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Research article

Extended dissipative analysis for memristive neural networks with two-delay components via a generalized delay-product-type Lyapunov-Krasovskii functional

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Abstract: In this study, we deal with the problem of extended dissipativity analysis for memristive neural networks (MNNs) with two-delay components. The goal is to get less conservative extended dissipativity criteria for delayed MNNs. An improved Lyapunov-Krasovskii functional (LKF) with some generalized delay-product-type terms is constructed based on the dynamic delay interval (DDI) method. Moreover, the derivative of the created LKF is estimated using the integral inequality technique, which includes the information of higher-order time-varying delay. Then, sufficient conditions are attained in terms of linear matrix inequalities (LMIs) to pledge the extended dissipative of MNNs via the new negative definite conditions of matrix-valued cubic polynomials. Finally, a numerical example is shown to prove the value and advantage of the presented approach.

Keywords: memristive neural networks; two-delay components; extended dissipativity; dynamic delay interval method
Mathematics Subject Classification: 93C10, 93D05, 93D30

1. Introduction

The idea of neural networks (NNs) was put forth using biological brain modeling. Numerous studies have used NNs for fault diagnosis, associative memory and multi-agent systems [1], making them a focus of research for scholars in recent decades. In 1971, according to the circuit's symmetry and completion principles, some scholars proposed that there may be a fourth basic element of the circuit that represents the link between charge and magnetic flux, which is "resistance with memory", namely memristor [2]. It was not until 2008 that this theory was confirmed by Hewlett-Packard Labs and the circuit element named memristor was first made using Tio_2 [3]. In addition, memristor is a kind of nonlinear resistance element and also has memory function. This means that after a power failure, it can also store the amount of charge that flowed through it at the previous moment [4]. Thus, when we use memristor instead of traditional resistance to realize NN, MNN is born. Compared with the traditional NN, the connection weight of MNN changes with the variation of state variables, and it can also remember the past state, which is more practical and can better mimic the structure and function of the human brain [5]. Therefore, MNNs have gradually become the focus of research in information processing, nonlinear systems and other fields. MNNs' dynamic behavior has become a hot topic of research [6–8].

Time delays are inevitable in some MNNs related to the slow speed at which neurons transmit signals [9, 10]. As everyone is aware, time delays are often the source of oscillation and instability [11–16]. In a practical MNN, signals transmitted from one point to another may experience several segments of networks, and the resulting time delays have different properties due to variable network transmission conditions [17]. Therefore, time delays are generally more than one in practical MNNs. Thus, the MNNs can take a form with additive delays or multiple delays. Therefore, the dynamic performance analysis of MNNs with additive or multiple delays has been investigated, such as stability analysis [18], stabilization [19, 20], synchronization [21–23], passivity analysis [24] and so on.

The theory of dissipativity was primally proposed by Willems [25], referring to supply rates and storage functions. Dissipativity refers to that more energy is being provided from outside than is being lost within the system. It is valuable in circuit systems. Moreover, dynamic control systems always need to attenuate external interference. Therefore, it is indispensable to use the extended dissipativity index to achieve the unity of dissipativity, passivity, H_∞ performance and $L_2 - L_\infty$ performance. The first attempt to meet this demand was made in [26]. In the past few years, the extended dissipativity anglysis of delayed MNNs has been adopted by an increasing number of researchers [27,28]. Although fruitful achievements have been acquired for extended dissipativity analysis of delayed MNNs, there is limited related literature concerned with this problem for delayed MNNs with two-delay components. Furthermore, although [28] has obtained some useful results, there is room to further explore.

We research the extended dissipativity analysis for MNNs with two-delay components. The major contributions are listed as follows.

1) By considering more information about the state vectors, delayed state vectors, and integral state vectors, an original improved delay-product-type LKF is constructed based on the dynamic delay interval method. In the constructed LKF, the augmented candidates enhance the connection between each state variable, which helps to reduce the conservatism.

2) When taking the derivative of the LKF, its negative definite condition not only includes some linear terms about delays, but also contains some square terms and cubic terms about delays, which have been neglected in most of the literature. Obviously, this type of LKF can take additional time-delay information into account and lead to less conservative results. Then, some sufficient extended dissipativity criteria are established in terms of LMIs using a matrix-valued cubic polynomial.

Notations: In this research paper, $\mathbb{R}^{m\times n}$ denotes the set of $m \times n$ real matrices; a block-diagonal matrix is denoted by the notation diag$\{\cdots\}$; the symbol 0 (I) stands for the zero-matrix (unit matrix);

$\mathcal{L} > 0$ signifies that $\mathcal{L}$ is a positive-define matrix; in a symmetric matrix, symmetric terms are denoted by $*$; Sym$\{\mathcal{R}\} = \mathcal{R} + \mathcal{R}^{\mathrm{T}}$.

2. Problem formulation

The two-delay MNN is described by:

$$\begin{cases} \dot{x}(t) = -\mathcal{W}(x(t))x(t) + C(x(t))g(x(t)) + \mathcal{D}(x(t))g(x(t-\kappa_1(t)-\kappa_2(t))) + \omega(t) \\ \mathcal{Y}(t) = x(t) + x(t-\kappa_1(t)-\kappa_2(t)) \end{cases}, \tag{2.1}$$

where $x(\cdot) = [x_1(\cdot), x_2(\cdot), ..., x_n(\cdot)]^{\mathrm{T}}$ is the state vector; $g(\cdot) = [g_1(\cdot), g_2(\cdot), ..., g_n(\cdot)]^{\mathrm{T}}$ is the neuron activation function; $\omega(t)$ is the external input; $\mathcal{Y}(t)$ is the output; $\mathcal{W}(x(t)) = \mathrm{diag}\{w_1(x_1(t)), w_2(x_2(t)), ..., w_n(x_n(t))\} > 0$ is the self-feedback connection weight matrix; $C(x(t)) = [c_{ij}(x_i(t))]_{n\times n}$ and $\mathcal{D}(x(t)) = [d_{ij}(x_i(t))]_{n\times n}$ are the memristor-based weights with :

$$w_i(x_i(t)) = \begin{cases} w_i^{(1)}, & \mathrm{sign}_{ij}\dot{g}_i(x_i(t)) - \dot{x}_i(t) < 0 \\ \text{unchanged}, & \mathrm{sign}_{ij}\dot{g}_i(x_i(t)) - \dot{x}_i(t) = 0 \\ w_i^{(2)}, & \mathrm{sign}_{ij}\dot{g}_i(x_i(t)) - \dot{x}_i(t) > 0 \end{cases},$$

$$c_{ij}(x_i(t)) = \begin{cases} c_{ij}^{(1)}, & \mathrm{sign}_{ij}\dot{g}_j(x_j(t)) - \dot{x}_i(t) < 0 \\ \text{unchanged}, & \mathrm{sign}_{ij}\dot{g}_j(x_j(t)) - \dot{x}_i(t) = 0 \\ c_{ij}^{(2)}, & \mathrm{sign}_{ij}\dot{g}_j(x_j(t)) - \dot{x}_i(t) > 0 \end{cases},$$

$$d_{ij}(x_i(t)) = \begin{cases} d_{ij}^{(1)}, & \mathrm{sign}_{ij}\dot{g}_j(x_j(t-\kappa(t))) - \dot{x}_i(t) < 0 \\ \text{unchanged}, & \mathrm{sign}_{ij}\dot{g}_j(x_j(t-\kappa(t))) - \dot{x}_i(t) > 0 \\ d_{ij}^{(2)}, & \mathrm{sign}_{ij}\dot{g}_j(x_j(t-\kappa(t))) - \dot{x}_i(t) = 0 \end{cases},$$

$$\mathrm{sign}_{ij} = \begin{cases} 1, & i \neq j \\ -1, & i = j \end{cases}$$

where $w_i^{(1)}, w_i^{(2)}, c_{ij}^{(1)}, c_{ij}^{(2)}, d_{ij}^{(1)}$ and $d_{ij}^{(2)}$ are known constants. Furthermore, "unchanged" denotes that the memristance retains its present value. It is easy to see that each weight varies between two different constant values, i.e., $w_i(x_i(t))$ can be either $w_i^{(1)}$ or $w_i^{(2)}$, likewise, $c_{ij}(x_i(t))$ and $d_{ij}(x_i(t))$ also have two options. In summary, the combination number of the possible form of $\mathcal{W}(x(t)), C(x(t))$ and $\mathcal{D}(x(t))$ is 2^{2n^2+n}, then, order these 2^{2n^2+n} cases in the following way:

$$(\mathcal{W}_1, C_1, \mathcal{D}_1), (\mathcal{W}_2, C_2, \mathcal{D}_2), ..., (\mathcal{W}_{2^{2n^2+n}}, C_{2^{2n^2+n}}, \mathcal{D}_{2^{2n^2+n}}).$$

