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Research article

Decay of unique global solution for 3D tropical climate model with partial dissipation

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Abstract: In this article, we studied the asymptotic behavior of weak solutions to the three-dimensional tropical climate model with one single diffusion $\mu\Lambda^{2\alpha}u$. We established that when $u_0 \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$, $(w_0, \theta_0) \in (L^2(\mathbb{R}^3))^2$ and $w \in L^\infty(0, \infty; W^{1-\alpha,\infty}(\mathbb{R}^3))$ with $\alpha \in (0, 1]$, the energy $\|u(t)\|_{L^2(\mathbb{R}^3)}$ vanishes and $\|w(t)\|_{L^2(\mathbb{R}^3)} + \|\theta(t)\|_{L^2(\mathbb{R}^3)}$ converges to a constant as time tends to infinity.

Keywords: tropical climate model; asymptotic behavior; energy vanish; weak solutions; one single diffusion
Mathematics Subject Classification: 35B40, 35Q35

1. Introduction

This paper is concerned with the following three-dimensional tropical climate model with partial fractional dissipation:

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \mu\Lambda^{2\alpha}u + \nabla\cdot(w\otimes w) + \nabla p = 0, \quad x\in\mathbb{R}^3,\ t>0,\\ \partial_t w + (u\cdot\nabla)w + \nabla\cdot(w\otimes u) + \nabla\theta = 0,\\ \partial_t\theta + (u\cdot\nabla)\theta + \nabla\cdot w = 0,\\ \nabla\cdot u = 0,\\ u(x,0) = u_0(x),\ \ w(x,0) = w_0(x),\ \ \theta(x,0) = \theta_0(x), \end{cases} \tag{1.1}$$

where $u = u(x,t)$, $w = w(x,t)$, $p = p(x,t)$ and $\theta = \theta(x,t)$ denote the barotropic mode of the velocity field, the first baroclinic mode of the velocity field, the scalar pressure and scalar temperature, respectively. The real parameters μ and α are nonnegative constants and $\Lambda := (-\Delta)^{\frac{1}{2}}$. The fractional operator Λ^r is defined via the Fourier transform as $\widehat{\Lambda^r f}(\xi) = |\xi|^r\widehat{f}(\xi)$.

System (1.1) is related to the following classical tropical climate model with full fractional dissipation:

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \mu\Lambda^{2\alpha}u + \nabla \cdot (w \otimes w) + \nabla p = 0, \quad x \in \mathbb{R}^3,\ t > 0, \\ \partial_t w + (u \cdot \nabla)w + \nu\Lambda^{2\beta}w + \nabla \cdot (w \otimes u) + \nabla\theta = 0, \\ \partial_t \theta + (u \cdot \nabla)\theta + \eta\Lambda^{2\gamma}\theta + \nabla \cdot w = 0, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \quad w(x,0) = w_0(x), \quad \theta(x,0) = \theta_0(x). \end{cases} \tag{1.2}$$

The non-dissipative case of system (1.2), namely, $\mu = \nu = \eta = 0$, was originally derived by Frierson, Majda and Pauluis [12] to study the interaction between large scale flow fields and precipitation in the tropical atmosphere. Subsequently, some mathematical problems concerning the primitive equation have been addressed extensively (see e.g. [4–7]).

Recently, researchers have extended the non-dissipative case of system (1.2) to the fully dissipative or partially dissipative cases. The fully dissipative case, that is, the coefficients $\mu,\ \nu,\ \eta > 0$, has attracted much attention, including well-posedness [9, 23, 24] and decay of solutions [15, 22]. It is worth noting that for the 2D case, when $\alpha = \beta = \gamma = 1$, Li and Xiao [15] showed that, for $(u_0, w_0, \theta_0) \in (H^2(\mathbb{R}^2))^3$ and $\nabla \cdot u_0 = 0$, it holds that

$$t^{\frac{s}{2}}\|(u, w, \theta)(t)\|_{H^s(\mathbb{R}^2)} \to 0, \text{ as } t \to \infty.$$

Meanwhile, for the n-dimensional space, when the initial data $(u_0, w_0, \theta_0) \in (L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n))^3$, Xie and Zhang [22] proved that

$$\|u(t)\|^2_{L^2(\mathbb{R}^n)} + \|v(t)\|^2_{L^2(\mathbb{R}^n)} + \|\theta(t)\|^2_{L^2(\mathbb{R}^n)} \le C(1 + t)^{-n/2},$$

and they claimed that by using the method of Heywood [14], it is possible to prove the existence and uniqueness (see [[22], Theorem 1.1 and Remark 1.1]).

Regarding the partly dissipative case of system (1.2), there are also many results, and for the well-posedness results, we can refer to [3, 8, 10, 16, 25]. In particular, in [25], the author considered strong solutions for the 3D case, when $\mu > 0,\ \nu = \eta = 0$, i.e., system (1.1), Zhu obtained the global regularity when $(u_0, w_0, \theta_0) \in (H^3(\mathbb{R}^3))^3$ with $\alpha \ge \frac{5}{2}$. While, references [3, 8, 10, 16] are concerned with the 2D case. However, with respect to the decay of solution to the partly dissipative case, to the best of our knowledge, there are no corresponding results, which is our motivation in this paper.

