AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(12): 30846–30874.
DOI: 10.3934/math.20231577
Received: 30 August 2023
Revised: 30 October 2023
Accepted: 05 November 2023
Published: 15 November 2023

Research article

Analysis of progressive Type-II censoring schemes for generalized power unit half-logistic geometric distribution

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Abstract: This study addresses the difficulties associated with parameter estimation in the generalized power unit half-logistic geometric distribution by employing a progressive Type-II censoring technique. The study uses a variety of methods, including maximum likelihood, maximum product of spacing, and Bayesian estimation. The work investigates Bayesian estimators taking into account a gamma prior and a symmetric loss function while working with observed data produced by likelihood and spacing functions. A full simulation experiment is carried out with varying sample sizes and censoring mechanisms in order to thoroughly evaluate the various estimation approaches. The highest posterior density approach is employed in the study to compute credible intervals for the parameters. Additionally, based on three optimal criteria, the study chooses the best progressive censoring scheme from a variety of rival methods. The study examines two real datasets in order to confirm the applicability of the generalized power unit half-logistic geometric distribution and the efficacy of the suggested estimators. The results show that in order to generate the necessary estimators, the maximum product of the spacing approach is better than the maximum likelihood method. Furthermore, as compared to traditional methods, the Bayesian strategy that makes use of probability and spacing functions produces estimates that are more satisfactory.

Keywords: generalized power unit half-logistic geometric distribution; estimation methods; progressive Type-II; optimal progressive Type-II censoring strategy; real data
Mathematics Subject Classification: 62F15, 62G20, 65C60

1. Introduction

Statistical analysis and the modeling of lifetime data are essential across various applied disciplines such as insurance, finance, biomedical research, and engineering. Consequently, a multitude of lifetime distributions have been introduced in these domains. Particularly, the modeling of datasets constrained within the range of (0, 1) has gained significant prominence in recent times. This approach has found widespread utility in addressing the survival and failure rates of products across diverse fields. As a result of its adaptability in handling probabilistic models of this nature, a plethora of unit distributions that are bounded within the interval (0, 1) have emerged. Furthermore, industries including medical, actuarial, and finance sectors are increasingly recognizing the indispensable value of these types of distributions.

Extensive efforts from statisticians have been directed toward comprehending the failure of components and units, particularly within the well-structured operating systems prevalent in industrial and mechanical engineering. Their investigation revolves around the observation of operating units until they encounter failure. Subsequently, the lifetimes of these units are recorded, followed by the application of statistical inference techniques to the accumulated data. This process culminates in the estimation of reliability and hazard functions for the entire system, leveraging the collected dataset. Despite these endeavors, situations arise where certain experimental units possess both high reliability and significant costs. In such cases, a practical necessity emerges to reduce the number of experimental units utilized as well as the duration of the lifetime experiments involving these units. To address this scenario, the progressive Type-II censoring (PT-IIC) scheme comes into play. This scheme offers a way to achieve robust estimations through lifetime experiments while safeguarding some experimental units from encountering failure. The progressive Type-II censoring scheme is frequently described as follows: Initially, the experimenter places n independent and identical units into the life measurement process. Upon the occurrence of the first failure, denoted as $x_{(1)}$, a random selection process removes R_1 units from the remaining $n-1$ surviving units. This process is reiterated for each subsequent failure event: At the time of the second failure, $x_{(2)}$, R_2 units are randomly chosen for removal from the surviving units, now numbering $n - R_1 - 2$. This pattern continues until the m-th failure transpires at time x_m, resulting in the extraction of $R_m = n - m - \sum_{i=1}^{m-1} R_i$ surviving units from the test. The collective set of r values, denoted as $R = (R_1, R_2, ..., R_m)$, characterizes the PT-IIC scheme. In contrast, Progressive Type-II right censoring involves a predefined censoring scheme R before the experiment's commencement. Interestingly, Type-II censoring can often be viewed as a specific instance of PT-IIC, where the scheme is represented as R =(0, 0, ..., $n - m$), as documented in [1–3]. This experimental setup concludes at the m^{th} failure, a predetermined event occurring at time t_m, and the value of R_m is computed as $n - m - \sum_{i=1}^{m-1} R_i$, (Figure 1).

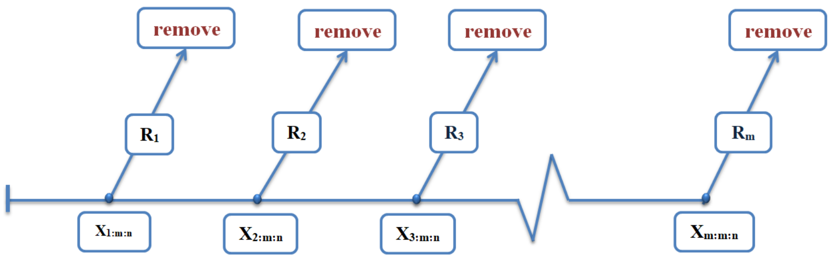

Figure 1. Schematic representation of the PT-IIC.

Let $x_{(1)}, x_{(2)}, ..., x_{(m)},\ 1 \leq m \leq n,$ be a PT-IIC sample observed from a lifetime test involving n units and $R_1, R_2, ..., R_m$ be the censoring scheme. The joint probability density function(PDF) of a PT-IIC sample is given by

$$L(x_{_{i:m:n}}; \theta, \alpha) = C \prod_{i=1}^{m} f\ (x_{_{i:m:n}}))\,[(1 - F(x_{_{i:m:n}}))]^{R_i}\,, \tag{1.1}$$

where C may be a constant defined as

$$C = n\,(n - R_1 - 1)\cdots(n - \sum_{i=1}^{m-1}(R_i + 1)).$$

See [1, 3, 4] for more details.

The attention has been on the development of PT-IIC over the last two to three decades. One can consult sources such as [2, 5–7] and others for insightful findings regarding this censoring scheme.

A substitute for the maximum likelihood estimation (MLE) approach for deducing the parameters of continuous uni-variate distributions was introduced by [8], termed the maximum product of spacing (MPS) method. This approach was put forth as a means to retain many of the properties inherent in maximum likelihood by substituting the likelihood function with a product of spacing. This technique was subsequently extended to parameter estimation using censored samples by various researchers. For complete samples, sources like [9–11] delve into this method. When dealing with Type-I and Type-II censored samples, the works of [12, 13] offer insights. The progressive Type-II censoring scheme (PT-IIC) is explored in studies such as [14, 15], while the adaptive progressive Type-II scheme is examined in sources like [16–18]. The organization of this paper is as follows: In Section 2, the generalized power unit half-logistic geometric distribution will be presented. Section 3 introduces the classical estimation methods. Bayesian estimates for the unknown parameters are obtained in Section 4. In Section 5, the numerical computations are analyzed. In Section 6, we present an optimal progressive censoring scheme and compare it with various alternative censoring schemes. The conclusions drawn from the findings are summarized in Section 7.

2. Generalized power unit half-logistic geometric distribution

Nasiru et al. [19] introduced a new generalized power unit half-logistic geometric (GPUHLG)distribution. The random variable X is said to follow a GPUHLG distribution if its probability density function (pdf) is expressed as:

$$f(x) = \frac{2\alpha\theta x^{\alpha-1}}{((\theta - 2)\,x^{\alpha} - \theta)^2}, \quad \theta, \alpha > 0,\ 0 < x < 1, \tag{2.1}$$

and the cumulative distribution function (cdf)

$$F(x) = 1 - \frac{\theta\,(1 - x^{\alpha})}{(2 - \theta)\,x^{\alpha} + \theta}, \quad \theta, \alpha > 0,\ 0 < x < 1. \tag{2.2}$$

Figure 2 shows the cdf and the pdf of the GPUHLG distribution at different values of θ and α.

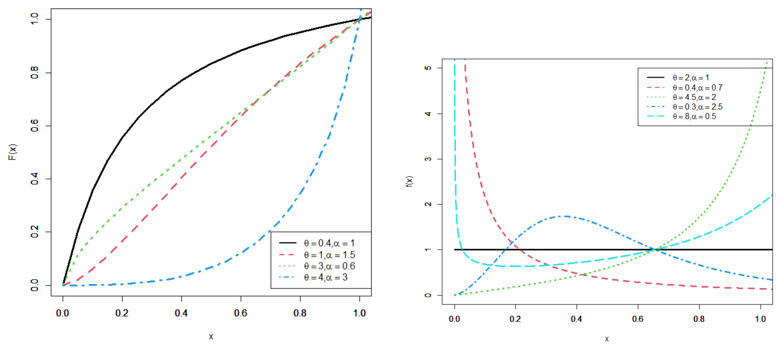

Figure 2. The cdf and pdf of the GPUHLG distribution at different values of θ and α.

Also, the survival and the hazard failure rate functions of the GPUHLG distribution are given respectively as

$$S(x) = 1 - F(x) = \frac{\theta(1 - x^{\alpha})}{(2 - \theta)\, x^{\alpha} + \theta}, \quad \theta, \alpha > 0, \ 0 < x < 1, \tag{2.3}$$

and

$$H(x) = \frac{f(x)}{S(x)} = \frac{2\alpha x^{\alpha-1}}{(x^{\alpha} - 1)\left((\theta - 2)\, x^{\alpha} - \theta\right)}, \quad \theta, \alpha > 0, \ 0 < x < 1. \tag{2.4}$$

Figure 3 shows the survival and the hazard rate functions of the GPUHLG distribution at different values of θ and α.

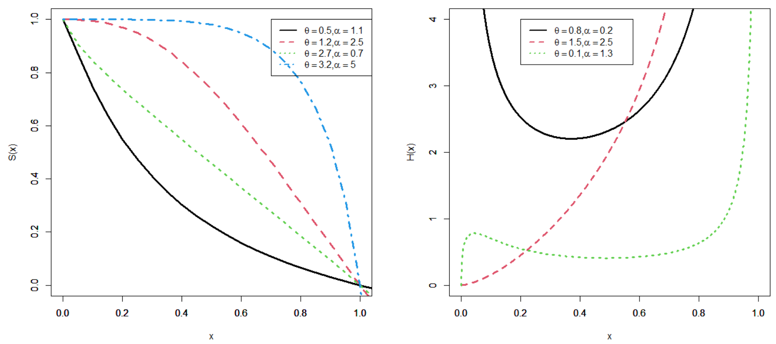

Figure 3. The survival and hazard rate functions of the GPUHLG distribution at different values of θ and α.

3. Classical estimation

In this section, the techniques of maximum likelihood estimation (MLEs) and maximum product of spacing estimation (MPSs) are employed to derive both point and interval estimators for the model parameter. The construction of interval estimators leverages the asymptotic characteristics of the MLEs and MPSs.

3.1. Maximum likelihood estimation technique

Consider a PT-IIC sample of size m, denoted by $x = x_{(i)}$, where $i = 1, \ldots, m$. This sample is acquired using the progressive censoring scheme S_i from the GPUHLG distribution, which is defined by the probability density function (pdf) and cumulative distribution function (cdf) shown in Eqs (2) and (3), respectively. By excluding the constant factor, the likelihood function of the GPUHLG distribution, accounting for the existence of PT-IIC, can be derived from Eqs (2), (3), and (1) as demonstrated below:

$$L \propto \left(2\alpha\theta^2\right)^m \prod_{i=1}^{m} \frac{x_{(i)}^{\alpha-1}}{\left((2-\theta)x_{(i)}^{\alpha}+\theta\right)^2} \left(\frac{1-x_{(i)}^{\alpha}}{(2-\theta)x_{(i)}^{\alpha}+\theta}\right)^{R_i}. \tag{3.1}$$

The log-likelihood function is given by

$$\begin{aligned} l = \log(L) \propto m\log(2\alpha) + 2m\log(\theta) + \sum_{i=1}^{m}\log(x_{(i)}^{\alpha-1}) - 2\sum_{i=1}^{m}\log((2-\theta)x_{(i)}^{\alpha}+\theta) \\ + \sum_{i=1}^{m} R_i\left(\log(1-x_{(i)}^{\alpha}) - \log((2-\theta)x_{(i)}^{\alpha}+\theta)\right). \end{aligned} \tag{3.2}$$

The derivatives of the log-likelihood function with respect to the parameters θ and α are presented as follows:

$$\frac{\partial l}{\partial\theta} = \frac{2m}{\theta} - 2\sum_{i=1}^{m}\frac{1-x_{(i)}^{\alpha}}{(2-\theta)x_{(i)}^{\alpha}+\theta} + \sum_{i=1}^{m}\frac{R_i(1-x_{(i)}^{\alpha})}{(2-\theta)x_{(i)}^{\alpha}+\theta}, \tag{3.3}$$

$$\begin{aligned} \frac{\partial l}{\partial\alpha} = & \frac{m}{\alpha} + \sum_{i=1}^{m}\log(x_{(i)}) - 2\sum_{i=1}^{m}\frac{(2-\theta)x_{(i)}^{\alpha}\log(x_{(i)})}{(2-\theta)x_{(i)}^{\alpha}+\theta} \\ & - \sum_{i=1}^{m} R_i\left(\frac{x_{(i)}^{\alpha}\log(x_{(i)})}{1-x_{(i)}^{\alpha}} + \frac{(2-\theta)x_{(i)}^{\alpha}\log(x_{(i)})}{(2-\theta)x_{(i)}^{\alpha}+\theta}\right). \end{aligned} \tag{3.4}$$

Equations (3.3) and (3.4) do not possess a readily available closed-form solution when equated to zero. Consequently, numerical methods using the Newton-Raphson algorithm implemented in the R programming language are employed to obtain solutions.