Then, $\mathcal{W}(x(t)), C(x(t))$ and $\mathcal{D}(x(t))$ must be one of the 2^{2n^2+n} cases at any fixed time t, which implies that, there exists $p \in N = \left\{1, 2, ..., 2^{2n^2+n}\right\}$ such that $\mathcal{W}(x(t)) = \mathcal{W}_p, C(x(t)) = C_p$ and $\mathcal{D}(x(t)) = \mathcal{D}_p$. Hence, system (2.1) may be expressed as

$$\begin{aligned} \dot{x}(t) &= \sum_{p=1}^{2^{2n^2+n}} \pi_i(t)[-\mathcal{W}_p x(t) + C_p g(x(t)) + \mathcal{D}_p g(x(t-\kappa_1(t)-\kappa_2(t))) + \omega(t)] \\ &= -\mathcal{W}(t)x(t) + C(t)g(x(t)) + \mathcal{D}(t)g(x(t-\kappa_1(t)-\kappa_2(t))) + \omega(t), \end{aligned} \tag{2.2}$$

where $\sum_{p=1}^{2^{2n^2+n}} \pi_p(t) = 1$ and $\pi_p(t) = \begin{cases} 1, & W(x(t)) = \mathcal{W}_p, C(x(t)) = C_p, \text{and } \mathcal{D}(x(t)) = \mathcal{D}_p, \\ 0, & \text{otherwise.} \end{cases}$

The time-varying delays, $\kappa_1(t)$ and $\kappa_2(t)$, are assumed to be continuous differentiable functions, which meet the two cases listed below:

$$Case\ 1:\ 0 \le \kappa_1(t) \le \kappa_{1M}, |\dot{\kappa}_1(t)| \le \overline{\mu}_1; 0 \le \kappa_2(t) \le \kappa_{2M}, |\dot{\kappa}_2(t)| \le \overline{\mu}_2. \tag{2.3}$$

$$Case\ 2:\ 0 \le \kappa_1(t) \le \kappa_{1M}, \dot{\kappa}_1(t) \le \overline{\mu}_1; 0 \le \kappa_2(t) \le \kappa_{2M}, \dot{\kappa}_2(t) \le \overline{\mu}_2. \tag{2.4}$$

Where κ_{iM} and $\overline{\mu}_i$ $(i = 1, 2)$ are scalars. And $g(\cdot)$ is continuous and satisfies

$$\phi_i^- \le \frac{g_i(\varrho_1) - g_i(\varrho_2)}{\varrho_1 - \varrho_2} \le \phi_i^+,\ i = 1, 2, \cdots, n, \tag{2.5}$$

where $\varrho_1 \neq \varrho_2$, $g_i(0) = 0$, ϕ_i^- and ϕ_i^+ are given real constants.

Definition 1. *[29]. For real symmetric matrices $\beth_1$, $\beth_2$, $\beth_3$ and $\beth_4 \in \mathbb{R}^{n\times n}$ with $\beth_1 \le 0$, $\beth_3 > 0$, $\beth_4 \geqslant 0$ and $(\|\beth_1\| + \|\beth_2\|)\|\beth_4\| = 0$, MNN (2.2) is extended dissipative if there exists a scalar δ such that:*

$$\int_0^{t^*} \Big(\mathcal{Y}^{\mathrm{T}}(t)\beth_1\mathcal{Y}(t) + 2\mathcal{Y}^{\mathrm{T}}(t)\beth_2\omega(t) + \omega^{\mathrm{T}}(t)\beth_3\omega(t)\Big)\, dt \geqslant \sup_{0\le k\le t^*} \mathcal{Y}^{\mathrm{T}}(k)\beth_4\mathcal{Y}(k) + \delta. \tag{2.6}$$

3. Main results

The examination of extended dissipativity for two-delay MNNs in two situations will be covered in this section, along with the establishment of numerous improved dissipativity criteria.

$$\mathcal{H}(t) = \alpha\kappa_1(t) + \beta\kappa_2(t),\quad (\alpha,\beta) \in \begin{cases} \aleph = \{[0,1]\times[0,1] - (0,0) \cup (1,1)\}, \overline{\mu}_1 + \overline{\mu}_2 < 1 \\ \aleph \cap \{(\alpha,\beta)|\alpha\overline{\mu}_1 + \beta\overline{\mu}_2 < 1\}, \overline{\mu}_1 + \overline{\mu}_2 > 1 \end{cases},$$

$$\kappa(t) = \kappa_1(t) + \kappa_2(t),\quad \kappa_M = \kappa_{1M} + \kappa_{2M},\quad K_1 = \mathrm{diag}\{\phi_1^+, \phi_2^+, \cdots, \phi_n^+\},\quad K_2 = \mathrm{diag}\{\phi_1^-, \phi_2^-, \cdots, \phi_n^-\},$$

$$\Im(t) = \Big[x^{\mathrm{T}}(t), x^{\mathrm{T}}(t-\mathcal{H}(t)), x^{\mathrm{T}}(t-\kappa_M), x^{\mathrm{T}}(t-\kappa(t)), g^{\mathrm{T}}(x(t)), g^{\mathrm{T}}(x(t-\kappa(t))), \upsilon_1^{\mathrm{T}}(t), \upsilon_2^{\mathrm{T}}(t), \upsilon_3^{\mathrm{T}}(t), \upsilon_4^{\mathrm{T}}(t), \omega^{\mathrm{T}}(t)\Big]^{\mathrm{T}},$$

$$\upsilon_1(t) = \int_{t-\mathcal{H}(t)}^{t} \frac{x(\nu)}{\mathcal{H}(t)}\mathrm{d}\nu,\quad \upsilon_2(t) = \int_{t-\kappa_M}^{t-\mathcal{H}(t)} \frac{x(\nu)}{\kappa_M - \mathcal{H}(t)}\mathrm{d}\nu,\quad \upsilon_3(t) = \int_{t-\mathcal{H}(t)}^{t}\int_{\nu}^{t} \frac{x(\epsilon)}{\mathcal{H}^2(t)}\mathrm{d}\epsilon\mathrm{d}\nu,$$

$$\upsilon_4(t) = \int_{t-\kappa_M}^{t-\mathcal{H}(t)}\int_{\nu}^{t-\mathcal{H}(t)} \frac{x(\epsilon)}{(h - \mathcal{H}(t))^2}\mathrm{d}\epsilon\mathrm{d}\nu,\quad e_i = \Big[0_{n\times(i-1)n},\ I_n,\ 0_{n\times(11-i)n}\Big]\ (i = 1, 2, \cdots, 11),$$

$$\Omega_1 = \Big\{\kappa_1(t) \in \{0, \kappa_{1M}\}, \kappa_2(t) \in \{0, \kappa_{2M}\}\Big\},\quad \Omega_2 = \Big\{\dot{\kappa}_1(t) \in \{-\overline{\mu}_1, \overline{\mu}_1\}, \dot{\kappa}_2(t) \in \{-\overline{\mu}_2, \overline{\mu}_2\}\Big\}. \tag{3.1}$$

We start by providing the dissipativity criterion for *Case* 1.