For more details of decay for other models, we could refer to the papers [1, 2, 11, 13, 18–21] and the references therein. Let us mention that Agapito and Schonbek [2] showed that for the MHD (magnetohydrodynamics) equation, the energy $\|u(t)\|_{L^2(\mathbb{R}^3)}$ vanishes and $\|B(t)\|_{L^2(\mathbb{R}^3)}$ converges to a constant as time tends to infinity when the initial data satisfies $(u_0, B_0) \in (L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)) \times (L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3))$.

Inspired by the works [2, 15, 22], in this paper we consider the decay of solutions to system (1.1). We mainly apply the Fourier splitting method to establish the decay of the high frequency part.

2. Preliminaries

2.1. Notation

We write $\|\cdot\|_p = \|\cdot\|_{L^p(\mathbb{R}^3)}$ for simplification and $\langle\cdot,\cdot\rangle$ stands for the L^2-inner product. If $u \in L^p(\mathbb{R}^3)$, we define its norm to be

$$\|u\|_{L^p(\mathbb{R}^3)} := \begin{cases} \left(\int_U |u|^p \, dx\right)^{1/p} & (1 \le p < \infty), \\ \sum_{|\alpha| \le k} \operatorname{ess\,sup}_U |u| & (p = \infty). \end{cases}$$

If $u \in W^{k,p}(\mathbb{R}^3)$, we define its norm to be

$$\|u\|_{W^{k,p}(\mathbb{R}^3)} := \begin{cases} \left(\sum_{|\alpha| \le k} \int_U |D^\alpha u|^p \, dx\right)^{1/p} & (1 \le p < \infty), \\ \sum_{|\alpha| \le k} \operatorname{ess\,sup}_U |D^\alpha u| & (p = \infty). \end{cases}$$

If $p = 2$, we usually write

$$H^k(\mathbb{R}^3) = W^{k,2}(\mathbb{R}^3) \ (k = 0, 1, \ldots).$$

The Fourier transform of a function f is denoted by

$$\hat{f} = f^\wedge = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} f(x) dx.$$

Various constants shall be denoted by C throughout the paper.

2.2. Generalized energy inequalities

Definition 2.1. *(Weak solution) Let $T > 0$. A function*

$$u \in L^\infty(0, \infty; L^2(\mathbb{R}^3)) \cap L^2(0, \infty; \dot{H}^\alpha(\mathbb{R}^3)), w \in L^\infty(0, \infty; L^2(\mathbb{R}^3)), \theta \in L^\infty(0, \infty; L^2(\mathbb{R}^3))$$

is called a weak solution to system (3.1) if (u, w, θ) satisfies

$$\begin{cases} \int_0^T \int_{\mathbb{R}^3} (u \cdot \partial_t \phi + (u \otimes u) : \nabla \phi + \mu \Lambda^{2\alpha} \phi \cdot u + (w \otimes w) : \nabla \phi + p(\nabla \cdot \phi)) dx dt = 0, \\ \int_0^T \int_{\mathbb{R}^3} (w \cdot \partial_t \phi + (u \otimes w) : \nabla \phi + (w \otimes u) : \nabla \phi + \theta(\nabla \cdot \phi)) dx dt = 0, \\ \int_0^T \int_{\mathbb{R}^3} (\theta \cdot \partial_t \phi + (\theta \otimes u) : \nabla \phi + w \cdot \nabla \phi) dx dt = 0, \\ \int_0^T \int_{\mathbb{R}^3} (u \cdot \nabla \phi) dx dt = 0, \\ \lim_{t \to 0} \int_{\mathbb{R}^3} u(x,t) \psi(x) dx = \int_{\mathbb{R}^3} u_0(x) \psi(x) dx, \\ \lim_{t \to 0} \int_{\mathbb{R}^3} w(x,t) \psi(x) dx = \int_{\mathbb{R}^3} w_0(x) \psi(x) dx, \\ \lim_{t \to 0} \int_{\mathbb{R}^3} \theta(x,t) \psi(x) dx = \int_{\mathbb{R}^3} \theta_0(x) \psi(x) dx, \end{cases} \tag{2.1}$$

for any test function $\phi \in C_0^\infty(\mathbb{R}^3 \times (0, T))$ and $\psi \in C_0^\infty(\mathbb{R}^3)$.

First, we give the following fundamental apriori L^2-estimates.

