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Research article

Enhanced bounds for Riemann-Liouville fractional integrals: Novel variations of Milne inequalities

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Abstract: In this research article, we present novel extensions of Milne type inequalities to the realm of Riemann-Liouville fractional integrals. Our approach involves exploring significant functional classes, including convex functions, bounded functions, Lipschitzian functions and functions of bounded variation. To accomplish our objective, we begin by establishing a crucial identity for differentiable functions. Leveraging this identity, we subsequently derive new variations of fractional Milne inequalities.

Keywords: Milne type inequalities; convex function; fractional integrals; bounded function; function of bounded variation
Mathematics Subject Classification: 26D07, 26D10, 26D15

1. Introduction

Over the years, numerous researchers have introduced various formulas for numerical integration and have extensively studied the error bounds associated with these formulas. In the field of mathematical inequalities, authors have also focused on deriving new error bounds using different classes of functions, such as convex functions, bounded functions, Lipschitzian functions, functions of bounded variation and more. Furthermore, they have investigated the error bounds for functions that are differentiable, twice differentiable or n-times differentiable. Additionally, some authors have established new bounds by employing the concept of fractional calculus.

The literature contains several significant integral inequalities, including Simpson, Trapezoid, midpoint and others. Many papers have been dedicated to extending and generalizing these integral inequalities. For instance, several trapezoid-type inequalities have been derived for differentiable convex functions [1], bounded functions [2], Lipschitzian functions [3], functions of bounded variation [4] and twice differentiable convex functions [5]. In papers [6, 7], the authors focused on fractional versions of trapezoid-type inequalities. Midpoint-type inequalities have also been obtained for differentiable convex functions [8], bounded functions and functions of bounded variation [9], twice differentiable convex functions [10] and fractional integrals [11–14]. Similarly, several papers have been dedicated to establishing Simpson-type inequalities [15–28].

Milne-type inequalities are a sort of mathematical inequity proposed by the British mathematician William John Millne. These inequalities have found applications in many areas of mathematics, including special function analysis [29], approximation theory [30] and numerical analysis [31]. Milne-type inequalities employ integrals to quantify the difference between a function and its approximation, making them helpful for bounding mistakes and analysing the correctness of numerical and analytical approaches [30]. Milne's open-type formula and Simpson's closed-type formula are two numerical integration methods that share similarities and differences. Both formulas approximate the definite integral of a function using a composite quadrature rule and require a uniformly spaced grid of sample points. However, Milne's formula excludes the first and last intervals, whereas Simpson's formula includes them. Despite these distinctions, both formulas adhere to similar conditions, such as the assumptions of function smoothness and integrability. The choice between the two methods depends on factors such as the characteristics of the function and the desired accuracy. A thorough understanding of their similarities and differences aids researchers in selecting the most appropriate numerical integration technique. Suppose that $F:[\sigma,\rho]\to\mathbb{R}$ is a four times continuously differentiable mapping on (σ,ρ), and let $\left\|F^{(4)}\right\|_{\infty}=\sup_{\kappa\in(\sigma,\rho)}\left|F^{(4)}(\kappa)\right|<\infty$. Then, one has the inequality [32]

$$\left|\frac{1}{3}\left[2F(\sigma)-F\left(\frac{\sigma+\rho}{2}\right)+2F(\rho)\right]-\frac{1}{\rho-\sigma}\int_{\sigma}^{\rho}F(\kappa)d\kappa\right|\leq\frac{7(\rho-\sigma)^{4}}{23040}\left\|F^{(4)}\right\|_{\infty}. \tag{1.1}$$

Definition 1.1. Let $F\in L_1[\sigma,\rho]$. The Riemann-Liouville fractional integrals $\mathfrak{J}_{\sigma+}^{\alpha}F$ and $\mathfrak{J}_{\rho-}^{\alpha}F$ of order $\alpha>0$ are defined by

$$\mathfrak{J}_{\sigma+}^{\alpha}F(\kappa)=\frac{1}{\Gamma(\alpha)}\int_{\sigma}^{\kappa}(\kappa-\eta)^{\alpha-1}F(\eta)d\eta,\quad \kappa>\sigma$$

and

$$\mathfrak{J}_{\rho-}^{\alpha}F(\kappa)=\frac{1}{\Gamma(\alpha)}\int_{\kappa}^{\rho}(\eta-\kappa)^{\alpha-1}F(\eta)d\eta,\quad \kappa<\rho,$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $\mathfrak{J}_{\sigma+}^{0}F(\kappa)=\mathfrak{J}_{\rho-}^{0}F(\kappa)=F(\kappa)$.

For more information and several properties of Riemann-Liouville fractional integrals, please refer to [33–38].