Theorem 1. *For given scalars $\kappa_{1M} \ge 0$, $\kappa_{2M} \ge 0$, $\overline{\mu}_1 \ge 0$, $\overline{\mu}_2 \ge 0$, $0 < \sigma < 1$ and symmetric matrices $\beth_1$, $\beth_2$, $\beth_3$ and $\beth_4 \in \mathbb{R}^{n\times n}$ satisfying Definition 1, MNN (2.2) satisfying (2.3) is extended dissipative if there exist matrices $P_1, P_2, P_3 \in \mathbb{R}^{3n\times 3n} > 0$, $Z_1, Z_2 \in \mathbb{R}^{3n\times 3n} > 0$, $Z_3 \in \mathbb{R}^{n\times n} > 0$, $R_1, R_3 \in \mathbb{R}^{n\times n} > 0$, $R_2 \in \mathbb{R}^{2n\times 2n} > 0$, any symmetric matrices $S_1, S_2 \in \mathbb{R}^{4n\times 4n}$, any matrices $S_3, S_4 \in \mathbb{R}^{4n\times 4n}$, $\mathcal{M} \in \mathbb{R}^{22n\times 11n}$ and diagonal matrices $\Lambda_1, \Lambda_2, \Lambda_3 \in \mathbb{R}^{n\times n} > 0$, $H_i = \mathrm{diag}\{\hbar_{i1}, \hbar_{i2}, \cdots, \hbar_{in}\} \in \mathbb{R}^{n\times n} > 0$ $(i = 1, 2)$ satisfying the following LMIs for $p \in N$, $(\kappa_1(t), \kappa_2(t)) \in \Omega_1$ and $(\dot{\kappa}_1(t), \dot{\kappa}_2(t)) \in \Omega_2$:*

$$\begin{bmatrix} \tilde{R}_2 - S_1 & -S_3 \\ * & \tilde{R}_2 \end{bmatrix} > 0,\quad \begin{bmatrix} \tilde{R}_2 & -S_4 \\ * & \tilde{R}_2 - S_2 \end{bmatrix} > 0, \tag{3.2}$$

$$\begin{bmatrix} \sigma P_1^* - \beth_4 & -\beth_4 \\ * & (1-\sigma)P_1^* - \beth_4 \end{bmatrix} > 0, \tag{3.3}$$

$$\Omega^p(\mathcal{H}(t), \dot{\mathcal{H}}(t)) < 0, \tag{3.4}$$

where

$$\Omega^p(\mathcal{H}(t), \dot{\mathcal{H}}(t)) = \begin{bmatrix} \Delta_0^p + \mathcal{H}(t)\Delta_1^p & \frac{\mathcal{H}(t)}{2}\Delta_2 \\ * & \mathcal{H}(t)\Delta_3 \end{bmatrix} + \mathrm{Sym}\left\{\mathcal{M}[-\mathcal{H}(t)I, I]\right\},$$

$$\begin{aligned}
\Delta_0^p = {} & \mathrm{Sym}\Big\{\Gamma_{1a}^{\mathrm{T}} P_1 \Gamma_{2a} + \Gamma_{3a}^{\mathrm{T}} P_2 \Gamma_{5a} + \Gamma_{4a}^{\mathrm{T}} P_3 \Gamma_{6a} + \Gamma_9^{\mathrm{T}} Z_2 \Gamma_{12a}\Big\} + \dot{\mathcal{H}}(t)\Gamma_{3a}^{\mathrm{T}} P_2 \Gamma_{3a} - \dot{\mathcal{H}}(t)\Gamma_{4a}^{\mathrm{T}} P_3 \Gamma_{4a} \\
& + \Gamma_7^{\mathrm{T}}(Z_1 + Z_2)\Gamma_7 - (1 - \dot{\mathcal{H}}(t))\Gamma_{8a}^{\mathrm{T}} Z_1 \Gamma_{8a} - \Gamma_{11a}^{\mathrm{T}} Z_2 \Gamma_{11a} + e_1^{\mathrm{T}} Z_3 e_1 - (1 - \dot{\kappa}(t)) e_4^{\mathrm{T}} Z_3 e_4 + \frac{\kappa_M^2}{2} e_{sp}^{\mathrm{T}} R_3 e_{sp} \\
& - \begin{bmatrix} \Gamma_{13a} \\ \Gamma_{14a} \end{bmatrix}^{\mathrm{T}} \mathfrak{R}_1 \begin{bmatrix} \Gamma_{13a} \\ \Gamma_{14a} \end{bmatrix} - \frac{1 - \dot{\mathcal{H}}(t)}{\alpha\kappa_{1M} + \beta\kappa_{2M}} \Upsilon_1^{\mathrm{T}} \tilde{R}_1 \Upsilon_1 + \kappa_M^2 \begin{bmatrix} e_{sp} \\ e_1 \end{bmatrix}^{\mathrm{T}} R_2 \begin{bmatrix} e_{sp} \\ e_1 \end{bmatrix} - \frac{\kappa_M}{\alpha\kappa_{1M} + \beta\kappa_{2M}} \Upsilon_1^{\mathrm{T}} \tilde{R}_{03} \Upsilon_1 \\
& - \Pi_1^{\mathrm{T}} \tilde{R}_3 \Pi_1 - \Pi_2^{\mathrm{T}} \tilde{R}_3 \Pi_2 + \mathrm{Sym}\Big\{e_5^{\mathrm{T}}(H_1 - H_2)e_{sp} + e_1^{\mathrm{T}}(K_1 H_2 - K_2 H_1)e_{sp} + (e_5 - K_2 e_1)^{\mathrm{T}} \Lambda_1 (K_1 e_1 - e_5) \\
& + (e_6 - K_2 e_4)^{\mathrm{T}} \Lambda_2 (K_1 e_4 - e_6) + [(e_5 - e_6) - K_2(e_1 - e_4)]^{\mathrm{T}} \Lambda_3 [K_1(e_1 - e_4) - (e_5 - e_6)]\Big\} \\
& - (e_1 + e_4)^{\mathrm{T}} \beth_1 (e_1 + e_4) - \mathrm{Sym}\Big\{(e_1 + e_4)^{\mathrm{T}} \beth_2 e_{11}\Big\} - e_{11}^{\mathrm{T}} \beth_3 e_{11},
\end{aligned}$$