The Fisher information matrix, which is required for obtaining the MLE and the corresponding asymptotic confidence intervals of the parameters, requires the second partial derivatives of the log-likelihood function with respect to the parameters. The Fisher matrix $\mathcal{F}$ is given by

$$\mathcal{F} = \begin{bmatrix} \frac{\partial^2\mathcal{L}}{\partial\theta^2} & \frac{\partial^2\mathcal{L}}{\partial\theta\partial\alpha} \\ \frac{\partial^2\mathcal{L}}{\partial\alpha\partial\theta} & \frac{\partial^2\mathcal{L}}{\partial\alpha^2}. \end{bmatrix}, \tag{3.5}$$

These matrices should be positive definite at the MLE estimates of the parameters. The 2nd partial derivatives of the log-likelihood function, which are needed for the Fisher information matrix, are given by

$$\frac{\partial^2 l}{\partial\theta^2} = \frac{-2m}{\theta^2} + 2\sum_{i=1}^{m}\frac{\left(1-x_{(i)}^{\alpha}\right)^2}{\left((2-\theta)x_{(i)}^{\alpha}+\theta\right)^2} + \sum_{i=1}^{m}\frac{\left(R_i(1-x_{(i)}^{\alpha}\right)^2}{\left((2-\theta)x_{(i)}^{\alpha}+\theta\right)^2}, \tag{3.6}$$

$$\frac{\partial^2 l}{\partial\alpha^2} = -\frac{m}{\alpha^2} - \sum_{i=1}^{m}\left(\frac{(2-\theta)\left(\log(x_{(i)})\right)^2 x_{(i)}^{\alpha}}{(2-\theta)x_{(i)}^{\alpha}+\theta} - \frac{(2-\theta)^2\left(\log(x_{(i)})\right)^2 x_{(i)}^{2\alpha}}{\left((2-\theta)x_{(i)}^{\alpha}+\theta\right)^2}\right)$$
$$+\sum_{i=1}^{m} R_i\left(\frac{-2\log(x_{(i)})x_{(i)}^{2\alpha}}{\left(1-x_{(i)}^{\alpha}\right)^2} - \frac{2\log(x_{(i)})x_{(i)}^{\alpha}}{1-x_{(i)}^{\alpha}} - \frac{2(2-\theta)\log(x_{(i)})x_{(i)}^{\alpha}}{(2-\theta)x_{(i)}^{\alpha}+\theta} + \frac{\alpha(2-\theta)^2\left(\log(x_{(i)})\right)^2 x_{(i)}^{2\alpha}}{\left((2-\theta)x_{(i)}^{\alpha}+\theta\right)^2}\right), \tag{3.7}$$

$$\frac{\partial^2 l}{\partial\theta\partial\alpha} = \frac{2m}{\theta} - 2\sum_{i=1}^{m}\frac{1-x_{(i)}^{\alpha}}{(2-\theta)x_{(i)}^{\alpha}+\theta} - \sum_{i=1}^{m}\frac{R_i(1-x_{(i)}^{\alpha})}{(2-\theta)x_{(i)}^{\alpha}+\theta}, \tag{3.8}$$

$$\frac{\partial^2 l}{\partial\alpha\partial\theta} = \frac{m}{\alpha} + \sum_{i=1}^{m}\log(x_{(i)}) - 2\sum_{i=1}^{m}\frac{(2-\theta)\log(x_{(i)})x_{(i)}^{\alpha}}{(2-\theta)x_{(i)}^{\alpha}+\theta}$$
$$-\sum_{i=1}^{m}R_i\left(\frac{\log(x_{(i)})x_{(i)}^{\alpha}}{1-x_{(i)}^{\alpha}} + \frac{(2-\theta)\log(x_{(i)})x_{(i)}^{\alpha}}{(2-\theta)x_{(i)}^{\alpha}+\theta}\right). \tag{3.9}$$

As was discussed above, the MLEs of the unknown parameters θ and α are not derived in closed forms. Therefore, the sampling distributions of the MLEs cannot be obtained analytically. Alternatively, we can compute the asymptotic confidence intervals of these parameters using one of the properties of the MLEs, which states that

$$(\hat{\theta},\hat{\alpha}) \sim N_2\left((\theta,\alpha),\hat{\mathcal{F}}^{-1}\right) \quad \text{as } n\to\infty,$$

where $\hat{\mathcal{F}}^{-1}$ is the inverse of $\mathcal{F}$ evaluated at the MLEs of the parameters, respectively.

3.2. Bootstrap confidence intervals

The preceding section underscored the challenges associated with calculating second-order derivatives for constructing asymptotic confidence intervals (ACIs) for the model's unknown parameters. Consequently, we turn our attention to employing bootstrapping techniques. Specifically, we consider the percentile bootstrap approach (Boot-p), as well as the bootstrap-t approach proposed by Efron [7], and the bootstrap-t method outlined by Hall [20].

3.2.1. Parametric Boot-p

(1) Utilizing the original data $x = x_{(1)}, x_{(2)}, ..., x_{(m)}$, maximize Eqs (8) and (9) to obtain $\hat{\theta}$ and $\hat{\alpha}$, respectively.
(2) Generate the PT-IIC sample $x^* = x^*_{(1)}, x^*_{(2)}, ..., x^*_{(m)}$ based on the pre-specified PT-IIC scheme $(R_1, R_2, ..., R_m)$ from the GPUHLG distribution with parameters $\hat{\theta}$ and $\hat{\alpha}$, using the algorithm detailed in Balakrishnan and Sandhu [4] and [3].
(3) Obtain the maximum likelihood estimates based on the bootstrap sample, denoting this estimate as $\hat{\psi}^*$, where in our case ψ could be θ and α.
(4) Repeat Steps (2) and (3) for a total of N bootstrap iterations, obtaining $\hat{\psi}^*_1, \hat{\psi}^*_2, ..., \hat{\psi}^*_{N\,boot}$, where $\hat{\psi}^*_i = \left(\hat{\theta}^*_i, \hat{\alpha}^*_i\right)$ and $i = 1, 2, 3, ..., N\,boot$.
(5) Arrange $\hat{\psi}^*_i$ in ascending order to obtain $\hat{\psi}^*_{(1)}, \hat{\psi}^*_{(2)}, ..., \hat{\psi}^*_{(N\,boot)}$.

Consider $G_1(z) = P(\hat{\psi}^* \leq z)$ to represent the cumulative distribution function of $\hat{\psi}^*$. Introduce $\hat{\psi}_{boot-p} = G_1^{-1}(z)$ for a given value of z. The estimated bootstrap-p $100(1-\gamma)\%$ confidence interval of $\hat{\psi}$ is then expressed as:

$$\left[\hat{\psi}_{boot-p}\left(\frac{\gamma}{2}\right), \hat{\psi}_{boot-p}\left(1-\frac{\gamma}{2}\right)\right]. \tag{3.10}$$

3.2.2. Parametric Boot-t

(1) The same as the parametric Boot-p.
(2) The same as the parametric Boot-p.
(3) The same as the parametric Boot-p.
(4) Utilizing the asymptotic variance-covariance matrix, calculate the matrix $I^{-1*}\left(\frac{\partial \ell}{\partial \theta}, \frac{\partial \ell}{\partial \alpha}\right)$.
(5) Calculate the statistic $T^{*\psi}$, defined as follows:

$$T^{*\psi} = \frac{\left(\hat{\psi}^* - \hat{\psi}\right)}{\sqrt{\widehat{var}\left(\hat{\psi}^*\right)}}.$$

(6) Repeat Steps 2 – 5, N-Boot times and obtain $T_1^{*\psi}, T_2^{*\psi}, ..., T_{N\,boot}^{*\psi}$.
(7) Arrange the values $T_1^{*\psi}, T_2^{*\psi}, ..., T_{N\,boot}^{*\psi}$ in ascending order to derive the ordered sequences $T_{(1)}^{*\psi}, T_{(2)}^{*\psi}, ..., T_{(N\,boot)}^{*\psi}$.

Let $G_2(z) = P(T^* \leq z)$ be the cumulative distribution function of T^* for given z. Define $\hat{\psi}_{boot-t} = \hat{\psi} + G_1^{-1}(z)\sqrt{\widehat{var}\left(\hat{\psi}^*\right)}$.

Then, the approximate bootstrap-t $100(1-\gamma)\%$ CI of $\hat{\psi} = \left(\hat{\theta}, \hat{\alpha}\right)$, is given by

$$\left[\hat{\psi}_{boot-t}\left(\frac{\gamma}{2}\right), \hat{\psi}_{boot-p}\left(1-\frac{\gamma}{2}\right)\right]. \tag{3.11}$$

3.3. Maximum product spacing estimation technique

A reliable alternative to the maximum likelihood approach is the maximum product spacing (MPS) method, which provides an approximation to the Kullback-Leibler information measure.

Examine a PT-IIC sample of size m, denoted as $x = x_{(i)}$, where i ranges from 1 to m. This sample is gathered using the progressive censoring scheme S_i from the GPUHLG population, described by the probability density function (pdf) and cumulative distribution function (cdf) outlined in Eqs (2) and (3) respectively. The probability spacing (PS) function, excluding the constant component, can be formulated within this framework by utilizing Eqs (2) and (3) as demonstrated below:

$$\begin{aligned} Gs(\theta, \alpha|data) &= \prod_{i=1}^{m+1}\left(F(x_{(i)}) - F(x_{(i-1)})\right)\prod_{i=1}^{m}(1-F(x_{(i)}))^{R_i} \\ &= \prod_{i=1}^{m+1}\left(\frac{\theta(1-x_{(i-1)}^{\alpha})}{(2-\theta)x_{(i-1)}^{\alpha}+\theta} - \frac{\theta(1-x_{(i)}^{\alpha})}{(2-\theta)x_{(i)}^{\alpha}+\theta}\right)\prod_{i=1}^{m}\left(\frac{\theta(1-x_{(i)}^{\alpha})}{(2-\theta)x_{(i)}^{\alpha}+\theta}\right)^{R_i}, \end{aligned} \tag{3.12}$$

and $g(\theta, \alpha|data) = \log(Gs(\theta, \alpha|data))$ can be obtained as

$$\begin{aligned} g(\theta, \alpha|data) = &\sum_{i=1}^{m+1} \log\left(\theta(1 - x_{(i-1)}^{\alpha})((2-\theta)x_{(i)}^{\alpha}) - \theta(1 - x_{(i)}^{\alpha})((2-\theta)x_{(i-1)}^{\alpha}))\right) \\ &- \sum_{i=1}^{m+1} \log\left((2-\theta)x_{(i-1)}^{\alpha} + \theta\right) - \sum_{i=1}^{m+1} \log\left((2-\theta)x_{(i)}^{\alpha} + \theta\right) \\ &+ \sum_{i=1}^{m+1} R_i \log(\theta(1 - x_{(i)}^{\alpha})) - \sum_{i=1}^{m+1} R_i \log((2-\theta)x_{(i)}^{\alpha} + \theta). \end{aligned} \tag{3.13}$$

Upon deriving the first derivative of the function $g(\theta, \alpha|\text{data})$ with respect to θ and α, we obtain:

$$\begin{aligned} \frac{\partial g(\theta, \alpha|data)}{\partial \theta} = &\sum_{i=1}^{m} \frac{R_i}{\theta} - \sum_{i=1}^{m+1} \frac{1 - x_{(i-1)}^{\alpha}}{(2-\theta)x_{(i-1)}^{\alpha} + \theta} - \sum_{i=1}^{m+1} \frac{1 - x_{(i)}^{\alpha}}{(2-\theta)x_{(i)}^{\alpha} + \theta} - \sum_{i=1}^{m} \frac{R_i(1 - x_{(i)}^{\alpha})}{(2-\theta)x_{(i)}^{\alpha} + \theta} \\ &+ \sum_{i=1}^{m+1} \frac{(1 - x_{(i-1)}^{\alpha})((2-\theta)x_{(i)}^{\alpha} + \theta) - (1 - x_{(i-1)}^{\alpha})((2-\theta)x_{(i-1)}^{\alpha} + \theta)}{\theta(1 - x_{(i-1)}^{\alpha})((2-\theta)x_{(i)}^{\alpha} + \theta) - \theta(1 - x_{(i)}^{\alpha})((2-\theta)x_{(i-1)}^{\alpha} + \theta)}, \end{aligned} \tag{3.14}$$

$$\begin{aligned} \frac{\partial g(\theta, \alpha|data)}{\partial \alpha} = &\sum_{i=1}^{m+1} \frac{\theta(2-\theta)x_{(i)}^{\alpha}(1 - x_{(i-1)}^{\alpha}) \log(x_{(i)}) + \theta x_{(i)}^{\alpha} \log(x_{(i)})((2-\theta)x_{(i-1)}^{\alpha} + \theta)}{\theta(1 - x_{(i-1)}^{\alpha})((2-\theta)x_{(i)}^{\alpha} + \theta) - \theta(1 - x_{(i)}^{\alpha})((2-\theta)x_{(i-1)}^{\alpha} + \theta)} \\ &- \sum_{i=1}^{m+1} \frac{\theta(2-\theta)x_{(i-1)}^{\alpha}(1 - x_{(i)}^{\alpha}) \log(x_{(i-1)}) + \theta x_{(i-1)}^{\alpha} \log(x_{(i-1)})((2-\theta)x_{(i)}^{\alpha} + \theta)}{\theta(1 - x_{(i-1)}^{\alpha})((2-\theta)x_{(i)}^{\alpha} + \theta) - \theta(1 - x_{(i)}^{\alpha})((2-\theta)x_{(i-1)}^{\alpha} + \theta)} \\ &- \sum_{i=1}^{m+1} \frac{(2-\theta)x_{(i-1)}^{\alpha} \log(x_{(i-1)})}{(2-\theta)x_{(i-1)}^{\alpha} + \theta} - \sum_{i=1}^{m} \frac{R_i x_{(i)}^{\alpha} \log(x_{(i)})}{1 - x_{(i)}^{\alpha}} - \sum_{i=1}^{m+1} \frac{(2-\theta)x_{(i)}^{\alpha} \log(x_{(i)})}{(2-\theta)x_{(i)}^{\alpha} + \theta} \\ &- \sum_{i=1}^{m} \frac{(2-\theta)R_i x_{(i)}^{\alpha} \log(x_{(i)})}{(2-\theta)x_{(i)}^{\alpha} + \theta}. \end{aligned} \tag{3.15}$$

Equations (3.14) and (3.15) lack closed-form analytical solutions when equated to zero. Consequently, numerical methods are employed to obtain solutions.

