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Research article

On the structure of finite groups associated to regular non-centralizer graphs

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Abstract: The non-centralizer graph of a finite group G is the simple graph Υ_G whose vertices are the elements of G with two vertices are adjacent if their centralizers are distinct. The induced non-centralizer graph of G is the induced subgraph of Υ_G on $G \setminus Z(G)$. A finite group is called regular (resp. induced regular) if its non-centralizer graph (resp. induced non-centralizer graph) is regular. In this paper we study the structure of regular groups and induced regular groups. We prove that if a group G is regular, then $G/Z(G)$ as an elementary 2-group. Using the concept of maximal centralizers, we succeeded in proving that if G is induced regular, then $G/Z(G)$ is a p-group. We also show that a group G is induced regular if and only if it is the direct product of an induced regular p-group and an abelian group.

Keywords: centralizers; finite groups; graph; regular
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1. Introduction

For standard terminology and notion in graph theory and group theory, we refer the reader to the textbooks of [1] and [2] respectively. Let Υ be a simple graph. The degree of a vertex x in Υ, denoted by $deg(x)$, is the number of vertices adjacent with x. Υ is said to be regular if all of its vertices have the same degree. If $deg(x) = n$, we say that Υ is n-regular (or regular of degree n).

Throughout this paper, G denotes a finite group. The order of a group G (respectively the order of x in G) is denoted by $|G|$ (respectively $o(x)$). The centralizer of x is $C_G(x) = \{y \in G \mid yx = xy\}$, and the center of G is $Z(G) = \{x \in G \mid xy = yx, \forall y \in G\}$. For an x in G, the coset $xZ(G)$ is denoted by $\overline{x}$. The set $Cent(G) = \{C_G(x) \mid x \in G\}$ is the set of distinct centralizers in G.

Over the last few decades in the theory of algebraic graphs, which deals with graphs associated to algebraic structure, it has become of great interest to many researchers in both fields of algebra and graph theory. Studying algebraic graphs usually aims to investigate the interplay between the algebraic structure and graph theory concepts. These investigations led to several interesting results and problems, see [3–10] and references therein.

Tolue in [11] introduced the notion of the non-centralizer graph of a finite group: The non-centralizer graph of a finite group G, denoted by Υ_G, is the simple graph whose vertices are the elements of G, where two vertices x and y are adjacent if $C_G(x) \neq C_G(y)$. The induced subgraph of Υ_G associated with the vertex set $G \backslash Z(G)$ is called the induced non-centralizer graph and denoted by $\Upsilon_{G\backslash Z(G)}$. It is clear that Υ_G is a complete $|Cent(G)|$-partite graph and $\Upsilon_{G\backslash Z(G)}$ is a complete $(|Cent(G)| - 1)$-partite graph. Classical properties of these graphs such as diameter, girth, domination and chromatic numbers, and independent set were studied in [11]. Also, the author showed that the induced non-centralizer graph and the non-commuting graph associated to an AC-group are isomorphic (for information on non-commuting graphs see [3, 12]). Interestingly, Tolue proved that Υ_G is 6-regular if and only if $G \cong D_8$ or Q_8, and Υ_G is not n-regular, for $n = 4, 5, 7, 8, 11, 13$, leading to the following conjecture.

Conjecture 1.1. *[11] Υ_G is not p-regular graph, where p is a prime integer.*

In this paper we prove this conjecture by showing that Υ_G in not n-regular if n is a prime power integer. This is a direct consequence of deeper results giving the structure of regular groups, such as Theorem 2.1 and Theorem 2.4.

In the last few decades, there has been much research interest in characterizing groups by properties of their centralizers, such as commutativity and the number of centralizer, see for instance [13–25]. Our work is no exception to this trend. Indeed, the sets $\beta_G(x) = \{y \in G \mid C_G(y) = C_G(x)\}$, $x \in G$, as well as the notion of maximal centralizers play key rolls in this paper.

We say that G is regular (resp. induced regular) if Υ_G is regular (resp. $\Upsilon_{G\backslash Z(G)}$ is regular). In Section 2, we study the regularity of the non-centralizer graph. We show that if G is regular then $G/Z(G)$ is elementary 2-group. We also prove that a group is regular if and only if it is the direct product of a regular 2-group and an abelian group. Following this result, the notion of reduced regular groups is introduced. Moreover, we observe that if G is n-regular then every centralizer in G is normal and $n + 2 \leq |G| \leq 4n/3$. We end this section with a table listing reduced n-regular 2-groups for all possible values of $n \leq 60$.

