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Research article

The maximum sum of the sizes of cross t-intersecting separated families

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Abstract: For a set X and an integer $r \geq 0$, let $\binom{X}{\leq r}$ denote the family of subsets of X that have at most r elements. Two families $\mathcal{A} \subset \binom{X}{\leq r}$ and $\mathcal{B} \subset \binom{X}{\leq s}$ are cross t-intersecting if $|A \cap B| \geq t$ for all $A \in \mathcal{A}, B \in \mathcal{B}$. In this paper, we considered the measures of cross t-intersecting families $\mathcal{A} \subset \binom{X}{\leq r}$, $\mathcal{B} \subset \binom{X}{\leq s}$, then we used this result to obtain the maximum sum of sizes of cross t-intersecting separated families.

Keywords: finite set; separated families; cross intersecting; generating set method; the shifting method
Mathematics Subject Classification: 05C35, 05D05

1. Introduction

For a set X, the power set of X (the set of subsets of X) is denoted by 2^X. For integer $r \geq 0$, the family of r-element subsets of X is denoted by $\binom{X}{r}$, and the family of subsets of X of size at most r is denoted by $\binom{X}{\leq r}$. Let $[n] = \{1, 2, \cdots, n\}$. For $\mathcal{F} \subset 2^{[n]}$ and $0 \leq i \leq n$, define

$$\mathcal{F}^{(i)} = \{F \in \mathcal{F} : |F| = i\}.$$

A family $\mathcal{F} \subset 2^X$ is said to be t-intersecting if $|F_1 \cap F_2| \geq t$ for every $F_1, F_2 \in \mathcal{F}$. If $\mathcal{A}, \mathcal{B} \subset 2^X$ are families such that $|A \cap B| \geq t$ for all $A \in \mathcal{A}, B \in \mathcal{B}$, then $\mathcal{A}$ and $\mathcal{B}$ are said to be cross t-intersecting. When $t = 1$, we usually omit t.

The following theorem by Erdös et al. is a basic result in the extremal set theory.

Theorem 1.1. *([1]) Let n, k and t be positive integers with $k \geq t \geq 1$ and let $\mathcal{F} \subset \binom{[n]}{k}$ be a t-intersecting family, then*

$$|\mathcal{F}| \leq \binom{n-t}{k-t}$$

for $n \geq n_0(k,t)$.

For $t = 1$, the exact value

$$n_0(k,t) = (k - t + 1)(t + 1) = 2k$$

was proved in [1]. For $t \geq 15$, it is due to [2]. Finally, Wilson [3] closed the gap $2 \leq t \leq 14$ with a proof valid for all t.

Hilton and Milner [4] obtained the maximum sum of sizes of cross intersecting families $\mathcal{A}, \mathcal{B} \subset \binom{[n]}{k}$, which was the first result on the sizes of cross intersecting families.

Theorem 1.2. *([4]) Let n and k be integers. Suppose that* $n \geq 2k$, $\mathcal{A}, \mathcal{B} \subset \binom{[n]}{k}$ *are cross intersecting and nonempty, then*

$$|\mathcal{A}| + |\mathcal{B}| \leq \binom{n}{k} - \binom{n-k}{k} + 1,$$

and the equality holds if $\mathcal{A} = \{[k]\}$ *and*

$$\mathcal{B} = \{B \in \binom{[n]}{k} : B \cap [k] \neq \emptyset\}.$$

It should be mentioned that Frankl and Tokushige [5] determined the maximum sum of the sizes of cross intersecting families $\mathcal{A} \subset \binom{[n]}{r}$ and $\mathcal{B} \subset \binom{[n]}{s}$, and the maximum of $|\mathcal{A}| + |\mathcal{B}|$ for cross t-intersecting families $\mathcal{A} \subset \binom{[n]}{r}$ and $\mathcal{B} \subset \binom{[n]}{s}$ were established in [6]. Recently, Borg and Feghli [7] solved the analogous maximum sum problem for the case where $\mathcal{A} \subset \binom{[n]}{\leq r}$ and $\mathcal{B} \subset \binom{[n]}{\leq s}$.

Theorem 1.3. *([7]) Let n, s and r be integers with* $n \geq 1$, $1 \leq r \leq s$. *Suppose that* $\mathcal{A} \subset \binom{[n]}{\leq r}$ *and* $\mathcal{B} \subset \binom{[n]}{\leq s}$ *are cross intersecting and nonempty, then*

$$|\mathcal{A}| + |\mathcal{B}| \leq 1 + \sum_{i=1}^{s} \left(\binom{n}{i} - \binom{n-r}{i} \right),$$

and the equality holds if $\mathcal{A} = \{[r]\}$ *and*

$$\mathcal{B} = \{B \in \binom{[n]}{\leq s} : B \cap [r] \neq \emptyset\}.$$

In this paper, we consider the cross t-intersecting families in the setting of measure. For a function w: $[k] \to \mathbb{R}_{>0}$ (the set of all positive reals) and a set $A \subset [k]$, we consider the measure $w(A) = w(|A|)$. Moreover, for $\mathcal{A} \subset 2^{[k]}$, let