In [29], Budak et al. established the first Milne inequality for convex functions in the case of Riemann-Liouville fractional integrals. This result represents a significant advancement in this particular research direction.

Theorem 1.1. *Let $F:[\sigma,\rho]\to\mathbb{R}$ be a differentiable mapping (σ,ρ) such that $F'\in L_1([\sigma,\rho])$. Moreover, suppose that the function $|F'|$ exhibits convexity over the interval $[\sigma,\rho]$ and $\alpha>0$. Then, we acquire the following Milne type inequalities for Riemann-Liouville fractional integrals*

$$\left|\frac{1}{3}\left[2F(\sigma)-F\left(\frac{\sigma+\rho}{2}\right)+2F(\rho)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha}}\left[\mathfrak{J}_{\sigma+}^{\alpha}F\left(\frac{\sigma+\rho}{2}\right)+\mathfrak{J}_{\rho-}^{\alpha}F\left(\frac{\sigma+\rho}{2}\right)\right]\right|$$
$$\leq \frac{\rho-\sigma}{12}\left(\frac{\alpha+4}{\alpha+1}\right)\left(|F'(\sigma)|+|F'(\rho)|\right).$$

In this research article, we embark on a comprehensive exploration of new variations of Milne-type inequalities for Riemann-Liouville fractional integrals. We establish fundamental identities for differentiable functions in Section 2, laying the groundwork for subsequent sections. Section 3 focuses on deriving Milne-type inequalities specifically for convex functions, while Section 4 presents the fractional Milne-type inequality for bounded functions. In Section 5, we extend our analysis to Lipschitzian functions, deriving fractional Milne-type inequalities tailored to this function class. Section 6 explores the derivation of fractional Milne-type inequalities for functions of bounded variation. Finally, in the Conclusion and Discussion (Section 7), we summarize our key findings and discuss their implications in advancing the understanding of integral inequalities across diverse mathematical contexts.

2. Some equalities

Let's begin with the following evaluated integrals, which will be utilized in obtaining our key findings:

$$\Upsilon_1(\alpha) = \int_0^{\frac{1}{2}}\left|\eta^{\alpha}-\frac{2}{3}\right|d\eta = \begin{cases} \frac{2\alpha}{\alpha+1}\left(\frac{2}{3}\right)^{\frac{1}{\alpha}+1}+\frac{1}{2^{\alpha+1}(\alpha+1)}-\frac{1}{3}, & 0<\alpha\leq\frac{\ln\frac{2}{3}}{\ln\frac{1}{2}} \\ \frac{1}{3}-\frac{1}{2^{\alpha+1}(\alpha+1)} & \alpha>\frac{\ln\frac{2}{3}}{\ln\frac{1}{2}}, \end{cases} \tag{2.1}$$
$$\Upsilon_2(\alpha) = \int_0^{\frac{1}{2}}\eta\left|\eta^{\alpha}-\frac{2}{3}\right|d\eta = \begin{cases} \frac{\alpha}{\alpha+2}\left(\frac{2}{3}\right)^{\frac{2}{\alpha}+1}+\frac{1}{2^{\alpha+2}(\alpha+2)}-\frac{1}{12}, & 0<\alpha\leq\frac{\ln\frac{2}{3}}{\ln\frac{1}{2}} \\ \frac{1}{12}-\frac{1}{2^{\alpha+2}(\alpha+2)} & \alpha>\frac{\ln\frac{2}{3}}{\ln\frac{1}{2}}, \end{cases}$$
$$\Upsilon_3(\alpha) = \int_{\frac{1}{2}}^{1}\left|\eta^{\alpha}-\frac{1}{3}\right|d\eta = \begin{cases} \frac{1}{\alpha+1}-\frac{1}{2^{\alpha+1}(\alpha+1)}-\frac{1}{6}, & 0<\alpha\leq\frac{\ln\frac{1}{3}}{\ln\frac{1}{2}} \\ \frac{2\alpha}{\alpha+1}\left(\frac{1}{3}\right)^{\frac{1}{\alpha}+1}+\frac{1}{2^{\alpha+1}(\alpha+1)}+\frac{1}{\alpha+1}-\frac{1}{2} & \alpha>\frac{\ln\frac{1}{3}}{\ln\frac{1}{2}}, \end{cases}$$
$$\Upsilon_4(\alpha) = \int_{\frac{1}{2}}^{1}\eta\left|\eta^{\alpha}-\frac{1}{3}\right|d\eta = \begin{cases} \frac{1}{\alpha+2}-\frac{1}{2^{\alpha+2}(\alpha+2)}-\frac{1}{8}, & 0<\alpha\leq\frac{\ln\frac{1}{3}}{\ln\frac{1}{2}} \\ \frac{\alpha}{\alpha+2}\left(\frac{1}{3}\right)^{\frac{2}{\alpha}+1}+\frac{1}{2^{\alpha+2}(\alpha+2)}+\frac{1}{\alpha+2}-\frac{5}{24} & \alpha>\frac{\ln\frac{1}{3}}{\ln\frac{1}{2}}. \end{cases}$$

Now, we prove the following identity for differentiable functions.

