AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(12): 30827–30845.
DOI: 10.3934/math.20231576
Received: 10 September 2023
Revised: 23 October 2023
Accepted: 06 November 2023
Published: 15 November 2023

Research article

Input-to-state stability for discrete-time switched systems by using Lyapunov functions with relaxed constraints

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Abstract: In this paper, input-to-state stability (ISS) is investigated for discrete-time time-varying switched systems. For a switched system with a given switching signal, the less conservative assumptions for ISS are obtained by using the defined weak multiple ISS Lyapunov functions (WMISSLFs). The considered switched system may contain some or all subsystems which do not possess ISS. Besides, for an ISS subsystem the introduced Lyapunov function could be increasing along the trajectory of the subsystem without input at some moments. Then for a switched system under any switching signal, the relaxed sufficient constraints for ISS are attained by using the defined weak common ISS Lyapunov functions. For this case, each subsystem of the considered system must be ISS. The proposed function may be increasing along the trajectory of each ISS subsystem of the considered system without input at some instants. The relationship between WMISSLFs for a switched system and the defined weak multiple Lyapunov functions for this switched system without input is set up. Three numerical examples are investigated to display the usefulness of the principal outcomes. According to the main conclusions, an intermittent controller is applied to ensure ISS for a discrete-time disturbed Chua's chaotic system.

Keywords: input-to-state stability; ISS Lyapunov function; switched system; intermittent control
Mathematics Subject Classification: 34D05, 37B25, 93C10, 93C55, 93D20

1. Introduction

The concept of input-to-state stability (ISS) for continuous-time dynamic systems was proposed by Sontag in [1]. Many interesting results with respect to ISS for continuous-time dynamic systems were discussed in references such as [2–5]. The ISS property for a discrete-time dynamic system was studied well in [6]. ISS Lyapunov functions with relaxed constraints for dynamic systems were discussed in [7]. The introduced Lyapunov function could increase at some times for the considered system without input. Since ISS is very useful for the analysis of the stability of dynamic systems, ISS for switched systems has been an interesting and meaningful topic for researchers. In [8], stability for a switched linear system was investigated. Then via multiple Lyapunov functions, the authors showed how to stabilize a switched linear system. By using Lyapunov-like functions with relatively mild requirements, in [9] a discrete-time switched system related to a given switching signal was proved to be asymptotically stable. In [10], a novel average dwelling time approach was designed to attain stability for a switched discrete-time system constituted by some subsystems which are not stable. For an asymptotically stable subsystem, the Lyapunov function must be decreasing along the trajectory of the state. By constructing a bounded function in regard to the average dwell time, the authors of [11] discussed ISS for a switched nonlinear system. The ISS property for switched nonlinear systems was explored in [12, 13]. In these papers, it was assumed that each subsystem is ISS. In [14], by constructing a hybrid ISS Lyapunov function, the authors attained ISS for switched systems consisting of some modes which are not stable. In these references, Lyapunov functions must be decreasing along the trajectory of an ISS subsystem without input. In [15], ISS for discrete-time time-invariant switched systems was investigated by utilizing Lyapunov functions with relatively weak conditions. The switched system may have subsystems which are not ISS. In [16], the authors proposed a formula related to the activation time and the average dwell time. Then, the formula was utilized to attain ISS for a continuous-time time-invariant switched system. The considered switched system may have some unstable subsystems. In [17], the authors utilized Lyapunov functions with relaxed conditions to investigate asymptotic stability for impulsive systems. In [18], via the multiple max-separable ISS Lyapunov function, the authors obtained ISS and stabilization state-feedback control was designed for switched nonlinear time-invariant positive systems under deterministic or random switching. In [19], a time-invariant system was stabilized to exponential ISS by designing aperiodic intermittent control. In [20], the authors designed periodic event-triggered control to make sure that nonlinear networked control systems are ISS. In [21], the authors studied how to input-state stabilize semilinear systems by designing aperiodical intermittent event-triggered control. The authors analyzed the ISS of multilayer coupled systems by introducing a periodic event-triggered control with a dynamic term in [22]. ISS was studied in [23] for time-varying nonlinear switched systems with multiple Lyapunov functions under relaxed constraints. The authors of [24] studied ISS for time-varying delayed switched systems. In [25], ISS was investigated for a discrete-time time-invariant switched nonlinear system. In these references, switched systems could have some subsystems which are not ISS. But ISS Lyapunov functions for ISS subsystems were used for the ISS of the considered system. In [26], the ISS for continuous-time time-varying dynamic systems with input was analyzed via Lyapunov functions with less conservative constraints. Inspired by this reference, we are interested in investigating ISS of discrete-time switched systems. Discrete-time switched systems are of interest since they are widely utilized to analyze practical phenomena in many application fields such as chemistry, finance and engineering. It is necessary to point out that in our manuscript non-ISS Lyapunov functions with certain constraints can be used to analyze the ISS of a system which may have non-ISS subsystems.

In this paper, we are going to investigate relaxed assumptions for the ISS of discrete-time time-varying switched nonlinear systems. For a switched system under a given switching, the less conservative sufficient requirements for ISS are obtained by using the defined WMISSLFs (Theorem 3.1). The key points of our results are listed as follows. Compared with the results of [14, 16, 24, 25], the introduced Lyapunov function along the trajectory of an ISS subsystem without input could increase at some instants. It is necessary to emphasize that we can apply Theorem 3.1 in ISS analysis for switched systems constituted by some or all subsystems which are not ISS. This point of view is demonstrated by Examples 2 and 3 from Section 4. The relationship between WMISSLFs for a switched system and the introduced weak multiple Lyapunov functions (WMLFs) for this switched system without input is described in Theorem 3.2. Proposition 3.1 discusses ISS for discrete-time time-varying switched systems with all ISS subsystems under any switching. The introduced Lyapunov function could increase along the trajectory of the sate at some instants for a system without input. According to Theorem 3.1, an intermittent controller is applied to ensure ISS for a discrete-time disturbed Chua's chaotic system.

The rest of the manuscript is organized as follows. In Section 2, we present the notations and definitions used in this paper. The problems studied in this paper are described. In Section 3, we attain the main results. For a switched system under a given switching signal, the less conservative sufficient conditions for ISS are proposed through the use of WMISSLFs (Theorem 3.1). For a switched system with any switching signal, the relaxed sufficient constraints for ISS are attained by using the introduced weak common ISS Lyapunov functions (WCISSLFs) (Proposition 3.1). Theorem 3.2 (Proposition 3.2) describes the relationship between WMISSLFs (WCISSLFs) for a switched system and WMLFs (WCLFs) for this switched system without input. The efficacy of the obtained outcomes is displayed by three numerical examples in Section 4. According to Theorem 3.1, in Section 5 intermittent control is applied to ensure ISS for a discrete-time disturbed Chua's chaotic system. Some concluding discussions are presented in Section 6.

2. Preliminaries

Notations utilized in this paper are listed as follows. $\mathbb{R}_+$ is for the set of nonnegative real numbers. $\mathbb{Z}_+$ represents the set of nonnegative integers. The n-dimension real Euclidean space is denoted by $\mathbb{R}^n$. The Euclidean norm of $x \in \mathbb{R}^n$ is represented by $|x|$. Given a function $u : \mathbb{Z}_+ \mapsto \mathbb{R}^m$, we let $|u|_\infty = \sup_{k\in\mathbb{Z}_+} u(k)$ denote the supremum norm of the function $u(k)$.