Lemma 2.1. Let $(u_0, w_0, \theta_0) \in (L^2(\mathbb{R}^3))^3$, then for any $t > 0$, the solution (u, w, θ) of system (1.1) satisfies

$$\|(u, w, \theta)(t)\|_2^2 + 2\int_0^t \mu\|\Lambda^\alpha u(\tau)\|_2^2 d\tau = \|(u_0, w_0, \theta_0)\|_2^2. \tag{2.2}$$

Proof. Multiplying $(1.1)_1$, $(1.1)_2$ and $(1.1)_3$ by u, w and θ, respectively and summing them up, we get after integrating by parts that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|u\|_2^2 + \|w\|_2^2 + \|\theta\|_2^2) + \mu\|\Lambda^\alpha u\|_2^2 \\
=& -\int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot u dx - \int_{\mathbb{R}^3}(u\cdot\nabla)w\cdot w dx - \int_{\mathbb{R}^3}(u\cdot\nabla)\theta\cdot\theta dx - \int_{\mathbb{R}^3}\nabla p\cdot u dx \\
&- \int_{\mathbb{R}^3}(w\cdot\nabla)u\cdot w dx - \int_{\mathbb{R}^3}\nabla\cdot(w\otimes w)\cdot u dx - \int_{\mathbb{R}^3}\nabla\theta\cdot w dx - \int_{\mathbb{R}^3}\nabla\cdot w\cdot\theta dx \\
=&0.
\end{aligned}$$

Integrating with respect to t, we get (2.2). □

Split the solution into low and high frequency parts as

$$\|u(t)\|_2^2 = \|\hat{u}(t)\|_2^2 \le \|\varphi\hat{u}(t)\|_2^2 + \|(1-\varphi)\hat{u}(t)\|_2^2,$$

where $\varphi(\xi)$ is a function in Fourier space to be chosen appropriately, to emphasize the low and high frequency of u.

Lemma 2.2. Let (u, w, θ) be a weak solution to system (1.1). Set $\varphi = e^{-|\xi|^{2\alpha}t}$, then,

$$\begin{aligned}
\|\varphi\hat{u}(t)\|_2^2 \le&\|\varphi e^{-\mu|\xi|^{2\alpha}(t-s)}\hat{u}(s)\|_2^2 + 2\int_s^t |\langle\widehat{u\cdot\nabla u}, \varphi^2 e^{-2\mu|\xi|^{2\alpha}(t-\tau)}\hat{u}\rangle| d\tau \\
&+ 2\int_s^t |\langle\nabla\cdot\widehat{(w\otimes w)}, \varphi^2 e^{-2\mu|\xi|^{2\alpha}(t-\tau)}\hat{u}\rangle| d\tau.
\end{aligned} \tag{2.3}$$

Proof. We take the Fourier transform of $(1.1)_1$, multiply it by $\varphi^2 e^{-2\mu|\xi|^{2\alpha}(t-s)}\hat{u}$ and integrate over $\mathbb{R}^3$ to yield

$$\int_{\mathbb{R}^3}[\partial_s u + (u\cdot\nabla)u + \mu\Lambda^{2\alpha}u + \nabla\cdot(w\otimes w) + \nabla p]^\wedge \cdot \varphi^2 e^{-2\mu|\xi|^{2\alpha}(t-s)}\hat{u} d\xi = 0. \tag{2.4}$$

Rewrite the first and third terms as

$$\begin{aligned}
&\int_{\mathbb{R}^3}\widehat{\partial_s u}\cdot\varphi^2 e^{-2\mu|\xi|^{2\alpha}(t-s)}\hat{u} d\xi \\
=&\int_{\mathbb{R}^3}\frac{1}{2}\frac{d}{ds}(|\hat{u}|^2\varphi^2 e^{-2\mu|\xi|^{2\alpha}(t-s)})d\xi - \mu\int_{\mathbb{R}^3}|\hat{u}|^2\varphi^2|\xi|^{2\alpha}e^{-2\mu|\xi|^{2\alpha}(t-s)}d\xi \\
=&\frac{1}{2}\frac{d}{ds}\|\varphi e^{-\mu|\xi|^{2\alpha}(t-s)}\hat{u}\|_2^2 - \mu\|\varphi|\xi|^\alpha e^{-\mu|\xi|^{2\alpha}(t-s)}\hat{u}\|_2^2,
\end{aligned} \tag{2.5}$$

$$\int_{\mathbb{R}^3}\mu\widehat{\Lambda^{2\alpha}u}\cdot\varphi^2 e^{-2\mu|\xi|^{2\alpha}(t-s)}\hat{u} d\xi = \mu\int_{\mathbb{R}^3}\varphi^2|\xi|^{2\alpha}e^{-2\mu|\xi|^{2\alpha}(t-s)}|\hat{u}|^2 d\xi = \mu\|\varphi|\xi|^\alpha e^{-\mu|\xi|^{2\alpha}(t-s)}\hat{u}\|_2^2. \tag{2.6}$$

Substituting (2.5) and (2.6) into (2.4), integrating over $[s, t]$ with respect to time yields (2.3). □