Section 3 is devoted to the regularity of $\Upsilon_{G\backslash Z(G)}$. Using the concept of maximal centralizers, we obtain many results on the structure of induced regular groups. We prove that if G is induced regular then $G/Z(G)$ is a p-group. We also show that a group G is induced regular if and only if it is the direct product of an induced regular p-group and an abelian group.

2. Regularity of Υ_G

We start this section by listing some known results that will be used later.

Proposition 2.1. *[18, Fact 2] If G is a non-abelian group, then $|Cent(G)| \geq 4$.*

Proposition 2.2. *[18, Fact 6] Let p be a prime. If $G/Z(G) \cong C_p \times C_p$, then $|Cent(G)| = p + 2$.*

The following proposition can be directly obtained from [17, Proposition 2.2] and its proof.

Proposition 2.3. *Let G be a non-abelian group, such that $[G : Z(G)] = p^3$, where p is the smallest prime dividing $|G|$. If $[G : C_G(x)] = p^2$ for all $x \in G \setminus Z(G)$, then $|Cent(G)| = p^2 + p + 2$. Otherwise $|Cent(G)| = p^2 + 2$.*

For every element x in G, define the set $\beta_G(x) = \{y \in G \mid C_G(y) = C_G(x)\}$. These sets form the parts of Υ_G. Clearly $deg(x) = |G| - |\beta_G(x)|$ and $\beta_G(e) = Z(G)$. Also, $\overline{x} \subseteq \beta_G(x)$ for all $x \in G$. So, $\beta_G(x)$ is a disjoint union of cosets of $Z(G)$. We also conclude that $|Z(G)|$ divides $deg(x)$ for every $x \in G$, and $|Cent(G)| \leq [G : Z(G)]$. The next proposition shows that Υ_G is regular if and only if for each x in G, $\beta_G(x)$ is a coset of $Z(G)$.

Proposition 2.4. *Let G be a non-abelian group. Then, G is regular if and only if $\beta_G(x) = \overline{x}$ for all $x \in G$.*

Proof. We know that $xZ(G) \subseteq \beta_G(x)$. So

$$\begin{aligned} G \text{ is regular} &\Longleftrightarrow deg(x) = deg(e) && \text{for all } x \in G \\ &\Longleftrightarrow |G| - |\beta_G(x)| = |G| - |Z(G)| && \text{for all } x \in G \\ &\Longleftrightarrow |\beta_G(x)| = |Z(G)| = |\overline{x}| && \text{for all } x \in G \\ &\Longleftrightarrow \overline{x} = \beta_G(x) && \text{for all } x \in G. \end{aligned}$$

□

Corollary 2.1. *Let G be a non-abelian group. Then, G is regular if and only if $|Cent(G)| = [G : Z(G)]$.*

Proof. From Proposition 2.4, we see that G is regular if and only if the number of parts of Υ_G is equal to the number of distinct cosets of $Z(G)$. Hence the result. □

A group G is called p-group (where p is a prime) if the order of G is a power of p. A p-group that is the direct product of copies of C_p is called an elementary abelian p-group. Is it well known in group theory that a group is an elementary abelian 2-group if and only if the order of each non-identity element in G is exactly 2. For convenience we call a group G elementary p-group if the order of each element in G is exactly p. With this convention, an elementary p-group is abelian if and only if it is the direct product of copies of C_p.

Theorem 2.1. *Let G be a non-abelian group. If G is regular then $G/Z(G)$ is an elementary abelian 2-group.*

Proof. Suppose G is regular, and let $x \in G$. By Proposition 2.4 we have $\beta_G(x^{-1}) = \overline{x}^{-1}$. Since $x \in \beta_G(x^{-1})$, we get $x = x^{-1}z$ for some $z \in Z(G)$, and so $x^2 = z \in Z(G)$. Therefore, $G/Z(G)$ is an elementary abelian 2-group. □

The converse of Theorem 2.1 is not true in general. For example, the group $G = \langle a, b, c \mid a^4 = b^4 = c^2 = e, ab = ba, cac^{-1} = a^{-1}, cbc^{-1} = b^{-1}\rangle$ is not regular even though $G/Z(G) \cong C_2 \times C_2 \times C_2$. In fact, using GAP we find that for each one of the groups with ID's $[32, 27], ..., [32, 35]$, $G/Z(G)$ is isomorphic to $C_2 \times C_2 \times C_2$, yet none of them are regular. The next two theorems show two cases in which the converse of Theorem 2.1 is true.