4. Bayesian estimation

Within this section, the Bayesian estimation (BE) technique is employed for the estimation of the parameters θ and α. These parameters are presumed to be independent and adhere to a gamma prior distribution characterized by parameters a and b.

The gamma prior density function takes the following shape:

$$\pi(u) = \frac{b^a}{\Gamma(a)} u^{a-1} e^{-ub}, \quad u, a, b > 0. \tag{4.1}$$

Subsequently, the joint prior density of θ and α can be expressed as follows:

$$\pi(\theta, \alpha) = \prod_{i=1}^{n} \pi(\theta)\pi(\alpha) \propto (\theta\alpha)^{a-1} e^{-(\theta+\alpha)b}. \tag{4.2}$$

The joint posterior distribution function according to the Bayesian procedure is given by

$$\pi(\theta, \alpha|\underline{x}) = \frac{\pi(\theta, \alpha)L(\underline{x})}{\int_0^\infty \int_0^\infty \pi(\theta, \alpha)L(\underline{x})d\theta d\alpha} \propto \pi(\theta, \alpha)L(\underline{x}). \tag{4.3}$$

Substituting from Eqs (3.1) and (4.2) into Eq (4.3), we get

$$\pi(\theta, \alpha|\underline{x}) \propto \theta^{a+2m-1}\alpha^{a+m-1}e^{-(\theta+\alpha)b)} \prod_{i=1}^{m} \frac{x_{(i)}^{\alpha-1}}{\left((2-\theta)x_{(i)}^{\alpha}+\theta\right)^2} \left(\frac{1-x_{(i)}^{\alpha}}{(2-\theta)x_{(i)}^{\alpha}+\theta}\right)^{R_i}. \tag{4.4}$$

4.1. Loss functions

The Bayesian estimator for a given function, denoted as $l(\phi)$, with respect to the squared error (SE) loss function, is defined as:

$$\hat{\phi}_{SE} = E\left[l(\phi)|\mathbf{x}\right] = \int_{\phi} l(\phi)\pi(\phi|x)d\phi. \tag{4.5}$$

The squared error (SE) loss function is a type of asymmetric loss function that assigns equal importance to both underestimation and overestimation. However, in various real-world scenarios, the gravity of underestimation might differ from that of overestimation, and the opposite could also be true. When dealing with such circumstances, a possible substitute for the SE loss function is the LINEX loss, characterized by:

$$\left(l(\phi), \hat{l}(\phi)\right) = \mathrm{e}^{\{\hat{l}(\phi)-l(\phi)\}} - v\left(\hat{l}(\phi) - l(\phi)\right) - 1.$$

In this context, when $v > 0$, it signifies a greater significance of overestimation compared to underestimation, whereas for $v < 0$, the opposite holds true. As v approaches zero, the loss function aligns with the standard squared error (SE) form. For a deeper understanding of this concept, additional information can be found in [21,22]. The Bayesian estimator (BE) for $l(\phi)$ under this loss function can be determined as follows:

$$\hat{\phi}_{LN} = E\left[\mathrm{e}^{\{-vl(\phi)\}}|\mathbf{x}\right] = -\frac{1}{v}\log\left[\int_{\phi} \mathrm{e}^{\{-vl(\phi)\}}\pi(\phi|x)d\phi\right]. \tag{4.6}$$

Observing Eqs (4.5) and (4.6), it becomes apparent that the resulting estimates cannot be transformed into concise analytical forms. To manage this, the Markov chain Monte Carlo(MCMC)method, as outlined in [23], is employed to numerically summarize the posterior distribution. This approach avoids the need for calculating the normalization constant and is executed using the R programming language, as described in [7]. Hence, our next step involves implementing the MCMC methodology and generating posterior samples via the Metropolis-Hastings algorithm. This enables us to acquire the desired Bayesian estimators (BEs).

4.2. Markov chain method

Markov chain Monte Carlo (MCMC)methods constitute a versatile simulation approach for obtaining samples from posterior distributions and calculating relevant posterior values. In fact, the MCMC samples can effectively encapsulate the full range of uncertainty regarding the parameter ϕ. By utilizing a kernel estimation technique on the posterior distribution, a comprehensive understanding can be obtained. For a more comprehensive exploration of MCMC principles, refer to sources such as [5,23–27].

Numerous methods exist for introducing random noise to create proposals, and a variety of approaches are available for the acceptance and rejection process. Techniques like Gibbs sampling and the Metropolis-Hastings algorithm are among the options for this purpose.

4.3. Metropolis-Hasting algorithm

To implement the Metropolis-Hastings (MH) algorithm for the GPUHLG distribution, certain elements must be established: a proposal distribution and initial values for the unknown parameters θ and α. For the proposal distribution, we opt for a bivariate normal distribution, denoted as $q((\theta', \alpha')|(\theta, \alpha)) \equiv N_2((\theta, \alpha), S_{\theta,\alpha})$, wherein $S_{\theta,\alpha}$ signifies the variance-covariance matrix. It is important to note that we must avoid generating negative observations, which are considered unacceptable. Regarding initial values, we employ the Maximum Likelihood Estimators (MLE) for θ and α, yielding $(\theta^{(0)}, \alpha^{(0)}) = (\hat{\theta}, \hat{\alpha})$. The selection of $S_{\theta,\alpha}$ is based on the asymptotic variance-covariance matrix $F^{-1}(\hat{\theta}, \hat{\alpha})$, with $F(.)$ representing the Fisher information matrix. It is worth noting that the choice of $S_{\theta,\alpha}$ holds significance in the MH algorithm, impacting the acceptance rate.

In this context, the sequential stages of the MH algorithm for drawing a sample from the posterior density, as indicated in Eq (22), unfold in the following manner:

Step 1. Initialize the value of η as $\eta^{(0)} = (\hat{\theta}, \hat{\alpha})$.

Step 2. For $i = 1, 2, \ldots, M$, iterate through the following process:

(1) Set $\eta = \eta^{(i-1)}$.
(2) Generate a fresh candidate parameter value δ from the bivariate normal distribution $N_2(\log \eta, S_{\theta,\alpha})$.
(3) Set $\eta' = \exp(\delta)$.
(4) Compute β using the formula $\beta = \frac{\pi(\eta'|x)}{\pi(\eta|x)}$, where $\pi(\cdot)$ represents the posterior density as defined in Eq (22).
(5) Generate a sample u from the uniform $U(0, 1)$; distribution.
(6) Accept or reject the new candidate η'

$$\begin{cases} \text{If} \quad u \leq \beta \quad \text{set} \quad \eta^{(i)} = \eta' \\ \text{otherwise} \quad \text{set} \quad \eta^{(i)} = \eta. \end{cases}$$

Ultimately, after obtaining a set of random samples of size M from the posterior density, it is common practice to discard a portion of the initial samples (burn-in), retaining the remaining samples for further analysis. Specifically, the Bayesian estimators (BEs) of the parameters θ and α using the squared error (SE) loss function, as outlined in Eq (4.5), can be computed as

$$\begin{aligned} \hat{\theta}_{SE} &= \frac{1}{M - l_B} \sum_{l=l_B}^{M} \theta^{(l)}, \\ \hat{\alpha}_{SE} &= \frac{1}{M - l_B} \sum_{l=l_B}^{M} \alpha^{(l)}. \end{aligned} \tag{4.7}$$

Moreover, the Bayesian estimators (BEs) for the parameters θ and α, employing the LINEX loss function, as provided in Eq (4.6), can be expressed as follows:

$$\begin{aligned}\hat{\theta}_{LN} &= -\frac{1}{v}\log\left[\frac{1}{M-l_B}\sum_{i=l_B}^{M} e^{\left\{-v\theta^{(i)}\right\}}\right]\\ \hat{\alpha}_{LN} &= -\frac{1}{v}\log\left[\frac{1}{M-l_B}\sum_{i=l_B}^{M} e^{\left\{-v\alpha^{(i)}\right\}}\right].\end{aligned} \tag{4.8}$$

Here, l_B denotes the count of burn-in samples.

4.4. Elicitation of hyper-parameters

The elicitation of the hyper-parameters will depend on informative priors. These informative priors are derived from the MLEs for (θ, α) by equating the mean and variance of $(\hat{\theta}^j, \hat{\alpha}^j)$ with those of the specified priors (Gamma priors). Here, $j = 1, 2, \ldots, k$, and k corresponds to the number of available samples from the GPUHLG distribution (Dey et al. [28]). By equating the moments of $(\hat{\theta}^j, \hat{\alpha}^j)$ with the moments of the gamma priors, the following equations are derived:

$$\frac{1}{k}\sum_{j=1}^{k}\hat{\theta}^j = \frac{a_1}{b_1} \quad , \quad \frac{1}{k-1}\sum_{j=1}^{k}\left(\hat{\theta}^j - \frac{1}{k}\sum_{j=1}^{k}\hat{\theta}^j\right)^2 = \frac{a_1}{b_1^2},$$

$$\frac{1}{k}\sum_{j=1}^{k}\hat{\alpha}^j = \frac{a_2}{b_2} \quad \text{and} \quad \frac{1}{k-1}\sum_{j=1}^{k}\left(\hat{\alpha}^j - \frac{1}{k}\sum_{j=1}^{k}\hat{\alpha}^j\right)^2 = \frac{a_2}{b_2^2}.$$

By solving the aforementioned equations, the estimated hyper-parameters can be expressed as follows:

$$\begin{aligned} a_1 &= \frac{\left(\frac{1}{k}\sum_{j=1}^{k}\hat{\theta}^j\right)^2}{\frac{1}{k-1}\sum_{j=1}^{k}\left(\hat{\theta}^j - \frac{1}{k}\sum_{j=1}^{k}\hat{\theta}^j\right)^2}, & b_1 &= \frac{\frac{1}{k}\sum_{j=1}^{k}\hat{\theta}^j}{\frac{1}{k-1}\sum_{j=1}^{k}\left(\hat{\theta}^j - \frac{1}{k}\sum_{j=1}^{k}\hat{\theta}^j\right)^2}\\ a_2 &= \frac{\left(\frac{1}{k}\sum_{j=1}^{k}\hat{\alpha}^j\right)^2}{\frac{1}{k-1}\sum_{j=1}^{k}\left(\hat{\alpha}^j - \frac{1}{k}\sum_{j=1}^{k}\hat{\alpha}^j\right)^2}, & b_2 &= \frac{\frac{1}{k}\sum_{j=1}^{k}\hat{\alpha}^j}{\frac{1}{k-1}\sum_{j=1}^{k}\left(\hat{\alpha}^j - \frac{1}{k}\sum_{j=1}^{k}\hat{\alpha}^j\right)^2}.\end{aligned} \tag{4.9}$$

4.5. Highest posterior density

We construct the highest posterior density (HPD) intervals for the unobservable parameters α and θ of the GPUHLG distribution within the context of the PT-IIC. These intervals are established using the samples acquired through the aforementioned MH approach from the previous section [29]. In the subsequent case study, let $\alpha^{(\delta)}$ and $\theta^{(\delta)}$ represent the δ-th quantiles of α and θ, respectively. In other words,

$$(\alpha^{(\delta)}, \theta^{(\delta)}) = \inf\{(\alpha, \theta) : \Pi((\alpha, \theta)|\mathbf{z}) \geq \delta\}.$$

Here, $0 < \delta < 1$, and $\Pi(\cdot)$ represents the posterior distribution of α and θ. Importantly, it is worth noting that for a specific set of α and θ, an effective estimator derived from simulating $\pi((\alpha, \theta)|\mathbf{z})$ can be computed as:

$$\Pi((\alpha,\theta)|\mathbf{z}) = \frac{1}{M-l_B}\sum_{i=l_B}^{M} I_{(\alpha,\theta)\leq(\alpha^{\cdot}\theta^{\cdot})}.$$