Lemma 2.1. *Let $F:[\sigma,\rho]\to\mathbb{R}$ be a differentiable mapping (σ,ρ) such that $F'\in L_1([\sigma,\rho])$. Then, for $\alpha>0$, the following equality holds:*

$$\frac{1}{3}\left[2F(\sigma)-F\left(\frac{\sigma+\rho}{2}\right)+2F(\rho)\right]-\frac{\Gamma(\alpha+1)}{2(\rho-\sigma)^{\alpha}}\left[\mathfrak{J}_{\sigma+}^{\alpha}F(\rho)+\mathfrak{J}_{\rho-}^{\alpha}F(\sigma)\right]$$
$$=\frac{\rho-\sigma}{2}\left[\mathcal{I}_1+\mathcal{I}_2-\mathcal{I}_3-\mathcal{I}_4\right],$$

where

$$\begin{aligned}\mathcal{I}_1 &= \int_0^{\frac{1}{2}}\left(\eta^\alpha-\frac{2}{3}\right)F'\left(\eta\rho+(1-\eta)\sigma\right)d\eta,\\ \mathcal{I}_2 &= \int_{\frac{1}{2}}^{1}\left(\eta^\alpha-\frac{1}{3}\right)F'\left(\eta\rho+(1-\eta)\sigma\right)d\eta,\\ \mathcal{I}_3 &= \int_0^{\frac{1}{2}}\left(\eta^\alpha-\frac{2}{3}\right)F'\left(\eta\sigma+(1-\eta)\rho\right)d\eta,\\ \mathcal{I}_4 &= \int_{\frac{1}{2}}^{1}\left(\eta^\alpha-\frac{1}{3}\right)F'\left(\eta\sigma+(1-\eta)\rho\right)d\eta.\end{aligned}$$

Proof. Using the integration by parts, we obtain

$$\begin{aligned}\mathcal{I}_1 &= \int_0^{\frac{1}{2}}\left(\eta^\alpha-\frac{2}{3}\right)F'\left(\eta\rho+(1-\eta)\sigma\right)d\eta \qquad (2.2)\\ &= \frac{1}{\rho-\sigma}\left(\eta^\alpha-\frac{2}{3}\right)F\left(\eta\rho+(1-\eta)\sigma\right)\Big|_0^{\frac{1}{2}}-\frac{\alpha}{\rho-\sigma}\int_0^{\frac{1}{2}}\eta^{\alpha-1}F\left(\eta\rho+(1-\eta)\sigma\right)d\eta\\ &= \frac{1}{\rho-\sigma}\left(\frac{1}{2^\alpha}-\frac{2}{3}\right)F\left(\frac{\sigma+\rho}{2}\right)+\frac{2}{3(\rho-\sigma)}F(\sigma)-\frac{\alpha}{\rho-\sigma}\int_0^{\frac{1}{2}}\eta^{\alpha-1}F\left(\eta\rho+(1-\eta)\sigma\right)d\eta,\end{aligned}$$

$$\begin{aligned}\mathcal{I}_2 &= \int_{\frac{1}{2}}^{1}\left(\eta^\alpha-\frac{1}{3}\right)F'\left(\eta\rho+(1-\eta)\sigma\right)d\eta \qquad (2.3)\\ &= \frac{1}{\rho-\sigma}\left(\eta^\alpha-\frac{1}{3}\right)F\left(\eta\rho+(1-\eta)\sigma\right)\Big|_{\frac{1}{2}}^{1}-\frac{\alpha}{\rho-\sigma}\int_{\frac{1}{2}}^{1}\eta^{\alpha-1}F\left(\eta\rho+(1-\eta)\sigma\right)d\eta\\ &= \frac{2}{3(\rho-\sigma)}F(\rho)-\frac{1}{\rho-\sigma}\left(\frac{1}{2^\alpha}-\frac{1}{3}\right)F\left(\frac{\sigma+\rho}{2}\right)-\frac{\alpha}{\rho-\sigma}\int_{\frac{1}{2}}^{1}\eta^{\alpha-1}F\left(\eta\rho+(1-\eta)\sigma\right)d\eta.\end{aligned}$$