Lemma 2.3. Assume (u, w, θ) is a weak solution to system (1.1). For $E(t) \in C^1(\mathbb{R}; \mathbb{R}_+)$ with $E(t) \geq 0$, then

$$\begin{aligned} E(t)\|(1-\varphi)\hat{u}(t)\|_2^2 =& E(s)\|(1-\varphi)\hat{u}(s)\|_2^2 + \int_s^t E'(\tau)\|(1-\varphi)\hat{u}\|_2^2 d\tau \\ &+ 2\int_s^t E(\tau)\langle \xi^{2\alpha}|\hat{u}|^2, (1-\varphi)\varphi\rangle d\tau - 2\mu \int_s^t E(\tau)\|(1-\varphi)\xi^{\alpha}\hat{u}\|_2^2 d\tau \\ &- 2\int_s^t E(\tau)\langle \widehat{u\cdot\nabla u}, (1-\varphi)^2\hat{u}\rangle d\tau - 2\int_s^t E(\tau)\langle \nabla\cdot\widehat{(w\otimes w)}, (1-\varphi)^2\hat{u}\rangle d\tau. \end{aligned} \tag{2.7}$$

Proof. We take the Fourier transform of $(1.1)_1$, multiply it by $E(t)(1-\varphi)^2\hat{u}$ and integrate over $\mathbb{R}^3$ to infer

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} E(t)|(1-\varphi)\hat{u}|^2 d\xi - \frac{1}{2}\int_{\mathbb{R}^3} E'(t)|(1-\varphi)\hat{u}|^2 d\xi \\ &- \int_{\mathbb{R}^3} E(t)\xi^{2\alpha}|\hat{u}|^2(1-\varphi)\varphi d\xi + \mu E(t)\int_{\mathbb{R}^3} |\xi|^{2\alpha}|(1-\varphi)\hat{u}|^2 d\xi \\ &+ E(t)\langle \widehat{u\cdot\nabla u}, (1-\varphi)^2\hat{u}\rangle + E(t)\langle \nabla\cdot\widehat{(w\otimes w)}, (1-\varphi)^2\hat{u}\rangle = 0. \end{aligned} \tag{2.8}$$

Integrating (2.8) over $[s, t]$ on time yields (2.7). □

2.3. Auxiliary estimates

In order to establish the estimate of high frequency parts, we need the following lemmas on the boundedness of $\hat{u}(\xi, t)$.

Lemma 2.4. Let (u, w, θ) be a weak solution to system (1.1) with the initial data $u_0 \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ and $(w_0, \theta_0) \in (L^2(\mathbb{R}^3))^2$, then we have

$$|\hat{u}(\xi, t)| \leq C(1 + |\xi|^{1-2\alpha}). \tag{2.9}$$

Proof. Taking the Fourier transform of the $(1.1)_1$ yields

$$\hat{u}_t + \mu|\xi|^{2\alpha}\hat{u} = H(\xi, t),$$

where

$$\begin{aligned} H(\xi, t) &= -\widehat{(u\cdot\nabla)u} - \nabla\cdot\widehat{(w\otimes w)} - \widehat{\nabla p} \\ &=: H_1 + H_2 + H_3. \end{aligned}$$

Thus,

$$\hat{u}(t) = e^{-\mu|\xi|^{2\alpha}t}\hat{u}(0) + \int_0^t e^{-\mu|\xi|^{2\alpha}(t-\tau)}H(\xi, \tau)d\tau.$$

For H_1, we get

$$|H_1| = |\widehat{(u\cdot\nabla)u}| = |\nabla\cdot\widehat{(u\otimes u)}| \leq |\xi|\|uu\|_1 \leq |\xi|\|u\|_2\|u\|_2 \leq |\xi|\|u_0\|_2^2 \leq C|\xi|. \tag{2.10}$$

Similarly,

$$|H_2| = |\nabla\cdot\widehat{(w\otimes w)}| \leq C|\xi|. \tag{2.11}$$

With respect to H_3, by taking the divergence of $(1.1)_1$, one has

$$\Delta p = -\nabla \cdot (u \cdot \nabla u) - \nabla \cdot (\nabla \cdot (w \otimes w)). \tag{2.12}$$

Taking the Fourier transform of (2.12) yields

$$|\xi|^2 \hat{p} \le |\nabla \cdot \widehat{\nabla \cdot (u} \otimes u)| + |\nabla \cdot (\widehat{\nabla \cdot (w} \otimes w))|,$$

which together with (2.10) and (2.11), it follows that

$$\hat{p} \le C. \tag{2.13}$$

Summing up (2.10), (2.11) and (2.13), we arrive at

$$|H(\xi, t)| \le C|\xi|.$$

Furthermore,

$$\begin{aligned} |\hat{u}(\xi, t)| &\le |\hat{u}(0)| + C|\xi| \int_0^t e^{-|\xi|^{2\alpha}(t-\tau)} d\tau \\ &\le C\|u_0\|_1 + C|\xi| \frac{1}{|\xi|^{2\alpha}} (1 - e^{-|\xi|^{2\alpha} t}) \\ &\le C(1 + |\xi|^{1-2\alpha}). \end{aligned}$$

Finally, we introduce the fractional Sobolev inequality.

Lemma 2.5. *[17] Let $0 \le k < l \le 1,\ 1 \le p < q < \infty$ satisfy $p(l-k) < n$ and $\frac{1}{q} = \frac{1}{p} - \frac{l-k}{n}$, then there exists a positive constant $C = C(n, p, q, k, l)$ such that*

$$\|f\|_{W^{k,q}(\mathbb{R}^n)} \le C\|f\|_{W^{l,p}(\mathbb{R}^n)}.$$

3. Results

Now, let us state our main result as follows.