$$\begin{aligned}
\Delta_1^p = {} & \mathrm{Sym}\Big\{\Gamma_{1a}^{\mathrm{T}} P_1 \Gamma_{2b} + \Gamma_{1b}^{\mathrm{T}} P_1 \Gamma_{2a} + \Gamma_{3b}^{\mathrm{T}} P_2 \Gamma_{5a} + \Gamma_{3a}^{\mathrm{T}} P_2 \Gamma_{5b} + \Gamma_{4b}^{\mathrm{T}} P_3 \Gamma_{6a} + \Gamma_{4a}^{\mathrm{T}} P_3 \Gamma_{6b} + \Gamma_9^{\mathrm{T}} Z_1 \Gamma_{10b} \\
& + \Gamma_9^{\mathrm{T}} Z_2 \Gamma_{12b} + \dot{\mathcal{H}}(t)\Gamma_{3a}^{\mathrm{T}} P_2 \Gamma_{3b} - \dot{\mathcal{H}}(t)\Gamma_{4a}^{\mathrm{T}} P_3 \Gamma_{4b} - (1 - \dot{\mathcal{H}}(t))\Gamma_{8a}^{\mathrm{T}} Z_1 \Gamma_{8b} - \Gamma_{11a}^{\mathrm{T}} Z_2 \Gamma_{11b} - \begin{bmatrix} \Gamma_{13a} \\ \Gamma_{14a} \end{bmatrix}^{\mathrm{T}} \mathfrak{R}_1 \begin{bmatrix} \Gamma_{13b} \\ \Gamma_{14b} \end{bmatrix}\Bigg\} \\
& + e_{sp}^{\mathrm{T}} R_1 e_{sp} + \frac{1}{\alpha\kappa_{1M} + \beta\kappa_{2M}} \Upsilon_1^{\mathrm{T}} \tilde{R}_{03} \Upsilon_1 - \begin{bmatrix} \Gamma_{13a} \\ \Gamma_{14a} \end{bmatrix}^{\mathrm{T}} \mathfrak{R}_2 \begin{bmatrix} \Gamma_{13a} \\ \Gamma_{14a} \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
\Delta_2 = {} & \mathrm{Sym}\left\{\Gamma_{1b}^{\mathrm{T}} P_1 \Gamma_{2b} + \Gamma_{1a}^{\mathrm{T}} P_1 \Gamma_{2c} + \Gamma_{3b}^{\mathrm{T}} P_2 \Gamma_{5b} + \Gamma_{4b}^{\mathrm{T}} P_3 \Gamma_{6b} + \Gamma_9^{\mathrm{T}} Z_1 \Gamma_{10c} + \Gamma_9^{\mathrm{T}} Z_2 \Gamma_{12c} - \begin{bmatrix} \Gamma_{13a} \\ \Gamma_{14a} \end{bmatrix}^{\mathrm{T}} \mathfrak{R}_2 \begin{bmatrix} \Gamma_{13b} \\ \Gamma_{14b} \end{bmatrix}\right\} \\
& + \dot{\mathcal{H}}(t)(\Gamma_{3b}^{\mathrm{T}} P_2 \Gamma_{3b} - \Gamma_{4b}^{\mathrm{T}} P_3 \Gamma_{4b}) - (1 - \dot{\mathcal{H}}(t))\Gamma_{8b}^{\mathrm{T}} Z_1 \Gamma_{8b} - \Gamma_{11b}^{\mathrm{T}} Z_2 \Gamma_{11b} - \begin{bmatrix} \Gamma_{13b} \\ \Gamma_{14b} \end{bmatrix}^{\mathrm{T}} \mathfrak{R}_1 \begin{bmatrix} \Gamma_{13b} \\ \Gamma_{14b} \end{bmatrix},
\end{aligned}$$

$$\Delta_3 = \mathrm{Sym}\Big\{\Gamma_{1b}^{\mathrm{T}} P_1 \Gamma_{2c}\Big\} - \begin{bmatrix} \Gamma_{13b} \\ \Gamma_{14b} \end{bmatrix}^{\mathrm{T}} \mathfrak{R}_2 \begin{bmatrix} \Gamma_{13b} \\ \Gamma_{14b} \end{bmatrix},$$

$$\Gamma_1^p = \mathcal{H}(t)\Gamma_{1b} + \Gamma_{1a}^p,\ \ \Gamma_2 = \mathcal{H}^2(t)\Gamma_{2c} + \mathcal{H}(t)\Gamma_{2b} + \Gamma_{2a},\ \ \Gamma_3 = \mathcal{H}(t)\Gamma_{3b} + \Gamma_{3a},\ \ \Gamma_4 = \mathcal{H}(t)\Gamma_{4b} + \Gamma_{4a},$$

$$\Gamma_5^p = \mathcal{H}(t)\Gamma_{5b}^p + \Gamma_{5a},\ \ \Gamma_6^p = \mathcal{H}(t)\Gamma_{6b}^p + \Gamma_{6a}^p,\ \ \Gamma_7 = \left[e_1^{\mathrm{T}}, e_1^{\mathrm{T}}, 0\right]^{\mathrm{T}},\ \ \Gamma_8 = \mathcal{H}(t)\Gamma_{8b} + \Gamma_{8a},\ \ \Gamma_9^p = \left[e_{sp}^{\mathrm{T}}, 0, e_1^{\mathrm{T}}\right]^{\mathrm{T}},$$

$$\Gamma_{10} = \mathcal{H}^2(t)\Gamma_{10c} + \mathcal{H}(t)\Gamma_{10b},\ \ \Gamma_{11} = \mathcal{H}(t)\Gamma_{11b} + \Gamma_{11a},\ \ \Gamma_{12} = \mathcal{H}^2(t)\Gamma_{12c} + \mathcal{H}(t)\Gamma_{12b} + \Gamma_{12a},$$

$$\Gamma_{13} = \mathcal{H}(t)\Gamma_{13b} + \Gamma_{13a},\ \Gamma_{14} = \mathcal{H}(t)\Gamma_{14b} + \Gamma_{14a},\ \Gamma_{1a}^p = \left[e_{sp}^{\mathrm{T}}, e_1^{\mathrm{T}} - e_3^{\mathrm{T}}, \kappa_M e_1^{T} - \kappa_M e_8^{\mathrm{T}}\right]^{\mathrm{T}},$$

$$\Gamma_{1b} = \left[0, 0, e_8^{\mathrm{T}} - e_7^{\mathrm{T}}\right]^{\mathrm{T}}, \Gamma_{2a} = \left[e_1^{\mathrm{T}}, \kappa_M e_8^{\mathrm{T}}, \kappa_M^2 e_{10}^{\mathrm{T}}\right]^{\mathrm{T}},\ \Gamma_{2b} = \left[0, e_7^{\mathrm{T}} - e_8^{\mathrm{T}}, \kappa_M e_7^{\mathrm{T}} - 2\kappa_M e_{10}^{\mathrm{T}}\right]^{\mathrm{T}},$$

$$\Gamma_{2c} = \left[0, 0, e_9^{\mathrm{T}} + e_{10}^{\mathrm{T}} - e_7^{\mathrm{T}}\right]^{\mathrm{T}}, \Gamma_{3a} = \left[e_1^{\mathrm{T}}, e_7^{\mathrm{T}}, 0\right]^{\mathrm{T}}, \Gamma_{3b} = \left[0, 0, e_9^{\mathrm{T}}\right]^{\mathrm{T}}, \Gamma_{4a} = \left[e_1^{\mathrm{T}}, e_8^{\mathrm{T}}, \kappa_M e_{10}^{\mathrm{T}}\right]^{\mathrm{T}},$$

$\Gamma_{4b} = \Big[0, 0, -e_{10}^{\mathrm{T}}\Big]^{\mathrm{T}}, \Gamma_{5a} = \Big[0, e_1^{\mathrm{T}} - (1-\dot{\mathcal{H}}(t))e_2^{\mathrm{T}} - \dot{\mathcal{H}}(t)e_7^{\mathrm{T}}, 0\Big]^{\mathrm{T}},$

$\Gamma_{5b}^p = \Big[e_{sp}^{\mathrm{T}}, 0, e_1^{\mathrm{T}} - (1-\dot{\mathcal{H}}(t))e_7^{\mathrm{T}} - \dot{\mathcal{H}}(t)e_9^{\mathrm{T}}\Big]^{\mathrm{T}}, \ \Gamma_{6b}^p = \Big[-e_{sp}^{\mathrm{T}}, 0, -(1-\dot{\mathcal{H}}(t))e_2^{\mathrm{T}} + e_8^{\mathrm{T}} - \dot{\mathcal{H}}(t)e_{10}^{\mathrm{T}}\Big]^{\mathrm{T}},$

$\Gamma_{6a}^p = \Big[\kappa_M e_{sp}^{\mathrm{T}}(1-\dot{\mathcal{H}}(t))e_2^{\mathrm{T}} - e_3^{\mathrm{T}} + \dot{\mathcal{H}}(t)e_8^{\mathrm{T}}, \ \kappa_M(1-\dot{\mathcal{H}}(t))e_2^{\mathrm{T}} - \kappa_M e_8^{\mathrm{T}} + \kappa_M \dot{\mathcal{H}}(t)e_{10}^{\mathrm{T}})\Big]^{\mathrm{T}}, \ \Gamma_{8a} = \Big[e_1^{\mathrm{T}}, e_2^{\mathrm{T}}, 0\Big]^{\mathrm{T}},$