$$w(\mathcal{A}) = \sum_{A \in \mathcal{A}} w(A).$$

Quite a few results for measures of intersecting and cross intersecting families are known [8–11]. In particular, Guapt et al. [10] determined the maximum sum of $\sum_{i \in [p]} w_i(\mathcal{F}_i)$ for the nonincreasing function w_i: $[k] \to \mathbb{R}_{\geq 0}$ and p-cross t-intersecting families $\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_p \subset 2^{[k]}$, where families $\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_p \subset 2^{[k]}$ are called p-cross t-intersecting if $|\cap_{i \in [p]} F_i| \geq t$ for all $F_i \in \mathcal{F}_i$, $i \in [p]$.

Given positive integers t, r, s, m, k with $t \le r \le k$ and $t \le m \le s \le k$, define the families

$$\mathcal{K}(m, r, t) = \left\{K \in \binom{[k]}{\le r} : |K \cap [m]| \ge t\right\},$$

$$\mathcal{S}(m, s) = \left\{S \in \binom{[k]}{\le s} : [m] \subset S\right\}.$$

It is easily checked that $\mathcal{K}(m, r, t)$ and $\mathcal{S}(m, s)$ are cross t-intersecting.

We first obtain the maximum sum of measures for cross t-intersecting families $\mathcal{A} \subset \binom{[n]}{\le r}$ and $\mathcal{B} \subset \binom{[n]}{\le s}$.

Theorem 1.4. *Let $\mathcal{A} \subset \binom{[k]}{\le r}$ and $\mathcal{B} \subset \binom{[k]}{\le s}$ be nonempty cross t-intersecting families with $k \ge r \ge s \ge t$. Let $w\colon [k] \to \mathbb{R}_{>0}$ be nonincreasing, then*

$$w(\mathcal{A}) + w(\mathcal{B}) \le \max\left\{\max_{t \le m \le s} w(\mathcal{K}(m, r, t)) + w(\mathcal{S}(m, s)), \max_{t \le m \le r} w(\mathcal{K}(m, s, t)) + w(\mathcal{S}(m, r))\right\}.$$

Let n and k be integers and

$$X = X_1 \uplus X_2 \uplus \cdots \uplus X_k, \ \ |X_i| = n.$$

We assume that the elements of X_i are ordered and let v_i denote its smallest elements. For $1 \le r \le k$, define

$$\mathcal{H}(n, k, r) = \left\{H \in \binom{X}{r} : |H \cap X_i| \le 1, 1 \le i \le k\right\}.$$

A family $\mathcal{F} \subset \mathcal{H}(n, k, r)$ is called a separated family. For $\mathcal{F}, \mathcal{G} \subset \mathcal{H}(n, k, r)$, we say that they are cross t-intersecting if $|F \cap G| \ge t$ for all $F \in \mathcal{F}, G \in \mathcal{G}$.

By applying Theorem 1.4 with a specific function $w\colon [k] \to \mathbb{R}_{>0}$, we obtain the maximum of $|\mathcal{F}|+|\mathcal{G}|$ of cross t-intersecting separated families $\mathcal{F}, \mathcal{G} \subset \mathcal{H}(n, k, r)$.

Theorem 1.5. *Let n, k, r and t be integers with $n \ge 2$, $k \ge r \ge t$. Suppose that $\mathcal{F}, \mathcal{G} \subset \mathcal{H}(n, k, r)$ are nonempty and cross t-intersecting, then*

$$|\mathcal{F}| + |\mathcal{G}| \le |\mathcal{F}_0| + |\mathcal{G}_0|,$$

where

$$\mathcal{F}_0 = \{F \in \mathcal{H}(n, k, r)\colon |F \cap \{v_1, v_2, \ldots, v_r\}| \ge t\}, \quad \mathcal{G}_0 = \{\{v_1, v_2, \ldots, v_r\}\}.$$

2. Proof of Theorem 1.4

The shifting technique will be used in this section. For $\mathcal{F} \subset 2^{[k]}$ and $1 \le i < j \le k$, define the shifting operation

$$S_{i,j}(\mathcal{F}) = \{S_{i,j}(F)\colon F \in \mathcal{F}\},$$

where

$$S_{i,j}(F) = \begin{cases} (F \setminus \{j\}) \cup \{i\}, & \text{if } j \in F, i \notin F \text{ and } (F \setminus \{j\}) \cup \{i\} \notin \mathcal{F}; \\ F, & \text{otherwise.} \end{cases}$$

It is well known ([12]) that $S_{i,j}$ maintains $|\mathcal{F}|$, the t-intersecting property and the cross t-intersecting property. We say that a family $\mathcal{F} \subset 2^{[k]}$ is initial if $S_{i,j}(\mathcal{F}) = \mathcal{F}$ for all $1 \leq i < j \leq k$. It is proved in [12] that by applying the shifting operation repeatedly, every family becomes an initial family.

