \in \mathbb{D},$$

then, $|f(z)| < 1$, $f(e^{i\theta}) = e^{i\theta}$, $f_z(z) = \frac{3-2|z|^2}{2}$, $f_{\bar{z}}(z) = -\frac{z^2}{2}$, $\Delta f(z) = 4 f_{z\bar{z}}(z) = -4z$ and $|f_z(z)| + |f_{\bar{z}}(z)| = \frac{3-|z|^2}{2}$. Therefore, f is a self-mapping of $\mathbb{D}$ satisfying that $|\Delta f(z)| \le 4(Df(z))^2$ and $f(0) = 0$. But $\mathrm{Re}(f_x(1)) = \mathrm{Re}(f_z(1) + f_{\bar{z}}(1)) = 0$. □

6. Proof of Theorem 5

Proof. Set $\Delta f(z) = h(z), z \in \mathbb{D}$, where $h(z) = l(z)(|Df(z)|^2 + 1)$ and $\|l\|_\infty \le \max\{a, b\}$. Since $h \in C(\mathbb{D})$, we get, by Lemma 9 that $\|h\|_\infty \le \max\{a, b\} \cdot (L^2(a, b, K) + 1)$. Then,

$$f(z) = P[k](z) - G[h](z), \tag{6.1}$$

where $f|_{\mathbb{T}} := k$. Next, the method is similar to the one given in [50]. But, here, we do not demand that $h \in C(\overline{\mathbb{D}})$, as we only need $h \in C(\mathbb{D})$. By virtue of (6.1) and Lemmas 8 and 10, we have

$$\begin{aligned} |f(z)| &= \left| P[k](z) - \frac{1-|z|^2}{1+|z|^2} \cdot P[k](0) + \frac{1-|z|^2}{1+|z|^2} \cdot P[k](0) - G[h](z) \right| \\ &\leq \left| P[k](z) - \frac{1-|z|^2}{1+|z|^2} \cdot P[k](0) \right| + \frac{1-|z|^2}{1+|z|^2} \cdot |P[k](0)| + |G[h](z)| \\ &\leq \frac{4}{\pi} \cdot \arctan|z| + \frac{1-|z|^2}{1+|z|^2} \cdot |G[h](0)| + |G[h](z)| \\ &\leq \frac{4}{\pi} \cdot \arctan|z| + \frac{1-|z|^2}{1+|z|^2} \cdot \frac{\|h\|_\infty}{4} + \frac{(1-|z|^2)\|h\|_\infty}{4} := M(z). \end{aligned} \tag{6.2}$$

Since f is differential at $z = 1$ and satisfies that $f(1) = 1$, we get

$$f(z) = 1 + f_z(z)(z-1) + f_{\overline{z}}(\overline{z}-1) + o(|z-1|).$$

Since $|f(z)|^2 \leq M^2(z)$,

$$2\mathrm{Re}(f_z(1-z) + f_{\overline{z}}(1-\overline{z})) \geq 1 - M^2(z) + o(|z-1|).$$

By letting $z = r \in (0,1)$ and $r \to 1^-$, we obtain

$$\begin{aligned} &2\mathrm{Re}(f_z(1) + f_{\overline{z}}(1)) \geq \lim_{r\to 1^-} \frac{1-M^2(z)}{1-r} \\ =&2 \lim_{r\to 1^-} \left(\frac{4}{\pi} \cdot \frac{1}{1+r^2} - \frac{r \cdot \|h\|_\infty}{(1+r^2)^2} - \frac{r \cdot \|h\|_\infty}{2} \right) \\ =&2\left(\frac{2}{\pi} - \frac{3\|h\|_\infty}{4} \right) \geq 2\left(\frac{2}{\pi} - \frac{3\max\{a,b\} \cdot (L^2(a,b,K)+1)}{4} \right) > 0. \end{aligned} \tag{6.3}$$

This finishes the proof. □

7. Conclusions

By establishing the general Schwarz-Pick lemma, which is a generalization of the result of Ruscheweyh [41] and sharp when $n = 1$ and $f(0) = 0$, we have obtained a Schwarz-Pick-type lemma for the holomorphic self-mapping of the unit disk with respect to the q-distance. We also have established the general Schwarz-Pick lemma for self-mappings of the unit disk satisfying the Poisson differential inequality. This obtained result improved the one in [47] by giving the explicit constant $L(a,b,K)$, and it showed that the result is still valid when we consider all of the cases for $q > 0$. As an application, it has been proven that this mapping is Lipschitz continuous with respect to the q-distance under certain conditions; Moreover, the corresponding explicit Lipschitz constant has been given. Next, we investigated the boundary Schwarz lemma for self-mappings of the unit disk satisfying the Poisson differential inequality. We found that those types of mappings are generally not associated with the boundary Schwarz lemma. With some additional conditions, we have established a boundary Schwarz lemma for self-mappings of the unit disk satisfying the Poisson differential inequality.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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