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Research article

Exploring the versatile properties and applications of multidimensional degenerate Hermite polynomials

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Abstract: In this study, we develop various features in special polynomials using the principle of monomiality, operational formalism, and other qualities. By utilizing the monomiality principle, new outcomes can be achieved while staying consistent with past knowledge. Furthermore, an explicit form satisfied by these polynomials is also derived. The emphasis of this study is to introduce the degenerate multidimensional Hermite polynomials (DMVHP) denoted as $\mathbb{H}_n^{[r]}(j_1, j_2, j_3, \cdots, j_r; \vartheta)$, which are closely related to the classical Hermite polynomials and are a significant class of orthogonal polynomials. The fundamental properties, such as symmetric identities for these polynomials are also established. An operational framework is also established for these polynomials.

Keywords: degenerate special polynomials; Hermite polynomials; monomiality principle; explicit form; operational formalism; symmetric identities
Mathematics Subject Classification: 11T23, 33B10, 33C45, 33E20, 33E30

1. Introduction and basic notions

Various mathematical disciplines, including enumerative combinatorics, algebraic combinatorics and applied mathematics, exhibit a keen interest in sequences of polynomials. For example, the utilization of Laguerre, Chebyshev, Legendre and Jacobi polynomials in solving specific ordinary differential equations within the realms of approximation theory and physics is well-documented. Hermite polynomials stand out as a significant subset within the family of polynomial sequences. The significance of Hermite polynomials lies in their versatility and wide-ranging applications across many fields of mathematics and science. They provide a powerful tool for solving problems and modeling complex systems, making them an essential part of many mathematical and scientific disciplines. The group of orthogonal polynomials known as Hermite polynomials was first developed by Hermite himself [1]. The Hermite polynomials are valuable because they possess various beneficial characteristics. One of their applications is to provide solutions to the Schrödinger equation and quantum harmonic oscillator in quantum mechanics. Additionally, they emerge in the study of random processes, particularly in the theory of Brownian motion. These polynomials find uses in numerical analysis, approximation theory and signal processing. They are commonly applied in the approximation of functions and the resolution of differential equations. These polynomials find applications in a wide range of fields, including mathematics, physics, engineering, and computer science. Hermite polynomials, for instance, play a crucial role in describing the wave functions of harmonic oscillators, forming the foundation for quantum theories related to light and research on hydrogen atoms. Furthermore, these polynomials are instrumental in deriving probability distribution functions for the kinetic energy of gas molecules and in analyzing the energy distribution of harmonic oscillators. Hermite polynomials find applications in computer science in various areas, especially in numerical methods, signal processing and computer vision. Hermite polynomials can be utilized in image and signal processing tasks, such as image compression, denoising and feature extraction. They have been used in wavelet transforms, where the polynomials form a basis set to represent signals or images in a sparse and efficient manner. Further, they are used in interpolation, which involves fitting a polynomial that passes through a set of given points and their corresponding derivatives. This can be useful for curve fitting, data smoothing and generating continuous functions from discrete data in computer graphics and animation. Also, these polynomials are employed in numerical methods for solving differential equations, integral equations and other mathematical problems that arise in computer simulations, scientific computing and computational modeling. In probability theory, Hermite polynomials are used in the context of the Hermite polynomial chaos expansion, which is a powerful tool for uncertainty quantification and propagation in computational models and simulations involving stochastic processes, see for example [2–6].

The significance of Hermite polynomials relies on their powerful properties. One of these properties is completeness, which means the possibility of representing any square-integrable real function in terms of a series of Hermite polynomials. These polynomials have significant applications in different fields of science, such as approximation theory and numerical analysis. Furthermore, they play a vital role in quantum mechanics, where they are considered a solution of the harmonic oscillator given in the Schrödinger equation. Moreover, they are efficient in studying random walks and Brownian motion. The Hermite polynomials also have applications in the biological and medical sciences. Numerous research endeavors have been dedicated to the development and exploration of distinctive features of degenerate special polynomials. This has been evidenced in studies such as [7–12]. More recently, in references [13–17], the authors introduced several doped polynomials of a specific nature and identified their unique characteristics and behaviors, which hold significant importance in engineering applications. These noteworthy properties encompass determinant representations, operational formalism, approximation attributes, Δ_h polynomial formulations, poly forms, degenerate configurations, summation theorems, approximation precision, explicit and implicit equations, and generative expressions.

Datolli [18] introduced the Hermite polynomials of three variables by the generating form:

$$\sum_{k=0}^{\infty} \mathbb{H}_k(u, v, w)\frac{\xi^k}{k!} = e^{u\xi+v\xi^2+w\xi^3}. \tag{1.1}$$

Exploring the degenerate manifestations of special functions holds a pivotal role in unraveling the mathematical underpinnings of various physical phenomena. These degenerate polynomials play a vital role in elucidating the dynamics of the quantum harmonic oscillator. Their versatile applications span the realms of science and engineering, with their value acknowledged in both theoretical mathematics and real-world implementations. As the theory of special functions advances, we anticipate future revelations and innovations on the horizon.

In the past few years, Ryoo and Hwang [8, 9] proposed two and three variable degenerate forms of Hermite polynomials:

$$\sum_{k=0}^{\infty} \mathcal{J}_k(u, v; \vartheta) \frac{t^k}{k!} = (1 + \vartheta)^{t(\frac{u+vt}{\vartheta})} \tag{1.2}$$

and

$$\sum_{k=0}^{\infty} \mathcal{F}_k(u, v, w; \vartheta) \frac{t^k}{k!} = (1 + \vartheta)^{t(\frac{u+vt+wt^2}{\vartheta})}. \tag{1.3}$$

In 1941, Steffenson [19] introduced the notion of Poweriod, where he presented the idea of monomiality. This idea was further developed by Dattoli [20, 21] and it has now evolved into a key instrument for the investigation of exceptional polynomials. Monomiality is the concept of expressing a polynomial set in view of monomials, which are the fundamental units of polynomials which enables a clear knowledge of the nurture and properties of these polynomials.