A family $\mathcal{A} \subset \binom{[k]}{\leq r}$ is called monotone if $A \in \mathcal{A}$, $B \supset A$ and $|B| \leq r$ imply $B \in \mathcal{A}$. Given a family $\mathcal{A} \subset \binom{[k]}{\leq r}$, let $\langle \mathcal{A} \rangle_r$ be the up-set of $\mathcal{A}$ defined by

$$\langle \mathcal{A} \rangle_r = \left\{ F \in \binom{[k]}{\leq r} \colon \text{ there exists } A \in \mathcal{A} \text{ such that } A \subset F \right\}.$$

Our proof is based on the generating set method, which follows [9,13]. We recall some well-known notions for the generating set method.

Let $\mathcal{A} \subset \binom{[k]}{\leq r}$ ($\mathcal{A} \neq \emptyset$ and $\mathcal{A} \neq \binom{[k]}{\leq r}$) be a monotone family. A generating set of $\mathcal{A}$ is a minimal set (for containment) $G \in \mathcal{A}$. The generating family of $\mathcal{A}$ consists of all generating sets of $\mathcal{A}$. The extent of $\mathcal{A}$ is the maximal element appearing in a generating set of $\mathcal{A}$. The boundary generating family of $\mathcal{A}$ consists of all generating sets of $\mathcal{A}$ containing its extent. For a monotone family $\mathcal{A} \subset \binom{[k]}{\leq r}$ with generating family $\mathcal{G}$, it is easy to see that $\mathcal{A} = \langle \mathcal{G} \rangle_r$.

The following result follows from the definitions of the generating family and initiality, and a detailed proof can be found in [14].

Lemma 2.1. *Let $\mathcal{A} \subset \binom{[k]}{\leq r}$ be a monotone initial family with extent $m \geq 2$, generating family $\mathcal{G}$ and boundary generating family $\bar{\mathcal{G}}$. For any $\mathcal{H} \subset \bar{\mathcal{G}}$, let*

$$\mathcal{G}' = \mathcal{G} \setminus \mathcal{H},\ \ \mathcal{G}'' = (\mathcal{G} \setminus \mathcal{H}) \cup \{H \setminus \{m\} \colon H \in \mathcal{H}\},$$

then

$$\mathcal{A} \setminus \langle \mathcal{G}' \rangle_r = \left\{ H \cup T \colon H \in \mathcal{H},\ T \in \binom{[m+1,k]}{\leq r - |H|} \right\} \tag{2.1}$$

and

$$\langle \mathcal{G}'' \rangle_r \setminus \mathcal{A} = \left\{ (H \setminus \{m\}) \cup T \colon H \in \mathcal{H},\ T \in \binom{[m+1,k]}{\leq r - |H| + 1} \right\}. \tag{2.2}$$

Now, we are in the position to prove the main result.

Proof of Theorem 1.4. Let $\mathcal{A} \subset \binom{[k]}{\leq r}$ and $\mathcal{B} \subset \binom{[k]}{\leq s}$ be nonempty cross t-intersecting families with $w(\mathcal{A}) + w(\mathcal{B})$ maximal. Since the shifting operator preserves the cross t-intersecting property and preserves $w(\mathcal{A}) + w(\mathcal{B})$, we may assume that both $\mathcal{A}$ and $\mathcal{B}$ are initial. By the maximality of $w(\mathcal{A}) + w(\mathcal{B})$, we may also assume that $\mathcal{A}$ and $\mathcal{B}$ are monotone.

Suppose that $\mathcal{A}$ has extent m_1, generating family $\mathcal{G}_1$ and boundary generating family $\bar{\mathcal{G}}_1$ and $\mathcal{B}$ has extent m_2, generating family $\mathcal{G}_2$ and boundary generating family $\bar{\mathcal{G}}_2$. Since $\mathcal{G}_1 \subset \mathcal{A}$ and $\mathcal{G}_2 \subset \mathcal{B}$, we see that $\mathcal{G}_1$ and $\mathcal{G}_2$ are cross t-intersecting.

Since $\mathcal{A}$ and $\mathcal{B}$ are nonempty and cross t-intersecting, $|F| \geq t$ for any $F \in \mathcal{A} \cup \mathcal{B}$. It follows that $m_1 \geq t$ and $m_2 \geq t$.

Claim 1. $m_1 = m_2$.