For a continuous function $\alpha : \mathbb{R}_+ \mapsto \mathbb{R}_+$, if α strictly increases and satisfies that $\alpha(0) = 0$, then we say that α is of class $\mathcal{K}$. A function $\alpha \in \mathcal{K}$ belongs to class $\mathcal{K}_\infty$ if $\lim_{s\to+\infty} \alpha(s) = +\infty$ holds. A function $\beta(s, t) : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ is of class $\mathcal{KL}$ if for any fixed $t \in \mathbb{R}_+$, $\beta(s, t)$ is a $\mathcal{K}$ function in the argument s, and for any fixed $s \in \mathbb{R}_+$, $\lim_{t\to+\infty} \beta(s, t) = 0$ is satisfied and $\beta(s, t)$ is strictly decreasing as the argument t increases.

In this paper, we are concerned about the following switched system

$$x(k + 1) = f_{\tau(k)}(k, x(k), u(k)),\ k \in \mathbb{Z}_+, \tag{2.1}$$

where $x(k) \in \mathbb{R}^n$ is the state of system (2.1) and $u(k) \in \mathbb{R}^m$ is an input. The admissible value function is $u \in \mathcal{U} = \{u : \mathbb{Z}_+ \mapsto \mathbb{R}^m\}$. The switching signal is determined by the function $\tau(k) : \mathbb{Z}_+ \mapsto \Delta = \{1, 2, \cdots, N\}$ $(N \geq 2)$. Let $0 = k_0 < k_1 < \cdots < k_r < \cdots$ denote the switching time instants.

For all $i \in \Delta$ and $k \in \mathbb{Z}_+$, it is required that $f_i(k, 0, 0) \equiv 0$ holds for all $i \in \Delta$. The function $f_i : \mathbb{Z}_+ \times \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n (i \in \Delta)$ is supposed to be Lipschitz continuous. Then there exist positive constants L_x and L_u satisfying

$$|f_i(k, x_1, u_1) - f_i(k, x_2, u_2)| \leq L_x|x_1 - x_2| + L_u|u_1 - u_2|, \tag{2.2}$$

for $x_1, x_2 \in \mathbb{R}^n$, $u_1, u_2 \in U$. The notation $x_\tau(k, x_0, u)$ is for the solution of system (2.1) in regard to an initial condition $x_0 = x(0)$, a given switching signal $\tau(k)$ and an input $u \in \mathcal{U}$. For convenience, $x(k, x_0, u)$ or $x(k)$ will be utilized hereafter instead of $x_\tau(k, x_0, u)$.

Since we are going to analyze ISS for system (2.1), the definition of ISS is recalled.

Definition 2.1. *System* (2.1) *is called input-to-state stable (ISS) if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that*

$$|x(k, x_0, u)| \leq \beta(|x_0|, k) + \gamma(|u|_\infty),$$

for any input $u \in \mathcal{U}$ and any initial condition x_0.

In what follows, we will utilize ISS Lyapunov functions with relaxed constraints to analyze ISS for system (2.1) with some or all non-ISS subsystems under a given switching signal or arbitrary switching signal.

3. Main results

In this part, for system (2.1) under a given switching signal, the less conservative sufficient constraints for ISS are investigated by using the subsequently defined WMISSLFs. For system (2.1) under any switching signal, the relaxed conditions for ISS are attained by using WCISSLFs introduced in the subsection.

Theorem 3.1. *For system* (2.1) *under a switching $\tau(k)$, if there exist positive constants $M, \mu_i \in \mathbb{R}_+$ and functions $V_i : \mathbb{R}_+ \times \mathbb{R}^n \mapsto \mathbb{R}_+$, $\alpha_1, \alpha_2, \rho \in \mathcal{K}_\infty$, $\varphi_i : \mathbb{Z}_+ \mapsto \mathbb{R}_+$ satisfying the following constraints for $i \in \Delta$, $x \in \mathbb{R}^n$, $u \in U_R$ and $k \in \mathbb{N}_+$,*

$$\alpha_1(|x|) \leq V_i(k, x) \leq \alpha_2(|x|), \tag{3.1}$$

$$V_i(k+1, f_i(k, x(k), u(k)) \leq \varphi_i(k) V_i(k, x(k)) + \rho(|u(k)|_\infty), \tag{3.2}$$

$$V_i(k, x(k)) \leq \mu_i V_j(k, x(k)), \ \ i, j \in \Delta, i \neq j, \tag{3.3}$$

$$0 < \varphi_i(k) \leq M. \tag{3.4}$$

Let $\mu = \max\limits_{i \in \Delta}\{\mu_i\}$ and $\varphi(k) = \varphi_{\tau(k)}(k)$ for $k \in \mathbb{Z}_+$. Furthermore, if we have positive constants $K \in \mathbb{Z}_+$ and $0 < \xi < 1$ ($\xi \in \mathbb{R}_+$) that satisfy the constraint

$$\mu^\chi \varphi(SK)\varphi(SK+1)\cdots\varphi(SK+K-1) \leq \xi, \ \ S \in \mathbb{N}_+, \tag{3.5}$$

where χ denotes the largest number of switching times during the time period $[SK, (S+1)K-1]$ (for any $S \in \mathbb{Z}_+$), then system (2.1) *is ISS under the given switching signal.*

Proof. Let $0 \leq k_0 < k_1 < \cdots < k_r < \cdots$ ($r \in \mathbb{Z}_+, r > 0$) denote the switching times. Then for $k+1 \in [k_r, k_{r+1})$ by utilizing the constraints (3.2) and (3.3), we have that

$$\begin{aligned} V_{\tau(k+1)}(k+1, x(k+1)) &\leq \varphi_{\tau(k)}(k)\mu^{N[k,k]}V_{\tau(k)}(k, x(k)) + \rho(|u|_\infty) \\ &\leq \varphi_{\tau(k)}(k)\varphi_{\tau(k)}(k-1)\mu^{N[k-1,k]}V_{\tau(k-1)}(k-1, x(k-1)) \\ &\quad + \varphi_{\tau(k)}(k)\mu^{N[k,k]}\rho(|u|_\infty) + \rho(|u|_\infty) \end{aligned}$$

$$\leq \prod_{j=k_r-1}^{k} \varphi(j)\mu^{N[k_r-1,k]} V_{\tau(k_r-1)}(k_r-1, x(k_r-1))$$
$$+ \sum_{j=k_r-1}^{k-1} \prod_{q=j+1}^{k} \varphi(q)\mu^{[j+1,k]}\rho(|u|_\infty) + \rho(|u|_\infty),$$

where $N[j,k]$ is for the switching times of $\tau(k)$ in $[j,k]$ with $j \in \mathbb{Z}_+$.

Then utilizing the constraints and the recursive method, we derive that

$$V_{\tau(k+1)}(k+1, x(k+1)) \leq \prod_{j=0}^{k} \varphi(j)\mu^{N[0,k]} V_{\tau(0)}(0, x(0)) + \sum_{j=0}^{k-1} \prod_{q=j+1}^{k} \varphi(q)\mu^{N[j+1,k]}\rho(|u|_\infty) + \rho(|u|_\infty),$$

where $N[0,k]$ denotes the switching times of $\tau(k)$ in $[0,k]$.