Similarly, we get

$$\begin{aligned}\mathcal{I}_3 &= \int_0^{\frac{1}{2}}\left(\eta^\alpha-\frac{2}{3}\right)F'\left(\eta\sigma+(1-\eta)\rho\right)d\eta \qquad (2.4)\\ &= -\frac{1}{\rho-\sigma}\left(\frac{1}{2^\alpha}-\frac{2}{3}\right)F\left(\frac{\sigma+\rho}{2}\right)-\frac{2}{3(\rho-\sigma)}F(\rho)+\frac{\alpha}{\rho-\sigma}\int_0^{\frac{1}{2}}\eta^{\alpha-1}F\left(\eta\sigma+(1-\eta)\rho\right)d\eta,\end{aligned}$$

and

$$\begin{aligned}\mathcal{I}_4 &= \int_{\frac{1}{2}}^{1}\left(\eta^\alpha-\frac{1}{3}\right)F'\left(\eta\sigma+(1-\eta)\rho\right)d\eta \qquad (2.5)\\ &= -\frac{2}{3(\rho-\sigma)}F(\sigma)+\frac{1}{\rho-\sigma}\left(\frac{1}{2^\alpha}-\frac{1}{3}\right)F\left(\frac{\sigma+\rho}{2}\right)+\frac{\alpha}{\rho-\sigma}\int_{\frac{1}{2}}^{1}\eta^{\alpha-1}F\left(\eta\sigma+(1-\eta)\rho\right)d\eta.\end{aligned}$$

By the equalities (2.2) and (2.3), we have

$$\begin{aligned}(\rho-\sigma)(\mathcal{I}_1+\mathcal{I}_2) &= \frac{2}{3}(F(\sigma)+F(\rho))-\frac{1}{3}F\left(\frac{\sigma+\rho}{2}\right)-\alpha\int_0^1\eta^{\alpha-1}F(\eta\rho+(1-\eta)\sigma)\,d\eta \\ &= \frac{2}{3}(F(\sigma)+F(\rho))-\frac{1}{3}F\left(\frac{\sigma+\rho}{2}\right)-\frac{\alpha}{(\rho-\sigma)^{\alpha}}\int_\sigma^\rho(\kappa-\sigma)^{\alpha-1}F(\kappa)\,d\kappa \\ &= \frac{2}{3}(F(\sigma)+F(\rho))-\frac{1}{3}F\left(\frac{\sigma+\rho}{2}\right)-\frac{\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha}}\mathfrak{J}^{\alpha}_{\sigma+}F(\rho).\end{aligned} \tag{2.6}$$

Similarly, by the equalities (2.4) and (2.5), we get

$$(\rho-\sigma)(\mathcal{I}_3+\mathcal{I}_4)=-\frac{2}{3}(F(\sigma)+F(\rho))+\frac{1}{3}F\left(\frac{\sigma+\rho}{2}\right)+\frac{\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha}}\mathfrak{J}^{\alpha}_{\rho-}F(\sigma). \tag{2.7}$$

The equalities (2.6) and (2.7) yield the following equality:

$$\frac{\rho-\sigma}{2}[\mathcal{I}_1+\mathcal{I}_2-\mathcal{I}_3-\mathcal{I}_4]=\frac{1}{3}\left[2F(\sigma)-F\left(\frac{\sigma+\rho}{2}\right)+2F(\rho)\right]-\frac{\Gamma(\alpha+1)}{2(\rho-\sigma)^{\alpha}}\left[\mathfrak{J}^{\alpha}_{\sigma+}F(\rho)+\mathfrak{J}^{\alpha}_{\rho-}F(\sigma)\right].$$

This is the end of the proof of Lemma 2.1. □

3. Milne type inequalities for convex functions

In this section, we prove some Milne type inequalities by using convex functions.

Theorem 3.1. *Let us consider that the conditions stated in Lemma 2.1 are satisfied. Additionally, suppose that the function $|F'|$ exhibits convexity over the interval $[\sigma,\rho]$. Then, we acquire the following Milne type inequalities for Riemann-Liouville fractional integrals*

$$\begin{aligned}&\left|\frac{1}{3}\left[2F(\sigma)-F\left(\frac{\sigma+\rho}{2}\right)+2F(\rho)\right]-\frac{\Gamma(\alpha+1)}{2(\rho-\sigma)^{\alpha}}\left[\mathfrak{J}^{\alpha}_{\sigma+}F(\rho)+\mathfrak{J}^{\alpha}_{\rho-}F(\sigma)\right]\right| \\ \leq\; &\frac{\rho-\sigma}{2}(\Upsilon_1(\alpha)+\Upsilon_3(\alpha))(|F'(\sigma)|+|F'(\rho)|),\end{aligned}$$

where Υ_1 and Υ_3 are defined as in (2.1).