Here, $I_{(\alpha,\theta)\leq(\alpha^{\cdot}\theta^{\cdot})}$ is the indicator function. The proper estimate is then determined as

$$\hat{\Pi}((\alpha,\theta)|\mathbf{z}) = \begin{cases} 0 & \text{if } (\alpha,\theta) < (\alpha_{(l_B)},\theta_{(l_B)}) \\ \sum_{j=l_B}^{i}\omega_j & \text{if } (\alpha_{(i)},\theta_{(i)}) < (\alpha,\theta) < (\alpha_{(i+1)},\theta_{(i+1)}) \\ 1 & \text{if } (\alpha,\theta) > (\alpha_{(M)},\theta_{(M)}) \end{cases}$$

where $\omega_j = \frac{1}{M-l_B}$ and $(\alpha_{(j)},\theta_{(j)})$ are the ordered values of (α_j,θ_j). Now, for $i = l_B,\ldots,M$, $(\alpha^{(\delta)},\theta^{(\delta)})$ may be estimated by

$$(\widetilde{\alpha}^{(\delta)},\widetilde{\theta}^{(\delta)}) = \begin{cases} (\alpha_{(l_B)},\theta_{(l-B)}) & \text{if} \quad \delta = 0 \\ (\alpha_{(i)},\theta_{(i)}) & \text{if } \sum_{j=l_B}^{i-1}\omega_j < \delta < \sum_{j=l_B}^{i}\omega_j. \end{cases}$$

Furthermore, let us determine a $100(1-\delta)\%$ HPD credible interval for α and θ:

$$HPD_j^{\alpha} = (\widetilde{\alpha}^{(\frac{j}{M})},\widetilde{\alpha}^{(\frac{j+(1-\delta)M}{M})}) \qquad \& \qquad HPD_j^{\lambda} = (\widetilde{\theta}^{(\frac{j}{M})},\widetilde{\theta}^{(\frac{j+(1-\delta)M}{M})})$$

for $j = l_B,\ldots,[\delta M]$, where $[a]$ represents indicates the largest integer $\leq a$. We need to choose HPD_{j^*} from one of many $HPD_j^{'}$s with the narrowest width.

5. Numerical computations and real data

The aim of this section is to assess and compare the efficiencies of the different estimation approaches discussed in the previous sections. To achieve this, a simulation study is conducted to observe the performances of the proposed methods and to gauge the statistical prowesses of the estimators within the framework of a PT-IIC scheme. Furthermore, a flood dataset is analyzed to offer a practical illustration. All calculations were executed using the *R* programming language.

5.1. Simulation study

In this subsection, a Monte Carlo simulation study is carried out to evaluate the performance of distinct estimation methods – namely, MLE, MPS, and BE – within the framework of the PT-IIC scheme applied to the GPUHLG distribution. We generate 1000 sets of random data from the GPUHLG distribution under the PT-IIC, employing parameters $\theta = 0.5$ and $\alpha = 1.5$. The configuration of the PT-IIC scheme is established through predetermined values of n and m, alongside various patterns for censoring items R_i, where $i = 1, 2, \ldots, m$, as detailed in Table 1. These patterns can be classified into four distinct cases.

- In the first pattern, the removal of items $(n-m)$ takes place during life testing, coinciding with the occurrence of the first failure item. This scenario is represented by patterns such as $\mathscr{R}_1, \mathscr{R}_5, \ldots, \mathscr{R}_{29}$.

- Conversely, the second pattern involves the removal of items occurring with the last m failure items, and this is exemplified by patterns like $\mathscr{R}_2, \mathscr{R}_6, \ldots, \mathscr{R}_{30}$.
- Moving to the third pattern, the removal of items happens at the median of the m items, as demonstrated by patterns $\mathscr{R}_3, \mathscr{R}_7, \ldots, \mathscr{R}_{31}$.
- Lastly, the final pattern arises when equal items are removed, whenever possible, at each m stage. This pattern is characterized by representations such as $\mathscr{R}_4, \mathscr{R}_8, \ldots, \mathscr{R}_{32}$.

Steps of the Monte Carlo simulation:

Step 1: Generate m sets of PT-IIC random data points from the GPUHLG(θ, α) distribution using the algorithm proposed by [3]. Use the removal pattern of items from Table 1.

Table 1. Patterns of item removal for varying values of n and m.

n	m	Censoring Scheme $(R_1, R_2, \ldots, R_m)$	Scheme
20	10	$(10, 0^{*9})$	$\mathscr{R}_1$
		$(0^{*9}, 10)$	$\mathscr{R}_2$
		$(0^{*4}, 5, 5, 0^{*4})$	$\mathscr{R}_3$
		(1^{*10})	$\mathscr{R}_4$
	15	$(5, 0^{*14})$	$\mathscr{R}_5$
		$(0^{*14}, 5)$	$\mathscr{R}_6$
		$(0^{*7}, 2, 3, 0^{*7})$	$\mathscr{R}_7$
		$(1^{*5}, 0^{*10})$	$\mathscr{R}_8$
30	20	$(10, 0^{*19})$	$\mathscr{R}_9$
		$(0^{*19}, 10)$	$\mathscr{R}_{10}$
		$(0^{*9}, 5, 5, 0^{*9})$	$\mathscr{R}_{11}$
		$(1^{*10}, 0^{*10})$	$\mathscr{R}_{12}$
	25	$(5, 0^{*24})$	$\mathscr{R}_{13}$
		$(0^{*24}, 5)$	$\mathscr{R}_{14}$
		$(0^{*12}, 5, 0^{*12})$	$\mathscr{R}_{15}$
		$(1^{*5}, 0^{*20})$	$\mathscr{R}_{16}$
40	20	$(20, 0^{*19})$	$\mathscr{R}_{17}$
		$(0^{*19}, 20)$	$\mathscr{R}_{18}$
		$(0^{*9}, 10, 10, 0^{*9})$	$\mathscr{R}_{19}$
		(1^{*20})	$\mathscr{R}_{20}$
	30	$(10, 0^{*29})$	$\mathscr{R}_{21}$
		$(0^{*29}, 10)$	$\mathscr{R}_{22}$
		$(0^{*14}, 5, 5, 0^{*14})$	$\mathscr{R}_{23}$
		$(1^{*10}, 0^{*20})$	$\mathscr{R}_{24}$
60	40	$(20, 0^{*39})$	$\mathscr{R}_{25}$
		$(0^{*39}, 20)$	$\mathscr{R}_{26}$
		$(0^{*19}, 10, 10, 0^{*19})$	$\mathscr{R}_{27}$
		$(1^{*20}, 0^{*20})$	$\mathscr{R}_{28}$
	50	$(10, 0^{*49})$	$\mathscr{R}_{29}$
		$(0^{*49}, 10)$	$\mathscr{R}_{30}$
		$(0^{*24}, 5, 5, 0^{*24})$	$\mathscr{R}_{31}$
		$(1^{*10}, 0^{*30})$	$\mathscr{R}_{32}$

As an example, the scheme $(0^{(4)}, 10)$ signifies the utilization of the censoring scheme $(0, 0, 0, 0, 10)$.

Step 2: Obtain MLE and MPS estimates for the parameters θ and α. Additionally, calculate the variance-covariance matrix of MLEs.

Step 3: Compute confidence interval estimates: Asy-CI, Boot-p, and Boot-t .

Step 4: Compute BEs using the MH algorithm as follows:

(1) Consider two scenarios for prior distributions. The first scenario involves an informative prior (INF), wherein hyper-parameter values are computed using Eq (4.9). Specifically, we generate 1000 complete samples, each consisting of 60 data points, from a GPUHLG($\theta = 0.5, \alpha = 1.5$) distribution as past samples and compute their MLEs $(\hat{\theta}, \hat{\alpha})$. Subsequently, by utilizing Eq (4.9), we can determine the hyper-parameter values as follows: $a_1 = 6.52$, $b_1 = 18.22$, $a_2 = 49.67$, and $b_2 = 27.50$.
(2) The second scenario involves a non-informative prior (Non-INF), where hyper-parameter values are set to $a_1 = b_1 = a_2 = b_2 = 0$. This leads to the prior distributions $\pi(\theta) = \frac{1}{\theta}$ and $\pi(\alpha) = \frac{1}{\alpha}$.
(3) Generate 10,000 samples of α and θ for both INF and Non-INF prior cases from the posterior density using MCMC and utilizing the MH algorithm. Use the initial MLEs and their variance-covariance matrix, along with the given PT-IIC data $x = (x_{(1)}, x_{(2)}, \ldots, x_{(m)})$.
(4) The initial 2,000 samples are discarded as burn-in from the overall set of 10,000 samples generated from the posterior density.
(5) Compute BEs of α and θ using various loss functions: SE and LINEX (with $v = -0.5(LN_1)$ and $v = 0.5(LN_2)$), as defined by Eqs (4.7) and (4.8).
(6) Finally, calculate the HPD interval using the posterior samples.

Step 5: Repeat Steps 1–4 a total of 1,000 times and save all the estimates.

Step 6: Calculate statistical metrics for point estimates: mean (Avg.) estimate and root mean square error (RMSE) estimate. These calculations can be carried out using the following formulas:

$$Avg.(\phi) = \frac{1}{1000} \sum_{l=1}^{1000} \hat{\phi} l,$$

$$RMSE(\phi) = \sqrt{\frac{1}{1000} \sum_{l=1}^{1000} \left(\hat{\phi}_l - \phi\right)^2}.$$

In this context, ϕ represents the parameter, while $\hat{\phi}$ denotes the estimated value of that parameter.

Step 7: Compute statistical performance measures for interval estimates: average interval length (AIL) and coverage probability (CP) in percentage.

To provide point estimations, we present the results of Avg. and RMSE estimates for various PT-IIC schemes in Tables 2 and 3, corresponding to $\theta = 0.5$ and $\alpha = 1.5$, respectively. In terms of interval estimation, Tables 4.a and 4.b display the outcomes for AILs and CPs for $\theta = 0.5$ and $\alpha = 1.5$, respectively.

From the results obtained for point estimation of distribution parameters, it is generally observed that an increase in both n and m leads to an improvement in Avg. estimates and its convergence towards the true parameter values. Additionally, we notice a decrease in the RMSEs as well. Regarding interval estimation, as n and m increase, we observe a reduction in the AILs for all interval estimation methods. Additionally, it is worth mentioning that the CP ranges from 90% to 99%. The confidence intervals can be ranked in terms of the efficiency of AILs as follows:

$$\text{HPD: INF} \geq \text{Boot-}t \geq \text{Asy-CI} \geq \text{Boot-}p \geq \text{HPD: Non-INF}.$$

In terms of the efficiency of proposed estimation methods, by comparing classical point estimation methods, we observe that the efficiency of the MLE method for the parameter θ is superior to that of the MPS estimations; and for the parameter α, we observe the opposite. Concerning the BEs methods using assumed loss functions, it is evident that the BEs using LN_1 loss function at $v = 0.5$ exhibits the highest efficiency, followed by the estimation using the SE loss function, and then the LN_2 loss function at $v = -0.5$. Moreover, when comparing the BEs using MCMC under INF and Non-INF approaches, there is a very clear indication that the INF prior case significantly outperforms the Non-INF prior one. In a broader sense, it can be concluded that the BEs using MCMC under INF case efficiency are superior among the assumed methods of classical and Bayes estimation.

Furthermore, it is worth noting that these conclusions pertain to a specific set of distribution parameters ($\theta = 0.5, \alpha = 1.5$). We recommend conducting further research on alternative parameter combinations and comparing the results obtained with those from our study.