We have that

$$\sum_{j=SK}^{(S+1)K-1} M^{j-SK+1}\mu^{N[j+1,(S+1)K-1]} \leq \begin{cases} KM^K\mu^\chi, & M > 1, \\ K\mu^\chi, & M \leq 1. \end{cases}$$

Since $KM^K\mu^\chi$ is bounded, we obtain a positive constant $A \in \mathbb{R}_+$ such that

$$A \geq \begin{cases} KM^K\mu^\chi, & M > 1, \\ K\mu^\chi, & M \leq 1. \end{cases}$$

It is evident that the equation $k = dK + c$ holds with $d, c \in \mathbb{Z}_+$ and $0 \leq c < K$. By using the condition (3.5), it is computed that

$$\begin{aligned} V_{\tau(k+1)}(k+1, x(k+1)) \leq{}& \prod_{j=0}^{k} \varphi(j)\mu^{N[0,k]} V_{\tau(0)}(0, x(0)) + \rho(|u|_\infty) + \underbrace{\sum_{j=k-1-(K-1)}^{k-1} \prod_{q=j+1}^{k} \varphi(q)\mu^{N[j+1,k]}}_{\leq A}\rho(|u|_\infty) \\ &+ \underbrace{\sum_{j=k-1-2(K-1)-1}^{k-1-(K-1)-1} \prod_{q=j+1}^{k} \varphi(q)\mu^{N[j+1,k+1]}\rho(|u|_\infty)}_{\leq A\xi} + \cdots + \underbrace{\sum_{j=0}^{r} \prod_{q=j+1}^{k} \varphi(q)\mu^{N[j+1,k]}\rho(|u|_\infty)}_{\leq \xi^d A} \\ \leq{}& \prod_{j=0}^{k} \varphi(j)\mu^{N[0,k]} V_{\tau(0)}(0, x(0)) + (A + \xi A + \cdots + \xi^d A)\rho(|u|_\infty) + \rho(|u|_\infty) \\ \leq{}& \prod_{j=0}^{k} \varphi(j)\mu^{N[0,k]} V_{\tau(0)}(0, x(0)) + (1 + \frac{1}{1-\xi}A)\rho(|u|_\infty). \end{aligned}$$

Based on the constraints (3.2) and (3.5), we attain the following inequalities

$$
\begin{aligned}
V_{\tau(k+1)}(k+1, x(k+1)) \le & \prod_{j=lK}^{k} \varphi(j)\mu^{N[lK,k]} \underbrace{\prod_{j=(l-1)K}^{lK-1} \varphi(j)\mu^{N[(l-1)K,lK-1]}}_{\le \xi} \cdots \\
& \underbrace{\prod_{j=0}^{K-1} \varphi(j)\mu^{N[0,K-1]}}_{\le \xi} V_{\tau(0)}(0, x(0)) + (1 + \frac{1}{1-\xi}A)\rho(|u|_\infty) \\
\le & M^{c+1}\mu^{\chi}\xi^{d} V_{\tau(0)}(0, x(0)) + (1 + \frac{1}{1-\xi}A)\rho(|u|_\infty) \\
\le & M^{c+1}\mu^{\chi}\frac{1}{\xi}\xi^{\frac{k}{K}}\alpha_2(|x_0|) + (1 + \frac{1}{1-\xi}A)\rho(|u|_\infty).
\end{aligned}
$$

Now we define the function

$$
\beta(s,t) = \begin{cases} \alpha_1^{-1}(2\mu\chi\frac{1}{\xi}\xi^{\frac{t}{K}}\alpha_2(s)), \ M \le 1, \ s,t \in \mathbb{R}_+, \\ \alpha_1^{-1}(2\mu\chi M^K\frac{1}{\xi}\xi^{\frac{t}{K}}\alpha_2(s)), \ M > 1, \ s,t \in \mathbb{R}_+. \end{cases}
$$

Because $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $0 < \xi < 1$, it is gained that the function α_1^{-1} is of $\mathcal{K}_\infty$. Moreover, for any fixed variable $t \in \mathbb{R}_+$ the above defined function $\beta(s,t)$ belongs to $\mathcal{K}$ which is related to the argument s, and for each fixed s the function $\beta(s,t)$ is decreasing as the argument $t \in \mathbb{R}_+$ increases and $\lim_{t\to+\infty}\beta(s,t) = 0$ holds. Thus β belongs to $\mathcal{KL}$.

Using the inequalities given by (3.1), the estimate of the norm of the state is obtained:

$$
|x(k)| \le \beta(|x_0|, k) + \alpha_1^{-1}(2(1 + \frac{1}{1-\xi}A)\rho(|u|_\infty)).
$$

Then based on Definition 2.1, the ISS for system (2.1) is attained. □

Remark 3.1. *(1) For system* (2.1) *with* $u \equiv 0$*, if the constraints of Theorem 3.1 hold, then the asymptotic stability of system* (2.1) *is derived. This conclusion is similar to Theorem 2 from [27].*

(2) It is clear that the inequality $\mu \ge 1$ *holds. To make sure that the constraint* (3.5) *is satisfied, we must require that the inequality* $\varphi(SK)\varphi(SK+1)\cdots\varphi(SK+K-1) \le \delta$ *with* $0 < \xi < 1$ *holds for* $S \in \mathbb{N}_+$*. We emphasize that it may hold that* $\varphi(SK+j) > 1$ *for some* $j \in \mathbb{Z}_+$*.*

(3) According to the constraints of Theorem 3.1, for a subsystem $x(k+1) = f_i(k, x(k), u(k))$ *(some* $i \in \Delta$*), since* $\varphi_i(k)$ *may be larger than* 1 *for all* $k \in \mathbb{Z}_+$*, this system may not be ISS. However, for system* (2.1) *with some or all subsystems which are not ISS, Theorem 3.1 can be used to analyze ISS. This point of view is illustrated in Examples 2–3 from Section 4. For an ISS subsystem,* V_i *may not be an ISS Lyapunov function for this system. This is demonstrated by Example 1 from Section 4.*

(4) Let $\Delta_s \cup \Delta_u = \Delta$*, and* $\Delta_s \cap \Delta_u = \emptyset$*. In Theorem 3.1, if the constraints* (3.2) *and* (3.4) *are replaced by*

$$
V_i(k+1, f_i(k, x(k), u(k))) \le \lambda_s V_i(k, x(k)) + \rho(|u|_\infty), i \in \Delta_s,
$$

$$V_i(k+1, f_i(k, x(k), u(k))) \le \lambda_u V_i(k, x(k)) + \rho(|u|_\infty),\ i \in \Delta_u,$$

where $0 \le \lambda_s < 1$ *and* $\lambda_u \ge 1$. *Then in order to ensure that the constraint* (3.5) *holds, it is necessary to require that we have positive constants* $r, \sigma_a \in \mathbb{R}_+$ *satisfying the following constraints*

$$\begin{aligned} \sigma_a &> \frac{-\ln \mu}{(1-r)\ln \lambda_s - r \ln \lambda_u}, \\ r &< \frac{-\ln \rho_s}{\ln \lambda_u - \ln \lambda_s}, \\ N_\tau[k, k+K] &\le \frac{K}{\sigma_a},\ k \in \mathbb{N}, \\ k^u[k, k+K] &\le rK,\ k \in \mathbb{N}, \end{aligned} \tag{3.6}$$

where $\mu = \max_{i \in \Delta}\{\mu_i\}$ *and* $k^u[k, k+K]$ *denotes the activation time period of the subsystems in* Δ_u *in* $[k, k+K]$. *Under these constraints, we derive the following:*

$$\begin{aligned} \mu^\chi \varphi(SK)\varphi(SK+1)\cdots\varphi(SK+K-1) &\le \mu^{\frac{k}{\sigma_a}} \lambda_u^{k^u} \lambda_s^{(1-r)k} \\ &\le \mu^{\frac{k}{\sigma_a}} \lambda_u^{rk} \lambda_s^{(1-r)k} = \xi < 1. \end{aligned}$$

Then according to the derivation of Theorem 3.1, we can attain ISS for system (2.1). *We have to point out that the above constraints are similar to those of [28, Theorem 2].*

Based on Theorem 3.1, for system (2.1) under a given switching, we introduce the concept of weak multiple input-to-state stability Lyapunov functions for switched system (2.1).