Theorem 2.2. *Let G be a non-abelian group. If $G/Z(G) \cong C_2 \times C_2$ then G is regular.*

Proof. Suppose that $G/Z(G) \cong C_2 \times C_2$. Then, by Proposition 2.2, $|Cent(G)| = 4 = [G : Z(G)]$. Hence, by Corollary 2.1, G is regular. □

Example 2.1. *For any positive integer $k \geq 3$, let $M(2^k)$ be the group defined by*

$$M(2^k) = \langle a, b \mid a^{2^{k-1}} = b^2 = 1,\ bab = a^{2^{k-2}+1} \rangle.$$

We have $Z(M(2^k)) = \langle a^2 \rangle$. So, $M(2^k)/Z(M(2^k)) \cong C_2 \times C_2$. Then $M(2^k)$ is regular of degree $3 \cdot 2^{k-2}$.

Theorem 2.3. *Let G be a non-abelian group such that $G/Z(G) \cong C_2 \times C_2 \times C_2$. Then G is regular if and only if $[G : C_G(x)] = 4$ for all non central elements x of G.*

Proof. Suppose G is regular. Then $|Cent(G)| = [G : Z(G)] = 8$. Now, for each x in $G \setminus Z(G)$, $[G : Z(G)] = [G : C_G(x)][C_G(x) : Z(G)]$, and so $[G : C_G(x)] \neq 1, 8$. If $[G : C_G(x)] = 2$, for some $x \in G$, then Proposition 2.3 implies that $|Cent(G)| = 6$, a contradiction. So $[G : C_G(x)] = 4$ for all $x \in G \setminus Z(G)$. Now for the converse, assume that $[G : C_G(x)] = 4$ for all $x \in G \setminus Z(G)$. Then Proposition 2.3 implies that $|Cent(G)| = 8 = [G : Z(G)]$. Hence, by Corollary 2.1, G is regular. □

Theorem 2.4. *Let G be a regular group. Then, for each $x \in G$, $C_G(x)$ is normal in G and $G/C_G(x)$ is isomorphic to a subgroup of $Z(G)$.*

Proof. Suppose G is regular, and let $x \in G$. Since $G/Z(G)$ is abelian, we can associate with every $g \in G$ a unique element z_{xg} in $Z(G)$ such that $xg = gxz_{xg}$. By Proposition 2.4, we have that $C_G(xz_{xg}) = C_G(x)$. So, we obtain that

$$g^{-1}C_G(x)g = C_G(g^{-1}xg) = C_G(g^{-1}gxz_{xg}) = C_G(xz_{xg}) = C_G(x).$$

Hence $C_G(x)$ is normal in G. Also, the mapping $\phi_x : G \longrightarrow Z(G)$ defined by $\phi_x(g) = z_{xg}$ is a homomorphism with $Ker(\phi_x) = C_G(x)$. Thus, $G/C_G(x) \cong Img(\phi_x)$. □

The following corollary proves Conjecture 1.1.

Corollary 2.2. *Let G be a non-abelian group. Then Υ_G is not n-regular, where n is a prime power integer.*

Proof. Suppose Υ_G is n-regular for some integer n. We know that $n = |G| - |Z(G)| = |Z(G)|([G : Z(G)] - 1)$. By Theorem 2.1 we get that G/Z is a 2-group, and so $[G : Z(G)] = 2^t$ for some integer $t \geq 1$. Hence, $[G : Z(G)] - 1$ is odd. On the other hand, since G be a non-abelian, there exists $x \in G \setminus Z(G)$, and from the relation $[G : Z(G)] = [G : C_G(x)][C_G(x) : Z]$, we see that $[G : C_G(x)] = 2^s$ for some $s \geq 1$. By Theorem 2.4, we know that $G/C_G(x)$ is isomorphic to a subgroup of $Z(G)$, therefore $[G : C_G(x)]$ divides $|Z(G)|$. So $|Z(G)|$ is even. Thus n can not be a power of prime. □

Theorem 2.5. *If G is a non-abelian group such that Υ_G is n-regular, then n is even, $|G| \equiv 0 \pmod 8$, and $n + 2 \leq |G| \leq 4n/3$.*