Table 2. Average estimate values and MSE under different PT-IIC schemes at $\theta = 0.5$.

n	m	Scheme		Classical		BE: Non-INF			BE: INF		
				MLE	MPS	SE	LN1	LN2	SE	LN1	LN2
20	10	$\mathscr{R}_1$	Avg.	0.4765	0.2380	0.9846	0.9935	0.4801	0.4454	0.4502	0.4407
			RMSE	0.4441	0.3613	2.0388	2.4955	0.5909	0.1304	0.1298	0.1311
		$\mathscr{R}_2$	Avg.	0.4091	0.3442	0.7972	0.8928	0.4799	0.4641	0.4686	0.4597
			RMSE	0.3798	0.3491	1.6437	2.2714	0.6062	0.1487	0.1493	0.1483
		$\mathscr{R}_3$	Avg.	0.4332	0.2202	0.9129	0.7361	0.4940	0.4576	0.4622	0.4531
			RMSE	0.4277	0.3734	1.8939	2.0204	0.6852	0.1508	0.1512	0.1505
		$\mathscr{R}_4$	Avg.	0.4361	0.2869	0.8717	1.0763	0.4961	0.4662	0.4709	0.4617
			RMSE	0.4019	0.3500	1.7581	2.5829	0.7831	0.1327	0.1331	0.1325
	15	$\mathscr{R}_5$	Avg.	0.4591	0.2873	0.7439	1.1386	0.5258	0.4764	0.4813	0.4717
			RMSE	0.3731	0.3332	1.3680	2.5534	0.6915	0.1004	0.1006	0.1005
		$\mathscr{R}_6$	Avg.	0.4389	0.3453	0.6972	1.0921	0.4856	0.4886	0.4933	0.4840
			RMSE	0.3344	0.3099	1.1001	2.4799	0.4603	0.1070	0.1078	0.1063
		$\mathscr{R}_7$	Avg.	0.4426	0.2826	0.7559	1.2912	0.4837	0.4850	0.4897	0.4803
			RMSE	0.3620	0.3343	1.3448	2.8913	0.4890	0.1117	0.1124	0.1111
		$\mathscr{R}_8$	Avg.	0.4538	0.2880	0.8059	1.1692	0.4735	0.4824	0.4873	0.4778
			RMSE	0.3740	0.3347	1.5222	2.6285	0.4553	0.1016	0.1021	0.1014
30	20	$\mathscr{R}_9$	Avg.	0.4485	0.3078	0.5471	0.9019	0.4331	0.5034	0.5082	0.4986
			RMSE	0.2936	0.2849	0.6704	1.7669	0.2984	0.0824	0.0838	0.0814
		$\mathscr{R}_{10}$	Avg.	0.4323	0.3668	0.5274	0.8331	0.4448	0.5238	0.5285	0.5193
			RMSE	0.2648	0.2602	0.6618	1.7338	0.3150	0.0926	0.0948	0.0906
		$\mathscr{R}_{11}$	Avg.	0.4362	0.3034	0.5479	0.9766	0.4466	0.5200	0.5248	0.5154
			RMSE	0.2899	0.2892	0.6457	2.0462	0.4408	0.0969	0.0991	0.0949
		$\mathscr{R}_{12}$	Avg.	0.4412	0.3081	0.5295	1.0421	0.4448	0.5151	0.5200	0.5104
			RMSE	0.2929	0.2870	0.6151	2.1183	0.4526	0.0904	0.0923	0.0887
	25	$\mathscr{R}_{13}$	Avg.	0.4434	0.3232	0.5299	0.8705	0.4386	0.5121	0.5167	0.5077

Continued on next page

n	m	Scheme		Classical		BE: Non-INF			BE: INF		
				MLE	MPS	SE	LN1	LN2	SE	LN1	LN2
			RMSE	0.2865	0.2784	0.6843	1.6652	0.2974	0.0737	0.0755	0.0722
		$\mathscr{R}_{14}$	Avg.	0.4369	0.3555	0.4885	0.7234	0.4391	0.5210	0.5255	0.5167
			RMSE	0.2657	0.2574	0.3718	1.3427	0.2772	0.0770	0.0792	0.0751
		$\mathscr{R}_{15}$	Avg.	0.4385	0.3222	0.5079	0.9652	0.4424	0.5209	0.5255	0.5165
			RMSE	0.2847	0.2789	0.4529	2.0063	0.3065	0.0795	0.0817	0.0775
		$\mathscr{R}_{16}$	Avg.	0.4422	0.3250	0.5143	0.8986	0.4416	0.5145	0.5190	0.5100
			RMSE	0.2873	0.2784	0.5201	1.7517	0.3040	0.0754	0.0772	0.0738
40	20	$\mathscr{R}_{17}$	Avg.	0.4543	0.3089	0.4947	0.7791	0.4277	0.4994	0.5041	0.4948
			RMSE	0.2894	0.2816	0.4169	1.4385	0.2869	0.0782	0.0793	0.0775
		$\mathscr{R}_{18}$	Avg.	0.4285	0.3771	0.4773	0.7596	0.4307	0.5296	0.5340	0.5252
			RMSE	0.2650	0.2618	0.3458	1.4479	0.2763	0.1024	0.1047	0.1002
		$\mathscr{R}_{19}$	Avg.	0.4374	0.3010	0.4890	0.8803	0.4314	0.5250	0.5297	0.5205
			RMSE	0.2922	0.2920	0.4109	1.7176	0.3209	0.1014	0.1037	0.0994
		$\mathscr{R}_{20}$	Avg.	0.4375	0.3392	0.4915	0.6960	0.4331	0.5245	0.5291	0.5200
			RMSE	0.2719	0.2687	0.5183	1.1427	0.3153	0.0941	0.0963	0.0921
	30	$\mathscr{R}_{21}$	Avg.	0.4398	0.3361	0.4652	0.5819	0.4318	0.5220	0.5264	0.5177
			RMSE	0.2580	0.2581	0.3050	0.6697	0.2642	0.0755	0.0778	0.0735
		$\mathscr{R}_{22}$	Avg.	0.4334	0.3748	0.4622	0.5112	0.4345	0.5368	0.5411	0.5326
			RMSE	0.2388	0.2363	0.2776	0.3785	0.2462	0.0816	0.0844	0.0791
		$\mathscr{R}_{23}$	Avg.	0.4348	0.3350	0.4699	0.6156	0.4344	0.5354	0.5398	0.5310
			RMSE	0.2577	0.2597	0.3094	1.0343	0.2655	0.0852	0.0879	0.0826
		$\mathscr{R}_{24}$	Avg.	0.4376	0.3377	0.4631	0.5999	0.4298	0.5294	0.5339	0.5251
			RMSE	0.2596	0.2588	0.3035	0.8299	0.2660	0.0793	0.0819	0.0769
60	40	$\mathscr{R}_{25}$	Avg.	0.4405	0.3549	0.4486	0.4854	0.4293	0.5313	0.5355	0.5272
			RMSE	0.2210	0.2267	0.2443	0.3704	0.2267	0.0717	0.0743	0.0693
		$\mathscr{R}_{26}$	Avg.	0.4368	0.3943	0.4521	0.4731	0.4369	0.5492	0.5532	0.5452
			RMSE	0.2004	0.2023	0.2185	0.2436	0.2081	0.0818	0.0849	0.0790
		$\mathscr{R}_{27}$	Avg.	0.4372	0.3552	0.4514	0.4806	0.4321	0.5487	0.5529	0.5446
			RMSE	0.2200	0.2282	0.2433	0.2829	0.2281	0.0827	0.0859	0.0797
		$\mathscr{R}_{28}$	Avg.	0.4383	0.3572	0.4509	0.4824	0.4314	0.5426	0.5469	0.5384
			RMSE	0.2205	0.2268	0.2436	0.3004	0.2270	0.0782	0.0813	0.0754
	50	$\mathscr{R}_{29}$	Avg.	0.4323	0.3614	0.4420	0.4608	0.4274	0.5402	0.5442	0.5363
			RMSE	0.1918	0.2080	0.2080	0.2308	0.1989	0.0768	0.0796	0.0742
		$\mathscr{R}_{30}$	Avg.	0.4316	0.3832	0.4419	0.4564	0.4298	0.5460	0.5498	0.5422
			RMSE	0.1816	0.1913	0.1930	0.2031	0.1871	0.0800	0.0828	0.0773
		$\mathscr{R}_{31}$	Avg.	0.4312	0.3624	0.4432	0.4611	0.4290	0.5475	0.5516	0.5436
			RMSE	0.1908	0.2071	0.2063	0.2214	0.1984	0.0814	0.0844	0.0786
		$\mathscr{R}_{32}$	Avg.	0.4321	0.3635	0.4414	0.4617	0.4269	0.5419	0.5460	0.5379
			RMSE	0.1920	0.2073	0.2087	0.2512	0.1997	0.0776	0.0805	0.0749

Table 3. Average estimate values and MSE under different PT-IIC schemes at $\alpha = 1.5$.

n	m	Scheme		Classical		BE: Non-INF			BE: INF		
				MLE	MPS	SE	LN1	LN2	SE	LN1	LN2
20	10	$\mathscr{R}_1$	Avg.	1.7721	2.3772	1.9696	2.0637	1.8813	1.1786	1.1833	1.1741
			RMSE	0.6475	1.1949	0.8568	0.9146	0.8111	0.3598	0.3559	0.3636
		$\mathscr{R}_2$	Avg.	1.8662	2.0620	1.9848	2.0582	1.9140	1.2272	1.2320	1.2225
			RMSE	0.7424	0.8617	0.8921	0.9282	0.8637	0.4044	0.4015	0.4073
		$\mathscr{R}_3$	Avg.	1.8394	2.3798	1.9934	2.0731	1.9169	1.2184	1.2232	1.2136

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n	m	Scheme		Classical		BE: Non-INF			BE: INF		
				MLE	MPS	SE	LN1	LN2	SE	LN1	LN2
			RMSE	0.7082	1.2040	0.8852	0.9245	0.8546	0.3489	0.3456	0.3521
		$\mathscr{R}_4$	Avg.	1.8076	2.1490	1.9565	2.0315	1.8842	1.1983	1.2030	1.1936
			RMSE	0.6521	0.9279	0.8278	0.8680	0.7953	0.3423	0.3385	0.3461
	15	$\mathscr{R}_5$	Avg.	1.7452	2.1438	1.8621	1.9333	1.7937	1.1700	1.1747	1.1655
			RMSE	0.5699	0.9062	0.7314	0.7689	0.7015	0.3429	0.3386	0.3472
		$\mathscr{R}_6$	Avg.	1.7552	1.9731	1.8415	1.9030	1.7818	1.1819	1.1866	1.1773
			RMSE	0.5667	0.7302	0.6866	0.7179	0.6613	0.3337	0.3295	0.3380
		$\mathscr{R}_7$	Avg.	1.7670	2.1367	1.8801	1.9459	1.8164	1.1849	1.1896	1.1803
			RMSE	0.5879	0.8991	0.7258	0.7592	0.6988	0.3332	0.3290	0.3374
		$\mathscr{R}_8$	Avg.	1.7505	2.1170	1.8693	1.9356	1.8046	1.1769	1.1815	1.1723
			RMSE	0.5712	0.8722	0.7035	0.7389	0.6750	0.3377	0.3334	0.3420
30	20	$\mathscr{R}_9$	Avg.	1.6654	1.9584	1.7773	1.8248	1.7310	1.1903	1.1947	1.1860
			RMSE	0.4334	0.6564	0.5360	0.5649	0.5110	0.3170	0.3128	0.3211
		$\mathscr{R}_{10}$	Avg.	1.6761	1.8228	1.7498	1.7903	1.7102	1.2080	1.2123	1.2037
			RMSE	0.4335	0.5349	0.5112	0.5334	0.4921	0.3042	0.3001	0.3082
		$\mathscr{R}_{11}$	Avg.	1.6814	1.9459	1.7806	1.8243	1.7377	1.2106	1.2150	1.2063
			RMSE	0.4442	0.6433	0.5415	0.5649	0.5217	0.3022	0.2982	0.3063
		$\mathscr{R}_{12}$	Avg.	1.6722	1.9322	1.7805	1.8245	1.7373	1.2053	1.2097	1.2010
			RMSE	0.4343	0.6262	0.5381	0.5625	0.5173	0.3051	0.3010	0.3092
30	25	$\mathscr{R}_{13}$	Avg.	1.6717	1.9147	1.7567	1.7981	1.7161	1.1986	1.2029	1.1943
			RMSE	0.4221	0.5972	0.5087	0.5337	0.4871	0.3065	0.3023	0.3106
		$\mathscr{R}_{14}$	Avg.	1.6674	1.8303	1.7325	1.7695	1.6961	1.2062	1.2105	1.2019
			RMSE	0.4060	0.5111	0.4729	0.4943	0.4542	0.2997	0.2956	0.3038
		$\mathscr{R}_{15}$	Avg.	1.6760	1.9057	1.7580	1.7979	1.7188	1.2070	1.2113	1.2027
			RMSE	0.4222	0.5868	0.5152	0.5376	0.4955	0.2990	0.2949	0.3031
		$\mathscr{R}_{16}$	Avg.	1.6708	1.9006	1.7512	1.7917	1.7115	1.2036	1.2080	1.1993
			RMSE	0.4174	0.5804	0.5006	0.5241	0.4802	0.3018	0.2976	0.3059
40	20	$\mathscr{R}_{17}$	Avg.	1.6423	1.9368	1.7615	1.8045	1.7198	1.2053	1.2095	1.2011
			RMSE	0.4002	0.6245	0.5011	0.5285	0.4772	0.3026	0.2986	0.3066
		$\mathscr{R}_{18}$	Avg.	1.6707	1.7935	1.7531	1.7878	1.7191	1.2422	1.2464	1.2380
			RMSE	0.4195	0.5051	0.4935	0.5120	0.4775	0.2803	0.2766	0.2840
		$\mathscr{R}_{19}$	Avg.	1.6661	1.9161	1.7755	1.8133	1.7383	1.2417	1.2459	1.2375
			RMSE	0.4163	0.6018	0.5165	0.5354	0.5006	0.2777	0.2740	0.2815
		$\mathscr{R}_{20}$	Avg.	1.6556	1.8383	1.7492	1.7841	1.7149	1.2351	1.2393	1.2310
			RMSE	0.4008	0.5261	0.4867	0.5055	0.4703	0.2806	0.2768	0.2844
	30	$\mathscr{R}_{21}$	Avg.	1.6506	1.8522	1.7232	1.7565	1.6905	1.2214	1.2256	1.2173
			RMSE	0.3815	0.5203	0.4436	0.4639	0.4255	0.2849	0.2809	0.2888
		$\mathscr{R}_{22}$	Avg.	1.6472	1.7643	1.7021	1.7306	1.6740	1.2325	1.2367	1.2284
			RMSE	0.3663	0.4364	0.4206	0.4370	0.4059	0.2737	0.2697	0.2776
		$\mathscr{R}_{23}$	Avg.	1.6552	1.8408	1.7213	1.7521	1.6909	1.2344	1.2386	1.2303
			RMSE	0.3830	0.5095	0.4434	0.4611	0.4277	0.2731	0.2692	0.2771
		$\mathscr{R}_{24}$	Avg.	1.6502	1.8340	1.7250	1.7561	1.6944	1.2308	1.2350	1.2267
			RMSE	0.3765	0.5002	0.4431	0.4613	0.4270	0.2757	0.2717	0.2796
60	40	$\mathscr{R}_{25}$	Avg.	1.6030	1.7586	1.6638	1.6874	1.6405	1.2544	1.2584	1.2504
			RMSE	0.2937	0.3927	0.3486	0.3630	0.3357	0.2522	0.2483	0.2560
		$\mathscr{R}_{26}$	Avg.	1.5999	1.6826	1.6436	1.6634	1.6239	1.2714	1.2753	1.2675
			RMSE	0.2813	0.3275	0.3297	0.3404	0.3200	0.2355	0.2318	0.2392
		$\mathscr{R}_{27}$	Avg.	1.6060	1.7448	1.6650	1.6865	1.6437	1.2727	1.2766	1.2688
			RMSE	0.2925	0.3799	0.3512	0.3628	0.3407	0.2343	0.2306	0.2380
		$\mathscr{R}_{28}$	Avg.	1.6029	1.7395	1.6603	1.6818	1.6391	1.2704	1.2744	1.2665