Definition 3.1. *For system* (2.1) *under a switching signal* $\tau(k)$, *we call* $\{V_i : \mathbb{R}_+ \times \mathbb{R}^n \mapsto \mathbb{R}_+, i \in \Delta\}$ *a weak multiple input-to-state stability Lyapunov function (WMISSLF) if the conditions* (3.1)–(3.3) *of Theorem 3.1 on the functions* V_i*s and the assumptions* (3.4) *and* (3.5) *imposed on* $\varphi_i, \mu_i (i \in \Delta)$ *hold.*

Remark 3.2. *(1) For system* (2.1) *with* $u \equiv 0$, $\{V_i : \mathbb{R}_+ \times \mathbb{R}^n \mapsto \mathbb{R}_+, i \in \Delta\}$ *from Definition 3.1 is called a weak multiple Lyapunov function (WMLF).*

(2) If the functions f_i $(i \in \Delta)$ *satisfy*

$$|f_i(k, x(k), u(k))| \le \varphi_i(k)|x(k)| + \rho(|u|_\infty),$$

where $\varphi_i(k)$, ρ *satisfy the constraints of Theorem 3.1, then* $V_i(k, x(k)) = V_j(k, x(k)) = |x|$ $(i, j \in \Delta, i \ne j)$ *is a WMISSLF for system* (2.1). *This conclusion indicates that under the above given constraints we may first check if* $|x|$ *is a WMISSLF for the considered examples in Section 4. It is noteworthy that* $\varphi_i \ne \varphi_j$ *may hold for* $i, j \in \Delta$, $i \ne j$.

The relationship between WMISSLFs for system (2.1) and WMLFs for system (2.1) without input is described as Theorem 3.2.

Theorem 3.2. *For system* (2.1) *with* $u \equiv 0$ *and a switching* $\tau(k)$, *if there exist a WMLF* $\{V_i : \mathbb{R}_+ \times \mathbb{R}^n \mapsto \mathbb{R}_+, i \in \Delta\}$ *and a positive constant* L *such that* $|V_i(k, x_1) - V_i(k, x_2)| \le L|x_1 - x_2|$ *for* $k \in \mathbb{Z}_+$ *and* $x_1, x_2 \in \mathbb{R}^n$, *then* $\{V_i : \mathbb{R}_+ \times \mathbb{R}^n \mapsto \mathbb{R}_+, i \in \Delta\}$ *is a WMISSLF for the switched system* (2.1) *under the switching signal* $\tau(k)$.

Proof. To demonstrate that $\{V_i : \mathbb{R}_+ \times \mathbb{R}^n \mapsto \mathbb{R}_+, i \in \Delta\}$ is a WMISSLF for system (2.1) with the given switching signal $\tau(k)$, we have to prove that the inequality (3.2) holds. Based on the assumptions, we have

$$V_i(k+1, f_i(k, x(k), 0)) \leq \varphi_i(k) V_i(k, x(k)).$$

Then we get that

$$\begin{aligned} V_i(k+1, f_i(k, x(k), u(k))) &\leq V_i(k+1, f_i(k, x(k), u(k))) + \varphi_i(k) V_i(k, x(k)) \\ &\quad - V_i(k+1, f_i(k, x(k), 0)) \\ &\leq \varphi_i(k) V_i(k, x(k)) + L|f_i(k, x(k), u(k)) - f_i(k, x(k), 0)| \\ &\leq \varphi_i(k) V_i(k, x(k)) + L L_u |u|_\infty. \end{aligned}$$

Hence we attain that the inequality (3.2) is satisfied. Therefore, $\{V_i : \mathbb{R}_+ \times \mathbb{R}^n \mapsto \mathbb{R}_+, i \in \Delta\}$ is a WMISSLF for system (2.1). □

Remark 3.3. *Theorem 3.2 shows one way to construct a WMISSLF for system* (2.1)*, i.e., construction of a WMLF with Lipschitz continuity for system* (2.1) *with* $u \equiv 0$*. The topic with respect to the computation of WMISSLFs for system* (2.1) *will be investigated in the future.*

For Theorem 3.1, if the equations $\varphi_i(k) \equiv \varphi_j(k)$ and $V_i(k, x(k)) = V_j(k, x(k))$ $(i \neq j, i, j \in \Delta)$ are required, then we have the following proposition.

Proposition 3.1. *For system* (2.1)*, if we have the functions* $V : \mathbb{R}_+ \times \mathbb{R}^n \mapsto \mathbb{R}_+$*,* $\alpha_1, \alpha_2, \rho \in \mathcal{K}_\infty$*,* $\varphi : \mathbb{Z}_+ \mapsto \mathbb{R}_+$ *and positive constants* $M, \mu_i \in \mathbb{R}_+$ *satisfying the following constraints for* $i \in \Delta$*,* $x \in \mathbb{R}^n$*,* $u \in U_R$ *and* $k \in \mathbb{N}_+$*,*

$$\alpha_1(|x|) \leq V(k, x) \leq \alpha_2(|x|), \tag{3.7}$$

$$V(k+1, f_i(k, x(k), u(k)) \leq \varphi(k) V(k, x(k)) + \rho(|u(k)|_\infty), \tag{3.8}$$

$$0 < \varphi(k) \leq M. \tag{3.9}$$

Moreover, if the positive constants $K \in \mathbb{Z}_+$ *and* $0 < \xi < 1$ *(*$\xi \in \mathbb{R}_+$*) satisfying the following inequality exist:*

$$\varphi(SK)\varphi(SK+1)\cdots\varphi(SK+K-1) \leq \xi, \;\; S \in \mathbb{N}_+, \tag{3.10}$$

then system (2.1) *is ISS under any switching signal.*

Proof. The conclusion can be obtained through a similar derivation of Theorem 3.1. □

Remark 3.4. *(1) It is worthy to point out that for an ISS subsystem, V from Proposition 3.1 may not be an ISS Lyapunov function, since for some* $i \in \Delta$*,* $\varphi > 1$ *can hold for some* $k \in \mathbb{Z}_+$*.*

(2) Compared to Theorem 3.1, the merit of Proposition 3.1 is that it can be used for the ISS analysis of switched system (2.1) *in regard to arbitrary switching signal. This point is illustrated via Example 1 in Section 4.*

(3) Under the constraints of Proposition 3.1, for system (2.1) *without input (*$u \equiv 0$*), the asymptotic stability is derived.*

Based on Proposition 3.1, we introduce the definition of weak common input-to-state stability Lyapunov functions for switched system (2.1).