Proof. By taking modulus in Lemma 2.1 and using the convexity of $|F'|$, we have

$$\begin{aligned}&\left|\frac{1}{3}\left[2F(\sigma)-F\left(\frac{\sigma+\rho}{2}\right)+2F(\rho)\right]-\frac{\Gamma(\alpha+1)}{2(\rho-\sigma)^{\alpha}}\left[\mathfrak{J}^{\alpha}_{\sigma+}F(\rho)+\mathfrak{J}^{\alpha}_{\rho-}F(\sigma)\right]\right| \\ \leq\; &\frac{\rho-\sigma}{2}[|\mathcal{I}_1|+|\mathcal{I}_2|+|\mathcal{I}_3|+|\mathcal{I}_4|].\end{aligned} \tag{3.1}$$

By convexity of $|F'|$, we have

$$|\mathcal{I}_1| \leq \int_0^{\frac{1}{2}}\left|\eta^{\alpha}-\frac{2}{3}\right||F'(\eta\rho+(1-\eta)\sigma)|\,d\eta \tag{3.2}$$

$$
\begin{aligned}
&\leq \int_0^{\frac{1}{2}} \left|\eta^\alpha - \frac{2}{3}\right| \left[\eta \left|F'(\rho)\right| + (1-\eta)\left|F'(\sigma)\right|\right] d\eta \\
&= \Upsilon_2(\alpha)\left|F'(\rho)\right| + \left(\Upsilon_1(\alpha) - \Upsilon_2(\alpha)\right)\left|F'(\sigma)\right|,
\end{aligned}
$$

and similarly

$$
|\mathcal{I}_3| \leq \int_0^{\frac{1}{2}} \left|\eta^\alpha - \frac{2}{3}\right| \left|F'(\eta\sigma + (1-\eta)\rho)\right| d\eta \leq \Upsilon_2(\alpha)\left|F'(\sigma)\right| + \left(\Upsilon_1(\alpha) - \Upsilon_2(\alpha)\right)\left|F'(\rho)\right|. \tag{3.3}
$$

Similarly, by advantange of the convexity of $|F'|$, we get

$$
\begin{aligned}
|\mathcal{I}_2| &\leq \int_{\frac{1}{2}}^1 \left|\eta^\alpha - \frac{1}{3}\right| \left|F'(\eta\rho + (1-\eta)\sigma)\right| d\eta \qquad (3.4) \\
&\leq \int_{\frac{1}{2}}^1 \left|\eta^\alpha - \frac{1}{3}\right| \left[\eta \left|F'(\rho)\right| + (1-\eta)\left|F'(\sigma)\right|\right] d\eta \\
&= \Upsilon_4(\alpha)\left|F'(\rho)\right| + \left(\Upsilon_3(\alpha) - \Upsilon_4(\alpha)\right)\left|F'(\sigma)\right|,
\end{aligned}
$$

and

$$
|\mathcal{I}_4| \leq \int_{\frac{1}{2}}^1 \left|\eta^\alpha - \frac{1}{3}\right| \left|F'(\eta\sigma + (1-\eta)\rho)\right| d\eta \leq \Upsilon_4(\alpha)\left|F'(\sigma)\right| + \left(\Upsilon_3(\alpha) - \Upsilon_4(\alpha)\right)\left|F'(\rho)\right|. \tag{3.5}
$$

By substituting the inequalities (3.2)–(3.5) in (3.1), then we obtain the desired result. □

Remark 3.1. Let us note that $\alpha = 1$ in Theorem 3.1. Then we have the following inequality

$$
\begin{aligned}
&\left|\frac{1}{3}\left[2F(\sigma) - F\left(\frac{\sigma+\rho}{2}\right) + 2F(\rho)\right] - \frac{1}{\rho-\sigma}\int_\sigma^\rho F(\eta)d\eta\right| \\
\leq\ &\frac{5(\rho-\sigma)}{24}\left(\left|F'(\sigma)\right| + \left|F'(\rho)\right|\right),
\end{aligned}
$$

which is given in [29].

Theorem 3.2. *Suppose that the assumptions of Lemma 2.1 hold. Suppose also that the mapping $|F'|^q$, $q > 1$ is convex on $[\sigma,\rho]$. Then, we have the following Milne type inequality for Riemann-Liouville fractional integrals*