$\Gamma_{8b} = \Big[0, 0, e_7^{\mathrm{T}}\Big]^{\mathrm{T}}, \ \Gamma_{10b} = \Big[e_1^{\mathrm{T}}, e_7^{\mathrm{T}}, 0\Big]^{\mathrm{T}}, \ \Gamma_{10c} = \Big[0, 0, e_9^{\mathrm{T}}\Big]^{\mathrm{T}}, \Gamma_{11a} = \Big[e_1^{\mathrm{T}}, e_3^{\mathrm{T}}, \kappa_M e_8^{\mathrm{T}}\Big]^{\mathrm{T}}, \Gamma_{11b} = \Big[0, 0, e_7^{\mathrm{T}} - e_8^{\mathrm{T}}\Big]^{\mathrm{T}},$

$\Gamma_{12a} = \Big[\kappa_M e_1^{\mathrm{T}}, \kappa_M e_8^{\mathrm{T}}, \kappa_M^2 e_{10}^{\mathrm{T}}\Big]^{\mathrm{T}}, \Gamma_{12b} = \Big[0, e_7^{\mathrm{T}} - e_8^{\mathrm{T}}, \kappa_M e_7^{\mathrm{T}} - 2\kappa_M e_{10}^{\mathrm{T}}\Big]^{\mathrm{T}}, \Gamma_{12c} = \Big[0, 0, e_9^{\mathrm{T}} + e_{10}^{\mathrm{T}} - e_7^{\mathrm{T}}\Big]^{\mathrm{T}},$

$\Gamma_{13a} = \Big[e_1^{\mathrm{T}} - e_2^{\mathrm{T}}, 0, e_1^{\mathrm{T}} + e_2^{\mathrm{T}} - 2e_7^{\mathrm{T}}, 0\Big]^{\mathrm{T}}, \Gamma_{13b} = \Big[0, e_7^{\mathrm{T}}, 0, e_7^{\mathrm{T}} - 2e_9^{\mathrm{T}}\Big]^{\mathrm{T}}, \ \Gamma_{14b} = \Big[0, -e_8^{\mathrm{T}}, 0, 2e_{10}^{\mathrm{T}} - e_8^{\mathrm{T}}\Big]^{\mathrm{T}},$

$\Gamma_{14a} = \Big[e_2^{\mathrm{T}} - e_3^{\mathrm{T}}, \kappa_M e_8^{\mathrm{T}}, e_2^{\mathrm{T}} + e_3^{\mathrm{T}} - 2e_8^{\mathrm{T}}, \kappa_M e_8^{\mathrm{T}} - 2\kappa_M e_{10}^{\mathrm{T}}\Big]^{\mathrm{T}}, \ e_{sp} = -\mathcal{W}_p e_1 + C_p e_5 + \mathcal{D}_p e_6 + e_{11} \ (p \in N),$

$\tilde{R}_1 = \mathrm{diag}\{R_1, 3R_1, 5R_1\}, \ \tilde{R}_2 = \mathrm{diag}\{R_2, 3R_2\}, \ \tilde{R}_3 = \mathrm{diag}\{2R_3, 4R_3\}, \ \tilde{R}_{03} = \mathrm{diag}\{R_3, 3R_3, 5R_3\},$

$P_1^* = [I, 0, 0] \ P_1 \ [I, 0, 0]^{\mathrm{T}}, \ \Upsilon_1 = \Big[e_1^{\mathrm{T}} - e_2^{\mathrm{T}}, e_1^{\mathrm{T}} + e_2^{\mathrm{T}} - 2e_7^{\mathrm{T}}, e_1^{\mathrm{T}} - e_2^{\mathrm{T}} + 6e_7^{\mathrm{T}} - 12e_9^{\mathrm{T}}\Big]^{\mathrm{T}},$

$$\Pi_i = \begin{bmatrix} e_i - e_{i+6} \\ e_i + 2e_{i+6} - 6e_{i+8} \end{bmatrix} (i = 1, 2), \mathfrak{R}_1 = \begin{bmatrix} \tilde{R}_2 + S_1 & S_4 \\ * & \tilde{R}_2 \end{bmatrix}, \mathfrak{R}_2 = \begin{bmatrix} -\frac{1}{\kappa_M}S_1 & \frac{1}{\kappa_M}S_3 - \frac{1}{\kappa_M}S_4 \\ * & \frac{1}{\kappa_M}S_2 \end{bmatrix}. \tag{3.5}$$

Proof. We choose LKF $\mathbb{V}(x_t, t) = \sum_{i=1}^4 \mathbb{V}_i(x_t, t)$, where

$$\mathbb{V}_1(x_t, t) = \lambda_1^{\mathrm{T}}(t)P_1\lambda_1(t) + \mathcal{H}(t)\lambda_2^{\mathrm{T}}(t)P_2\lambda_2(t) + (\kappa_M - \mathcal{H}(t))\lambda_3^{\mathrm{T}}(t)P_3\lambda_3(t),$$

$$\mathbb{V}_2(x_t, t) = \int_{t-\mathcal{H}(t)}^t \lambda_4^{\mathrm{T}}(t, \nu)Z_1\lambda_4(t, \nu)\mathrm{d}\nu + \int_{t-\kappa_M}^t \lambda_4^{\mathrm{T}}(t, \nu)Z_2\lambda_4(t, \nu)\mathrm{d}\nu + \int_{t-\kappa(t)}^t x^{\mathrm{T}}(\nu)Z_3 x(\nu)\mathrm{d}\nu,$$

$$\mathbb{V}_3(x_t, t) = \int_{-\mathcal{H}(t)}^0 \int_{t+\theta}^t \dot{x}^{\mathrm{T}}(\nu)R_1\dot{x}(\nu)\mathrm{d}\nu\mathrm{d}\theta + \kappa_M \int_{-\kappa_M}^0 \int_{t+\theta}^t \lambda_5^{\mathrm{T}}(\nu)R_2\lambda_5(\nu)\mathrm{d}\nu\mathrm{d}\theta$$
$$+ \int_{-\kappa_M}^0 \int_{\theta}^0 \int_{t+\epsilon}^t \dot{x}^{\mathrm{T}}(\nu)R_3\dot{x}(\nu)\mathrm{d}\nu\mathrm{d}\epsilon\mathrm{d}\theta,$$

$$\mathbb{V}_4(x_t, t) = 2\sum_{j=1}^n \int_0^{x_{j(t)}} [\hbar_{1j}(g_j(\nu) - \phi_j^- \nu) + \hbar_{2j}(\phi_j^+ \nu - g_j(\nu))]\mathrm{d}\nu,$$

with $\lambda_1(t) = \Big[x^{\mathrm{T}}(t), \ \int_{t-\kappa_M}^t x^{\mathrm{T}}(\nu)\mathrm{d}\nu, \ \int_{t-\kappa_M}^t \int_\nu^t x^{\mathrm{T}}(\epsilon)\mathrm{d}\epsilon\mathrm{d}\nu\Big]^{\mathrm{T}}, \lambda_2(t) = \Big[x^{\mathrm{T}}(t), \ \upsilon_1^{\mathrm{T}}(t), \ \mathcal{H}(t)\upsilon_3^{\mathrm{T}}(t)\Big]^{\mathrm{T}},$
$\lambda_3(t) = \Big[x^{\mathrm{T}}(t), \ \upsilon_2^{\mathrm{T}}(t), \ (\kappa_M - \mathcal{H}(t))\upsilon_4^{\mathrm{T}}(t)\Big]^{\mathrm{T}}, \lambda_4(t) = \Big[x^{\mathrm{T}}(t), \ x^{\mathrm{T}}(\nu), \ \int_\nu^t x^{\mathrm{T}}(u)\mathrm{d}u\Big]^{\mathrm{T}}, \lambda_5(t) = \Big[\dot{x}^{\mathrm{T}}(t), \ x^{\mathrm{T}}(t)\Big]^{\mathrm{T}}.$