In the study of special polynomials, the two operators $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}$ are very important. They perform as multiplicative and derivative operators for $b_k(\xi)$,$k \in \mathbb{N}$ polynomial set, which allow constructing new polynomials using some existing ones. Note that the multiplicative property of the operator $\hat{\mathcal{M}}$ is defines as

$$b_{k+1}(\xi) = \hat{\mathcal{M}}\{b_k(\xi)\}, \tag{1.4}$$

which generates a new polynomial $b_{k+1}(\xi)$ from $b_k(\xi)$. In the same way, the relation

$$k\, b_{k-1}(\xi) = \hat{\mathcal{D}}\{b_k(\xi)\}, \tag{1.5}$$

provides the derivative property of the operator $\hat{\mathcal{D}}$.

Incorporating the fundamental principle of monomiality and operational rules into the examination of special polynomials offers a promising avenue for the creation of hybrid special polynomials. These hybrid polynomials serve as solutions to a wide array of mathematical challenges and have a significant impact on various applications in physics, engineering and computer science. The properties of special polynomials, such as orthogonality, interpolation and recursive relationships, make them valuable tools for solving a wide range of computational problems in computer economics. Their ability to approximate complex functions and represent economic data efficiently helps in understanding economic phenomena, making predictions and supporting decision-making processes. Researchers and practitioners continue to explore new ways to apply special polynomials to address emerging challenges in the field of computer economics, see for example [6, 22–25].

The Eqs (1.4) and (1.5) are part of the theory of quasi-monomials and Weyl groups, which have applications in various branches of mathematical physics, such as the representation theory, algebraic geometry and quantum field theory. Therefore, a set of polynomials $\{b_k(\xi)\}_{m\in\mathbb{N}}$ in view of operators represented by expressions (1.4) and (1.5) is mentioned by name quasi-monomial and fulfills the axiom:

$$[\hat{\mathcal{D}}, \hat{\mathcal{M}}] = \hat{\mathcal{D}}\hat{\mathcal{M}} - \hat{\mathcal{M}}\hat{\mathcal{D}} = \hat{1}, \tag{1.6}$$

thus presents a Weyl group structure.

In specific, $b_k(\xi)$ is a solution to the differential equation described by:

$$\hat{\mathcal{M}}\hat{\mathcal{D}}\{b_k(\xi)\} = k\, b_k(\xi), \tag{1.7}$$

where the operators $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}$ are subject to differential recognition. This equation implies that the quasi-monomials act as eigenfunctions of the operator $\hat{\mathcal{M}}\hat{\mathcal{D}}$ with an associated eigenvalue of k. For certain selections of $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}$, this differential equation can be explicitly solved, leading to clear expressions for the quasi-monomials, represented as:

$$b_k(\xi) = \hat{\mathcal{M}}^k\,\{1\}, \tag{1.8}$$

with the initial condition $b_0(\xi) = 1$. Equation (1.8) offers a recursive method for calculating the quasi-monomials through iterative application of the operator $\hat{\mathcal{M}}$ to the identity operator $\hat{I}$. Given the initial condition $b_0(\xi) = 1$, $b_k(\lambda)$ in exponential form can be expressed as:

$$e^{w\hat{\mathcal{M}}}\{1\} = \sum_{k=0}^{\infty} b_k(\xi)\frac{w^k}{k!}, \quad |w| < \infty \ , \tag{1.9}$$

by usage of identity (1.8).

The formula (1.9) serves as a generating formula that represents the quasi-monomials as a power series in the variable w, with the coefficients being determined by the quasi-monomials themselves. This formula facilitates the precise computation of quasi-monomials and finds applications in the examination of specific classes of special functions. Generally, the quasi-monomials theory and Weyl groups are employed to analyze the characterizations of certain functions and have many applications in mathematical physics [26–28].

The recent advancement in the theory of special functions has led to the development of multi-variable and multi-index special functions. These functions involve several variables and multiple indices, which provide a more flexible and efficient way of solving complex problems. Multi-variable special functions are useful for solving problems in quantum mechanics, statistical physics, and fluid dynamics, among others. They have wide-ranging applications in several fields, including engineering, computer science, and finance [2, 5, 29, 30].

In light of the importance of these degenerate special polynomials of mathematical physics and motivated by the previous discussion, we construct a multidimensional degenerate form of Hermite polynomials, which can be expressed in the following generating formula:

$$(1+\vartheta)^{\xi\left(\frac{j_1+j_2\xi+j_3\xi^2+\cdots+j_n\xi^{n-1}}{\vartheta}\right)} = \sum_{n=0}^{\infty} \mathbb{H}_n^{[r]}(j_1, j_2, j_3, \cdots, j_r; \vartheta)\frac{\xi^n}{n!}. \tag{1.10}$$

Accordingly, we study and derive their characterization. The paper is organized into several sections, each of which focuses on a different aspect of degenerate multidimensional Hermite polynomials. Section 2 provides an in-depth analysis of the structure of these polynomials and examines their fundamental properties. This section is crucial for establishing a solid foundation upon which subsequent sections can build. Further, the quasi-monomial properties of the degenerate multidimensional Hermite polynomials are derived. These properties are essential for understanding how polynomials behave and can be used in various applications. Section 3 is devoted to the establishment of symmetric identities that are fulfilled by these polynomials. The identification of these identities is a critical step in furthering our understanding of these polynomials and their properties. In Section 4, operational formalism is introduced, which provides a powerful tool for deriving the extended and generalized forms of these polynomials. This section is significant because it enables mathematicians to explore the behavior of polynomials in new and previously unexplored ways. Finally, the paper concludes with a summary of the findings presented in the previous sections. Together, the sections build upon one another to provide a comprehensive overview of degenerate multidimensional Hermite polynomials and their properties, making a significant contribution to the field of mathematics.