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n	m	Scheme		Classical		BE: Non-INF			BE: INF		
				MLE	MPS	SE	LN1	LN2	SE	LN1	LN2
			RMSE	0.2875	0.3722	0.3404	0.3524	0.3296	0.2364	0.2327	0.2401
	50	$\mathscr{R}_{29}$	Avg.	1.6029	1.7340	1.6472	1.6676	1.6270	1.2656	1.2696	1.2617
			RMSE	0.2684	0.3534	0.3059	0.3182	0.2947	0.2406	0.2368	0.2444
		$\mathscr{R}_{30}$	Avg.	1.5979	1.6890	1.6329	1.6511	1.6148	1.2744	1.2783	1.2705
			RMSE	0.2559	0.3099	0.2854	0.2962	0.2754	0.2319	0.2282	0.2356
		$\mathscr{R}_{31}$	Avg.	1.6025	1.7255	1.6466	1.6663	1.6271	1.2742	1.2782	1.2703
			RMSE	0.2660	0.3449	0.3082	0.3200	0.2975	0.2321	0.2284	0.2358
		$\mathscr{R}_{32}$	Avg.	1.6008	1.7235	1.6465	1.6662	1.6270	1.2721	1.2760	1.2682
			RMSE	0.2633	0.3415	0.3039	0.3158	0.2931	0.2342	0.2305	0.2379

Table 4.a. AILs and CPs (in %) under different PT-IIC schemes at $\theta = 0.5$

n	m	Scheme	Asy-CI		Boot-p		Boot-t		HPD: Non-INF		HPD: INF	
			AIL	CP	AIL	CP	AIL	CP	AIL	CP	AIL	CP
20	10	$\mathscr{R}_1$	1.4189	95.9	1.5415	95.9	0.7608	88.4	4.8626	95.0	0.5204	99.1
		$\mathscr{R}_2$	1.1619	96.1	1.2247	98.2	0.6802	89.5	3.2465	95.1	0.6088	97.7
		$\mathscr{R}_3$	1.3180	96.5	1.4333	97.4	0.5603	91.3	3.9636	95.1	0.6048	99.5
		$\mathscr{R}_4$	1.2576	96.4	1.3209	95.5	0.8746	92.0	3.6755	95.0	0.5620	99.2
	15	$\mathscr{R}_5$	1.2277	95.6	1.2477	96.1	0.6493	92.6	2.6191	95.1	0.3386	99.5
		$\mathscr{R}_6$	1.1148	95.5	1.1275	96.5	0.5890	90.4	2.3954	95.0	0.3972	99.7
		$\mathscr{R}_7$	1.1773	95.2	1.1608	96.8	0.6258	89.5	2.8261	95.1	0.4248	99.7
		$\mathscr{R}_8$	1.2193	95.3	1.2128	95.9	0.5943	88.9	3.0876	95.1	0.3578	99.2
30	20	$\mathscr{R}_9$	1.0927	96.7	1.0943	95.1	0.6490	91.3	1.3973	95.1	0.2841	99.5
		$\mathscr{R}_{10}$	0.9981	96.3	0.9243	97.7	0.5944	93.0	1.1855	95.2	0.3108	98.7
		$\mathscr{R}_{11}$	1.0667	96.1	1.0269	98.2	0.5948	92.1	1.4141	95.0	0.3176	99.3
		$\mathscr{R}_{12}$	1.0805	96.1	1.0777	98.2	0.6145	89.9	1.3295	95.0	0.3026	98.3
	25	$\mathscr{R}_{13}$	1.0243	95.1	0.9469	96.3	0.6274	95.1	1.2047	95.2	0.2698	99.1
		$\mathscr{R}_{14}$	0.9742	96.0	0.8937	95.4	0.6363	92.3	1.0922	95.1	0.2675	98.0
		$\mathscr{R}_{15}$	1.0104	95.7	0.9589	97.3	0.6613	91.1	1.1901	95.2	0.2783	98.4
		$\mathscr{R}_{16}$	1.0229	95.1	0.9508	96.1	0.6376	90.6	1.1745	95.0	0.2655	98.7
40	20	$\mathscr{R}_{17}$	1.0921	96.7	1.0707	95.2	0.6702	91.7	1.1492	95.1	0.2697	98.8
		$\mathscr{R}_{18}$	0.9883	96.7	0.9653	95.2	0.5982	93.2	1.0353	95.1	0.3432	99.9
		$\mathscr{R}_{19}$	1.0724	96.3	1.0343	97.8	0.6204	93.0	1.1892	95.0	0.3265	99.9
		$\mathscr{R}_{20}$	1.0230	96.5	0.9883	95.0	0.6683	92.6	1.0704	95.1	0.3059	99.9
	30	$\mathscr{R}_{21}$	0.9616	96.3	0.8855	95.4	0.6230	94.4	0.9886	96.0	0.2726	99.6
		$\mathscr{R}_{22}$	0.9058	96.0	0.7910	95.0	0.6209	94.7	0.9068	95.2	0.2673	98.8
		$\mathscr{R}_{23}$	0.9501	95.9	0.8675	98.0	0.5978	95.1	1.0082	95.1	0.2681	98.3
		$\mathscr{R}_{24}$	0.9603	95.9	0.8720	97.5	0.6431	94.9	0.9898	95.2	0.2638	99.1
60	40	$\mathscr{R}_{25}$	0.8883	95.7	0.7760	98.3	0.6047	93.4	0.8285	95.3	0.2492	97.9
		$\mathscr{R}_{26}$	0.8094	96.8	0.6893	96.5	0.6022	96.0	0.7593	95.9	0.2528	97.7
		$\mathscr{R}_{27}$	0.8834	96.4	0.7516	98.1	0.6123	95.2	0.8392	95.2	0.2500	98.7
		$\mathscr{R}_{28}$	0.8865	96.3	0.7366	95.7	0.5882	95.6	0.8417	95.9	0.2486	96.9
	50	$\mathscr{R}_{29}$	0.8015	96.8	0.6950	95.0	0.6217	91.8	0.7084	96.3	0.2395	97.3
		$\mathscr{R}_{30}$	0.7498	96.7	0.6383	97.1	0.5683	94.6	0.6648	95.9	0.2446	97.9
		$\mathscr{R}_{31}$	0.7953	96.9	0.6675	96.4	0.5653	95.0	0.7297	95.2	0.2433	97.3
		$\mathscr{R}_{32}$	0.8015	96.9	0.6834	94.2	0.5587	93.5	0.7211	95.3	0.2435	97.7

Table 4.b. AILs and CPs (in %) under different PT-IIC schemes at $\alpha = 1.5$.

n	m	Scheme	Asy-CI		Boot-p		Boot-t		HPD: Non-INF		HPD: INF	
			AIL	CP	AIL	CP	AIL	CP	AIL	CP	AIL	CP
20	10	$\mathscr{R}_1$	2.3107	96.1	2.1366	94.0	2.0951	93.1	2.7333	97.2	0.3973	95.1
		$\mathscr{R}_2$	2.3761	95.6	2.3699	95.1	2.1587	96.4	2.7816	97.9	0.6140	95.1
		$\mathscr{R}_3$	2.4083	95.7	2.2643	96.0	2.3124	92.4	2.8226	97.5	0.5531	95.1
		$\mathscr{R}_4$	2.2859	96.3	2.1960	98.0	2.2039	93.4	2.6721	97.0	0.4768	95.0
	15	$\mathscr{R}_5$	2.0615	95.7	1.8408	95.2	1.8885	92.9	2.3849	97.7	0.2278	96.0
		$\mathscr{R}_6$	1.9783	95.9	1.8272	94.9	1.8646	91.8	2.2227	97.2	0.2610	95.2
		$\mathscr{R}_7$	2.0682	95.7	1.8650	95.0	1.9044	93.2	2.3365	97.6	0.3140	95.1
		$\mathscr{R}_8$	2.0407	95.5	1.8657	95.0	1.9896	92.9	2.2907	96.9	0.2497	95.5
30	20	$\mathscr{R}_9$	1.7057	96.7	1.5668	97.3	1.6037	94.6	1.7739	96.0	0.2086	97.1
		$\mathscr{R}_{10}$	1.6217	95.9	1.4839	96.7	1.6509	94.0	1.6737	96.7	0.2075	96.5
		$\mathscr{R}_{11}$	1.6993	96.4	1.5414	97.0	1.6649	93.9	1.7660	96.7	0.2179	96.1
		$\mathscr{R}_{12}$	1.6768	96.3	1.5368	96.3	1.6548	96.4	1.7178	97.6	0.2140	96.4
	25	$\mathscr{R}_{13}$	1.5994	97.5	1.4491	97.1	1.5323	93.1	1.6573	96.8	0.1837	97.9
		$\mathscr{R}_{14}$	1.5289	97.1	1.4021	98.3	1.4219	93.0	1.5820	96.0	0.1882	96.4
		$\mathscr{R}_{15}$	1.5880	97.1	1.4487	98.0	1.4920	90.6	1.6958	96.9	0.1899	96.4
		$\mathscr{R}_{16}$	1.5784	97.3	1.4300	97.5	1.5311	94.3	1.6675	96.5	0.1898	96.7
40	20	$\mathscr{R}_{17}$	1.6089	96.7	1.4261	93.2	1.4952	94.2	1.5914	96.8	0.2241	97.1
		$\mathscr{R}_{18}$	1.5611	96.0	1.4436	94.2	1.5665	92.3	1.5812	97.2	0.2495	95.9
		$\mathscr{R}_{19}$	1.6019	96.5	1.4577	92.0	1.5229	94.5	1.6767	97.1	0.2508	96.0
		$\mathscr{R}_{20}$	1.5287	96.3	1.4226	96.4	1.4079	95.6	1.5991	96.9	0.2252	96.1
	30	$\mathscr{R}_{21}$	1.4275	96.7	1.2919	98.0	1.3891	94.0	1.4599	96.1	0.2088	97.2
		$\mathscr{R}_{22}$	1.3438	96.4	1.2161	97.3	1.2368	96.0	1.3826	95.5	0.2077	95.9
		$\mathscr{R}_{23}$	1.4091	96.0	1.2930	98.8	1.3777	94.6	1.4867	97.1	0.2088	96.3
		$\mathscr{R}_{24}$	1.3959	96.4	1.2764	97.4	1.2923	94.7	1.4635	95.5	0.2031	96.9
60	40	$\mathscr{R}_{25}$	1.1996	97.7	1.0889	95.4	1.1660	93.3	1.1783	95.7	0.2253	98.4
		$\mathscr{R}_{26}$	1.1140	97.3	1.0045	96.1	1.0582	94.7	1.0678	96.3	0.2103	96.9
		$\mathscr{R}_{27}$	1.1718	97.7	1.0669	96.0	1.1131	94.8	1.1786	95.9	0.2103	96.4
		$\mathscr{R}_{28}$	1.1594	97.9	1.0421	96.7	1.0855	95.8	1.1774	96.1	0.2136	97.9
	50	$\mathscr{R}_{29}$	1.1143	98.0	1.0070	97.4	1.0414	94.0	1.0306	96.0	0.2038	98.0
		$\mathscr{R}_{30}$	1.0557	98.1	0.9460	95.8	0.9877	98.1	0.9522	97.2	0.1999	97.6
		$\mathscr{R}_{31}$	1.0961	98.3	0.9732	96.1	1.0344	95.4	1.0251	97.2	0.2044	97.6
		$\mathscr{R}_{32}$	1.0912	98.0	0.9802	95.6	1.0566	96.2	1.0332	96.7	0.2008	97.7

5.2. *Illustrative example*

Suppose that one can generate a random sample following scheme number 30 ($n = 60, m = 50$, and $\mathscr{R}_{30} = (0^{*49}, 10)$), assuming the two parameters of the GPUHLG distribution as $\theta = 0.5, \alpha = 1.5$. The generated samples are provided in Table 5, and upon examining them, we find that they are ordered and bounded from zero to one, as specified in the distribution range.