Proof. Suppose that $m_1 \neq m_2$. By symmetry assume that $m_1 > m_2 \geq t$, then let

$$\mathcal{G}_1' = (\mathcal{G}_1 \setminus \bar{\mathcal{G}}_1) \cup \{G \setminus \{m_1\} \colon G \in \bar{\mathcal{G}}_1\}.$$

Note that $m_1 \notin B$ for any $B \in \mathcal{G}_2$, then $\mathcal{G}_1'$ and $\mathcal{G}_2$ are cross t-intersecting, implying that $\langle \mathcal{G}_1' \rangle_r, \mathcal{B} = \langle \mathcal{G}_2 \rangle_s$ are also cross t-intersecting. Since $\langle \mathcal{G}_1' \rangle_r \supsetneq \mathcal{A}$ and $w(j) > 0$, we see that

$$w(\langle \mathcal{G}_1' \rangle_r) + w(\mathcal{B}) > w(\mathcal{A}) + w(\mathcal{B}).$$

This contradicts the maximality of $w(\mathcal{A}) + w(\mathcal{B})$. □

Let $m_1 = m_2 = m$. We may further assume that $\mathcal{A} \subset \binom{[k]}{\leq r}$ and $\mathcal{B} \subset \binom{[k]}{\leq s}$ are nonempty cross t-intersecting families with $w(\mathcal{A}) + w(\mathcal{B})$ maximal and m minimal. That is, for any cross t-intersecting families $\mathcal{A}' \subset \binom{[k]}{\leq r}$, $\mathcal{B}' \subset \binom{[k]}{\leq s}$ with

$$w(\mathcal{A}) + w(\mathcal{B}) = w(\mathcal{A}') + w(\mathcal{B}'),$$

if $\mathcal{A}'$ and $\mathcal{B}'$ have extent m', then $m \leq m'$ holds.

Claim 2. *If $A \in \bar{\mathcal{G}}_1$ and $B \in \bar{\mathcal{G}}_2$ satisfy $|A \cap B| = t$, then $A \cup B = [m]$ and $|A| + |B| = m + t$.*

Proof. Note that $\{m\} \in A \cap B$. We show that $A \cup B = [m]$ follows from initiality. Indeed, $x \in [m] \setminus (A \cup B)$ would imply

$$A' := (A \setminus \{m\}) \cup \{x\} \in \mathcal{A}$$

and

$$|A' \cap B| = |(A \cap B) \setminus \{m\}| = t - 1;$$

a contradiction. Now, $|A| + |B| = m + t$ follows from $|A| + |B| = |A \cap B| + |A \cup B|$. □

If $m = t$, then

$$\mathcal{G}_1 = \mathcal{G}_2 = \{[t]\}.$$

It implies that both $\mathcal{A}$ and $\mathcal{B}$ are t-star. Thus,

$$w(\mathcal{A}) + w(\mathcal{B}) = w(\mathcal{K}(t, r, t)) + w(\mathcal{S}(t, s)),$$

and we are done.

Now, we may assume that $m \geq t + 1$ and distinguish the four cases.

Case 1. If $\bar{\mathcal{G}}_1^{(t)} \neq \emptyset$, then

$$[t-1] \cup \{m\} \in \bar{\mathcal{G}}_1^{(t)} \subset \mathcal{A}.$$

By initiality, we have

$$[t], [t-1] \cup \{t+1\}, [t-1] \cup \{t+2\}, \cdots, [t-1] \cup \{m\} \in \mathcal{A}.$$

Since $\mathcal{A}$ and $\mathcal{B}$ are cross t-intersecting, then $[m] \subset B$ for any $B \in \mathcal{B}$. It follows $\mathcal{G}_2 = \{[m]\}$ and $t \leq m \leq s$. Since $\mathcal{B}$ is monotone, then

$$\mathcal{B} = \langle \mathcal{G}_2 \rangle_s = \mathcal{S}(m, s).$$

By the maximality of $w(\mathcal{A}) + w(\mathcal{B})$, we infer that $\mathcal{A} = \mathcal{K}(m, r, t)$ and

$$w(\mathcal{A}) + w(\mathcal{B}) = w(\mathcal{K}(m, r, t)) + w(\mathcal{S}(m, s)).$$

Note that in this case, $t \le m \le s$ holds. Thus,

$$w(\mathcal{A}) + w(\mathcal{B}) \le \max_{t\le m\le s} w(\mathcal{K}(m, r, t)) + w(\mathcal{S}(m, s)).$$

Case 2. If $\bar{\mathcal{G}}_2^{(t)} \neq \emptyset$, by a similar argument as in Case 1, we obtain that

$$w(\mathcal{A}) + w(\mathcal{B}) = w(\mathcal{K}(m, s, t)) + w(\mathcal{S}(m, r))$$

and $t \le m \le r$. Thus,

$$w(\mathcal{A}) + w(\mathcal{B}) \le \max_{t\le m\le r} w(\mathcal{K}(m, s, t)) + w(\mathcal{S}(m, r)).$$

Case 3. There exists a, b with $a + b = m + t$ and exactly one of $\bar{\mathcal{G}}_1^{(a)}$ and $\bar{\mathcal{G}}_2^{(b)}$ is nonempty.