2. Degenerate multidimensional Hermite polynomials

The new class of degenerate multidimensional Hermite polynomials is examined in this section. Moreover, these polynomials are also produced with a number of properties. We defined degenerate multidimensional Hermite polynomials by the generating relation:

$$(1+\vartheta)^{\xi\left(\frac{j_1+j_2\xi+j_3\xi^2+\cdots+j_n\xi^{n-1}}{\vartheta}\right)}=\sum_{n=0}^{\infty}\mathbb{H}_n^{[r]}(j_1,j_2,j_3,\cdots,j_r;\vartheta)\frac{\xi^n}{n!} \tag{2.1}$$

or

$$(1+\vartheta)^{\frac{j_1\xi}{\vartheta}}(1+\vartheta)^{\frac{j_2\xi^2}{\vartheta}}(1+\vartheta)^{\frac{j_3\xi^3}{\vartheta}}\cdots(1+\vartheta)^{\frac{j_r\xi^r}{\vartheta}}=\sum_{n=0}^{\infty}\mathbb{H}_n^{[r]}(j_1,j_2,j_3,\cdots,j_r;\vartheta)\frac{\xi^n}{n!}. \tag{2.2}$$

Theorem 2.1. *The degenerate multidimensional Hermite polynomials* $\mathbb{H}_n^{[r]}(j_1,j_2,j_3,\cdots,j_r;\vartheta)$*,* $n\in\mathbb{N}$ *can be determined in terms of the power series expansion of the product* $(1+\vartheta)^{\frac{j_1\xi}{\vartheta}}(1+\vartheta)^{\frac{j_2\xi^2}{\vartheta}}$ $(1+\vartheta)^{\frac{j_3\xi^3}{\vartheta}}\cdots(1+\vartheta)^{\frac{j_r\xi^r}{\vartheta}}$*, that is*

$$\begin{aligned}(1+\vartheta)^{\frac{j_1\xi}{\vartheta}}(1+\vartheta)^{\frac{j_2\xi^2}{\vartheta}}(1+\vartheta)^{\frac{j_3\xi^3}{\vartheta}}\cdots(1+\vartheta)^{\frac{j_r\xi^r}{\vartheta}}&=\mathbb{H}_0^{[r]}(j_1,j_2,j_3,\cdots,j_r;\vartheta)+\mathbb{H}_1^{[r]}(j_1,j_2,\\ j_3,\cdots,j_r;\vartheta)\,\frac{\xi}{1!}+\mathbb{H}_2^{[r]}(j_1,j_2,j_3,\cdots,j_r;\vartheta)\frac{\xi^2}{2!}&+\cdots+\mathbb{H}_n^{[r]}(j_1,j_2,j_3,\cdots,j_r;\vartheta)\frac{\xi^m}{m!}+\cdots.\end{aligned} \tag{2.3}$$

Proof. Using the Newton series for finite differences at $j_1=j_2=\cdots=j_r=0$ and ordering the product of the functions $(1+\vartheta)^{\frac{j_1\xi}{\vartheta}}(1+\vartheta)^{\frac{j_2\xi^2}{\vartheta}}(1+\vartheta)^{\frac{j_3\xi^3}{\vartheta}}\cdots(1+\vartheta)^{\frac{j_r\xi^r}{\vartheta}}$ according to the powers of ξ, we obtain a series of special polynomials denoted by $\mathbb{H}_n^{[r]}(j_1,j_2,j_3,\cdots,j_r;\vartheta)$. These polynomials are expressed in Eq (2.3) as coefficients of $\frac{\xi^m}{m!}$, and they serve as the generating function for degenerate multidimensional Hermite polynomials. □

Next, we find the explicit form satisfied by the degenerate multidimensional Hermite polynomials $\mathbb{H}_n^{[r]}(j_1,j_2,j_3,\cdots,j_r;\vartheta)$, $n\in\mathbb{N}$ by proving the succeeding result:

Theorem 2.2. *The degenerate multidimensional Hermite polynomials $\mathbb{H}_n^{[r]}(j_1, j_2, j_3, \cdots, j_r; \vartheta)$, $n \in \mathbb{N}$ satisfy the succeeding explicit form:*

$$\mathbb{H}_n^{[r]}(j_1, j_2, j_3, \cdots, j_r; \vartheta) = n! \sum_{k=0}^{[\frac{n}{k}]} \left(\frac{j_r}{\vartheta} \log(1+\vartheta)\right)^k \frac{\mathbb{H}_{n-kr}^{[r]}(j_1, j_2, j_3, \cdots, j_{r-1}; \vartheta)}{k!\,(n-rk)!}. \tag{2.4}$$

Proof. The generating function (2.2) can be written as:

$$\left\{(1+\vartheta)^{\frac{j_1\xi}{\vartheta}}(1+\vartheta)^{\frac{j_2\xi^2}{\vartheta}}(1+\vartheta)^{\frac{j_3\xi^3}{\vartheta}}\cdots(1+\vartheta)^{\frac{j_r\xi^r}{\vartheta}}\right\} = \sum_{n=0}^{\infty} \mathbb{H}_n^{[r]}(j_1, j_2, j_3, \cdots, j_{r-1}; \vartheta)\frac{\xi^n}{n!} \sum_{k=0}^{\infty} \frac{\left(\frac{j_r}{\vartheta} \log(1+\vartheta)\right)^k \xi^{rk}}{k!}. \tag{2.5}$$

substituting $n \to n-rk$ in the right hand side of previous expression and using the Cauchy product rule in the resultant expression, result (2.2) is proved. □

Further, we establish the multiplicative and derivative operators for degenerate multidimensional Hermite polynomials $\mathbb{H}_n^{[r]}(j_1, j_2, j_3, \cdots, j_r; \vartheta)$, by proving the results listed as follows:

Theorem 2.3. *For DMVHP $\mathbb{H}_n^{[r]}(j_1, j_2, j_3, \cdots, j_r; \vartheta)$, the following multiplicative and derivative operators hold true:*

$$\hat{\mathbb{M}}_{\mathbb{H}^{[r]}} = \left(j_1 \frac{log(1+\vartheta)}{\vartheta} + 2j_2\frac{\partial}{\partial j_1} + 3j_3\Big(\frac{\vartheta}{log(1+\vartheta)}\Big)\frac{\partial^2}{\partial j_1^2} + \cdots + r j_r\Big(\frac{\vartheta}{log(1+\vartheta)}\Big)^{r-1}\frac{\partial^{r-1}}{\partial j_1^{r-1}}\right) \tag{2.6}$$

and

$$\hat{\mathbb{D}}_{\mathbb{H}^{[r]}} = \frac{\vartheta}{log(1+\vartheta)}\, D_{j_1}. \tag{2.7}$$