Definition 3.2. *For system* (2.1)*, a function* $V : \mathbb{R}_+ \times \mathbb{R}^n \mapsto \mathbb{R}_+$ *is called a weak common input-to-state stability Lyapunov function (WCISSLF) for system* (2.1)*, if the constraints* (3.7) *and* (3.8) *from Proposition 3.1 on the function V and the assumptions* (3.9) *and* (3.10) *imposed on* φ *are satisfied.*

Remark 3.5. *We call a WCISSLF* $V : \mathbb{R}_+ \times \mathbb{R}^n \mapsto \mathbb{R}_+$ *from Definition 3.2 a weak common Lyapunov function(WCLF) for system* (2.1) *with* $u \equiv 0$.

The relationship between the WCISSLFs for system (2.1) and WCLFs for system (2.1) without input is set up via Proposition 3.2.

Proposition 3.2. *If there exist a WCLF* $V : \mathbb{R}_+ \times \mathbb{R}^n \mapsto \mathbb{R}_+$ *for system* (2.1) *with* $u \equiv 0$ *and a positive constant* L_1 *such that* $|V(k, x_1) - V(k, x_2)| \leq L_1|x_1 - x_2|$ *for* $k \in \mathbb{Z}_+$, $x_1, x_2 \in \mathbb{R}^n$*, then V is a WCISSLF for system* (2.1).

Proof. This result is obtained via a derivation similar to that for Theorem 3.2. □

Remark 3.6. *Proposition 3.2 provides one way to construct a WCISSLF for system* (2.1)*, i.e., construction of a Lipschitz continuous WCLF for system* (2.1) *with* $u \equiv 0$*. We will investigate how to compute the WCISSLF for system* (2.1) *in the future.*

4. Numerical examples

In this part, we are going to analyze ISS for three examples by applying our main results.

4.1. Example 1

We study a switched system described by

$$x(k+1) = f_{\tau(k)}(k, x(k), u(k)), \tag{4.1}$$

where $x(k) \in \mathbb{R}$, $u(k) \in \mathbb{R}$, $\tau(k) \in \Delta = \{1, 2\}$, and

$$\begin{aligned} f_1(k, x(k), u(k)) &= \frac{6 + 2k^2}{1 + 4k^2 + x^2(k)} x(k) + \frac{1}{4} \sin^2(\frac{k\pi}{4}) u(k), \\ f_2(k, x(k), u(k)) &= (\frac{1}{2} + \sin^2(\frac{k\pi}{4})) x(k) + 0.2u(k). \end{aligned}$$

Now we use Proposition 3.1 to investigate ISS for system (4.1) with an arbitrary switching signal. We check if a WCISSLF candidate $V(k, x(k)) = |x(k)|$ satisfies all requirements from Proposition 3.1. By simple calculation, we have the inequalities

$$\begin{aligned} V(k+1, f_1(k, x(k), u(k))) &\leq \frac{6 + 2k^2}{1 + 4k^2} |x(k)| + 0.2|u(k)|, \\ V(k+1, f_2(k, x(k), u(k))) &\leq (\frac{1}{2} + \sin^2(\frac{k\pi}{4})) |x(k)| + 0.2|u(k)|. \end{aligned}$$

Define $\varphi : \mathbb{Z}_+ \mapsto \mathbb{R}_+$ as follows

$$\varphi(k) = \max\{\frac{6 + 2k^2}{1 + 4k^2}, (\frac{1}{2} + \sin^2(\frac{k\pi}{4}))\}.$$

It is clear that $\varphi(k) \leq 6$ is satisfied and the inequality (3.10) holds with $K = 37$ and $\xi = 0.9234$. Thus

$$V(k + 1, f_i(k, x(k), u(k))) \leq \varphi(k)V(k, x(k)) + 0.2|u(k)|_\infty, i \in \Delta. \tag{4.2}$$

Hence the requirements of Proposition 3.1 hold. Therefore, system (4.1) under an arbitrary switching signal is ISS. Figure 1 shows the ISS for system (4.1) with $x_0 = 0.5$, $u(k) = rand(1)$ and the following switching signal

$$\tau(k) = \begin{cases} 1, & k = 2s, s \in \mathbb{Z}_+, \\ 2, & k \neq 2s, s \in \mathbb{Z}_+. \end{cases} \tag{4.3}$$

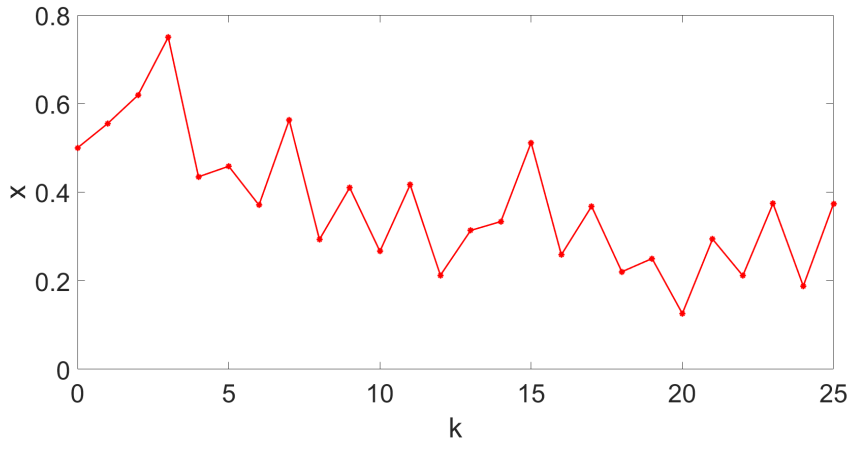

Figure 1. State of system (4.1) related to the switching signal (4.3), $x_0 = 0.5$ and $u(k) = rand(1)$.

Figure 2 displays the asymptotic stability of system (4.1) without input. Example 1 demonstrates how to analyze ISS for switched systems with any switching signal.

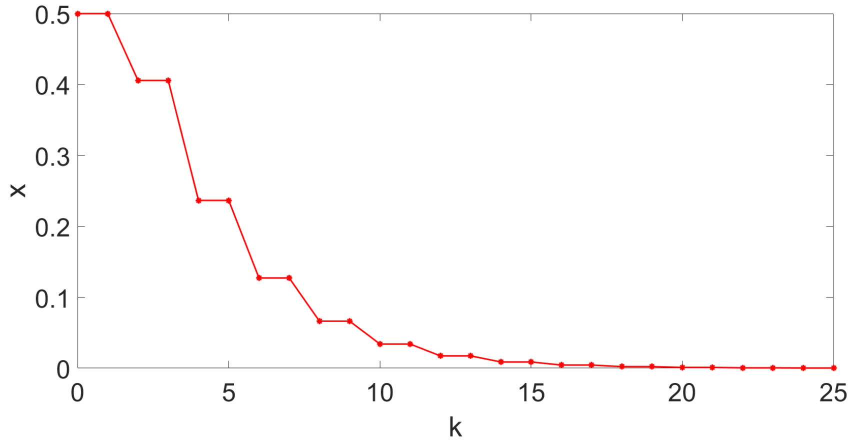

Figure 2. State of system (4.1) without input, but with $x_0 = 0.5$ and the switching signal (4.3).