Without loss of generality, assume that $\bar{\mathcal{G}}_1^{(a)} \neq \emptyset$. By Case 1, we assume that $a \ge t + 1$. Let

$$\mathcal{G}'_1 = (\mathcal{G}_1 \setminus \bar{\mathcal{G}}_1^{(a)}) \cup \{G \setminus \{m\} \colon G \in \bar{\mathcal{G}}_1^{(a)}\}.$$

By Claim 2 we infer that $\mathcal{G}'_1$ and $\mathcal{G}_2$ are cross t-intersecting. Thus $\langle\mathcal{G}'_1\rangle_r$ and $\mathcal{B}$ are cross t-intersecting, then $\mathcal{A} \subsetneq \langle\mathcal{G}'_1\rangle_r$ and $w(j) > 0$ for $j \in [k]$ imply

$$w(\langle\mathcal{G}'_1\rangle_r) + w(\mathcal{B}) > w(\mathcal{A}) + w(\mathcal{B})$$

contradicting the maximality of $w(\mathcal{A}) + w(\mathcal{B})$.

Case 4. For all $a, b \ge t + 1$ with $a + b = m + t$, both $\bar{\mathcal{G}}_1^{(a)}$ and $\bar{\mathcal{G}}_2^{(b)}$ are nonempty or both $\bar{\mathcal{G}}_1^{(a)}$ and $\bar{\mathcal{G}}_2^{(b)}$ are empty.

By Cases 1–3, we may assume that $\bar{\mathcal{G}}_1^{(t)}, \bar{\mathcal{G}}_1^{(m)}, \bar{\mathcal{G}}_2^{(t)}$ and $\bar{\mathcal{G}}_2^{(m)}$ are empty in this case. We claim that $\mathcal{G}_1 \setminus \bar{\mathcal{G}}_1$ and $\mathcal{G}_2 \setminus \bar{\mathcal{G}}_2$ are nonempty. $\bar{\mathcal{G}}_1^{(m)} = \emptyset$ and $\mathcal{G}_1 \neq \emptyset$ imply that there exists $G \in \mathcal{G}_1 \subset \mathcal{A}$ and $|G| < m$. Since $\mathcal{A}$ is initial, $A' = [|G|] \in \mathcal{A}$ and $m \notin A'$, then A' (or a subset of A') belongs to $\mathcal{G}_1 \setminus \bar{\mathcal{G}}_1$. Similarly, $\mathcal{G}_2 \setminus \bar{\mathcal{G}}_2$ is nonempty.

Since $\bar{\mathcal{G}}_1, \bar{\mathcal{G}}_2 \neq \emptyset$, there exists $a, b \ge t + 1$ with $a + b = m + t$ and both $\bar{\mathcal{G}}_1{}^{(a)}$ and $\bar{\mathcal{G}}_2^{(b)}$ are nonempty. Let

$$\mathcal{G}'_1 = (\mathcal{G}_1 \setminus \bar{\mathcal{G}}_1^{(a)}) \cup \{G \setminus \{m\} \colon G \in \bar{\mathcal{G}}_1^{(a)}\},\ \mathcal{G}'_2 = \mathcal{G}_2 \setminus \bar{\mathcal{G}}_2^{(b)}$$

and

$$\mathcal{G}''_1 = \mathcal{G}_1 \setminus \bar{\mathcal{G}}_1^{(a)},\ \mathcal{G}''_2 = (\mathcal{G}_2 \setminus \bar{\mathcal{G}}_2^{(b)}) \cup \{G \setminus \{m\} \colon G \in \bar{\mathcal{G}}_2^{(b)}\}.$$

By Claim 2, $\langle\mathcal{G}'_1\rangle_r$ and $\langle\mathcal{G}'_2\rangle_s$ are cross t-intersecting and $\langle\mathcal{G}''_1\rangle_r$ and $\langle\mathcal{G}''_2\rangle_s$ are cross t-intersecting. By Lemma 2.1 we have

$$w(\langle\mathcal{G}'_1\rangle_r) + w(\langle\mathcal{G}'_2\rangle_s) = w(\mathcal{A}) + w(\mathcal{B}) - |\bar{\mathcal{G}}_2^{(b)}| \sum_{b\le j\le s} \binom{k-m}{j-b} w(j) + |\bar{\mathcal{G}}_1^{(a)}| \sum_{a-1\le j\le r} \binom{k-m}{j-a+1} w(j) \quad (2.3)$$

and

$$w(\langle \mathcal{G}_1'' \rangle_r) + w(\langle \mathcal{G}_2'' \rangle_s) = w(\mathcal{A}) + w(\mathcal{B}) - |\bar{\mathcal{G}}_1^{(a)}| \sum_{a \le j \le r} \binom{k-m}{j-a} w(j) + |\bar{\mathcal{G}}_2^{(b)}| \sum_{b-1 \le j \le s} \binom{k-m}{j-b+1} w(j). \quad (2.4)$$