$$
\begin{aligned}
&\left|\frac{1}{3}\left[2F(\sigma) - F\left(\frac{\sigma+\rho}{2}\right) + 2F(\rho)\right] - \frac{\Gamma(\alpha+1)}{2(\rho-\sigma)^\alpha}\left[\mathfrak{I}^\alpha_{\sigma+}F(\rho) + \mathfrak{I}^\alpha_{\rho-}F(\sigma)\right]\right| \qquad (3.6) \\
\leq\ &\frac{\rho-\sigma}{2}\left\{\left(\int_0^{\frac{1}{2}}\left|\eta^\alpha - \frac{2}{3}\right|^p d\eta\right)^{\frac{1}{p}}\right. \\
&\times\left[\left(\frac{\left|F'(\rho)\right|^q + 3\left|F'(\sigma)\right|^q}{8}\right)^{\frac{1}{q}} + \left(\frac{\left|F'(\sigma)\right|^q + 3\left|F'(\rho)\right|^q}{8}\right)^{\frac{1}{q}}\right] \\
&+\left(\int_{\frac{1}{2}}^1\left|\eta^\alpha - \frac{1}{3}\right|^p d\eta\right)^{\frac{1}{p}}
\end{aligned}
$$

$$\times\left[\left(\frac{3\left|F'(\rho)\right|^{q}+\left|F'(\sigma)\right|^{q}}{8}\right)^{\frac{1}{q}}+\left(\frac{3\left|F'(\sigma)\right|^{q}+\left|F'(\rho)\right|^{q}}{8}\right)^{\frac{1}{q}}\right]\Bigg\},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By applying Hölder inequality and by using the convexity of $|F'|^q$, we obtain

$$\begin{aligned}
|\mathcal{I}_1| &\leq \int_0^{\frac{1}{2}}\left|\eta^{\alpha}-\frac{2}{3}\right|\left|F'(\eta\rho+(1-\eta)\sigma)\right|d\eta \qquad (3.7)\\
&\leq \left(\int_0^{\frac{1}{2}}\left|\eta^{\alpha}-\frac{2}{3}\right|^{p}d\eta\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|F'(\eta\rho+(1-\eta)\sigma)\right|^{q}d\eta\right)^{\frac{1}{q}}\\
&\leq \left(\int_0^{\frac{1}{2}}\left|\eta^{\alpha}-\frac{2}{3}\right|^{p}d\eta\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left[\eta\left|F'(\rho)\right|^{q}+(1-\eta)\left|F'(\sigma)\right|^{q}\right]d\eta\right)^{\frac{1}{q}}\\
&= \left(\int_0^{\frac{1}{2}}\left|\eta^{\alpha}-\frac{2}{3}\right|^{p}d\eta\right)^{\frac{1}{p}}\left(\frac{\left|F'(\rho)\right|^{q}+3\left|F'(\sigma)\right|^{q}}{8}\right)^{\frac{1}{q}}.
\end{aligned}$$

Similar way, we obtain

$$\begin{aligned}
|\mathcal{I}_2| &\leq \int_{\frac{1}{2}}^{1}\left|\eta^{\alpha}-\frac{1}{3}\right|\left|F'(\eta\rho+(1-\eta)\sigma)\right|d\eta \qquad (3.8)\\
&\leq \left(\int_{\frac{1}{2}}^{1}\left|\eta^{\alpha}-\frac{1}{3}\right|^{p}d\eta\right)^{\frac{1}{p}}\left(\frac{3\left|F'(\rho)\right|^{q}+\left|F'(\sigma)\right|^{q}}{8}\right)^{\frac{1}{q}},
\end{aligned}$$

$$\begin{aligned}
|\mathcal{I}_3| &\leq \int_0^{\frac{1}{2}}\left|\eta^{\alpha}-\frac{2}{3}\right|\left|F'(\eta\sigma+(1-\eta)\rho)\right|d\eta \qquad (3.9)\\
&\leq \left(\int_0^{\frac{1}{2}}\left|\eta^{\alpha}-\frac{2}{3}\right|^{p}d\eta\right)^{\frac{1}{p}}\left(\frac{\left|F'(\sigma)\right|^{q}+3\left|F'(\rho)\right|^{q}}{8}\right)^{\frac{1}{q}},
\end{aligned}$$

and

$$\begin{aligned}
|\mathcal{I}_4| &\leq \int_{\frac{1}{2}}^{1}\left|\eta^{\alpha}-\frac{1}{3}\right|\left|F'(\eta\sigma+(1-\eta)\rho)\right|d\eta \qquad (3.10)\\
&\leq \left(\int_{\frac{1}{2}}^{1}\left|\eta^{\alpha}-\frac{1}{3}\right|^{p}d\eta\right)^{\frac{1}{p}}\left(\frac{3\left|F'(\sigma)\right|^{q}+\left|F'(\rho)\right|^{q}}{8}\right)^{\frac{1}{q}}.
\end{aligned}$$

If we put the inequalities (3.7)–(3.10) in (3.1), then we obtain the required inequality (3.6). □