Theorem 3.1. *Let $\alpha \in (0, 1]$ and $u_0 \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$, $(w_0, \theta_0) \in (L^2(\mathbb{R}^3))^2$. Assume that there is a weak solution of system (1.1) satisfying*

$$w \in L^\infty(0, \infty; L^2(\mathbb{R}^3)) \cap L^\infty(0, \infty; W^{1-\alpha,\infty}(\mathbb{R}^3)),$$

then we have

$$\lim_{t\to\infty} \|u(t)\|_{L^2(\mathbb{R}^3)} = 0,\ \ \lim_{t\to\infty}(\|w(t)\|_{L^2(\mathbb{R}^3)} + \|\theta(t)\|_{L^2(\mathbb{R}^3)}) = C$$

for some absolute constant C.

Lemma 3.1. *(Low frequency decay) Let (u, w, θ) be a weak solution to system (1.1). Assume $(u_0, w_0, \theta_0) \in (L^2(\mathbb{R}^3))^3$. Setting $\varphi(\xi) = e^{-|\xi|^{2\alpha} t}$, we deduce*

$$\lim_{t\to\infty} \|\varphi \hat{u}(t)\|_2 = 0.$$

Proof. The generalized energy inequality (2.3) implies

$$\begin{aligned}
\|\varphi\hat{u}(t)\|_2^2 \leq & \|\varphi e^{-\mu|\xi|^{2\alpha}(t-s)}\hat{u}(s)\|_2^2 + 2\int_s^t |\langle \widehat{u\cdot\nabla u}, \varphi^2 e^{-2\mu|\xi|^{2\alpha}(t-\tau)}\hat{u}\rangle| d\tau \\
& + 2\int_s^t |\langle \nabla\cdot\widehat{(w\otimes w)}, \varphi^2 e^{-2\mu|\xi|^{2\alpha}(t-\tau)}\hat{u}\rangle| d\tau \\
=: & \sum_{i=1}^3 I_i.
\end{aligned}$$

For I_1, it follows that

$$\limsup_{t\to\infty} I_1 = \limsup_{t\to\infty} \|\varphi e^{-\mu|\xi|^{2\alpha}(t-s)}\hat{u}(s)\|_2^2 = 0. \tag{3.1}$$

Regarding the term I_2 by Hölder, Hausdorff-Young and Sobolev inequalities, the facts that φ^2 is a rapidly decreasing function of $|\xi|$ and $\|u(t)\|_2$ is bounded for all the time, we infer, for $\alpha \in (0, 1]$,

$$\begin{aligned}
I_2 = & 2\int_s^t |\langle \widehat{u\cdot\nabla u}, \varphi^2 e^{-2\mu|\xi|^{2\alpha}(t-\tau)}\hat{u}\rangle| d\tau \\
= & 2\int_s^t |\langle |\xi_i|\widehat{u_iu_j}, \varphi^2 e^{-2\mu|\xi|^{2\alpha}(t-\tau)}\hat{u}\rangle| d\tau \\
= & 2\int_s^t |\langle \widehat{u_iu_j}, |\xi|^{1-\alpha}\varphi^2 e^{-2\mu|\xi|^{2\alpha}(t-\tau)}|\xi|^\alpha\hat{u}\rangle| d\tau \\
\leq & C\int_s^t \|\widehat{u_iu_j}\|_{\frac{3}{\alpha}} \||\xi|^{1-\alpha}\varphi^2\|_{\frac{6}{3-2\alpha}} \|e^{-2\mu|\xi|^{2\alpha}(t-\tau)}|\xi|^\alpha\hat{u}\|_2 d\tau \\
\leq & C\int_s^t \|u\otimes u\|_{\frac{3}{3-\alpha}} \|\Lambda^\alpha u\|_2 d\tau \\
\leq & C\int_s^t \|u\|_2\|u\|_{\frac{6}{3-2\alpha}} \|\Lambda^\alpha u\|_2 d\tau \\
\leq & C\int_s^t \|\Lambda^\alpha u\|_2^2 d\tau.
\end{aligned} \tag{3.2}$$

Similar to the estimation of I_2, we get for I_3 that

$$\begin{aligned}
I_3 = & 2\int_s^t |\langle \nabla\cdot\widehat{(w\otimes w)}, \varphi^2 e^{-2\mu|\xi|^{2\alpha}(t-\tau)}\hat{u}\rangle| d\tau \\
= & 2\int_s^t |\langle |\xi_i|\widehat{(w_iw_j)}, \varphi^2 e^{-2\mu|\xi|^{2\alpha}(t-\tau)}\hat{u}\rangle| d\tau \\
\leq & C\int_s^t \|\widehat{w_iw_j}\|_\infty \|\xi\varphi^2\|_2 \|e^{-2\mu|\xi|^{2\alpha}(t-\tau)}\hat{u}\|_2 d\tau \\
\leq & C\int_s^t \|w\otimes w\|_1 \|\xi\varphi^2\|_2 \|u\|_2 d\tau \\
\leq & C\int_s^t \|w\|_2^2 \|\xi\varphi^2\|_2 d\tau \\
\leq & C\int_s^t \|\xi\varphi^2\|_2 d\tau.
\end{aligned} \tag{3.3}$$