Since $w(j-1) \ge w(j) > 0$, we obtain

$$\begin{aligned} \sum_{a-1 \le j \le r} \binom{k-m}{j-a+1} w(j) - \sum_{a \le j \le r} \binom{k-m}{j-a} w(j) &= \sum_{a \le j \le r+1} \binom{k-m}{j-a} w(j-1) - \sum_{a \le j \le r} \binom{k-m}{j-a} w(j) \\ &\ge \binom{k-m}{r+1-a} w(r) \\ &\ge 0. \end{aligned} \quad (2.5)$$

Similarly,

$$\sum_{b-1 \le j \le s} \binom{k-m}{j-b+1} w(j) - \sum_{b \le j \le s} \binom{k-m}{j-b} w(j) \ge 0. \quad (2.6)$$

Adding (2.3) and (2.4) and using (2.5) and (2.6), we obtain that

$$\begin{aligned} &\frac{1}{2} \left(w(\langle \mathcal{G}_1' \rangle_r) + w(\langle \mathcal{G}_2' \rangle_s) + w(\langle \mathcal{G}_1'' \rangle_r) + w(\langle \mathcal{G}_2'' \rangle_s) \right) - (w(\mathcal{A}) + w(\mathcal{B})) \\ &= \frac{|\bar{\mathcal{G}}_1^{(a)}|}{2} \left(\sum_{a-1 \le j \le r} \binom{k-m}{j-a+1} w(j) - \sum_{a \le j \le r} \binom{k-m}{j-a} w(j) \right) \\ &\quad + \frac{|\bar{\mathcal{G}}_2^{(b)}|}{2} \left(\sum_{b-1 \le j \le s} \binom{k-m}{j-b+1} w(j) - \sum_{b \le j \le s} \binom{k-m}{j-b} w(j) \right) \\ &\ge 0. \end{aligned} \quad (2.7)$$

By the maximality of $w(\mathcal{A}) + w(\mathcal{B})$, we have

$$w(\langle \mathcal{G}_1' \rangle_r) + w(\langle \mathcal{G}_2' \rangle_s) \le w(\mathcal{A}) + w(\mathcal{B}), \quad w(\langle \mathcal{G}_1'' \rangle_r) + w(\langle \mathcal{G}_2'' \rangle_s) \le w(\mathcal{A}) + w(\mathcal{B}).$$

Combining (2.7), we obtain the following claim.

Claim 3. *If both $\bar{\mathcal{G}}_1^{(a)}$ and $\bar{\mathcal{G}}_2^{(b)}$ are nonempty and $a + b = m + t$, $a, b \ge t + 1$, then*

$$w(\langle \mathcal{G}_1' \rangle_r) + w(\langle \mathcal{G}_2' \rangle_s) = w(\mathcal{A}) + w(\mathcal{B}),$$

where

$$\mathcal{G}_1' = (\mathcal{G}_1 \setminus \bar{\mathcal{G}}_1^{(a)}) \cup \{G \setminus \{m\} \colon G \in \bar{\mathcal{G}}_1^{(a)}\}, \quad \mathcal{G}_2' = \mathcal{G}_2 \setminus \bar{\mathcal{G}}_2^{(b)}.$$

Now, we make the foregoing operation for all nonempty pairs $\bar{\mathcal{G}}_1^{(a)}$ and $\bar{\mathcal{G}}_2^{(b)}$ with $a + b = m + t$ and we will obtain a pair of new generating families. Define

$$\mathcal{G}_1^* = (\mathcal{G}_1 \setminus \bar{\mathcal{G}}_1) \cup \{G \setminus \{m\} \colon G \in \bar{\mathcal{G}}_1\}, \quad \mathcal{G}_2^* = \mathcal{G}_2 \setminus \bar{\mathcal{G}}_2.$$

We claim that $\langle \mathcal{G}_1^* \rangle_r$ and $\langle \mathcal{G}_2^* \rangle_s$ are cross t-intersecting. Note that $\mathcal{G}_1 \subset \mathcal{G}_1^*$ and $\mathcal{G}_2^* \subset \mathcal{G}_2$. For $G \in \mathcal{G}_1^* \setminus \mathcal{G}_1$ and $F \in \mathcal{G}_2^*$, we have $m \notin G$, $m \notin F$ and $G \cup \{m\} \in \mathcal{G}_1$. Since $\mathcal{G}_1$ and $\mathcal{G}_2$ are cross t-intersecting,

$$|G \cap F| = |(G \cup \{m\}) \cap F| \geq t,$$

then $\mathcal{G}_1^*$ and $\mathcal{G}_2^*$ are cross t-intersecting. Thus, $\langle \mathcal{G}_1^* \rangle_r$ and $\langle \mathcal{G}_2^* \rangle_s$ are cross t-intersecting.