Remark 4.1. *In [25], for an ISS subsystem, the function V is an ISS Lyapunov function. However, it is clear that the function $V(k, x)$ is not an ISS Lyapunov function for each subsystem of system* (4.1)*, since it increases at some times for each subsystem without input.*

4.2. Example 2

We investigate the ISS for a switched system

$$x(k+1) = f_i(k, x(k), u(k)), \quad i = 1, 2, \tag{4.4}$$

with $x = (x_1, x_2)^\top \in \mathbb{R}^2$, $u = (u_1, u_2)^\top \in \mathbb{R}^2$.

The functions denoted by f_i are

$$f_1(k, x(k), u(k)) = \begin{cases} (\frac{1}{5} + \sin^2(\frac{k\pi}{2}))x_1(k) + u_1(k), \\ (\frac{1}{5} + \sin^2(\frac{k\pi}{2}))x_2(k) + u_2(k). \end{cases}$$

$$f_2(k, x(k), u(k)) = \begin{cases} x_1(k) + u_1(k), \\ (1 + \cos^2(k))x_2(k). \end{cases}$$

The switching signals are determined by

$$\tau(k) = \begin{cases} 1, & k \in [k_s, k_s + 2), \\ 2, & k \in [k_s + 2, k_s + 4), \end{cases}$$

where $k_0 = 0$, $k_{s+1} = k_s + 4$ and $s \in \mathbb{Z}_+$.

For the subsystem described by (4.5), it is clear that we cannot get asymptotic stability for system (4.5) without input. Therefore, the subsystem (4.5) is not ISS. Thus system (4.4) is not ISS for any switching signal.

$$x(k+1) = f_2(k, x(k), u(k)). \tag{4.5}$$

Then using Theorem 3.1, we analyze the ISS for system (4.4). We examine if a WMISSLF candidate $\{V_1(k,x) = V_2(k,x) = |x|_1 = |x_1| + |x_2|\}$ meets the requirements of Theorem 3.1. Here in order to make it easy to calculate, 1-norm $|\cdot|_1$ is used. We compute that

$$V_1(k+1, f_1(k, x(k), u(k))) \leq (\frac{1}{5} + \sin^2(\frac{k\pi}{2}))V_1(k, x(k)) + |u(k)|_1,$$
$$V_2(k+1, f_2(k, x(k), u(k))) = |x_1(k+1)| + |x_2(k+1)| \leq 2V_2(k, x(k)) + |u(k)|_1.$$

Thus we have

$$\varphi_1(k) = \frac{1}{5} + \sin^2(\frac{k\pi}{2}),\ \ \varphi_2(k) = 2.$$

Let

$$\varphi(k) = \begin{cases} \varphi_1(k) = \frac{1}{5} + \sin^2(\frac{k\pi}{2}),\ \ k \in [k_s, k_s+2), \\ \varphi_2(k) = 2,\ \ k \in [k_s+2, k_s+4). \end{cases}$$

Then it holds that $\varphi(k) \leq 2$ for $k \in \mathbb{Z}_+$. The constraint (3.5) is satisfied with $K = 4$ and $\xi = \frac{24}{25}$. Therefore, the assumptions of Theorem 3.1 hold. Hence system (4.4) is ISS (see Figure 3). The asymptotic stability of system (4.4) without input is demonstrated by Figure 4.

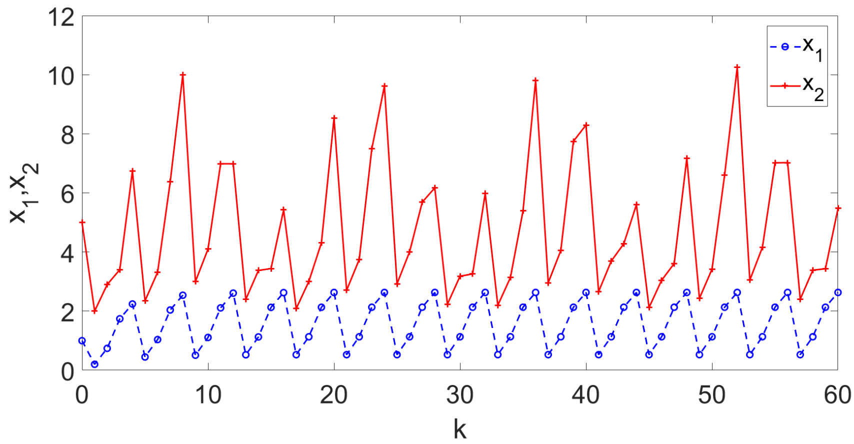

Figure 3. State of system (4.4) given $x_0 = (1, 5)^\top$ and $u(k) = (\sin^2(\frac{k\pi}{4}), \cos^2(\frac{k\pi}{4}))^\top$.

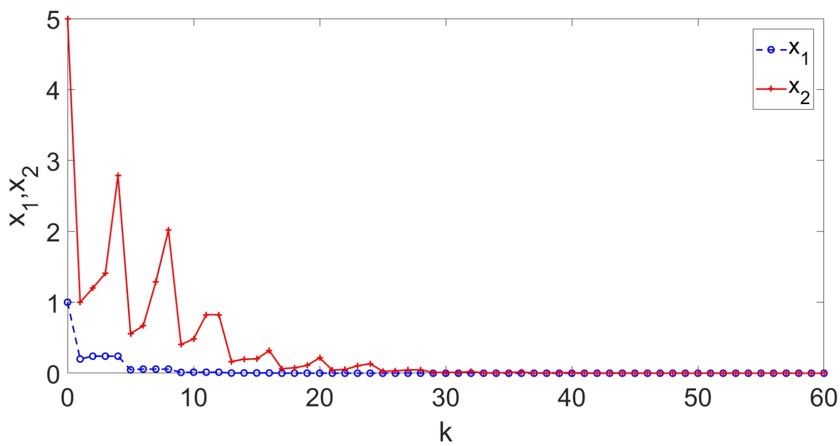

Figure 4. State of system (4.4) in connection with $x_0 = (1, 5)^\top$ but without input.

Remark 4.2. *It is necessary to point out that the subsystem* (4.5) *of system* (4.4) *is not ISS and that* V_1 *increases at some moments for the second subsystem without input. However, via Theorem 3.1, ISS of system* (4.4) *is attained under the given switching signal.*

4.3. Example 3

We evaluate ISS for the following switched system

$$x(k+1) = f_{\tau(k)} = (k, x(k), u(k)), \tag{4.6}$$

with $x(k) \in \mathbb{R}^2, u(k) \in \mathbb{R}^2$ and

$$\tau(k) = \begin{cases} 1, k \in [k_s, k_s + 2), \\ 2, k \in [k_s + 2, k_s + 4), \end{cases}$$

where $k_0 = 0$, $k_{s+1} = k_s + 4$ and $s \in \mathbb{Z}_+$.