Straightforward computations show that

$$\begin{aligned}\|\xi\varphi^2\|_2^2 &= \int_{\mathbb{R}^3} |\xi|^2 e^{-4|\xi|^{2\alpha}\tau} d\xi \\ &\le C\int_0^\infty r^2 e^{-4r^{2\alpha}\tau} r^2 dr \\ &= C\tau^{-\frac{5}{2\alpha}} \int_0^\infty a^4 e^{-4a^{2\alpha}} da \\ &\le C\tau^{-\frac{5}{2\alpha}}.\end{aligned} \tag{3.4}$$

Summing up (3.2)–(3.4), one has

$$I_2 + I_3 \le C\left(\int_s^t \|\Lambda^\alpha u\|_2^2 d\tau + \int_s^t \tau^{-\frac{5}{4\alpha}} d\tau\right). \tag{3.5}$$

This together with $\int_0^\infty \|\Lambda^\alpha u\|_2^2 d\tau$ being finite, it follows that

$$\lim_{t\to\infty}(I_2 + I_3) \le \lim_{s\to\infty}\lim_{t\to\infty} C\int_s^t (\|\Lambda^\alpha u\|_2^2 + \tau^{-\frac{5}{4\alpha}})d\tau = 0, \quad \text{for } 0 < \alpha \le 1. \tag{3.6}$$

Combining (3.1) and (3.6), we conclude

$$\lim_{t\to\infty} \|\varphi\hat{u}(t)\|_2 = 0.$$

Lemma 3.2. *(High frequency decay) Let (u, w, θ) be a weak solution to system* (1.1)*. Assume $u_0 \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$, $(w_0, \theta_0) \in (L^2(\mathbb{R}^3))^2$ and $w \in L^\infty(0, \infty; W^{1-\alpha,\infty}(\mathbb{R}^3))$. Setting $\varphi = e^{-|\xi|^{2\alpha}t}$, then,*

$$\lim_{t\to\infty} \|(1-\varphi)\hat{u}(t)\|_2 = 0.$$

Proof. To obtain the high frequency decay, we first rewrite (2.7) as

$$\begin{aligned}&E(t)\|(1-\varphi)\hat{u}(t)\|_2^2 \\ =&E(s)\|(1-\varphi)\hat{u}(s)\|_2^2 + \int_s^t E'(\tau)\|(1-\varphi)\hat{u}\|_2^2 d\tau - 2\mu\int_s^t E(\tau)\|(1-\varphi)\xi^\alpha\hat{u}\|_2^2 d\tau \\ &- 2\int_s^t E(\tau)\langle\widehat{u\cdot\nabla u}, (1-\varphi)^2\hat{u}(\tau)\rangle d\tau - 2\int_s^t E(\tau)\langle\nabla\cdot\widehat{(w\otimes w)}, (1-\varphi)^2\hat{u}(\tau)\rangle d\tau \\ &+ 2\int_s^t E(\tau)\langle\xi^{2\alpha}|\hat{u}|^2, (1-\varphi)\varphi\rangle d\tau \\ =:&E(s)\|(1-\varphi)\hat{u}(s)\|_2^2 + \sum_{i=1}^5 K_i.\end{aligned}$$

In what follows, we deal with the terms K_1 and K_2 by the Fourier splitting method. Denote the ball

$\chi(\varepsilon)=\{\xi\in\mathbb{R}^3:|\xi|\le G(\varepsilon)\}$, where the radius $G(\varepsilon)$ will be determined later, then we infer

$$
\begin{aligned}
&K_1+K_2\\
=&\int_s^t E'(\tau)\|(1-\varphi)\hat{u}\|_2^2d\tau-2\mu\int_s^t E(\tau)\|(1-\varphi)\xi^\alpha\hat{u}\|_2^2d\tau\\
\le&\int_s^t E'(\tau)\int_{\chi(\varepsilon)}|(1-\varphi)\hat{u}|^2d\xi d\tau+\int_s^t E'(\tau)\int_{\mathbb{R}^3\backslash\chi(\varepsilon)}|(1-\varphi)\hat{u}|^2d\xi d\tau\\
&-2\mu\int_s^t E(\tau)\int_{\mathbb{R}^3\backslash\chi(\varepsilon)}|(1-\varphi)\xi^\alpha\hat{u}|^2d\xi d\tau-2\mu\int_s^t E(\tau)\int_{\chi(\varepsilon)}|(1-\varphi)\xi^\alpha\hat{u}|^2d\xi d\tau\\
\le&\int_s^t E'(\tau)\int_{\chi(\varepsilon)}|(1-\varphi)\hat{u}|^2d\xi d\tau+\int_s^t E'(\tau)\int_{\mathbb{R}^3\backslash\chi(\varepsilon)}|(1-\varphi)\hat{u}|^2d\xi d\tau\\
&-2\mu\int_s^t E(\tau)\int_{\mathbb{R}^3\backslash\chi(\varepsilon)}|(1-\varphi)\xi^\alpha\hat{u}|^2d\xi d\tau\\
\le&\int_s^t E'(\tau)\int_{\chi(\varepsilon)}|(1-\varphi)\hat{u}|^2d\xi d\tau+\int_s^t[E'(\tau)-2\mu E(\tau)G^{2\alpha}(\varepsilon)]\int_{\mathbb{R}^3\backslash\chi(\varepsilon)}|(1-\varphi)\hat{u}|^2d\xi d\tau.
\end{aligned}
$$