Proof. Differentiating (2.2) with respect to j_1 partially, we find

$$\begin{aligned}&\frac{\partial}{\partial j_1}\left\{(1+\vartheta)^{\frac{j_1\xi}{\vartheta}}(1+\vartheta)^{\frac{j_2\xi^2}{\vartheta}}(1+\vartheta)^{\frac{j_3\xi^3}{\vartheta}}\cdots(1+\vartheta)^{\frac{j_r\xi^r}{\vartheta}}\right\} = \\ &\frac{\xi}{\vartheta}\, log(1+\vartheta)\left\{(1+\vartheta)^{\frac{j_1\xi}{\vartheta}}(1+\vartheta)^{\frac{j_2\xi^2}{\vartheta}}(1+\vartheta)^{\frac{j_3\xi^3}{\vartheta}}\cdots(1+\vartheta)^{\frac{j_r\xi^r}{\vartheta}}\right\}.\end{aligned} \tag{2.8}$$

Thus (2.8) can be written in the form of identity as

$$\begin{aligned}&\frac{\vartheta}{log(1+\vartheta)}\frac{\partial}{\partial j_1}\left\{(1+\vartheta)^{\frac{j_1\xi}{\vartheta}}(1+\vartheta)^{\frac{j_2\xi^2}{\vartheta}}(1+\vartheta)^{\frac{j_3\xi^3}{\vartheta}}\cdots(1+\vartheta)^{\frac{j_r\xi^r}{\vartheta}}\right\} = \\ &\xi\left\{(1+\vartheta)^{\frac{j_1\xi}{\vartheta}}(1+\vartheta)^{\frac{j_2\xi^2}{\vartheta}}(1+\vartheta)^{\frac{j_3\xi^3}{\vartheta}}\cdots(1+\vartheta)^{\frac{j_r\xi^r}{\vartheta}}\right\}\end{aligned} \tag{2.9}$$

and further simplified in the form

$$\frac{\vartheta}{log(1+\vartheta)}\frac{\partial}{\partial j_1}\left\{\sum_{n=0}^{\infty}\mathbb{H}_n^{[r]}(j_1, j_2, j_3, \cdots, j_r; \vartheta)\frac{\xi^n}{n!}\right\} = \xi\left\{\sum_{n=0}^{\infty}\mathbb{H}_n^{[r]}(j_1, j_2, j_3, \cdots, j_r; \vartheta)\frac{\xi^n}{n!}\right\}. \tag{2.10}$$

On taking partial differentiation of (2.2) on both sides w.r.t. ξ, it follows that

$$\frac{\partial}{\partial\xi}\left\{(1+\vartheta)^{\frac{j_1\xi}{\vartheta}}(1+\vartheta)^{\frac{j_2\xi^2}{\vartheta}}(1+\vartheta)^{\frac{j_3\xi^3}{\vartheta}}\cdots(1+\vartheta)^{\frac{j_r\xi^r}{\vartheta}}\right\}=\frac{\partial}{\partial\xi}\left\{\sum_{n=0}^{\infty}\mathbb{H}_n^{[r]}(j_1,j_2,j_3,\cdots,j_r;\vartheta)\frac{\xi^n}{n!}\right\}, \tag{2.11}$$

which further can be written as

$$\begin{gathered}\left(\frac{j_1}{\vartheta}+2\frac{j_2}{\vartheta}\xi+3\frac{j_3}{\vartheta}\xi^2+\cdots+r\frac{j_r}{\vartheta}\xi^{r-1}\right)\left\{(1+\vartheta)^{\frac{j_1\xi}{\vartheta}}(1+\vartheta)^{\frac{j_2\xi^2}{\vartheta}}(1+\vartheta)^{\frac{j_3\xi^3}{\vartheta}}\cdots(1+\vartheta)^{\frac{j_r\xi^r}{\vartheta}}\right\}\\ =\sum_{n=0}^{\infty}\ n\ \mathbb{H}_n^{[r]}(j_1,j_2,j_3,\cdots,j_r;\vartheta)\frac{\xi^{n-1}}{n!}.\end{gathered} \tag{2.12}$$

Thus, in view of expression (2.10), expression (2.12) can be written as

$$\begin{gathered}\left(\frac{j_1}{\vartheta}+2\frac{j_2}{log(1+\vartheta)}\frac{\partial}{\partial j_1}+3j_3\frac{\vartheta}{(log(1+\vartheta))^2}\frac{\partial^2}{\partial {j_1}^2}+\cdots+rj_r\frac{\vartheta^{r-2}}{(log(1+\vartheta))^{r-1}}\right)\left\{(1+\vartheta)^{\frac{j_1\xi}{\vartheta}}(1+\vartheta)^{\frac{j_2\xi^2}{\vartheta}}\right.\\ \left.\times(1+\vartheta)^{\frac{j_3\xi^3}{\vartheta}}\cdots(1+\vartheta)^{\frac{j_r\xi^r}{\vartheta}}\right\}=\sum_{n=0}^{\infty}\ n\ \mathbb{H}_n^{[r]}(j_1,j_2,j_3,\cdots,j_r;\vartheta)\frac{\xi^{n-1}}{n!}.\end{gathered} \tag{2.13}$$

Inserting the r.h.s. of (2.2) on the r.h.s. of previous expression and simplifying, we obtain

$$\begin{gathered}\left(j_1\frac{log(1+\vartheta)}{\vartheta}+2j_2\frac{\partial}{\partial j_1}+3j_3\Big(\frac{\vartheta}{log(1+\vartheta)}\Big)\frac{\partial^2}{\partial {j_1}^2}+\cdots+rj_r\Big(\frac{\vartheta}{log(1+\vartheta)}\Big)^{r-2}\frac{\partial^{r-1}}{\partial {j_1}^{r-1}}\right)\\ \times\left\{\sum_{n=0}^{\infty}\mathbb{H}_n^{[r]}(j_1,j_2,j_3,\cdots,j_r;\vartheta)\frac{\xi^n}{n!}\right\}=\sum_{n=0}^{\infty}\ n\ \mathbb{H}_n^{[r]}(j_1,j_2,j_3,\cdots,j_r;\vartheta)\frac{\xi^{n-1}}{n!}.\end{gathered} \tag{2.14}$$

For, $n\to n+1$ in the r.h.s. of (2.14) and comparing exponents of $\frac{\xi^n}{n!}$, assertion (2.6) is obtained. Moreover, in view of identity expression (2.10), we find

$$\frac{\vartheta}{log(1+\vartheta)}\ \frac{\partial}{\partial j_1}\left\{\sum_{n=0}^{\infty}\mathbb{H}_n^{[r]}(j_1,j_2,j_3,\cdots,j_r;\vartheta)\frac{\xi^n}{n!}\right\}=\left\{\sum_{n=0}^{\infty}\mathbb{H}_n^{[r]}(j_1,j_2,j_3,\cdots,j_r;\vartheta)\frac{\xi^{n+1}}{n!}\right\}. \tag{2.15}$$