Table 5. Simulated random data for illustrative example.

0.0321	0.0591	0.0697	0.0880	0.1156	0.1383	0.1767	0.1867	0.1979	0.2214
0.2374	0.2408	0.2487	0.2709	0.2728	0.2921	0.3040	0.3068	0.3071	0.3481
0.3563	0.3599	0.3949	0.4030	0.4215	0.4298	0.4531	0.4627	0.4651	0.4741
0.4947	0.5421	0.5430	0.5535	0.5623	0.5656	0.5827	0.6006	0.6165	0.6260
0.6267	0.6358	0.6530	0.6821	0.7341	0.7648	0.7798	0.8341	0.9472	0.9705

Hence, we obtained the estimates of parameters (θ, α), respectively, as follows:

- Classical estimation point: MLE: (0.4208, 1.9353) and MPS: (0.3471, 2.1086).
- BE point: BE Non-INF: (0.4266, 1.9847) and BE Non-INF: (0.6496, 1.3310).

The convergence of MCMC estimates using the MH algorithm can be demonstrated in Figures 4 and 5. These figures include trace plots and histograms, respectively, for each estimated parameter, θ and α, under two prior scenarios: Non-INF and INF. These graphs illustrate the normality of generated posterior samples for INF priors for both parameters. Additionally, for parameter α in the case of Non-INF priors, the posterior samples also exhibit normal distribution. However, for parameter θ under Non-INF priors, the posterior samples do not follow a normal distribution.

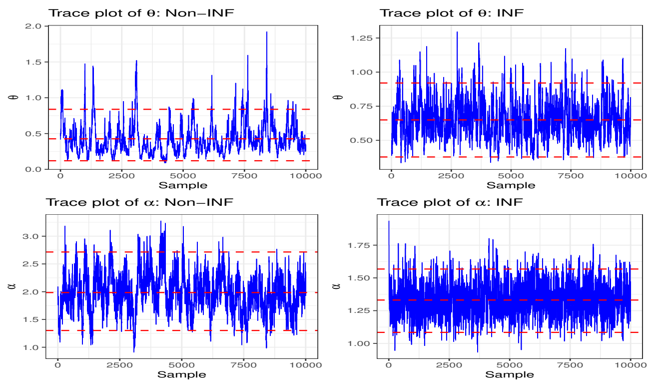

Figure 4. Trace plot of MCMC samples for simulated data.

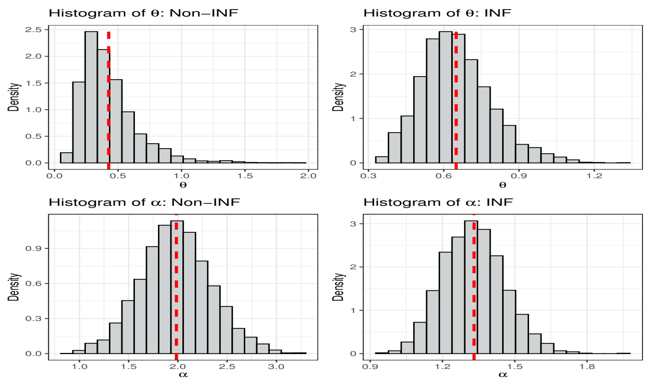

Figure 5. Histogram of MCMC samples for simulated data.

5.3. *Real data analysis*

A real dataset is analyzed to offer illustrative instances and to assess the statistical effectiveness of MLE, MPS, and BEs for the GPUHLG distribution under various PT-IIC schemes.

The following dataset consists of 20 flood observations and was previously analyzed by [30]. The dataset is provided below:

Table 6.a. Data set of flood data for 20 observations.

0.2650	0.2690	0.2970	0.3150	0.3235	0.3380	0.3790	0.3790	0.3920	0.4020
0.4120	0.4160	0.4180	0.4230	0.4490	0.4840	0.4940	0.6130	0.6540	0.7400

To begin with, it is crucial to determine whether the GPUHLG distribution is a suitable choice for analyzing the provided dataset. This involves calculating the MLEs for the parameters (θ, α) and evaluating various goodness-of-fit criteria, including the negative log-likelihood criterion (NLC), Akaike information criterion (AIC), Bayesian information criterion (BIC), the Kolmogorov-Smirnov (K-S) test statistic and its corresponding p-value. These criteria are then compared with those obtained from alternative distributions, such as the Weibull (We), inverse gamma (IGa), beta, Kumaraswamy (Kum), and generalized exponential (GEx) distributions. Lower values of these criteria, along with larger p-values, indicate a better fit. The findings are presented in Table 6.b, which includes parameter estimates and goodness-of-fit statistics. The results from Table 6.b indicate that, among the compared distributions, the GPUHLG distribution serves as an appropriate model for the provided dataset. Consequently, the dataset can be effectively analyzed using this distribution, with the MLEs calculated as $\hat{\theta} = 0.0054$ and $\hat{\alpha} = 6.4977$.

Table 6.b. Evaluation of the goodness of fit for the provided data set.

Pdf	Estimate		NLC	AIC	BIC	K-S	P-value
GPUHLG	0.0054	6.4977	-16.1649	-28.3298	-26.3384	0.1177	0.9447
GEx	57.5089	0.0908	-16.1383	-28.2766	-26.2852	0.1217	0.9285
IGa	14.5702	5.7347	-15.7329	-27.4659	-25.4744	0.1271	0.9032
We	3.5258	0.4688	-13.2640	-22.5280	-20.5365	0.1987	0.4084
beta	6.7564	9.1108	-14.0622	-24.1244	-22.1330	0.1987	0.4081
Kum	3.3633	11.7902	-12.8660	-21.0265	-19.7409	0.2109	0.3359

For a visual evaluation of the compatibility between the provided dataset and the chosen distribution, graphical representations can be highly informative. One common approach is to juxtapose the empirical cumulative distribution function (CDF) with the fitted CDFs for alternative distributions such as Weibull (We), inverse gamma (IGa), beta, Kumaraswamy (Kum), and generalized exponential (GEx). Moreover, a histogram can be illustrated alongside fitted probability density function (pdf) lines for the same set of distributions. Figure 6 illustrates these plotted curves for the CDFs and pdfs of the provided dataset in comparison with their respective distributions. These visualizations clearly underscore that the GPUHLG distribution aligns more favorably with the data compared to the other considered distributions, at least within the context of this particular dataset.

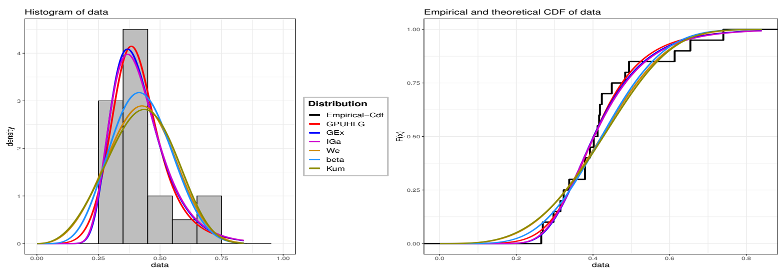

Figure 6. The density and empirical cdf for given real data set with corresponding distributions

Using the original dataset, we generate eight PT-IIC samples. These samples are created with two distinct numbers of stages, specifically, $m = 10$ and $m = 15$, while following the item removal plan detailed in Table 1. Furthermore, we examine a situation where complete sampling cases are considered, where $n = m = 20$ and $R_1 = R_2 = \ldots = R_m = 0$.

In Table 7.a, we compute estimates (Est.) and standard errors (St.Er) through classical estimation methods, specifically, MLEs and MPS. These estimations are carried out for the parameters θ and α, considering varying PT-IIC patterns based on the provided real data set. Furthermore, we calculate BEs using the MH algorithm with the Non-INF prior. While generating samples from the posterior distribution using MH, we initialize the values of (θ, α) as $(\theta^{(0)}, \alpha^{(0)}) = (\hat{\theta}, \hat{\alpha})$, where $\hat{\theta}$ and $\hat{\alpha}$ represent the MLEs of the parameters θ and α, respectively. Subsequently, we discard the initial 2000 burn-in samples from a total of 10,000 samples generated from the posterior density. BEs are then derived using different loss functions, including SE, LN_1 with $v = -0.5$, and LN_1 with $v = 0.5$, as defined by Eqs (4.7) and (4.8). Additionally, Table 7.b presents the lower and upper bounds of confidence intervals for the parameters θ and α using various interval estimation methods: Asy-CI, *Boot.p*, *Boot.t*, and HPD.

Table 7.a. Classical and BE point estimates and standard error for given real data set under different PT-IIC schemes.

n	m	Scheme			Classical		BE: MCMC		
					MLE	MPS	SEL	LN1	LN2
20	10	$\mathscr{R}_1$	θ	Est.	0.0136	0.0045	0.0104	0.0106	0.0102
				St.Er	0.0201	0.0048	0.0285	0.0302	0.0277
			α	Est.	6.0779	7.3007	9.3911	11.2794	7.4286
				St.Er	1.5177	1.2496	2.9862	3.1285	2.9233
		$\mathscr{R}_2$	θ	Est.	0.0356	0.0315	0.0036	0.0037	0.0036
				St.Er	0.0490	0.0462	0.0056	0.0059	0.0057
			α	Est.	5.4831	5.7276	8.9865	10.0021	8.2808
				St.Er	1.5587	1.6848	1.8498	1.9445	1.8511
		$\mathscr{R}_3$	θ	Est.	0.0293	0.0130	0.0758	0.0792	0.0727
				St.Er	0.0419	0.0201	0.1144	0.1057	0.1152
			α	Est.	5.6885	6.6484	5.7250	6.4814	5.0367
				St.Er	1.5768	1.7430	1.7178	1.6895	1.7214
		$\mathscr{R}_4$	θ	Est.	0.0166	0.0059	0.0297	0.0305	0.0289
				St.Er	0.0256	0.0069	0.0554	0.0289	0.0551
			α	Est.	5.8469	6.9524	6.7183	7.6361	5.8656
				St.Er	1.5669	1.2739	1.9392	1.8330	1.9531
	15	$\mathscr{R}_5$	θ	Est.	0.0049	0.0026	0.0018	0.0018	0.0018
				St.Er	0.0068	0.0011	0.0027	0.0028	0.0026
			α	Est.	6.5072	7.1806	8.5326	9.2548	7.8853
				St.Er	1.3548	0.6462	1.6942	1.5213	1.7025
		$\mathscr{R}_6$	θ	Est.	0.0664	0.0554	0.0550	0.0559	0.0541
				St.Er	0.0653	0.0580	0.0596	0.0622	0.0589
			α	Est.	4.4391	4.6638	5.0685	5.3604	4.7860
				St.Er	1.0389	1.1172	1.0750	1.0973	1.1163
		$\mathscr{R}_7$	θ	Est.	0.0110	0.0049	0.0225	0.0229	0.0220
				St.Er	0.0146	0.0044	0.0431	0.0458	0.0420
			α	Est.	5.9344	6.7844	6.0844	6.5910	5.6045
				St.Er	1.3495	0.9911	1.4046	1.4709	1.3821
		$\mathscr{R}_8$	θ	Est.	0.0058	0.0020	0.0017	0.0017	0.0017
				St.Er	0.0081	0.0003	0.0032	0.0033	0.0031
			α	Est.	6.6795	7.8283	8.7660	9.3073	8.2410
				St.Er	1.4660	0.4950	1.4727	1.5300	1.4465
20	Complete		θ	Est.	0.0054	0.0044	0.0053	0.0053	0.0053
				St.Er	0.0067	0.0035	0.0001	0.0001	0.0001
			α	Est.	6.4977	6.1805	6.5046	6.5420	6.4684
				St.Er	1.2688	0.9650	0.3838	0.3831	0.3871

Table 7.b. Different interval estimates for given real data set under different PT-IIC schemes.

n	m	Scheme		Asy-CI	Boot-p	Boot-t	HPD: Non-INF
20	10	$\mathscr{R}_1$	θ	(0.0000, 0.0530)	(0.0001, 0.1115)	(0.0000, 1.9533)	(0.0000, 0.0636)
			α	(3.1032, 9.0526)	(4.3132, 12.1106)	(0.0000, 13.3042)	(4.1305, 14.3879)
		$\mathscr{R}_2$	θ	(0.0000, 0.1317)	(0.0000, 0.0937)	(0.0000, 1.3141)	(0.0000, 0.0117)
			α	(2.4281, 8.5380)	(4.1321, 15.1568)	(0.0000, 15.1879)	(6.1364, 13.4784)
		$\mathscr{R}_3$	θ	(0.0000, 0.1115)	(0.0001, 0.2209)	(0.0000, 0.6662)	(0.0004, 0.3388)
			α	(2.5981, 8.7789)	(4.1054, 10.9713)	(3.7289, 17.0075)	(2.3982, 9.0098)
		$\mathscr{R}_4$	θ	(0.0000, 0.0668)	(0.0000, 0.2016)	(0.0000, 22.2738)	(0.0001, 0.1472)
			α	(2.7758, 8.9180)	(3.7858, 13.4185)	(0.0000, 12.7341)	(3.1100, 10.0225)
	15	$\mathscr{R}_5$	θ	(0.0000, 0.0182)	(0.0001, 0.0292)	(0.0000, 0.2996)	(0.0000, 0.0069)
			α	(3.8518, 9.1627)	(4.9720, 10.2566)	(0.0000, 13.8741)	(5.7423, 11.6252)
		$\mathscr{R}_6$	θ	(0.0000, 0.1944)	(0.0004, 0.1331)	(0.0000, 0.2994)	(0.0031, 0.1831)
			α	(2.4029, 6.4753)	(3.6820, 10.0842)	(3.5814, 9.6642)	(2.9210, 7.0976)
		$\mathscr{R}_7$	θ	(0.0000, 0.0396)	(0.0001, 0.0971)	(0.0000, 1.7671)	(0.0003, 0.0721)
			α	(3.2894, 8.5794)	(4.1532, 10.3699)	(0.0000, 16.7183)	(3.6938, 9.2988)
		$\mathscr{R}_8$	θ	(0.0000, 0.0217)	(0.0000, 0.0556)	(0.0000, 0.8662)	(0.0000, 0.0065)
			α	(3.8062, 9.5529)	(4.6246, 12.0765)	(0.0000, 12.2860)	(5.9434, 11.6350)
20	Complete		θ	(0.0000, 0.0184)	(0.0001, 0.0387)	(0.0000, 0.3996)	(0.0051, 0.0054)
			α	(4.0108, 8.9845)	(4.6717, 10.5509)	(0.0000, 13.0766)	(5.7553, 7.2060)

6. Optimal progressive Type-II censoring scheme

In the preceding sections, we have deliberated upon the classical and BEs of unknown parameters within the context of the GPUHLG distribution when samples are procured using the PT-IIC approach. Consequently, to execute a life-testing experiment following the PT-IIC scheme, it becomes imperative to possess foreknowledge of the values of n, m, and $(R_1, R_2, \ldots, R_m)$. However, in various reliability and life testing studies, practical considerations should select the optimum PT-IIC scheme from a class of possible schemes. This problem was first discussed in detail by [6], which considered the problem of determining the optimal censoring plan via various set-ups. The problem of comparing two different censoring schemes has received a lot of interest from various researchers. See, for example, [31–36].