The functions f_i $(i = 1, 2)$ are described by

$$\begin{aligned} f_1(k, x(k), u(k)) &= 3x(k) + u(k), k \in \mathbb{Z}_+, \\ f_2(k, x(k), u(k)) &= c(k)x(k) + 3u(k), k \in \mathbb{Z}_+, \text{ with} \\ c(k) &= \begin{cases} 5, k \in [k_s, k_s + 2),\ s \in \mathbb{Z}_+, \\ \frac{1}{4}, k \in [k_s + 2, k_s + 4),\ s \in \mathbb{Z}_+. \end{cases} \end{aligned}$$

For the subsystem $x(k+1) = f_i(k, x(k), u(k))$ $(i = 1, 2)$ with $u \equiv 0$, we can not attain asymptotic stability. Hence each subsystem of system (4.6) is not ISS. However, we can analyze the ISS of system (4.6) by utilizing Theorem 3.1. Let $\{V_1(k, x(k)) = |x|, V_2(k, x(k)) = \frac{5}{6}|x|\}$ be a WMISSLF candidate for system (4.6). Under the given constraints, we have

$$V_1(k+1, x(k+1)) \leq 3V(k, x(k)) + 3|u|_\infty, \tag{4.7}$$

$$V_1(k+1, x(k+1)) \leq c(k)V(k, x(k)) + 3|u|_\infty, \tag{4.8}$$

where $c(k) = \begin{cases} 5, k \in [k_s, k_s + 2),\ s \in \mathbb{Z}_+, \\ \frac{1}{4}, k \in [k_s + 2, k_s + 4),\ s \in \mathbb{Z}_+. \end{cases}$

Then we obtain that $\mu = \frac{6}{5}$, $M = 5$ and

$$\varphi(k) = \begin{cases} 3, k \in [k_s, k_s + 2),\ s \in \mathbb{Z}_+, \\ \frac{1}{4}, k \in [k_s + 2, k_s + 4),\ s \in \mathbb{Z}_+. \end{cases}$$

We check that the inequality (3.5) holds with $K = 4$, $\chi = 2$ and $\xi = \frac{324}{400}$. Thus all constraints of Theorem 3.1 are satisfied. Therefore, under the given switching signal, system (4.6) is ISS. This is shown in Figure 5. Figure 6 displays that system (4.6) without input is asymptotically stable.

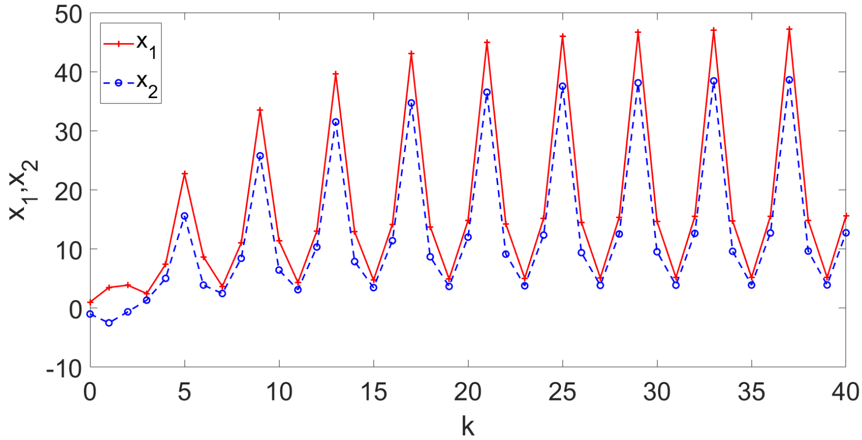

Figure 5. State of system (4.6) given $x_0 = (1, -1)^\top$ and $u(k) = (\sin^2(\frac{k\pi}{4}), \cos^2(\frac{k\pi}{4}))^\top$.

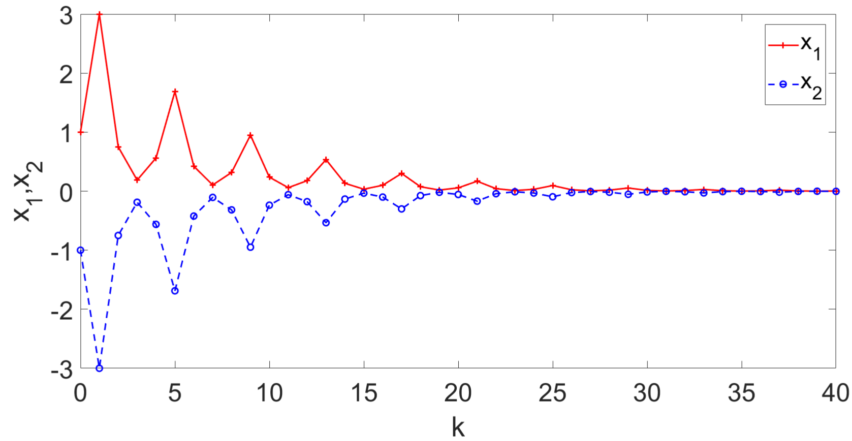

Figure 6. State of system (4.6) given $x_0 = (1, -1)^\top$ and $u(k) = 0$.

Remark 4.3. *According to the above analysis, it is clear that subsystems of system* (4.6) *are not ISS. Hence Example 3 demonstrates that we can use Theorem 3.1 to analyze ISS for switched systems constituted by all subsystems which are not ISS.*

5. Intermittent control of a chaotic system

By using Theorem 3.1, in this section we are going to design an intermittent controller to ensure ISS for a discrete-time disturbed Chua's chaotic system described by

$$x(k+1) = Ax(k) + G(x(k)) + u(k) + w(k), \tag{5.1}$$

where $A = \begin{pmatrix} -\alpha(1+b)+1 & \alpha & 0 \\ 1 & 0 & 1 \\ 0 & -\beta & 1 \end{pmatrix}$, $G(x) = \begin{pmatrix} g(x) \\ 0 \\ 0 \end{pmatrix}$, $x = (x_1, x_2, x_3)^\top \in \mathbb{R}^3$, $g(x) = -\frac{\alpha(a-b)(|x_1+1|-|x_1-1|)}{2}$ and constants $\alpha, \beta, a < b < 0$. For system (5.1), the control input is $u : \mathbb{R}^n \mapsto \mathbb{R}^n$ and the external disturbance is $w : \mathbb{R}^n \mapsto \mathbb{R}^n$.

The intermittent control is designed as follows:

$$u(k)=\begin{cases}Hx(k)+(-\alpha(a-b),0,0)^\top, \ x_1<-1, \bar{k}_i\le k<\bar{k}_i+T_i,\\ Hx(k)+(\alpha(a-b)x_1,0,0)^\top, \ 1\ge x_1\ge -1, \bar{k}_i\le k<\bar{k}_i+T_i,\\ Hx(k)+(\alpha(a-b),0,0)^\top, \ x_1>1, \bar{k}_i\le k<\bar{k}_i+T_i,\\ 0, \ \bar{k}_i+T_i\le k<\bar{k}_{i+1}, \ i\in\mathbb{Z}_+,\end{cases} \tag{5.2}$$

where H is a constant control matrix, $\bar{k}_i$ is the beginning time of the ith intermittent control and $0 < T_i \le \bar{k}_{i+1} - \bar{k}_i$ is the ith time period during which the control u is activated. It is required that T_i and $\bar{k}_{i+1} - \bar{k}_i$ are bounded and $\bar{k}_0 = 0$. Let $T_m = \min_{i\in\mathbb{Z}_+}\{T_i\}$ and $\bar{k}_m = \max_{i\in\mathbb{Z}_+}\{\bar{k}_{i+1} - \bar{k}_i - T_i\}$. It is clear that the switching times are $\bar{k}_i, \bar{k}_i + T_i$ ($i \in \mathbb{Z}_+$). Then we consider system (5.1) with the input control (5.2) as a switched system composed of two subsystems. Then we are going to stabilize system (5.1) to ISS by applying the intermittent controller (5.2) and Theorem 3.1.