Proof. Corollary 2.1 yields that $[G : Z(G)]$ is dividable by 4, and Theorem 2.4 implies that $|Z(G)|$ is even. Hence $|G|$ is divisible by 8. Since $|G| = \frac{[G:Z(G)]n}{[G:Z(G)]-1}$, from Proposition 2.1 we get $[G : Z(G)] \geq 4$, and so we obtain that $|G| \leq 4n/3$. In addition, $2 \leq |Z(G)| = |G| - n$. Therefore, $n + 2 \leq |G| \leq 4n/3$. □

Theorem 2.6. *Let $k \geq 3$ and $t \geq 1$ be integers and let G be a group of order $2^k(2t - 1)$. Then, G is regular if and only if $G \cong H \times A$, where A is an abelian group of order $2t - 1$ and H is a regular 2-group of order 2^k.*

Proof. Suppose G is regular. Then $G/Z(G)$ is a 2-group, which yields $|Z(G)| = 2^s(2t-1)$ for some $1 \leq s \leq k$. Let A be the subgroup of $Z(G)$ of order $2t-1$ and let H be a 2-sylow subgroup of G. Now, $HA = AH$. So, $HA \leq G$. Also, we have $H \cap A = \{e\}$ (this is because H is a 2- group and $|A|$ is odd). Thus, $|HA| = |H||A| = |G|$, and so $G = HA$. Moreover, since $ha = ah$ for all $a \in A$ and $h \in H$, we have $HA \cong H \times A$. It remains to be shown that H is regular. Since $A \leq Z(G)$, we obtain that for any two elements $x, y \in H$, $C_G(x) = C_G(y)$ if and only if $C_H(x) = C_H(y)$. So, for each $x \in H, \beta_H(x) = \beta_G(x) \cap H$. Now, $Z(G) = Z(H)A$, and by Proposition 2.4 we have $\beta_G(x) = \overline{x}$. Therefore,

$$\beta_H(x) = \beta_G(x) \cap H = \overline{x} \cap H = xZ(H)A \cap H = xZ(H).$$

Then, Proposition 2.4 yields regularity of H.

Now we prove the converse. Assume that $G = H \times A$ where A is an abelian and H is regular 2-group. We have $Z(G) = Z(H) \times Z(A) = Z(H) \times A$. Also, for each $(h, a) \in G$, $C_G(h, a) = C_H(h) \times C_A(a) = C_H(h) \times A$. Combining this we obtain

$$\beta_G(h, a) = \beta_H(h) \times A = hZ(H) \times aA = (h, a)(Z(H) \times A) = (h, a)Z(G).$$

Therefore, by Proposition 2.4, G is regular. □

Example 2.2. *Let A be an abelian group of order $k \geq 1$ and let $G = D_8 \times A$ or $G = Q_8 \times A$. Then, Υ_G is $K_{2k,2k,2k,2k}$ which is $6k$-regular.*

We get the following two remarks from Theorem 2.6 and its proof.

Remark 2.1. *The sylow subgroup H is normal in G, and the abelian subgroup A is the direct product of all other sylow subgroups. So all sylow subgroups of G are normal, and G is isomorphic to the direct product of its sylow subgroups.*

Remark 2.2. *Characterizing regular groups reduces to studying regular 2-groups that are not the direct product of a regular group and an abelian group.*

Definition 2.1. *A regular 2-group is called reduced regular if it is not the direct product of a regular group and an abelian group.*

Using Theorem 2.5, we can find for a fixed value of n all n-regular and reduced n-regular groups. For example, for $n = 6$, Theorem 2.5 implies that $|G|$ must be 8. Among the groups of order 8, we find that Q_8, D_8 are the only 6-regular groups and both of them are reduced regular. For the case $n = 10$, we obtain that $|G| = 12$. But, $|G|$ must also be divisible by 8. So, there are no 10-regular groups. For $n = 12$ we find that $|G| = 16$. After investigating the groups of order 16, we find that six of them are 10-regular, and four out of these six are reduced 10-regular. Table 1 lists all reduced n-regular 2-groups with $n \leq 60$.