For, $n\to n-1$ in r.h.s. of above expression (2.15) and comparing exponents of $\frac{\xi^n}{n!}$, assertion (2.7) is obtained. □

Next, we establish the differential equation satisfied by DMVHP $\mathbb{H}_n^{[r]}(j_1,j_2,j_3,\cdots,j_r;\vartheta)$ by demonstrating the result listed below:

Theorem 2.4. *The DMVHP* $\mathbb{H}_n^{[r]}(j_1,j_2,j_3,\cdots,j_r;\vartheta)$ *satisfies the differential equation:*

$$\begin{gathered}\left(j_1\frac{log(1+\vartheta)}{\vartheta}\frac{\partial}{\partial j_1}+2j_2\frac{\partial^2}{\partial {j_1}^2}+3j_3\Big(\frac{\vartheta}{log(1+\vartheta)}\Big)\frac{\partial^3}{\partial j_1^3}+\cdots+rj_r\Big(\frac{\vartheta}{log(1+\vartheta)}\Big)^{r-1}\frac{\partial^r}{\partial j_1^r}\right.\\ \left.-n\frac{log(1+\vartheta)}{\vartheta}\right)=0.\end{gathered} \tag{2.16}$$

Proof. By operation of expressions (2.6) and (2.7), and in view of expression (1.7), assertion (2.16) is verified.

□

3. Symmetric identities

In the current section, we establish the symmetrical relations for the DMVHP $\mathbb{H}_n^{[r]}(j_1, j_2, j_3, \cdots, j_r; \vartheta)$ by demonstrating the result listed below:

Theorem 3.1. *For, $\mathfrak{A} \neq \mathfrak{B}$, with $\mathfrak{A}, \mathfrak{B} > 0$, it follows that:*

$$\mathfrak{A}^n \mathbb{H}_n^{[r]}(\mathfrak{B} j_1, \mathfrak{B}^2 j_2, \mathfrak{B}^3 j_3, \cdots, \mathfrak{B}^r j_r; \vartheta) = \mathfrak{B}^n \mathbb{H}_n^{[r]}(\mathfrak{A} j_1, \mathfrak{A}^2 j_2, \mathfrak{A}^3 j_3, \cdots, \mathfrak{A}^r j_r; \vartheta). \tag{3.1}$$

Proof. Given that $\mathfrak{A} \neq \mathfrak{B}$, with $\mathfrak{A}, \mathfrak{B} > 0$, we proceed in the manner designated as:

$$\mathbb{R}(\xi; j_1, j_2, j_3, \cdots, j_r; \vartheta) = (1+\vartheta)^{\frac{\mathfrak{A}\mathfrak{B} j_1 \xi}{\vartheta}} (1+\vartheta)^{\frac{\mathfrak{A}^2\mathfrak{B}^2 j_2 \xi^2}{\vartheta}} (1+\vartheta)^{\frac{\mathfrak{A}^3\mathfrak{B}^3 j_3 \xi^3}{\vartheta}} \cdots (1+\vartheta)^{\frac{\mathfrak{A}^r\mathfrak{B}^r j_r \xi^r}{\vartheta}}. \tag{3.2}$$

Thus, the previous expression (3.2) $\mathbb{R}(\xi; j_1, j_2, j_3, \cdots, j_r; \vartheta)$ is symmetric in $\mathfrak{A}$ and $\mathfrak{B}$. Consequently, we write

$$\begin{aligned} \mathbb{R}(\xi; j_1, j_2, j_3, \cdots, j_r; \vartheta) &= \mathbb{H}_n^{[r]}(\mathfrak{A} j_1, \mathfrak{A}^2 j_2, \mathfrak{A}^3 j_3, \cdots, \mathfrak{A}^r j_r; \vartheta) \frac{(\mathfrak{B}\xi)^n}{n!} \\ &= \mathfrak{B}^n\, \mathbb{H}_n^{[r]}(\mathfrak{A} j_1, \mathfrak{A}^2 j_2, \mathfrak{A}^3 j_3, \cdots, \mathfrak{A}^r j_r; \vartheta) \frac{\xi^n}{n!}. \end{aligned} \tag{3.3}$$

Therefore, it follows that

$$\begin{aligned} \mathbb{R}(\xi; j_1, j_2, j_3, \cdots, j_r; \vartheta) &= \mathbb{H}_n^{[r]}(\mathfrak{B} j_1, \mathfrak{B}^2 j_2, \mathfrak{B}^3 j_3, \cdots, \mathfrak{B}^r j_r; \vartheta) \frac{(\mathfrak{A}\xi)^n}{n!} \\ &= \mathfrak{A}^n\, \mathbb{H}_n^{[r]}(\mathfrak{B} j_1, \mathfrak{B}^2 j_2, \mathfrak{B}^3 j_3, \cdots, \mathfrak{B}^r j_r; \vartheta) \frac{\xi^n}{n!}. \end{aligned} \tag{3.4}$$

On comparing the exponents of same powers of ξ in the previous expressions (3.3) and (3.4), the assertion (3.1) is achieved. □

Theorem 3.2. *For, $\mathfrak{A} \neq \mathfrak{B}$, with $\mathfrak{A}, \mathfrak{B} > 0$, we find*

$$\sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} \binom{k}{m} \mathfrak{A}^k \mathfrak{B}^{n+1-k} \mathbb{H}_{k-m}^{[r]}(\mathfrak{B} j_1, \mathfrak{B}^2 j_2, \mathfrak{B}^3 j_3, \cdots, \mathfrak{B}^r j_r; \vartheta) \mathbb{P}_{n-k}(\mathfrak{A}-1; \vartheta) =$$

$$\sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} \binom{k}{m} \mathfrak{B}^k \mathfrak{A}^{n+1-k} \mathbb{H}_{k-m}^{[r]}(\mathfrak{A} j_1, \mathfrak{A}^2 j_2, \mathfrak{A}^3 j_3, \cdots, \mathfrak{A}^r j_r; \vartheta) \mathbb{P}_{n-k}(\mathfrak{B}-1; \vartheta). \tag{3.5}$$