Taking $E(t)=e^{\varepsilon t}$ and $G(\varepsilon)=(\frac{\varepsilon}{2\mu})^{\frac{1}{2\alpha}}$, indicates that $E'(t)-2\mu E(t)G^{2\alpha}(\varepsilon)=0$. Thus, we have

$$
K_1+K_2\le\int_s^t E'(\tau)\int_{\chi(\varepsilon)}|(1-\varphi)\hat{u}|^2d\xi d\tau,\tag{3.7}
$$

which yields by Lemma 2.4 that

$$
\begin{aligned}
\int_{\chi(\varepsilon)}|(1-\varphi)\hat{u}|^2d\xi\le C\int_{\chi(\varepsilon)}(1+|\xi|^{1-2\alpha})^2d\xi\le C\int_{\chi(\varepsilon)}(1+|\xi|^{2-4\alpha})d\xi\\
\le C\int_0^{G(\varepsilon)}(1+r^{2-4\alpha})r^2dr\le C(\varepsilon^{\frac{3}{2\alpha}}+\varepsilon^{\frac{5-4\alpha}{2\alpha}}).
\end{aligned}\tag{3.8}
$$

In order to estimate K_3 by Hölder, Hausdorff-Young and Sobolev inequalities and $\alpha\in(0,1]$, we get

$$
\begin{aligned}
K_3=&-2\int_s^t E(\tau)\langle\widehat{u\cdot\nabla u},(1-\varphi)^2\hat{u}(\tau)\rangle d\tau\\
=&-2\int_s^t E(\tau)\langle\widehat{u\cdot\nabla u},((1-\varphi)^2-1)\hat{u}(\tau)\rangle d\tau\\
\le&C\int_s^t E(\tau)|\langle\widehat{u\otimes u},|\xi|^{1-\alpha}(\varphi^2-2\varphi)|\xi|^\alpha\hat{u}\rangle|d\tau\\
\le&C\int_s^t E(\tau)\|\widehat{u\otimes u}\|_{\frac{3}{\alpha}}\||\xi|^{1-\alpha}(\varphi^2-2\varphi)\|_{\frac{6}{3-2\alpha}}\||\xi|^\alpha\hat{u}\|_2d\tau\\
\le&C\int_s^t E(\tau)\|u\otimes u\|_{\frac{3}{3-\alpha}}\|\Lambda^\alpha u\|_2d\tau\\
\le&C\int_s^t E(\tau)\|u\|_2\|u\|_{\frac{6}{3-2\alpha}}\|\Lambda^\alpha u\|_2d\tau\\
\le&C\int_s^t E(\tau)\|\Lambda^\alpha u\|_2^2d\tau.
\end{aligned}\tag{3.9}
$$

Similarly,

$$
\begin{aligned}
K_4 \leq & C\int_s^t E(\tau)|\langle|\xi|^{1-\alpha}\widehat{w\otimes w},(1-\varphi)^2\xi^\alpha\hat{u}\rangle|d\tau \\
\leq & C\int_s^t E(\tau)\||\xi|^{1-\alpha}\widehat{w\otimes w}\|_2\|\xi^\alpha\hat{u}\|_2d\tau \\
= & C\int_s^t E(\tau)\|\widehat{\Lambda^{1-\alpha}w}*\widehat{w}\|_2\|\xi^\alpha\hat{u}\|_2d\tau \\
= & C\int_s^t E(\tau)\|\mathcal{F}^{-1}(\widehat{\Lambda^{1-\alpha}w}*\widehat{w})\|_2\|\xi^\alpha\hat{u}\|_2d\tau \\
= & C\int_s^t E(\tau)\|\mathcal{F}^{-1}(\widehat{\Lambda^{1-\alpha}w})\cdot\mathcal{F}^{-1}(\widehat{w})\|_2\|\xi^\alpha\hat{u}\|_2d\tau \\
= & C\int_s^t E(\tau)\|w\Lambda^{1-\alpha}w\|_2\|\Lambda^\alpha u\|_2d\tau \\
\leq & C\int_s^t E(\tau)\|w\|_2\|\Lambda^{1-\alpha}w\|_\infty\|\Lambda^\alpha u\|_2d\tau \\
\leq & C\int_s^t E(\tau)\|\Lambda^\alpha u\|_2d\tau \\
\leq & C\left(\int_s^t E(\tau)^2d\tau\right)^{\frac{1}{2}}\left(\int_s^t\|\Lambda^\alpha u\|_2^2d\tau\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.10}
$$