Taking the derivative of $\mathbb{V}(x_t, t)$, we have

$$\dot{\mathbb{V}}_1(x_t, t) = \mathfrak{I}^{\mathrm{T}}(t)\Big[2\Gamma_1^{p\mathrm{T}}P_1\Gamma_2 + \dot{\mathcal{H}}(t)\Gamma_3^{\mathrm{T}}P_2\Gamma_3 - \dot{\mathcal{H}}(t)\Gamma_4^{\mathrm{T}}P_3\Gamma_4 + 2\Gamma_3^{\mathrm{T}}P_2\Gamma_5^p + 2\Gamma_4^{\mathrm{T}}P_3\Gamma_6^p\Big]\mathfrak{I}(t), \tag{3.6}$$

$$\dot{\mathbb{V}}_2(x_t, t) = \mathfrak{I}^{\mathrm{T}}(t)\Big[\Gamma_7^{\mathrm{T}}(Z_1 + Z_2)\Gamma_7 - (1 - \dot{\mathcal{H}}(t))\Gamma_8^{\mathrm{T}}Z_1\Gamma_8 + 2\Gamma_9^{p\mathrm{T}}Z_1\Gamma_{10} - \Gamma_{11}^{\mathrm{T}}Z_2\Gamma_{11}$$
$$+ 2\Gamma_9^{p\mathrm{T}}Z_2\Gamma_{12} + e_1^{\mathrm{T}}Z_3e_1 - (1 - \dot{\kappa}(t))e_4^{\mathrm{T}}Z_3e_4\Big]\mathfrak{I}(t), \tag{3.7}$$

$$\dot{\mathbb{V}}_3(x_t, t) = \mathfrak{I}^{\mathrm{T}}(t)\left(\mathcal{H}(t)e_{sp}^{\mathrm{T}}R_1e_{sp} + \kappa_M^2\begin{bmatrix} e_{sp} \\ e_1 \end{bmatrix}^{\mathrm{T}} R_2 \begin{bmatrix} e_{sp} \\ e_1 \end{bmatrix} + \frac{\kappa_M^2}{2}e_{sp}^{\mathrm{T}}R_3e_{sp}\right)\mathfrak{I}(t)$$
$$- (1 - \dot{\mathcal{H}}(t))\int_{t-\mathcal{H}(t)}^t \dot{x}^{\mathrm{T}}(\nu)R_1\dot{x}(\nu)\mathrm{d}\nu - \kappa_M\int_{t-\kappa_M}^t \lambda_5^{\mathrm{T}}(\nu)R_2\lambda_5(\nu)\mathrm{d}\nu$$

$$
\begin{aligned}
&-\int_{t-\kappa_M}^{t-\mathcal{H}(t)}\int_{\nu}^{t-\mathcal{H}(t)}\dot{x}^{\mathrm{T}}(\epsilon)R_3\dot{x}(\epsilon)\mathrm{d}\epsilon\mathrm{d}\nu-\int_{t-\mathcal{H}(t)}^{t}\int_{\nu}^{t}\dot{x}^{\mathrm{T}}(\epsilon)R_3\dot{x}(\epsilon)\mathrm{d}\epsilon\mathrm{d}\nu\\
&-(\kappa_M-\mathcal{H}(t))\int_{t-\mathcal{H}(t)}^{t}\dot{x}^{\mathrm{T}}(\epsilon)R_3\dot{x}(\epsilon)\mathrm{d}\epsilon, \qquad (3.8)
\end{aligned}
$$

$$
\dot{\mathbb{V}}_4(x_t,t)=2\mathfrak{I}^{\mathrm{T}}(t)\Big[e_5^{\mathrm{T}}(H_1-H_2)e_{sp}+e_1^{\mathrm{T}}(K_1H_2-K_2H_1)e_{sp}\Big]\mathfrak{I}(t), \qquad (3.9)
$$

where $\Gamma_i(i=1,2,...,12)$ are defined in (3.5).

Through Lemma 5.1 in [30], the R_1 and R_3-dependent integral term in (3.8) can be rebounded as

$$
-(1-\dot{\mathcal{H}}(t))\int_{t-\mathcal{H}(t)}^{t}\dot{x}^{\mathrm{T}}(\nu)R_1\dot{x}(\nu)\mathrm{d}\nu\leq-\frac{1-\dot{\mathcal{H}}(t)}{\alpha\kappa_{1M}+\beta\kappa_{2M}}\mathfrak{I}^{T}(t)\Upsilon_1^{\mathrm{T}}\tilde{R}_1\Upsilon_1\mathfrak{I}(t), \qquad (3.10)
$$

$$
-(\kappa_M-\mathcal{H}(t))\int_{t-\mathcal{H}(t)}^{t}\dot{x}^{\mathrm{T}}(\nu)R_3\dot{x}(\nu)\mathrm{d}\nu\leq-\frac{\kappa_M-\mathcal{H}(t)}{\alpha\kappa_{1M}+\beta\kappa_{2M}}\mathfrak{I}^{\mathrm{T}}(t)\Upsilon_1^{\mathrm{T}}\tilde{R}_{03}\Upsilon_1\mathfrak{I}(t), \qquad (3.11)
$$

$$
-\int_{t-\mathcal{H}(t)}^{t}\int_{\nu}^{t}\dot{x}^{\mathrm{T}}(\epsilon)R_3\dot{x}(\epsilon)\mathrm{d}\epsilon\mathrm{d}\nu\leq-\mathfrak{I}^{\mathrm{T}}(t)\Pi_1^{\mathrm{T}}\tilde{R}_3\Pi_1\mathfrak{I}(t), \qquad (3.12)
$$

$$
-\int_{t-\kappa_M}^{t-\mathcal{H}(t)}\int_{\nu}^{t-\mathcal{H}(t)}\dot{x}^{\mathrm{T}}(\epsilon)R_3\dot{x}(\epsilon)\mathrm{d}\epsilon\mathrm{d}\nu\leq-\mathfrak{I}^{\mathrm{T}}(t)\Pi_2^{\mathrm{T}}\tilde{R}_3\Pi_2\mathfrak{I}(t). \qquad (3.13)
$$

With conditions (3.2), by using Corollary 5 in [31] and Lemma 2 in [32] to rebound the R_2-dependent integral term in (3.8), one yields

$$
\begin{aligned}
-\kappa_M\int_{t-\kappa_M}^{t}\lambda_5^{\mathrm{T}}(\nu)R_2\lambda_5(\nu)\mathrm{d}\nu&=-\kappa_M\int_{t-\mathcal{H}(t)}^{t}\lambda_5^{\mathrm{T}}(\nu)R_2\lambda_5(\nu)\mathrm{d}\nu-\kappa_M\int_{t-\kappa_M}^{t-\mathcal{H}(t)}\lambda_5^{\mathrm{T}}(\nu)R_2\lambda_5(\nu)\mathrm{d}\nu\\
&\leq-\mathfrak{I}^{\mathrm{T}}(t)\left(\frac{\kappa_M}{\mathcal{H}(t)}\Gamma_{13}^{\mathrm{T}}\tilde{R}_2\Gamma_{13}+\frac{\kappa_M}{\kappa_M-\mathcal{H}(t)}\Gamma_{14}^{\mathrm{T}}\tilde{R}_2\Gamma_{14}\right)\mathfrak{I}(t)\\
&\leq-\mathfrak{I}^{\mathrm{T}}(t)\begin{bmatrix}\Gamma_{13}\\ \Gamma_{14}\end{bmatrix}^{\mathrm{T}}\Big(\mathfrak{R}_1+\mathcal{H}(t)\mathfrak{R}_2\Big)\begin{bmatrix}\Gamma_{13}\\ \Gamma_{14}\end{bmatrix}\mathfrak{I}(t). \qquad (3.14)
\end{aligned}
$$