Proof. As it is given that $\mathfrak{A} \neq \mathfrak{B}$, with $\mathfrak{A}, \mathfrak{B} > 0$, we proceed by manner designated as:

$$\begin{aligned} \mathbb{S}(\xi; j_1, j_2, j_3, \cdots, j_r; \vartheta) &= \mathfrak{A}\mathfrak{B}\xi (1+\vartheta)^{\frac{\mathfrak{A}\mathfrak{B} j_1 \xi}{\vartheta}} (1+\vartheta)^{\frac{\mathfrak{A}^2\mathfrak{B}^2 j_2 \xi^2}{\vartheta}} (1+\vartheta)^{\frac{\mathfrak{A}^3\mathfrak{B}^3 j_3 \xi^3}{\vartheta}} \cdots \\ &\times \frac{(1+\vartheta)^{\frac{\mathfrak{A}^r\mathfrak{B}^r j_r \xi^r}{\vartheta}} \left((1+\vartheta)^{\frac{\mathfrak{A}\mathfrak{B}\xi}{\vartheta}} - 1 \right)}{\left((1+\vartheta)^{\frac{\mathfrak{A}\xi}{\vartheta}} - 1 \right) \left((1+\vartheta)^{\frac{\mathfrak{B}\xi}{\vartheta}} - 1 \right)}. \end{aligned} \tag{3.6}$$

By continuing in the same manner as in the above theorem, assertion (3.5) is proved.

□

Theorem 3.3. *For, $\mathfrak{A} \neq \mathfrak{B}$, with $\mathfrak{A}, \mathfrak{B} > 0$, we find*

$$\mathfrak{A}^n \mathbb{H}_n^{[r]}(\mathfrak{B} j_1, \mathfrak{B}^2 j_2, \mathfrak{B}^3 j_3, \cdots, \mathfrak{B}^r j_r) = \mathfrak{B}^n \mathbb{H}_n^{[r]}(\mathfrak{A} j_1, \mathfrak{A}^2 j_2, \mathfrak{A}^3 j_3, \cdots, \mathfrak{A}^r j_r). \tag{3.7}$$

Proof. As far as $\mathfrak{A} \neq \mathfrak{B}$, with $\mathfrak{A}, \mathfrak{B} > 0$, we proceed in the manner designated as:

$$\mathbb{G}(\xi; j_1, j_2, j_3, \cdots, j_r; \vartheta) = (1+\vartheta)^{\frac{\mathfrak{A}\mathfrak{B}j_1\xi}{\vartheta}}(1+\vartheta)^{\frac{\mathfrak{A}^2\mathfrak{B}^2j_2\xi^2}{\vartheta}}(1+\vartheta)^{\frac{\mathfrak{A}^3\mathfrak{B}^3j_3\xi^3}{\vartheta}}\cdots(1+\vartheta)^{\frac{\mathfrak{A}^r\mathfrak{B}^rj_r\xi^r}{\vartheta}}. \tag{3.8}$$

Thus, the expression (3.7) $\mathbb{G}(\xi; j_1, j_2, j_3, \cdots, j_r; \vartheta)$ is symmetric in $\mathfrak{A}$ and $\mathfrak{B}$.

□

Theorem 3.4. *For, $\mathfrak{A} \neq \mathfrak{B}$, with $\mathfrak{A}, \mathfrak{B} > 0$, we find*

$$\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}\mathfrak{A}^k\mathfrak{B}^{n+1-k}\mathcal{B}_n(\vartheta)\mathbb{H}_{k-m}^{[r]}(\mathfrak{B}j_1, \mathfrak{B}^2j_2, \mathfrak{B}^3j_3, \cdots, \mathfrak{B}^rj_r; \vartheta)\sigma_{n-k}(\mathfrak{A}-1;\vartheta) =$$

$$\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}\mathfrak{B}^k\mathfrak{A}^{n+1-k}\mathcal{B}_n(\vartheta)\mathbb{H}_{k-m}^{[r]}(\mathfrak{A}j_1, \mathfrak{A}^2j_2, \mathfrak{A}^3j_3, \cdots, \mathfrak{A}^rj_r; \vartheta)\sigma_{n-k}(\mathfrak{B}-1;\vartheta). \tag{3.9}$$

Proof.

$$\mathbb{G}(\xi; j_1, j_2, j_3, \cdots, j_r; \vartheta) = \mathfrak{A}\mathfrak{B}\xi\,(1+\vartheta)^{\frac{\mathfrak{A}\mathfrak{B}j_1\xi}{\vartheta}}(1+\vartheta)^{\frac{\mathfrak{A}^2\mathfrak{B}^2j_2\xi^2}{\vartheta}}(1+\vartheta)^{\frac{\mathfrak{A}^3\mathfrak{B}^3j_3\xi^3}{\vartheta}}\cdots(1+\vartheta)^{\frac{\mathfrak{A}^r\mathfrak{B}^rj_r\xi^r}{\vartheta}}$$
$$\times\frac{\left((1+\vartheta)^{\frac{\mathfrak{A}\mathfrak{B}\xi}{\vartheta}}-1\right)}{\left((1+\vartheta)^{\frac{\mathfrak{A}\xi}{\vartheta}}-1\right)\left((1+\vartheta)^{\frac{\mathfrak{B}\xi}{\vartheta}}-1\right)}. \tag{3.10}$$