K_5 can be estimated as

$$
K_5=2\int_s^t E(\tau)\langle\xi^{2\alpha}|\hat{u}|^2,(1-\varphi)\varphi\rangle d\tau\leq C\int_s^t E(\tau)\|\Lambda^\alpha u\|_2^2. \tag{3.11}
$$

Putting (3.7)–(3.11) into (2.7), we deduce

$$
\begin{aligned}
\|(1-\varphi)\hat{u}(t)\|_2^2\leq & \frac{E(s)}{E(t)}\|(1-\varphi)\hat{u}(s)\|_2^2+C\int_s^t\frac{E(\tau)}{E(t)}\|\Lambda^\alpha u\|_2^2d\tau \\
& +\frac{C}{E(t)}\left(\int_s^t E(\tau)^2d\tau\right)^{\frac{1}{2}}\left(\int_s^t\|\Lambda^\alpha u\|_2^2d\tau\right)^{\frac{1}{2}}+\frac{C}{E(t)}(\varepsilon^{\frac{3}{2\alpha}}+\varepsilon^{\frac{5-4\alpha}{2\alpha}}).
\end{aligned}
$$

Now, we first pass the limit $t\to\infty$,

$$
\begin{aligned}
& \lim_{t\to\infty}\|(1-\varphi)\hat{u}(t)\|_2^2 \\
\leq & \lim_{t\to\infty}\frac{E(s)}{E(t)}\|(1-\varphi)\hat{u}(s)\|_2^2+\lim_{t\to\infty}C\int_s^t\frac{E(\tau)}{E(t)}\|\Lambda^\alpha u\|_2^2d\tau \\
& +\lim_{t\to\infty}\frac{C}{E(t)}\left(\int_s^t E(\tau)^2d\tau\right)^{\frac{1}{2}}\left(\int_s^t\|\Lambda^\alpha u\|_2^2d\tau\right)^{\frac{1}{2}}+\frac{C}{E(t)}(\varepsilon^{\frac{3}{2\alpha}}+\varepsilon^{\frac{5-4\alpha}{2\alpha}}) \\
\leq & \lim_{t\to\infty}e^{\varepsilon(s-t)}\|u_0\|_2^2+C\int_s^\infty\|\Lambda^\alpha u\|_2^2d\tau+\frac{C}{\sqrt{\varepsilon}}\left(\int_s^\infty\|\Lambda^\alpha u\|_2^2d\tau\right)^{\frac{1}{2}}+C(\varepsilon^{\frac{3}{2\alpha}}+\varepsilon^{\frac{5-4\alpha}{2\alpha}}) \\
\leq & C\int_s^\infty\|\Lambda^\alpha u\|_2^2d\tau+\frac{C}{\sqrt{\varepsilon}}\left(\int_s^\infty\|\Lambda^\alpha u\|_2^2d\tau\right)^{\frac{1}{2}}+C(\varepsilon^{\frac{3}{2\alpha}}+\varepsilon^{\frac{5-4\alpha}{2\alpha}}),
\end{aligned}
$$

and then pass the limit $s \to \infty$,

$$\begin{aligned}\lim_{t\to\infty} \|(1-\varphi)\hat{u}(t)\|_2^2 &\le \lim_{s\to\infty}\left(C\int_s^\infty \|\Lambda^\alpha u\|_2^2 d\tau + \frac{C}{\sqrt{\varepsilon}}\left(\int_s^\infty \|\Lambda^\alpha u\|_2^2 d\tau\right)^{\frac{1}{2}} + C(\varepsilon^{\frac{3}{2\alpha}} + \varepsilon^{\frac{5-4\alpha}{2\alpha}})\right) \\ &\le C(\varepsilon^{\frac{3}{2\alpha}} + \varepsilon^{\frac{5-4\alpha}{2\alpha}}).\end{aligned}$$

Since $\varepsilon > 0$ can be chosen arbitrarily small, it implies that $\lim_{t\to\infty} \|(1-\varphi)\hat{u}(t)\|_2 = 0$.

Combining Lemmas 3.1 and 3.2 yields

$$\lim_{t\to\infty} \|u(t)\|_2 = 0. \tag{3.12}$$

For the limit of $\|w(t)\|_2 + \|\theta(t)\|_2$, set

$$\zeta(t) = \|u(t)\|_2 + \|w(t)\|_2 + \|\theta(t)\|_2.$$

By Lemma 2.1 and (3.12), we know that $\zeta(t)$ is nonnegative and decreasing. Therefore, there exists a nonnegative constant C such that $\zeta(t) \to C$ as $t \to \infty$. Since $\|u(t)\|_2 \to 0$, it follows that

$$\|w(t)\|_2 + \|\theta(t)\|_2 \to C, \quad \text{as} \quad t \to \infty.$$

This completes the proof of Theorem 3.1. □

4. Conclusions

For the energy decay problem of the tropical climate model, we refered to the decay of solution of the fully dissipative case by Li, Xiao [15] and Xie, Zhang [22]. However, with respect to the decay of solution to the partly dissipative case, to the best of our knowledge, there are no corresponding results, which was our motivation in this paper.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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