Table 1. Reduced n-regular 2-groups with $n \leq 60$.

n	Reduced n-Regular 2-Groups
6	Q_8, D_8
12	[16,3], [16,4], [16,6], [16,13]
24	[32,2], [32,4], [32,5], [32,12], [32,17], [32,24], [32,38]
30	[32,49], [32,50]
48	[64,3], [64,17], [64,27], [64,29], [64,44],
	[64,51], [64,57], [64, 86], [64, 112], [64, 185]
56	[64, 73], [64, 74], [64, 75], [64, 76], [64, 77]
	[64, 78], [64, 79], [64, 80], [64, 81],[64, 82]
60	[64, 199], [64, 200], [64, 201], [64, 226], [64, 227],
	[64, 228], [64, 229], [64, 230], [64, 231], [64, 232],
	[64, 233], [64, 234], [64, 235], [64, 236], [64, 237],
	[64, 238], [64, 239], [64, 240], [64, 249], [64, 266]

3. Regularity of $\Upsilon_{G\setminus Z(G)}$

Definition 3.1. *Let G be a non-abelian group. A centralizer $C_G(x)$ is called maximal if it is not contained in any other proper centralizer.*

Proposition 3.1. *Let G be a group. If $C_G(x)$ is a maximal centralizer in G, then $\beta_G(x) \cup Z(G)$ is a subgroup of G.*

Proof. Since $C_G(y) = C_G(y^{-1})$, we get that $y \in \beta_G(x) \cup Z(G)$ if and only if $y^{-1} \in \beta_G(x) \cup Z(G)$. Now, let $y, w \in \beta_G(x)$. Since $C_G(y) = C_G(x) = C_G(w)$, we get $C_G(x) \subseteq C_G(yw)$. So, by maximality of $C_G(x)$, we obtain that $C_G(x) = C_G(yw)$ or $C_G(yw) = G$. In both cases yw belongs to $\beta_G(x) \cup Z(G)$. Therefore, $\beta_G(x) \cup Z(G)$ is a subgroup. □

Proposition 3.2. *Let G be an induced regular group, and let $C_G(x)$ be a maximal centralizer such that $C_G(x) \neq \beta_G(x) \cup Z(G)$. Then, there is an element y in $C_G(x) \setminus \beta_G(x)$ such that $o(\overline{y})$ is prime.*

Proof. Assume that for each $y \in C_G(x) \setminus \beta_G(x)$, $o(\overline{y})$ is not prime. We claim that for each $w \in C_G(x) \setminus (\beta_G(x) \cup Z(G))$, $C_G(w) \subseteq C_G(x)$. Indeed, take a prime divisor p of $o(\overline{w})$, and let $y_0 = w^{\frac{o(\overline{w})}{p}}$. Then, $o(\overline{y_0}) = p$ and thus $y_0 \in \beta_G(x)$. Hence $C_G(w) \subseteq C_G(y_0) = C_G(x)$, as we claimed. Now let $y \in C_G(x) \setminus (\beta_G(x) \cup Z(G))$. We claim that $y\beta_G(x) \subseteq \beta_G(y)$. By Proposition 3.1, $\beta_G(x) \cup Z(G)$ is a group. This implies that, for all $g \in \beta_G(x)$, $yg \notin \beta_G(x) \cup Z(G)$. Now let $g \in \beta_G(x)$, we want to show that $C_G(gy) = C_G(y)$. Clearly $C_G(g) \cap C_G(y) \subseteq C_G(gy)$, and so $C_G(y) = C_G(x) \cap C_G(y) = C_G(g) \cap C_G(y) \subseteq C_G(gy)$. For the other inclusion, let $h \in C_G(gy)$, then $hgy = gyh$. Also since $gy \in C_G(x) \setminus (\beta_G(x) \cup Z(G))$, according to our first claim we get $C_G(gy) \subseteq C_G(x)$. So $h \in C_G(x) = C_G(g)$, and so $hgy = ghy$. Thus we have $hy = yh$, which yields $h \in C_G(y)$. This proves our claim. Since G is an induced regular group, we get that $|y\beta_G(x)| = |\beta_G(y)|$, and so $y\beta_G(x) = \beta_G(y)$. This implies that $1 \in \beta_G(x)$, a contradiction. Therefore, there must be $y \in C_G(x) \setminus \beta_G(x)$ such that $o(\overline{y})$ is prime. □

Proposition 3.3. *Let G be an induced regular group, and let $C_G(x)$ be a maximal centralizer in G. If there exists an element y in $C_G(x) \setminus \beta_G(x)$ such that $o(\overline{y})$ is a prime $p \neq 2$, then $\dfrac{\beta_G(x) \cup Z(G)}{Z(G)}$ is an elementary $p-$group.*