Corollary 3.1. *Let us consider $\alpha=1$ in Theorem 3.2. Then, we have the following inequality*

$$\left|\frac{1}{3}\left[2F(\sigma)-F\left(\frac{\sigma+\rho}{2}\right)+2F(\rho)\right]-\frac{1}{\rho-\sigma}\int_{\sigma}^{\rho}F(\eta)d\eta\right|$$

$$
\begin{aligned}
&\leq \frac{\rho-\sigma}{2}\left(\frac{4^{p+1}-1}{6^{p+1}(p+1)}\right)^{\frac{1}{p}} \\
&\quad\times\left\{\left(\frac{|F'(\rho)|^q+3|F'(\sigma)|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{|F'(\sigma)|^q+3|F'(\rho)|^q}{8}\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\frac{3|F'(\rho)|^q+5|F'(\sigma)|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{3|F'(\sigma)|^q+|F'(\rho)|^q}{8}\right)^{\frac{1}{q}}\right\}.
\end{aligned}
$$

Theorem 3.3. *Assume that the assumptions of Lemma 2.1 hold. If the mapping $|F'|^q$, $q\geq 1$, is convex on $[\sigma,\rho]$, then we have the following Milne type inequalities for Riemann-Liouville fractional integrals*

$$
\begin{aligned}
&\left|\frac{1}{3}\left[2F(\sigma)-F\left(\frac{\sigma+\rho}{2}\right)+2F(\rho)\right]-\frac{\Gamma(\alpha+1)}{2(\rho-\sigma)^\alpha}\left[\mathfrak{I}^\alpha_{\sigma+}F(\rho)+\mathfrak{I}^\alpha_{\rho-}F(\sigma)\right]\right| \qquad (3.11)\\
&\leq \frac{\rho-\sigma}{2}\left\{(\Upsilon_1(\alpha))^{1-\frac{1}{q}}\left(\Upsilon_2(\alpha)|F'(\rho)|^q+(\Upsilon_1(\alpha)-\Upsilon_2(\alpha))|F'(\sigma)|^q\right)^{\frac{1}{q}}\right. \\
&\quad+(\Upsilon_3(\alpha))^{1-\frac{1}{q}}\left(\Upsilon_4(\alpha)|F'(\rho)|^q+(\Upsilon_3(\alpha)-\Upsilon_4(\alpha))|F'(\sigma)|^q\right)^{\frac{1}{q}} \\
&\quad+(\Upsilon_1(\alpha))^{1-\frac{1}{q}}\left(\Upsilon_2(\alpha)|F'(\sigma)|^q+(\Upsilon_1(\alpha)-\Upsilon_2(\alpha))|F'(\rho)|^q\right)^{\frac{1}{q}} \\
&\quad\left.+(\Upsilon_3(\alpha))^{1-\frac{1}{q}}\left(\Upsilon_4(\alpha)|F'(\sigma)|^q+(\Upsilon_3(\alpha)-\Upsilon_4(\alpha))|F'(\rho)|^q\right)^{\frac{1}{q}}\right\},
\end{aligned}
$$

where Υ_1-Υ_4 are defined as in (2.1).

Proof. By applying the inequality of the power-mean and by using the convexity of $|F'|^q$, we obtain

$$
\begin{aligned}
|\mathcal{I}_1| &\leq \int_0^{\frac{1}{2}}\left|\eta^\alpha-\frac{2}{3}\right||F'(\eta\rho+(1-\eta)\sigma)|\,d\eta \qquad (3.12)\\
&\leq \left(\int_0^{\frac{1}{2}}\left|\eta^\alpha-\frac{2}{3}\right|d\eta\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\left|\eta^\alpha-\frac{2}{3}\right||F'(\eta\rho+(1-\eta)\sigma)|^q\,d\eta\right)^{\frac{1}{q}} \\
&\leq (\Upsilon_1(\alpha))^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\left|\eta^\alpha-\frac{2}{3}\right|\left[\eta|F'(\rho)|^q+(1-\eta)|F'(\sigma)|^q\right]d\eta\right)^{\frac{1}{q}} \\
&= (\Upsilon_1(\alpha))^{1-\frac{1}{q}}\left(\Upsilon_2(\alpha)|F'(\rho)|^q+(\Upsilon_1(\alpha)-\Upsilon_2(\alpha))|F'(\sigma)|^q\right)^{\frac{1}{q}}.
\end{aligned}
$$