In order to identify the most appropriate PT-IIC scheme, we assess an information measure through a specific set of criteria. These criteria for optimal sampling are contingent upon the variance-covariance matrix $\mathcal{F}^{-1}$ of the maximum likelihood estimators (MLEs), as formulated in Eq (3.5), and can be articulated in a subsequent manner:

Criterion 1: Minimizing the determinant of $\mathcal{F}^{-1}$:

$$\det\left[\mathcal{F}^{-1}\right] = \mathrm{var}(\hat{\theta})\,\mathrm{var}(\hat{\alpha}) - (\mathrm{cov}(\hat{\theta}, \hat{\alpha}))^2.$$

Criterion 2: Minimizing the trace of $(\mathcal{F}^{-1})$:

$$\mathrm{tr}\left[\mathcal{F}^{-1}\right] = \mathrm{var}(\hat{\theta}) + \mathrm{var}(\hat{\alpha}).$$

Criterion 3: This criterion relies on the choice of u and aims to minimize the variance of the logarithm of the MLE of the u-th quantile (denoted as $\log(\hat{T}_u)$), where $0 < u < 1$. The u-th quantile of the GPUHLG distribution is given by

$$T_u = \left(\frac{2 - 2\theta + u\theta}{u\theta}\right)^{\frac{-1}{\alpha}}.$$

Consequently, the logarithm of T_u is expressed as:

$$\log(T_u) = \frac{-1}{\alpha}\left[\log(2-2\theta+u\theta) - \log(u\theta)\right].$$

By utilizing the delta method, an approximation of the variance of $\log(\hat{T}_u)$ is derived as:

$$\text{Var}(\log(\hat{T}_u)) = \left[\nabla \log(\hat{T}_u)\right]^T \mathcal{F}^{-1}\left[\nabla \log(\hat{T}_u)\right].$$

Here, $\left[\nabla \log(\hat{T}_u)\right]^T$ represents the gradient of $\log(T_u)$ concerning the parameters θ and α, evaluated at $\theta = \hat{\theta}$ and $\alpha = \hat{\alpha}$. The partial derivatives of $\log(T_u)$ are:

$$\frac{\partial \nabla \log(T_u)}{\partial \theta} = \frac{-1}{\alpha}\left[\frac{-2+u}{2-2\theta+u\theta} - \frac{1}{\theta}\right], \frac{\partial \nabla \log(T_u)}{\partial \alpha} = \frac{1}{\alpha^2}\left[\log(2-2\theta+u\theta) - \log(u\theta)\right].$$

This leads to the expression for the variance of $\log(\hat{T}_u)$:

$$\text{Var}(\log(\hat{T}_u)) = \left[\frac{\partial \nabla \log(T_u)}{\partial \theta} \; \frac{\partial \nabla \log(T_u)}{\partial \alpha}\right]\begin{bmatrix} \text{var}(\hat{\theta}) & \text{cov}(\hat{\theta},\hat{\alpha}) \\ \text{cov}(\hat{\alpha},\hat{\theta}) & \text{var}(\hat{\alpha}) \end{bmatrix}\begin{bmatrix} \frac{\partial \nabla \log(T_u)}{\partial \theta} \\ \frac{\partial \nabla \log(T_u)}{\partial \alpha} \end{bmatrix}.$$

It is known that the optimal sampling scheme for PT-IIC is the one that attains the lowest value in any of the criteria mentioned above. To assess the efficacy of the suggested optimal criteria across various PT-IIC schemes, we will conduct a Monte Carlo simulation and also consider the provided real data set.

Monte-Carlo Simulation: The simulation method was utilized while considering the identical steps performed in the simulation section. Specifically, the initial two steps were employed, involving the estimation of MLEs for the GPUHLG distribution and obtaining the asymptotic variances of MLEs (the Fisher information matrix), with parameters $\theta = 0.5$ and $\alpha = 1.5$, across various PT-IIC patterns as outlined in Table 1.

We conducted simulations across 1000 iterations and subsequently computed the Avg. value for each criterion, as outlined in Table 8.a. Generally, we observe that as the n or m increases, the criterion value tends to decrease. Furthermore, we notice that the specific patterns: $\mathscr{R}_2, \mathscr{R}_6, \ldots, \mathscr{R}_{30}$, where items are removed towards the end of the m stages, yield lower values. This implies that these patterns are particularly advantageous for the sampling of PT-IIC. Regarding the comparison between criteria themselves, we believe that they differ in terms of calculation methodology. Therefore, the value of one criterion does not hold significance in relation to another criterion. However, for the Criterion 3, it is possible to compare results based on variations in u. We observe that as the value of u increases, the value of the criterion decreases.

Real data application: Using the previous application "Real Data Analysis" section, we considered the first eight PT-IIC schemes, as outlined in Table 1, utilizing the provided real data set. By utilizing the variance-covariance matrix of the MLEs, it is possible to compute the values of the three criteria for all conceivable selections of n, m, and schemes $\mathscr{R}l$, where $l = 1, 2, \ldots, 8$, as well as the comprehensive sampling approach where $m = n$. The outcomes are presented in Table 8.b. Notably, we observe that the optimal schemes are $\mathscr{R}2$ and $\mathscr{R}_6$.

Table 8.a. Optimal censoring scheme under simulated data from GPUHLG distribution at $(\theta, \alpha) = (0.5, 1.5)$.

n	m	Scheme	Criterion 1	Criterion 2	Criterion 3		
					$u = 0.25$	$u = 0.5$	$u = 0.75$
20	10	$\mathscr{R}_1$	0.03506	0.83682	0.11268	0.07976	0.05729
		$\mathscr{R}_2$	0.01217	0.64069	0.06442	0.04715	0.04140
		$\mathscr{R}_3$	0.01998	0.79471	0.07253	0.06173	0.05435
		$\mathscr{R}_4$	0.01523	0.62961	0.07420	0.05620	0.04626
	15	$\mathscr{R}_5$	0.01297	0.49835	0.07986	0.05236	0.03773
		$\mathscr{R}_6$	0.00896	0.44197	0.06528	0.04037	0.02931
		$\mathscr{R}_7$	0.01077	0.50202	0.06663	0.04458	0.03420
		$\mathscr{R}_8$	0.01328	0.52928	0.07711	0.05219	0.03785
30	20	$\mathscr{R}_9$	0.00700	0.35165	0.06254	0.04127	0.02893
		$\mathscr{R}_{10}$	0.00384	0.28795	0.04490	0.02887	0.02167
		$\mathscr{R}_{11}$	0.00480	0.33571	0.04582	0.03353	0.02681
		$\mathscr{R}_{12}$	0.00559	0.33819	0.05100	0.03700	0.02830
	25	$\mathscr{R}_{13}$	0.00432	0.28031	0.05162	0.03272	0.02292
		$\mathscr{R}_{14}$	0.00333	0.25036	0.04523	0.02778	0.01949
		$\mathscr{R}_{15}$	0.00373	0.27659	0.04456	0.02914	0.02169
		$\mathscr{R}_{16}$	0.00433	0.28334	0.04939	0.03218	0.02287
40	20	$\mathscr{R}_{17}$	0.00603	0.31757	0.05918	0.04126	0.02910
		$\mathscr{R}_{18}$	0.00272	0.27014	0.03429	0.02544	0.02131
		$\mathscr{R}_{19}$	0.00429	0.33148	0.03842	0.03314	0.02797
		$\mathscr{R}_{20}$	0.00353	0.28271	0.03848	0.02951	0.02350
	30	$\mathscr{R}_{21}$	0.00294	0.22649	0.04286	0.02747	0.01906
		$\mathscr{R}_{22}$	0.00188	0.19031	0.03415	0.02132	0.01522
		$\mathscr{R}_{23}$	0.00223	0.21707	0.03406	0.02338	0.01786
		$\mathscr{R}_{24}$	0.00266	0.22059	0.03809	0.02631	0.01922
60	40	$\mathscr{R}_{25}$	0.00154	0.16089	0.03263	0.02116	0.01457
		$\mathscr{R}_{26}$	0.00087	0.13226	0.02284	0.01473	0.01079
		$\mathscr{R}_{27}$	0.00109	0.15160	0.02398	0.01768	0.01377
		$\mathscr{R}_{28}$	0.00120	0.15151	0.02599	0.01897	0.01425
	50	$\mathscr{R}_{29}$	0.00104	0.13316	0.02634	0.01657	0.01144
		$\mathscr{R}_{30}$	0.00077	0.11752	0.02294	0.01409	0.00969
		$\mathscr{R}_{31}$	0.00085	0.12752	0.02335	0.01529	0.01111
		$\mathscr{R}_{32}$	0.00096	0.12957	0.02508	0.01646	0.01153

Table 8.b. Optimal censoring scheme for given real data set.

n	m	Scheme	Criterion 1	Criterion 2	Criterion 3		
					$u = 0.25$	$u = 0.5$	$u = 0.75$
20	10	$\mathscr{R}_1$	5.44E-03	2.53956	0.00627	0.00688	0.01037
		$\mathscr{R}_2$	**5.41E-04**	**1.33875**	0.02476	**0.00472**	**0.00364**
		$\mathscr{R}_3$	1.87E-03	1.94060	0.00908	0.00952	0.01424
		$\mathscr{R}_4$	8.52E-03	2.40327	**0.00616**	0.00637	0.01045
	15	$\mathscr{R}_5$	3.10E-05	1.66920	0.00621	0.00561	0.00789
		$\mathscr{R}_6$	**1.03E-05**	**1.07840**	0.00936	**0.00383**	**0.00284**
		$\mathscr{R}_7$	1.31E-04	1.56659	0.00660	0.00573	0.00532
		$\mathscr{R}_8$	1.77E-05	2.09906	**0.00476**	0.00443	0.00667
20	Complete		4.46E-07	1.21885	0.00250	0.00199	0.00262

7. Conclusions

In the field of distribution theory, a continuous effort is dedicated to generalizing existing distributions. This pursuit aims to create more robust and adaptable models that can be applied to a wide array of scenarios. To achieve this goal, a multitude of methods are explored, as evidenced by a wealth of literature. The validity and practicality of the chosen distribution in fitting the given data significantly impact the subsequent analysis and empirical findings. This paper centers on addressing the challenge of estimating unknown parameters within the context of a GPUHLG distribution under a PT-IIC scheme. Our approach encompasses both classical and Bayesian perspectives. We derived MLEs, MPS, Asy-CI estimates, and bootstrap confidence intervals for the unidentified parameters of the GPUHLG distribution. Additionally, we employed MCMC by utilizing MH algorithm to calculate BE under both symmetric and asymmetric loss functions, accompanied by their corresponding HPD interval estimates. We explored methods for selecting hyper-parameter values for the INF prior case. The simulation study revealed that BEs under the INF prior consistently outperform each of the classical estimates as well as BEs under the Non-INF prior case. We also identified the optimal censoring scheme for life testing experiments, considering three criteria measures, a crucial aspect for practitioners in the field of reliability. The flood data set was employed for all estimations within our research study as a real data application. Future research directions could involve delving into neurotrophic statistics applied to the GPUHLG distribution. Furthermore, there is potential to model COVID-19 data using various progressive censoring schemes, presenting an avenue for further investigation.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This research is supported by researcher support project number (RSPD2023R860), King Saud University, Riyadh, Saudi Arabia.

Conflicts of Interest

The authors declare no conflicts of interest.

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