Proof. Let $H_x = \beta_G(x) \cup Z(G)$ and let $B = \{w \in \beta_G(x) \mid o(\overline{w}) \neq p\}$. For $w \in B$, $C_G(w) \cap C_G(y) \subseteq C_G(wy)$. Also, $C_G(wy) \subseteq C_G((wy)^p) = C_G(w^p) = C_G(w)$. Then, $C_G(w) \cap C_G(y) = C_G(wy)$. Hence, we obtain that $yB \subseteq \beta_G(w_0y)$ for some $w_0 \in B$. Let $T = \langle\{w \in H_x \mid o(\overline{w}) = p\}\rangle$. Then, $T/Z(G)$ is an elementary p-subgroup of $H_x/Z(G)$. We have $y(H_x \setminus T) \subseteq \beta_G(w_0y)$ and $y^{-1}(H_x \setminus T) \subseteq \beta_G(w_0y)$. If $y(H_x \setminus T)$ and $y^{-1}(H_x \setminus T)$ are not disjoint then $\overline{yu} = \overline{y}^{-1}\overline{v}$ for some $u, v \in H_x \setminus T$. This implies that $y^2 \in \overline{v}\,\overline{u}^{-1} \subseteq \beta_G(x) \cup Z(G)$. But also, since $y^p \in Z(G)$ and $p \neq 2$, we get that $y \in \beta_G(x)$, a contradiction. So, $y(H_x \setminus T)$ and $y^{-1}(H_x \setminus T)$ are disjoint. This implies that

$$2|y(H_x \setminus T)| = |y(H_x \setminus T)| + |y^{-1}(H_x \setminus T)| \leq |\beta_G(w_0y)| < |H_x|.$$

So $2(|H_x| - |T|) < |H_x|$. This implies that $|H_x| < 2|T|$, which yields $[H_x : T] = 1$. Thus $H_x = T$, and hence $H_x/Z(G)$ is an elementary p-group. □

Theorem 3.1. *If G is a non-abelian induced regular group, then $G/Z(G)$ is a p-group.*

Proof. Let G be a non-abelian induced regular group. If $C_G(x) = \beta_G(x) \cup Z(G)$, for all $x \in G$, then G is an AC-group. Thus, by [11, Theorem 2.11] and [3, Proposition 2.6], we have $G = P \times A$, where P is a p-group and A is an abelian group. Hence, $\dfrac{G}{Z(G)} = \dfrac{P \times A}{Z(P) \times A} \cong \dfrac{P}{Z(P)}$, and so it is a p-group. Now suppose that G is not an AC-group. Let $C_G(x_0)$ be a non-abelian maximal centralizer of G and let $H_{x_0} = \beta_G(x_0) \cup Z(G)$. By Proposition 3.2, there exists an element y in $C_G(x_0) \setminus H_{x_0}$ such that $o(\overline{y})$ is prime p. The next goal is to show the following claim.

Claim: $H_{x_0}/Z(G)$ is a p-group.

If there is an element y in $C_G(x_0) \setminus H_{x_0}$ such that $o(\overline{y}) = p \neq 2$, then by Proposition 3.3, $H_{x_0}/Z(G)$ is an elementary p-group. Now suppose that for each element y in $C_G(x_0) \setminus H_x$, $o(\overline{y})$ is either 2 or not prime. In this case we show that $H_{x_0}/Z(G)$ is a 2-group. This is achieved in three steps.

Step 1: We show that $C_G(x_0)/H_{x_0}$ is a 2-group. For simplicity, we will denote the coset yH_{x_0} in $C_G(x_0)$ by $\tilde{y}$. Assume there is $y \in C_G(x_0)$ such that $o(\tilde{y})$ is divisible by a prime $p \neq 2$, and let $y_0 = y^{\frac{o(\tilde{y})}{p}}$. Clearly, $o(\tilde{y}_0) = p$ and $x_0y_0 \notin H_{x_0}$. Now, y_0^p and $(x_0y_0)^p$ belong to H_{x_0}. But, since $o(\overline{y}_0)$ and $o(\overline{x_0y_0})$ are either 2 or not prime, we have y_0^p and $(x_0y_0)^p$ belong to $\beta_G(x_0)$. Therefore, $C_G(x_0y_0) \cup C_G(y_0) \subseteq C_G((x_0y_0)^p) \cup C_G((y_0)^p) = C_G(x_0)$. Using the same method as in the proof of Proposition 3.2 one can easily show that $C_G(wy_0) = C_G(x_0) \cap C_G(y_0) = C_G(y_0)$ for all $w \in \beta_G(x)$. Thus, we obtain that $y_0\beta_G(x_0) \subseteq \beta(y_0)$. From the fact that $y_0 \notin y_0\beta_G(x_0)$, we get that $|\beta_G(y_0)| > |y_0\beta_G(x_0)| = |\beta_G(x_0)|$, a contradiction. Therefore $C_G(x_0)/H_{x_0}$ is a 2-group.