Claim 3 shows that

$$w(\langle \mathcal{G}_1^* \rangle_r) + w(\langle \mathcal{G}_2^* \rangle_s) = w(\mathcal{A}) + w(\mathcal{B}).$$

Moreover, $m \notin G$ for all $G \in \mathcal{G}_1^* \cup \mathcal{G}_2^*$, then the extents of $\langle \mathcal{G}_1^* \rangle_r$ and $\langle \mathcal{G}_2^* \rangle_s s$ are less than m, contradicting the minimality of m. □

3. Cross t-intersecting separated families

The shifting operation can also be used in separated families. Let $\mathcal{F} \subset \mathcal{H}(n,k,r)$ be a separated family on $X = X_1 \uplus X_2 \uplus \cdots \uplus X_k$. Recall that the elements of X_i are linearly ordered for each $1 \leq i \leq k$ and v_i is the minimal element of X_i. The shift $S_{x,y}$ is allowed to apply on $\mathcal{F}$ only if x, y are in the same X_i and $x < y$. A separated family $\mathcal{F} \subset \mathcal{H}(n,k,r)$ is called initial if $S_{x,y}(\mathcal{F}) = \mathcal{F}$ for all x, y in the same X_i and $x < y$. Similarly, by applying the allowed shifting operation repeatedly, every separated family becomes an initial family.

For a set $H \in \mathcal{H}(n,k,r)$, define

$$A(H) = \{i : H \cap X_i = \{v_i\}\}.$$

For $\mathcal{F} \subset \mathcal{H}(n,k,r)$, let

$$\mathcal{A}(\mathcal{F}) = \{A(H) : H \in \mathcal{F}\},$$

then

$$\mathcal{A}(\mathcal{F}) \subset \binom{[k]}{\leq r}.$$

The following reduction Lemma will be used in the proofs. Frankl and Füredi [15] showed the result for an intersecting family $\mathcal{F}$, and we gave a cross t-intersecting version of the reduction lemma in [14].

Lemma 3.1. *(reduction lemma [14, 15]) Suppose that $\mathcal{F}, \mathcal{G} \subset \mathcal{H}(n,k,r)$ are cross t-intersecting and both initial, then $\mathcal{A}(\mathcal{F})$ and $\mathcal{A}(\mathcal{G})$ are cross t-intersecting.*

When $r = s$, we may obtain the maximum value of $w(\mathcal{A}) + w(\mathcal{B})$ from Theorem 1.4 by some calculations.

Proposition 3.2. *Let k, r and t be positive integers with $k \geq r \geq t$, and $\mathcal{A}, \mathcal{B} \subset \binom{[k]}{\leq r}$ be nonempty and cross t-intersecting. Let w: $[k] \to \mathbb{R}_{>0}$ be nonincreasing, then*

$$w(\mathcal{A}) + w(\mathcal{B}) \leq w(\mathcal{K}(r,r,t)) + w(\mathcal{S}(r,r)),$$

where

$$\mathcal{S}(r,r) = \{[r]\}, \quad \mathcal{K}(r,r,t) = \left\{F \in \binom{[k]}{\leq r} : |F \cap [r]| \geq t\right\}.$$

Proof. We will prove the result by showing that for $t \leq m < r$, the following inequality holds:

$$w(\mathcal{K}(m,r,t)) + w(\mathcal{S}(m,r)) \leq w(\mathcal{K}(m+1,r,t)) + w(\mathcal{S}(m+1,r)).$$

By the definition of $\mathcal{S}(m,r)$ for $j+m \leq r$, the subfamily of $\mathcal{S}(m,r)$ consisting of the elements of $\mathcal{S}(m,r)$ with size $j+m$ is

$$\mathcal{S}(m,r)^{(j+m)} = \left\{[m] \cup R\colon R \in \binom{[k]\setminus[m]}{j}\right\}.$$

We obtain that

$$\begin{aligned}
w(\mathcal{S}(m,r)) - w(\mathcal{S}(m+1,r)) &= \sum_{j=0}^{r-m} w(j+m)\binom{k-m}{j} - \sum_{j=0}^{r-m-1} w(j+m+1)\binom{k-m-1}{j} \\
&= \sum_{j=0}^{r-m} w(j+m)\left(\binom{k-m-1}{j} + \binom{k-m-1}{j-1}\right) - \sum_{j=1}^{r-m} w(j+m)\binom{k-m-1}{j-1} \\
&= \sum_{j=0}^{r-m} w(j+m)\binom{k-m-1}{j}. \qquad (3.1)
\end{aligned}$$

Recall the definition of $\mathcal{K}(m,r,t)$. It is easy to see that $\mathcal{K}(m,r,t) \subset \mathcal{K}(m+1,r,t)$, and

$$\mathcal{K}(m+1,r,t) \setminus \mathcal{K}(m,r,t) = \left\{F \in \binom{[k]}{\leq r}\colon |F \cap [m]| = t-1 \text{ and } m+1 \in F\right\}.$$

Thus we have

$$w(\mathcal{K}(m+1,r,t)) - w(\mathcal{K}(m,r,t)) = \sum_{j=0}^{r-t} w(j+t)\binom{m}{t-1}\binom{k-m-1}{j}. \qquad (3.2)$$