Based on (2.5), one has

$$
2\Big[g(x(t))-K_2x(t)\Big]^{\mathrm{T}}\Lambda_1\Big[K_1x(t)-g(x(t))\Big]\geqslant 0, \qquad (3.15)
$$

$$
2\Big[g(x(t-\kappa(t)))-K_2x(t-\kappa(t))\Big]^{\mathrm{T}}\Lambda_2\Big[K_1x(t-\kappa(t))-g(x(t-\kappa(t)))\Big]\geqslant 0, \qquad (3.16)
$$

$$
2\Big[g(x(t))-g(x(t-\kappa(t)))-K_2(x(t)-x(t-\kappa(t)))\Big]^{\mathrm{T}}\Lambda_3\Big[K_1(x(t)-x(\kappa(t)))-g(x(t))+g(x(t-\kappa(t)))\Big]\geqslant 0. \qquad (3.17)
$$

Recommending the cost function $J_{t^*}=\int_0^{t^*}(\mathcal{Y}^{\mathrm{T}}(t)\sqsupset_1\mathcal{Y}(t)+2\mathcal{Y}^{\mathrm{T}}(t)\sqsupset_2\omega(t)+\omega^{\mathrm{T}}(t)\sqsupset_3\omega(t))\mathrm{d}t$. It can be readily derived from (3.6) to (3.17) that

$$
\int_0^{t^*}\dot{\mathbb{V}}(x_t)\mathrm{d}t-J_{t^*}\leq\int_0^{t^*}\mathfrak{I}^{\mathrm{T}}(t)\Theta(\mathcal{H}(t))\mathfrak{I}(t)\mathrm{d}t, \qquad (3.18)
$$

where $\Theta(\mathcal{H}(t)) = \Delta_3\mathcal{H}^3(t) + \Delta_2\mathcal{H}^2(t) + \Delta_1\mathcal{H}(t) + \Delta_0$. Then, it follows Lemma 3 in [33] that if (3.4) meets for $(\kappa_1(t), \kappa_2(t)) \in \Omega_1$ and $(\dot{\kappa}_1(t), \dot{\kappa}_2(t)) \in \Omega_2$, $\Theta(\mathcal{H}(t)) < 0$ holds, which implies $\dot{\mathbb{V}}(x_t, t) - J_{t^*} \leq 0$. Integrating two sides of the above-mentioned inequality from 0 to t gains

$$\int_0^t J_{t^*} \mathrm{d}\alpha \geqslant \mathbb{V}(t) - \mathbb{V}(0) \geqslant x^{\mathrm{T}}(t)P_1^* x(t) + \delta, \tag{3.19}$$

where P_1^* is defined in (3.5).

Next, two cases will be considered in the proof. First, if $\|\beth_4\| = 0$, (3.19) means that for any $t^* \geqslant 0$

$$\int_0^{t^*} J_{t^*} \mathrm{d}\alpha \geqslant x^{\mathrm{T}}(t^*)P_1^* x(t^*) + \delta \geqslant \delta, \tag{3.20}$$

this indicates that Definition 1 is accurate. If $\|\beth_4\| \neq 0$, as mentioned in (2.3), we can figure out that the matrices $\beth_1 = 0$, $\beth_2 = 0$ and $\beth_3 > 0$, therefore, for any $t^* \geq t \geq 0$,

$$\int_0^{t^*} J_{t^*} \mathrm{d}\alpha \geqslant \int_0^{t} J_{t^*} \mathrm{d}\alpha \geqslant x^{\mathrm{T}}(t)P_1^* x(t) + \delta, \tag{3.21}$$

when $t > \kappa(t)$, $0 < t - \kappa(t) \leq t^*$. Therefore,

$$\int_0^{t^*} J_{t^*} \mathrm{d}\alpha \geqslant x^{\mathrm{T}}(t-\kappa(t))P_1^* x(t-\kappa(t)) + \delta. \tag{3.22}$$

For a positive constant $0 < \sigma < 1$, we have

$$\int_0^{t^*} J_{t^*} \mathrm{d}\alpha \geqslant \delta + \sigma x^{\mathrm{T}}(t)P_1^* x(t) + (1-\sigma)x^{\mathrm{T}}(t-\kappa(t))P_1^* x(t-\kappa(t)). \tag{3.23}$$

Taking note of the fact

$$\begin{aligned} \mathcal{Y}^{\mathrm{T}}(t)\beth_4\mathcal{Y}(t) = &-\begin{bmatrix} x(t) \\ x(t-\kappa(t)) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \sigma P_1^* - \beth_4 & -\beth_4 \\ * & (1-\sigma)P_1^* - \beth_4 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\kappa(t)) \end{bmatrix} \\ &+ \sigma x^{\mathrm{T}}(t)P_1^* x(t) + (1-\sigma)x^{\mathrm{T}}(t-\kappa(t))P_1^* x(t-\kappa(t)). \end{aligned} \tag{3.24}$$

If (3.3) is satisfied, then

$$\mathcal{Y}^{\mathrm{T}}(t)\beth_4\mathcal{Y}(t) \leq \sigma x^{\mathrm{T}}(t)P_1^* x(t) + (1-\sigma)x^{\mathrm{T}}(t-\kappa(t))P_1^* x(t-\kappa(t)). \tag{3.25}$$

It is explicit that, for any $t^* \geq t \geq 0$, one has $\int_0^{t^*} J_{t^*} \mathrm{d}\alpha \geqslant \mathcal{Y}^{\mathrm{T}}(t)\beth_4\mathcal{Y}(t) + \delta$. Consequently, (2.6) meets for any $t^* \geq 0$. On the basis of the above analysis, whether $\|\beth_4\| = 0$ or $\|\beth_4\| \neq 0$, system (2.1) with (2.3) is extended dissipative. This completes the proof. □

Remark 1. *An improved LKF, which incorporates the information of state variables and integral state variables, is designed to heighten the association between various state variables in this paper. By multiplying time delays with an augmented single integral term, a novel delay-product-type LKF* $\mathbb{V}_1(x_t, t)$ *is constructed. When taking the derivative of the LKF, its negative definite condition not only includes some linear terms about delays, but also contains some square terms and cubic terms about delays. Obviously, it takes additional time-delay information into account. This type of LKF can lead to less conservative outcomes. These time-delay information have not been considered in most of the literature in the same domain [28].*

Remark 2. *If no new state variables are added, we discover that the time derivative of the LKF is the cubic polynomial about the time-varying delay. Then, we find in [33] that Lemma 3 skillfully settle the problem by regulating the positions of matrices Δ_i . Moreover, the computation complexity and the LMIs' dimension are decreased.*

Remark 3. *Upon most of existing achievements in regard to the analysis of dissipativity for MNNs with two-delay components, the integral terms constructed in LKFs usually were selected as $\int_{t-\kappa_1(t)}^{t} \mathcal{U}^{\mathrm{T}} N \mathcal{U} \mathrm{d}s$ or $\int_{t-\kappa_2(t)}^{t} \mathcal{U}^{\mathrm{T}} N \mathcal{U} \mathrm{d}s$, which means that the delay intervals were fixed [28]. To consider the two-delay components information more comprehensively, more single integral terms or double integral terms need to be established into the LKFs, which significantly increases the complexity of the obtained results. In this research paper, we develop a dynamic delay interval method. With the help of this method, the lower and upper limits of the integral terms can be converted to variable ones. It is obvious that our constructed integral terms are with more freedom and contain more general cases, thereby can take more time-delay information into account.*

Theorem 2 will be used to explain the situation where the bottom bound of the delays are unknown.