Step 2: We show that for all $y \notin H_{x_0}$, $o(\overline{y})$ is a power of 2. Let $y \notin H_{x_0}$, and suppose that $o(\overline{y}) = 2^k(2t-1)$. If y^{2t-1} belongs to $\beta_G(x_0)$, then $(\tilde{y})^{2t-1} = \tilde{1}$ in $C_G(x_0)/H_{x_0}$, and so by Step 1, we get that 2 divides $2t-1$, a contradiction. Thus, $y^{2t-1} \notin \beta_G(x)$. Now suppose that p is a prime divisor of $2t-1$ and let $y_p = y^{\frac{2t-1}{p}}$. Then, $o(\overline{y}_p) = p$, and so $y_p \in \beta_G(x_0)$. This implies that 2 divides $(2t-1)/p$, a contradiction. Thus, $2t-1 = 1$, and hence $o(\overline{y})$ is a power of 2.

Step 3: We show that $\frac{H_{x_0}}{Z(G)}$ is a 2-group. Assume that there is an element $w \in H_{x_0}$ such that $o(\overline{w}) = p \neq 2$. By Proposition 3.2, there is $y \in C_G(x_0) \setminus H_{x_0}$ such that $o(\overline{y}) = 2$. Then we have $wy \notin H_{x_0}$ and

$o(\overline{wy}) = 2p$, which contradicts the result in Step 2. Hence, $\frac{H_{x_0}}{Z(G)}$ is a 2-group. This completes the proof of the claim.

Now let $C_G(x)$ be an arbitrary maximal centralizer in G, and let $H_x = \beta_G(x) \cup Z(G)$. Then, from the regularity of $\Upsilon_{G\setminus Z(G)}$, we obtain that $|H_x| = |H_{x_0}|$. So, $|H_x|$ is a p-group. Now we come to the last step in the proof, which is showing that $G/Z(G)$ is a p-group. Assume that $[G : Z(G)]$ is divisible by a prime $q \neq p$. Let $y \in G$ such that $o(\overline{y}) = q$, and let $C_G(x)$ be maximal centralizer that contains $C_G(y)$. Again, following the same argument as in the proof of Proposition 3.2, one can show that $y\beta_G(x) \subseteq \beta_G(y)$. But since $y \notin y\beta_G(x)$, we obtain that $|\beta_G(x)| = |y\beta_G(x)| < |\beta_G(y)|$, which contradicts the fact that G is induced regular. Therefore, $G/Z(G)$ is a p-group. □

Proposition 3.4. *Let G be an induced regular group. If $[G : Z(G)] = p^q$ where p and q are primes, then $G/Z(G)$ is an elementary p-group. Moreover, for each $x \in G$, $|\beta_G(x)| = (p-1)|Z(G)|$.*

Proof. Suppose $[G : Z(G)] = p^q$ for some $q \geq 2$. Consider a maximal centralizer $C_G(x)$ in G. Then, $|\beta_G(x)|/|Z(G)| = p^s - 1$ for some $1 \leq s \leq q-1$ and $(|Cent(G)| - 1)(p^s - 1) = p^q - 1$, and hence $p^s - 1$ divides $p^q - 1$. This implies that s divides q. Thus, if q is prime, then $s = 1$, which completes the proof. □

Corollary 3.1. *Let G be an induced regular group of odd order. If $C_G(x) \neq \beta_G(x) \cup Z(G)$ for all $x \in G$, then $G/Z(G)$ is an elementary p-group.*

Proof. This result follows directly from Proposition 3.3. □

Proposition 3.5. *Let G be a non-abelian group. If there is a prime p such that $G/Z(G) \cong C_p \times C_p$, then G is induced regular.*