By $r-t \geq r-m$ and $w(j+t) \geq w(j+m)$, it follows that

$$\sum_{j=0}^{r-t} w(j+t)\binom{m}{t-1}\binom{k-m-1}{j} \geq \sum_{j=0}^{r-m} w(j+m)\binom{k-m-1}{j}. \qquad (3.3)$$

By (3.1)–(3.3), we obtain that

$$w(\mathcal{K}(m,r,t)) + w(\mathcal{S}(m,r)) \leq w(\mathcal{K}(m+1,r,t)) + w(\mathcal{S}(m+1,r)).$$

Therefore,

$$\max_{t\leq m\leq r} w(\mathcal{K}(m,r,t)) + w(\mathcal{S}(m,r)) = w(\mathcal{K}(r,r,t)) + w(\mathcal{S}(r,r)).$$

By Theorem 1.4, the proposition holds. □

Using the reduction lemma and assigning specific measures in Proposition 3.2, we may obtain the maximum of $|\mathcal{F}| + |\mathcal{G}|$ for cross t-intersecting separated families $\mathcal{F}, \mathcal{G} \subset \mathcal{H}(n,k,r)$.

Proof of Theorem 1.5. Let $\mathcal{F}, \mathcal{G} \subset \mathcal{H}(n,k,r)$ be cross t-intersecting families with maximal $|\mathcal{F}| + |\mathcal{G}|$. Since the shifting operating preserves the cross t-intersecting property, we may assume that both $\mathcal{F}$ and $\mathcal{G}$ are initial. By the Reduction lemma, $\mathcal{A} = \mathcal{A}(\mathcal{F})$ and $\mathcal{B} = \mathcal{A}(\mathcal{G})$ are cross t-intersecting. Moreover,

$$\mathcal{A}, \mathcal{B} \subset \binom{[k]}{\leq r}.$$

By the maximality of $|\mathcal{F}|+|\mathcal{G}|$, we further assume that $\mathcal{F}$ and $\mathcal{G}$ form a saturated pair; that is, adding further sets would destroy the cross t-intersecting property, then

$$|\mathcal{F}| = \sum_{0\leq i\leq r} |\mathcal{A}^{(i)}|\binom{k-i}{r-i}(n-1)^{r-i}, \quad |\mathcal{G}| = \sum_{0\leq i\leq r} |\mathcal{B}^{(i)}|\binom{k-i}{r-i}(n-1)^{r-i}.$$

To put it another way, if cross t-intersecting pairs $\mathcal{A}, \mathcal{B} \subset \binom{[k]}{\leq r}$ are given, then we can uniquely construct cross t-intersecting and saturated pairs $\mathcal{F}, \mathcal{G} \subset \mathcal{H}(n,k,r)$.

Thus,

$$|\mathcal{F}| + |\mathcal{G}| = \sum_{0\leq i\leq r} |\mathcal{A}^{(i)}|\binom{k-i}{r-i}(n-1)^{r-i} + \sum_{0\leq i\leq r} |\mathcal{B}^{(i)}|\binom{k-i}{r-i}(n-1)^{r-i}.$$

Let

$$w(i) = \binom{k-i}{r-i}(n-1)^{r-i}, \quad i \in [k],$$

then it is easy to see that

$$|\mathcal{F}| + |\mathcal{G}| = w(\mathcal{A}) + w(\mathcal{B}).$$

Since $n \geq 2$ and $k \geq r$, we have $w(i) > 0$. By

$$w(i) = \binom{k-i}{r-i}(n-1)^{r-i} \geq \binom{k-i-1}{r-i-1}(n-1)^{r-i-1} = w(i+1),$$

then $w(i)$ is nonincreasing. Applying Proposition 3.2 with

$$w(i) = \binom{k-i}{r-i}(n-1)^{r-i}, \quad i \in [k],$$

we obtain that

$$\begin{aligned} |\mathcal{F}| + |\mathcal{G}| &\leq \max\left\{w(\mathcal{A}) + w(\mathcal{B})\colon \mathcal{A}, \mathcal{B} \subset \binom{[k]}{\leq r} \text{ are cross } t\text{-intersecting}\right\} \\ &\leq w(\mathcal{K}(r,r,t)) + w(\mathcal{S}(r,r)) \\ &= |\mathcal{F}_0| + |\mathcal{G}_0|, \end{aligned}$$

where

$$\mathcal{F}_0 = \{F \in \mathcal{H}(n,k,r)\colon |F \cap \{v_1, v_2, \ldots, v_r\}| \geq t\}$$

and

$$\mathcal{G}_0 = \{\{v_1, v_2, \ldots, v_r\}\}.$$

Thus, the theorem holds. □

4. Conclusions

In this paper, we discussed the measures of cross t-intersecting families. By applying the main result with a specific weight function w, we obtained the maximum sum of the sizes of cross t-intersecting separated families.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares there is no conflict of interest.

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