Theorem 2. *For given scalars $\kappa_{1M} \geq 0$, $\kappa_{2M} \geq 0$, $\overline{\mu}_1 \geq 0$, $\overline{\mu}_2 \geq 0$, $0 < \sigma < 1$ and symmetric matrices $\beth_1$, $\beth_2$, $\beth_3$ and $\beth_4 \in \mathbb{R}^{n\times n}$ satisfying Definition 1, MNN (2.2) satisfying (2.4) is extended dissipative if there exist matrices $P_1 \in \mathbb{R}^{3n\times 3n} > 0$, $Z_1, Z_2 \in \mathbb{R}^{3n\times 3n} > 0$, $Z_3 \in \mathbb{R}^{n\times n} > 0$, $R_1, R_3 \in \mathbb{R}^{n\times n} > 0$, $R_2 \in \mathbb{R}^{2n\times 2n} > 0$, $\mathcal{M} \in \mathbb{R}^{22n\times 11n}$, any symmetric matrices $S_1, S_2 \in \mathbb{R}^{4n\times 4n}$, any matrices $S_3, S_4 \in \mathbb{R}^{4n\times 4n}$ and diagonal matrices $\Lambda_1, \Lambda_2, \Lambda_3 \in \mathbb{R}^{n\times n} > 0$, $H_i = \mathrm{diag}\{\hbar_{i1}, \hbar_{i2}, \cdots, \hbar_{in}\} \in \mathbb{R}^{n\times n} > 0$ $(i = 1, 2)$ satisfying LMIs (3.2) and (3.3) and $\Omega^{*}(\mathcal{H}(t), \dot{\mathcal{H}}(t)) < 0$ for $p \in N$, $(\kappa_1(t), \kappa_2(t)) \in \Omega_1$ and $\dot{\kappa}_1(t) = \overline{\mu}_1, \dot{\kappa}_2(t) = \overline{\mu}_2$, where $\Omega^{p*}(\mathcal{H}(t), \dot{\mathcal{H}}(t))$ is given by taking $P_2 = 0$ and $P_3 = 0$ of $\Omega^{p}(\mathcal{H}(t), \dot{\mathcal{H}}(t))$ in Theorem 1.*

4. A numerical example

A numerical example presented in this part aims to reveal the validity and preponderance of the proposed methods.

Example 1. *Consider delayed MNN (2.2) with:*

$$C(t) = \begin{bmatrix} c_{11}(x_1(t)) & c_{12}(x_1(t)) \\ c_{21}(x_2(t)) & -1.1 \end{bmatrix}, \quad \mathcal{D}(t) = \begin{bmatrix} -0.01 & d_{12}(x_1(t)) \\ d_{21}(x_2(t)) & d_{22}(x_2(t)) \end{bmatrix},$$

$$\mathcal{W}(t) = \mathrm{diag}\{w_1(x_1(t)),\ w_2(x_1(t))\},\ K_1 = \mathrm{diag}\{0.9,\ 0.9\},\ K_2 = \mathrm{diag}\{-0.1,\ -0.1\},$$

$$w_1(x_1(t)) \in \{5, 4\},\ w_2(x_1(t)) \in \{4, 5\},\ c_{11}(x_1(t)) \in \{0.5, 0.3\},\ c_{12}(x_1(t)) \in \{1.4, 1.6\},$$

$$c_{21}(x_2(t)) \in \{-0.7, -0.3\},\ d_{12}(x_1(t)) \in \{0.3, 0.6\},\ d_{21}(x_2(t)) \in \{-0.3, -0.2\},\ d_{22}(x_2(t)) \in \{0.09, 0.1\}.$$

The analysis of extended dissipativity for delayed MNN (2.1) is covered in this example. The dissipativity, the passivity, the H_∞ performance and the $L_2 - L_\infty$ performance are united by the extended dissipativity.

Choosing $\kappa_{1M} = 0.4$, $\kappa_{2M} = 0.2$, $\overline{\mu}_1 = 0.2$ and $\overline{\mu}_2 = 0.1$, specifically,

1) $\mathcal{Q}$, $\mathcal{S}$, $\mathcal{R}$−dissipativity: Let $\beth_1 = \mathcal{Q}$, $\beth_2 = \mathcal{S}$, $\beth_3 = \mathcal{R} - \gamma_0 I$, $\beth_4 = 0$, the allowable maximum dissipativity level γ_0^* that pledge MNN (2.1) strictly $\mathcal{Q}, \mathcal{S}, \mathcal{R} - \gamma-$ dissipative is displayed in Table 1.

2) H_∞ performance: When $\beth_1 = -I$, $\beth_2 = 0$, $\beth_3 = \gamma^2 I$, $\beth_4 = 0$, the H_∞ performance is obtained and given in Table 2.

3) Passivity: By choosing $\beth_1 = 0$, $\beth_2 = I$, $\beth_3 = \gamma I$, $\beth_4 = 0$, the passivity performance is derived and the allowable minimum passivity performance γ^* is given in Table 3.

4) $l_2 - l_\infty$ performance: When $\beth_1 = 0$, $\beth_2 = 0$, $\beth_3 = \gamma^2 I$, $\beth_4 = I$, the $l_2 - l_\infty$ performance is obtained and the allowable minimum $l_2 - l_\infty$ performance γ^* is given in Table 4.

Choosing $g(x(t)) = [0.9\tanh(t), 0.9\tanh(t)]$, $x(0) = [0.1, -0.1]^{\mathrm{T}}$, $\kappa_1(t) = 0.2\sin(t) + 0.2, \kappa_2(t) = -0.1\cos(t) + 0.1$. In this example, four performance derived by this paper and Theorem 3.1 [28] are displayed in Tables 1–4, respectively. If the system is dissipative, then it must be asymptotic stable. The state trajectory simulation figure is depicted in Figure 1. As can be seen from Figure 1, the final state of the MNNs (2.1) converges to zero, verifying the aymptotically stability of the simulated MNN. The outcomes of this study are superior to those of Theorem 3.1 [28], as seen in Tables 1–4, which demonstrates the merits of the analysis strategy proposed in this study.

Table 1. Maximum dissipativity performance γ_0^*.

Criteria	$\kappa_{1M} = 0.4, \kappa_{2M} = 0.2$
Theorem 1	0.1488
Theorem 2	0.1488
Theorem 3.1 [28]	0.0072

Table 2. Minimum H_∞ performance γ^*.

Criteria	$\kappa_{1M} = 0.4, \kappa_{2M} = 0.2$
Theorem 1	0.6805
Theorem 2	0.6777
Theorem 3.1 [28]	2.6455

Table 3. Minimum passivity performance γ^*.

Criteria	$\kappa_{1M} = 0.4, \kappa_{2M} = 0.2$
Theorem 1	0.2313
Theorem 2	0.2313
Theorem 3.1 [28]	2.0949

Table 4 Minimum $l_2 - l_\infty$ performance γ^*.

Criteria	$\kappa_{1M} = 0.4, \kappa_{2M} = 0.2$
Theorem 1	0.6685
Theorem 2	0.6637
Theorem 3.1 [28]	–

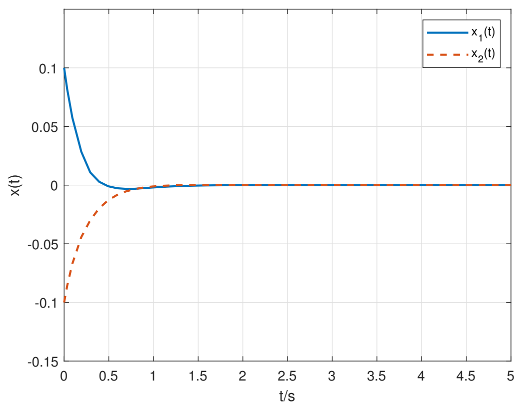

Figure 1. The state trajectory of system (2.1) with $\omega(t) = 0$.

5. Conclusions

We have explored the extended dissipativity analysis of MNNs with two additive time-delays. To consider the effects of these two delays more comprehensively, the DDI method is used to build a novel augmented LKF. When taking the derivative of the constructed LKF, its negative definite condition not only contains some linear terms about delays, but also contains some square terms and cubic terms about delay. Obviously, this condition contains more information about time delays, but cannot be solved directly. Then, by utilizing the inequality technique, some less conservative extended dissipativity criteria have been firsthand gained in terms of LMIs. Finally, an example has been provided to prove the viability of our presented approaches. From the work we have done, in future work, the study of extended dissipative state estimation for MNNs is our main research direction.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence(AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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