Proof. Suppose that $G/Z(G) \cong C_p \times C_p$. Then, by Proposition 2.2, $\Upsilon_{G\setminus Z(G)}$ has $p+1$ parts. Let $\beta_G(x_1), \cdots, \beta_G(x_{p+1})$ be the parts $\Upsilon_{G\setminus Z(G)}$. Then

$$\frac{1}{|Z(G)|}\sum_{i=1}^{p+1} |\beta_G(x_i)| = [G : Z(G)] - 1 = p^2 - 1.$$

On the other hand, $o(\overline{x}_i) = p$ for all i. Then for each $i \in \{1, \cdots, p+1\}$, $\bigcup_{j=1}^{p-1} \overline{x}_i^j \subseteq \beta_G(x_i)$, which implies that $\frac{|\beta_G(x_i)|}{|Z(G)|} \geq (p-1)$ for all i. If we assume that $\frac{|\beta_G(x_i)|}{|Z(G)|} > (p-1)$ for some i, we obtain that

$$p^2 - 1 = \frac{1}{|Z(G)|}\sum_{i=1}^{p+1} |\beta_G(x_i)| > (p+1)(p-1) = p^2 - 1,$$

a contradiction. Thus, $\frac{|\beta_G(x_i)|}{|Z(G)|} = (p-1)$ for all i, and hence $\Upsilon_{G\setminus Z(G)}$ is regular. □

Theorem 3.2. *A group G is induced regular if and only if $G \cong H \times A$ where A is an abelian group, and H is an induced regular p-group for some prime p.*

Proof. Suppose G is induced regular. Then $G/Z(G)$ is a p-group. Let $|G| = p^k m$, where $(m, p) = 1$. Then, $|Z(G)| = p^s m$ for some $1 \leq s \leq k$. Let A be the subgroup of $Z(G)$ of order m and let H be a p-sylow subgroup of G. Now, $HA = AH$. So, $HA \leq G$. Also, we have $H \cap A = \{e\}$ (because

$(|H|, |A|) = 1)$, and hence $|HA| = |H||A| = |G|$. Therefor $G = HA$. Moreover, since $ha = ah$ for all $a \in A$ and $h \in H$, we have $HA \cong H \times A$. It remains to be shown that H is induced regular. Fix x in $H \setminus Z(H)$ and let $s = |\beta_G(x)|/|Z(G)|$. Since $A \leq Z(G)$, we obtain that y belongs to $\beta_G(x)$ if and only if $C_H(x) = C_H(y)$. So, $\beta_H(x) = \beta_G(x) \cap H$. Also, since $Z(G) = Z(H)A$, there are $y_1, \cdots, y_s$ in H such that $\beta_G(x) = \bigsqcup_{i=1}^{s} y_i Z(G)$. So,

$$\beta_H(x) = \beta_G(x) \cap H = \left(\bigsqcup_{i=1}^{s} y_i Z(G) \right) \cap H = \bigsqcup_{i=1}^{s} y_i (Z(G) \cap H) = \bigsqcup_{i=1}^{s} y_i Z(H).$$

Thus, $|\beta_H(x)| = \dfrac{|\beta_G(x)|}{|Z(G)|} |Z(H)|$. Therefore, H is induced regular.

Now we prove the converse. Assume that $G = H \times A$ where A is an abelian group, H is an induced regular group of order p^k. Now, $Z(G) = Z(H) \times Z(A) = Z(H) \times A$. Also, for each $(h, a) \in G$, $C_G(h, a) = C_H(h) \times C_A(a) = C_H(h) \times A$. So $\beta_G(h, a) = \beta_H(h) \times A$, for all $(h, a) \in G$. Therefore G is induced regular. □

4. Conclusions

In this paper we introduced the notion of regular and induced regular groups using the non-centralizer graph of a group. We were able to prove that if G is regular, then $G/Z(G)$ is an elementary 2-group and the group is regular if and only if it is the direct product of a regular 2-group and an abelian group. Regarding the regularity of $\Upsilon_{G \setminus Z(G)}$, we applied the the concept of maximal centralizers to prove that, if G is induced regular, then $G/Z(G)$ is a p-group. We also show that a group G is induced regular if and only if it is the direct product of an induced regular p-group and an abelian group.

It is worth mentioning that there are several induced regular groups G for which $G/Z(G)$ is not abelian, such as the groups with GAP ID's $[243, 2]$ to $[243, 9]$. On the other hand, we checked many groups and we could not find one induced regular group G for which $G/Z(G)$ is not elementary. Based on these observations and the results obtained in this paper, one may conjecture that if G is an induced regular group, then $G/Z(G)$ is an elementary p-group, which amounts to proving the following problem:

Problem 4.1. *Let G be an induced regular group, and let $C_G(x)$ be a maximal centralizer such that $C_G(x) = \beta_G(x) \cup Z(G)$ or $(\beta_G(x) \cup Z(G))/Z(G)$ is 2-group. Then, $\beta_G(x) \cup Z(G)$ is an elementary p-group.*

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest.

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