The above expression (3.10) can be expressed as

$$\mathbb{G}(\xi; j_1, j_2, j_3, \cdots, j_r; \vartheta) = \frac{\mathfrak{A}\mathfrak{B}\xi}{\left((1+\vartheta)^{\frac{\mathfrak{A}\xi}{\vartheta}}-1\right)}(1+\vartheta)^{\frac{\mathfrak{A}\mathfrak{B}j_1\xi}{\vartheta}}(1+\vartheta)^{\frac{\mathfrak{A}^2\mathfrak{B}^2j_2\xi^2}{\vartheta}}(1+\vartheta)^{\frac{\mathfrak{A}^3\mathfrak{B}^3j_3\xi^3}{\vartheta}}$$
$$\times\cdots(1+\vartheta)^{\frac{\mathfrak{A}^r\mathfrak{B}^rj_r\xi^r}{\vartheta}}\frac{\left((1+\vartheta)^{\frac{\mathfrak{A}\mathfrak{B}\xi}{\vartheta}}-1\right)}{\left((1+\vartheta)^{\frac{\mathfrak{B}\xi}{\vartheta}}-1\right)}$$
$$= \mathfrak{B}\sum_{m=0}^{\infty}\mathcal{B}_m(\vartheta)\frac{(\mathfrak{A}\xi)^m}{m!}\sum_{k=0}^{\infty}\mathbb{H}_k^{[r]}(\mathfrak{B}j_1, \mathfrak{B}^2j_2, \mathfrak{B}^3j_3, \cdots, \mathfrak{B}^rj_r; \vartheta)\frac{(\mathfrak{A}\xi)^k}{k!}\sum_{n=0}^{\infty}\sigma_n(\mathfrak{A}-1;\vartheta)\frac{(\mathfrak{B}\xi)^n}{n!}$$
$$= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}\mathfrak{A}^k\mathfrak{B}^{n+1-k}\mathcal{B}_n(\vartheta)\mathbb{H}_{k-m}^{[r]}(\mathfrak{B}j_1, \mathfrak{B}^2j_2, \mathfrak{B}^3j_3, \cdots, \mathfrak{B}^rj_r; \vartheta)\sigma_{n-k}(\mathfrak{A}-1;\vartheta)\right]\frac{\xi^n}{n!}. \tag{3.11}$$

In the similar fashion we can write

$$\mathbb{G}(\xi; j_1, j_2, j_3, \cdots, j_r; \vartheta) = \frac{\mathfrak{A}\mathfrak{B}\xi}{\left((1+\vartheta)^{\frac{\mathfrak{B}\xi}{\vartheta}}-1\right)}(1+\vartheta)^{\frac{\mathfrak{A}\mathfrak{B}j_1\xi}{\vartheta}}(1+\vartheta)^{\frac{\mathfrak{A}^2\mathfrak{B}^2j_2\xi^2}{\vartheta}}(1+\vartheta)^{\frac{\mathfrak{A}^3\mathfrak{B}^3j_3\xi^3}{\vartheta}}$$
$$\times\cdots(1+\vartheta)^{\frac{\mathfrak{A}^r\mathfrak{B}^rj_r\xi^r}{\vartheta}}\frac{\left((1+\vartheta)^{\frac{\mathfrak{A}\mathfrak{B}\xi}{\vartheta}}-1\right)}{\left((1+\vartheta)^{\frac{\mathfrak{A}\xi}{\vartheta}}-1\right)}$$
$$= \mathfrak{A}\sum_{m=0}^{\infty}\mathcal{B}_m(\vartheta)\frac{(\mathfrak{B}\xi)^m}{m!}\sum_{k=0}^{\infty}\mathbb{H}_k^{[r]}(\mathfrak{A}j_1, \mathfrak{A}^2j_2, \mathfrak{A}^3j_3, \cdots, \mathfrak{A}^rj_r; \vartheta)\frac{(\mathfrak{B}\xi)^k}{k!}\sum_{n=0}^{\infty}\sigma_n(\mathfrak{B}-1;\vartheta)\frac{(\mathfrak{A}\xi)^n}{n!}$$
$$= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}\mathfrak{B}^k\mathfrak{A}^{n+1-k}\mathcal{B}_n(\vartheta)\mathbb{H}_{k-m}^{[r]}(\mathfrak{A}j_1, \mathfrak{A}^2j_2, \mathfrak{A}^3j_3, \cdots, \mathfrak{A}^rj_r; \vartheta)\sigma_{n-k}(\mathfrak{B}-1;\vartheta)\right]\frac{\xi^n}{n!}. \tag{3.12}$$

Comparing the same exponents of expression (3.11) and (3.12), assertion (3.9) is established.

□

4. Operational formalism

Operational techniques are indeed powerful tools used to generate new families of polynomials, especially in the context of doped-type polynomials. These techniques involve applying certain differential or difference operators to a given parental polynomial, which yields a new polynomial with related properties. These operators can be used to generate families of orthogonal polynomials that are related to classical orthogonal polynomials but possess additional symmetries. Furthermore, operational techniques are useful in umbral calculus, which involves introducing a new variable that satisfies certain algebraic properties. This allows for the construction of new polynomial families that are related to classical polynomials, but with additional properties. For instance, the Bernoulli polynomials can be written in terms of an umbral variable, which allows for the construction of new families of polynomials known as the Sheffer sequences. Thus, operational techniques are powerful tools for generating new families of polynomials that have properties related to the parental polynomial, and they have important applications in a variety of mathematical and physical contexts.

Differentiating successively (2.2) w.r.t. $j_1, j_2, j_3, \cdots, j_r$, we find

$$D_{j_1}\Big\{\mathbb{H}_n^{[r]}(j_1, j_2, j_3, \cdots, j_r; \vartheta)\Big\} = \frac{log(1+\vartheta)}{\vartheta}\, n \,\Big\{\mathbb{H}_{n-1}^{[r]}(j_1, j_2, j_3, \cdots, j_r; \vartheta)\Big\}, \tag{4.1}$$

$$D^2_{j_1}\Big\{\mathbb{H}_n^{[r]}(j_1, j_2, j_3, \cdots, j_r; \vartheta)\Big\} = \Big(\frac{log(1+\vartheta)}{\vartheta}\Big)^2 n(n-1) \,\Big\{\mathbb{H}_{n-1}^{[r]}(j_1, j_2, j_3, \cdots, j_r; \vartheta)\Big\}, \tag{4.2}$$

$$D^3_{j_1}\Big\{\mathbb{H}_n^{[r]}(j_1, j_2, j_3, \cdots, j_r; \vartheta)\Big\} = \Big(\frac{log(1+\vartheta)}{\vartheta}\Big)^3 n(n-1)(n-2) \,\Big\{\mathbb{H}_{n-1}^{[r]}(j_1, j_2, j_3, \cdots, j_r; \vartheta)\Big\}, \tag{4.3}$$

$$\vdots \qquad\qquad \vdots$$

$$\begin{aligned} D^r_{j_1}\Big\{\mathbb{H}_n^{[r]}(j_1, j_2, j_3, \cdots, j_r; \vartheta)\Big\} = \Big(\tfrac{log(1+\vartheta)}{\vartheta}\Big)^r n(n-1)(n-2)\cdots(n-r+1) \\ \times \Big\{\mathbb{H}_{n-1}^{[r]}(j_1, j_2, j_3, \cdots, j_r; \vartheta)\Big\}. \end{aligned} \tag{4.4}$$