By imposing the controller (5.2) on system (5.1), we will study the system described by

$$x(k+1)=\begin{cases}Ax(k)+Hx(k)+w(k), \ \bar{k}_i\le k<\bar{k}_i+T_i,\\ Ax(k)+G(x(k))+w(k), \ \bar{k}_i+T_i\le k<\bar{k}_{i+1}, \ i\in\mathbb{Z}_+.\end{cases} \tag{5.3}$$

Let $\{V_1(k,x) = V_2(k,x) = |x(k)|\}$ be a WMISSLF candidate. Then we have

$$|x(k+1)|\le\begin{cases}|A+H||x(k)|+|w|_\infty, \ \bar{k}_i\le k<\bar{k}_i+T_i,\\ (|A|+\alpha(b-a))|x(k)|+|w|_\infty, \ \bar{k}_i+T_i\le k<\bar{k}_{i+1}, \ i\in\mathbb{Z}_+.\end{cases}$$

According to Theorem 3.1 and Proposition 3.1, the following conclusion can be obtained.

Proposition 5.1. *For system* (5.1) *with the input control* (5.2)*, if we get a matrix H satisfying*

$$(|A|+\alpha(b-a))^{\bar{k}_m}|A+H|^{T_m}=\xi<1, \tag{5.4}$$

then the constraint (3.5) *with* $K = \bar{k}_m + T_m$ *from Theorem 3.1 is satisfied. Furthermore, system* (5.1) *with the input control* (5.2) *has ISS under the disturbance w.*

5.1. Simulation results

In this section, we investigate again system (5.1) with $\alpha = 9.2156$, $\beta = 15.9946$, $a = -1.2495$, $b = -0.75735$ and the disturbance

$$w(k)=2(\sin^2(k),\cos^2(k),rand(1))^\top.$$

These constraints are the same as those of Example 5.1 of [19].

Let $\bar{k}_m = 2$ and $T_m = 3$. In order to ensure that the condition (5.4) holds, we let

$$H=\begin{pmatrix}0.1+\alpha(1+b)-1 & -\alpha & 0\\ -1 & 0.1 & -1\\ 0 & \beta & 0.1-1\end{pmatrix}.$$

It is evident that $A+H=\begin{pmatrix}0.1 & 0 & 0\\ 0 & 0.1 & 0\\ 0 & 0 & 0.1\end{pmatrix}$, $|A|+\alpha(b-a)=23.0258$ and $(|A|+\alpha(b-a))^{\bar{k}_m}\times(0.1)^{T_m}=\xi=0.5302$.

Based on Theorem 3.1, system (5.1) is stabilized to ISS under the intermittent control (5.2) with the above given control matrix H. This is shown in Figure 7. The asymptotic stability of system (5.3) without the disturbance is displayed in Figure 8.

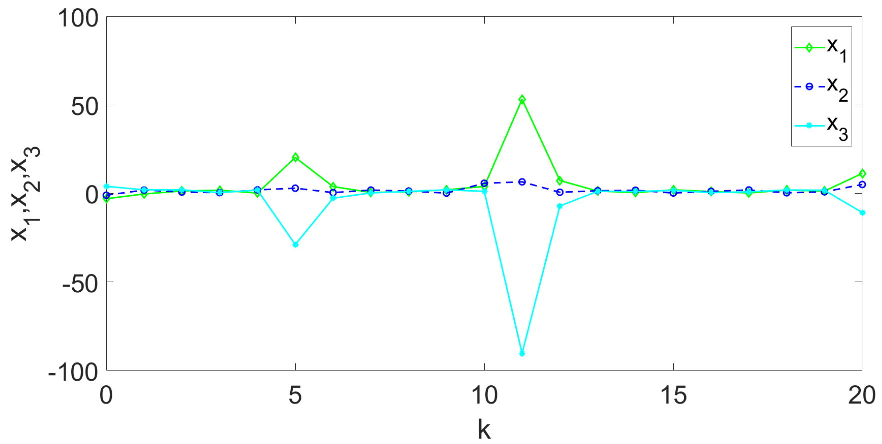

Figure 7. Sate of x for system (5.3) with the initial condition $x_0=(-3,-1,4)^{\top}$, $T_0=5,\bar{k}_1=6,T_1=4,\bar{k}_2=12,T_2=8,\bar{k}_3=20$.

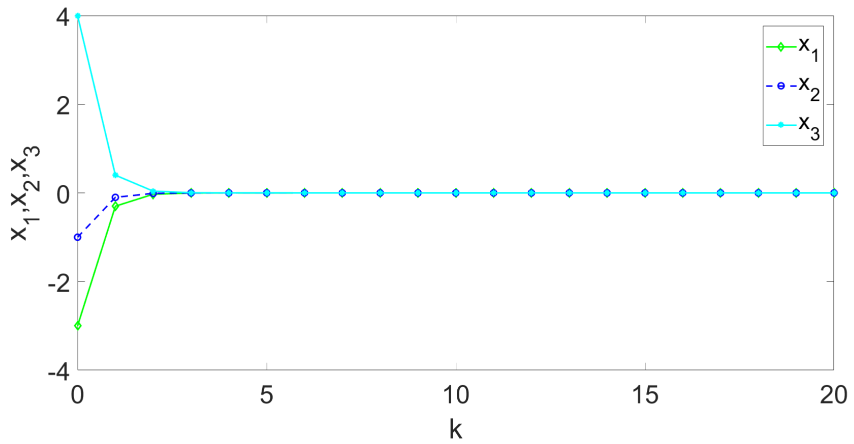

Figure 8. Sate of x for system (5.3) without the disturbance, but with the initial condition $x_0=(-3,-1,4)^{\top}$, $T_0=5,\bar{k}_1=6,T_1=4,\bar{k}_2=12,T_2=8,\bar{k}_3=20$.

6. Conclusions

For system (2.1) under a given switching signal, the less conservative sufficient requirements for ISS were obtained by using the defined WMISSLFs (Theorem 3.1). The key properties of Theorem 3.1 are described as follows. Compared with the results of [14, 16, 24, 25], the considered system can consist of all or some subsystems which are not ISS. Besides, the introduced function for an ISS subsystem could be increasing along the trajectory of the subsystem without input at some moments. Then for system (2.1) under any switching signal, the relaxed sufficient assumptions for ISS were attained by using the introduced WCISSLFs (Proposition 3.1). For this case, each subsystem of system (2.1) should be ISS. The function V from Proposition 3.1 may be increasing along the trajectory of each ISS subsystem of system (2.1) without input at some moments. The relationship between WMISSLFs for system (2.1) and WMLFs for system (2.1) without input was set up in Theorem 3.2. We also discussed the relationship between the WCISSLFs for system (2.1) and WCLFs for system (2.1) without input in Proposition 3.2. The efficacy of the main outcomes was illustrated through the three presented numerical examples. Based on Theorem 3.1 an intermittent controller was designed to ensure ISS for a discrete-time disturbed Chua's chaotic system. Proposition 5.1 describes the constraints which the control matrix H should satisfy. The efficiency of the designed intermittent control was demonstrated through the given simulation results. We will investigate how to construct WMISSLFs (WCISSLFs) for system (2.1) based on WMLFs (WCLFs) for system (2.1) without input in the future.

Use of AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported in part by the National Natural Science Foundation of China [NSFC11701533].

Conflict of interest

The author declares no conflicts of interest in this paper.

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