Also,

$$D_{j_2}\Big\{\mathbb{H}_n^{[r]}(j_1, j_2, j_3, \cdots, j_r; \vartheta)\Big\} = \frac{log(1+\vartheta)}{\vartheta}\, n(n-1) \,\Big\{\mathbb{H}_{n-1}^{[r]}(j_1, j_2, j_3, \cdots, j_r; \vartheta)\Big\}, \tag{4.5}$$

$$D_{j_3}\Big\{\mathbb{H}_n^{[r]}(j_1, j_2, j_3, \cdots, j_r; \vartheta)\Big\} = \frac{log(1+\vartheta)}{\vartheta}\, n(n-1)(n-2) \,\Big\{\mathbb{H}_{n-1}^{[r]}(j_1, j_2, j_3, \cdots, j_r; \vartheta)\Big\} \tag{4.6}$$

$$\vdots \qquad\qquad \vdots$$

$$D_{j_r}\Big\{\mathbb{H}_n^{[r]}(j_1, j_2, j_3, \cdots, j_r; \vartheta)\Big\} = \frac{log(1+\vartheta)}{\vartheta}\, n(n-1)(n-2)\cdots(n-r+1)\Big\{\mathbb{H}_{n-1}^{[r]}(j_1, j_2, j_3, \cdots, j_r; \vartheta)\Big\} \tag{4.7}$$

respectively.

In consideration of Eqs (4.1)–(4.7), it follows that $\mathbb{H}_{n-1}^{[r]}(j_1, j_2, j_3, \cdots, j_r; \vartheta)$ are the solutions of the equations:

$$\frac{\vartheta}{log(1+\vartheta)}\, D^2_{j_1}\Big\{\mathbb{H}_{n-1}^{[r]}(j_1, j_2, j_3, \cdots, j_r; \vartheta)\Big\} = D_{j_2}\Big\{\mathbb{H}_{n-1}^{[r]}(j_1, j_2, j_3, \cdots, j_r; \vartheta)\Big\}, \tag{4.8}$$

$$\left(\frac{\vartheta}{log(1+\vartheta)}\right)^2 D_{j_1}^3\left\{\mathbb{H}_{n-1}^{[r]}(j_1, j_2, j_3, \cdots, j_r; \vartheta)\right\} = D_{j_3}\left\{\mathbb{H}_{n-1}^{[r]}(j_1, j_2, j_3, \cdots, j_r; \vartheta)\right\} \tag{4.9}$$

$$\vdots \qquad\qquad\qquad \vdots$$

and

$$\left(\frac{\vartheta}{log(1+\vartheta)}\right)^{r-1} D_{j_1}^r\left\{\mathbb{H}_{n-1}^{[r]}(j_1, j_2, j_3, \cdots, j_r; \vartheta)\right\} = D_{j_r}\left\{\mathbb{H}_{n-1}^{[r]}(j_1, j_2, j_3, \cdots, j_r; \vartheta)\right\} \tag{4.10}$$

respectively, under the following initial condition

$$\mathbb{H}_n^{[r]}(j_1, 0, 0, \cdots, 0; \vartheta) = \mathbb{H}_n(j_1; \vartheta). \tag{4.11}$$

Therefore, in view of expressions (4.8)–(4.11), it follows that

$$\begin{aligned}\mathbb{H}_{n-1}^{[r]}(j_1, j_2, j_3, \cdots, j_r; \vartheta) = \exp\Bigg(&\frac{j_2\ \vartheta}{log(1+\vartheta)}\ D_{j_1}^2 + j_3\left(\frac{\vartheta}{log(1+\vartheta)}\right)^2 D_{j_1}^3 + \cdots \\ &+ j_r\left(\frac{\vartheta}{log(1+\vartheta)}\right)^{r-1} D_{j_1}^r\Bigg)\{\mathbb{H}_n(j_1; \vartheta)\}.\end{aligned} \tag{4.12}$$

Given the aforementioned perspective, it is possible to use the operational rule (4.12) to create the polynomials $\mathbb{H}_n^{[r]}(j_1, j_2, j_3, \cdots, j_r; \vartheta)$ from the degenerate polynomials $\mathbb{H}_n(j_1; \vartheta)$.

5. Conclusions

The study successfully provides and develops a novel set of degenerate multidimensional Hermite polynomials (DMVHP) denoted as $\mathbb{H}_n^{[r]}(j_1, j_2, j_3, \cdots, j_r; \vartheta)$. These polynomials are closely linked to the classical Hermite polynomials. The generating function for these polynomials is given in Theorem 2.1. The development of these polynomials is achieved by utilizing the monomiality principle and operational formalism, which allows for the exploration of new outcomes while maintaining consistency with existing knowledge. The quasi-monomial properties are given in the form of Theorem 2.2 and differential equation as Theorem 2.3. The Results 2.2 and 2.3 verifies the quasi-monomial properties. Furthermore, we establish the fundamental properties of the DMVHP, including symmetric identities, thereby enhancing our understanding of their behavior and mathematical characteristics. The introduction of an operational framework produced in Section 4 for these polynomials provides a useful tool for further research and applications in various fields. Therefore, this research contributes to the advancement of special polynomials and opens up new avenues for exploring their applications in mathematics, physics, engineering, and other related disciplines. The DMVHP and their properties established in this study add valuable knowledge to the existing literature, and they have the potential to facilitate future investigations in various mathematical and scientific contexts.

The operational formalism discussed earlier has significant implications for future research in mathematics. One area that can benefit from this approach is the derivation of extended and generalized forms of polynomials. Moreover, this method can be used to investigate families of differential equations, integral equations, recurrence relations, shift operators, and determinant forms. With the help of operational formalism, it is possible to obtain new insights into the properties and behavior of these mathematical constructs. These observations and investigations have the potential to lead to a deeper understanding of various mathematical concepts, as well as to the development of new mathematical tools and techniques. Therefore, operational formalism is a valuable tool for mathematicians and researchers who are interested in exploring and expanding the boundaries of mathematical knowledge.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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