Similary, we get

$$
\begin{aligned}
|\mathcal{I}_2| &\leq \int_{\frac{1}{2}}^1\left|\eta^\alpha-\frac{1}{3}\right||F'(\eta\rho+(1-\eta)\sigma)|\,d\eta \qquad (3.13)\\
&\leq (\Upsilon_3(\alpha))^{1-\frac{1}{q}}\left(\Upsilon_4(\alpha)|F'(\rho)|^q+(\Upsilon_3(\alpha)-\Upsilon_4(\alpha))|F'(\sigma)|^q\right)^{\frac{1}{q}},
\end{aligned}
$$

$$
\begin{aligned}
|\mathcal{I}_3| &\leq \int_0^{\frac{1}{2}}\left|\eta^\alpha-\frac{2}{3}\right||F'(\eta\sigma+(1-\eta)\rho)|\,d\eta \qquad (3.14)\\
&\leq (\Upsilon_1(\alpha))^{1-\frac{1}{q}}\left(\Upsilon_2(\alpha)|F'(\sigma)|^q+(\Upsilon_1(\alpha)-\Upsilon_2(\alpha))|F'(\rho)|^q\right)^{\frac{1}{q}},
\end{aligned}
$$

and

$$\begin{aligned} |\mathcal{I}_4| &\leq \int_{\frac{1}{2}}^{1} \left|\eta^{\alpha} - \frac{1}{3}\right| |F'(\eta\sigma + (1-\eta)\rho)| \, d\eta \\ &\leq (\Upsilon_3(\alpha))^{1-\frac{1}{q}} \left(\Upsilon_4(\alpha) |F'(\sigma)|^q + (\Upsilon_3(\alpha) - \Upsilon_4(\alpha)) |F'(\rho)|^q\right)^{\frac{1}{q}}. \end{aligned} \tag{3.15}$$

If we substitute the inequalities (3.12)–(3.15) in (3.1), then we obtain

$$\begin{aligned} &\left|\frac{1}{3}\left[2F(\sigma) - F\left(\frac{\sigma+\rho}{2}\right) + 2F(\rho)\right] - \frac{\Gamma(\alpha+1)}{2(\rho-\sigma)^{\alpha}}\left[\mathfrak{I}_{\sigma+}^{\alpha}F(\rho) + \mathfrak{I}_{\rho-}^{\alpha}F(\sigma)\right]\right| \\ \leq\; & \frac{\rho-\sigma}{2}\Big\{(\Upsilon_1(\alpha))^{1-\frac{1}{q}}\left(\Upsilon_2(\alpha)|F'(\rho)|^q + (\Upsilon_1(\alpha) - \Upsilon_2(\alpha))|F'(\sigma)|^q\right)^{\frac{1}{q}} \\ & + (\Upsilon_3(\alpha))^{1-\frac{1}{q}}\left(\Upsilon_4(\alpha)|F'(\rho)|^q + (\Upsilon_3(\alpha) - \Upsilon_4(\alpha))|F'(\sigma)|^q\right)^{\frac{1}{q}} \\ & + (\Upsilon_1(\alpha))^{1-\frac{1}{q}}\left(\Upsilon_2(\alpha)|F'(\sigma)|^q + (\Upsilon_1(\alpha) - \Upsilon_2(\alpha))|F'(\rho)|^q\right)^{\frac{1}{q}} \\ & + (\Upsilon_3(\alpha))^{1-\frac{1}{q}}\left(\Upsilon_4(\alpha)|F'(\sigma)|^q + (\Upsilon_3(\alpha) - \Upsilon_4(\alpha))|F'(\rho)|^q\right)^{\frac{1}{q}}\Big\}, \end{aligned}$$

which proves the inequality (3.11). □

Remark 3.2. If we take $\alpha = 1$ in Theorem 3.3, then we get the following inequality

$$\begin{aligned} &\left|\frac{1}{3}\left[2F(\sigma) - F\left(\frac{\sigma+\rho}{2}\right) + 2F(\rho)\right] - \frac{1}{\rho-\sigma}\int_{\sigma}^{\rho} F(\eta)d\eta\right| \\ \leq\; & \frac{5}{24}(\rho-\sigma)\left[\left(\frac{|F'(\rho)|^q + 4|F'(\sigma)|^q}{5}\right)^{\frac{1}{q}} + \left(\frac{4|F'(\rho)|^q + |F'(\sigma)|^q}{5}\right)^{\frac{1}{q}}\right], \end{aligned}$$

which is given in [29, Remark 2].

4. Fractional Milne type inequality for bounded functions

In this section, we present some fractional Milne type inequalities for bounded functions.

Theorem 4.1. *Suppose that the assumptions of Lemma 2.1 hold. If there exist $m, M \in \mathbb{R}$ such that $m \leq F'(\eta) \leq M$ for $\eta \in [\sigma, \rho]$, then we have the following Milne type inequality for Riemann-Liouville fractional integrals*

$$\begin{aligned} &\left|\frac{1}{3}\left[2F(\sigma) - F\left(\frac{\sigma+\rho}{2}\right) + 2F(\rho)\right] - \frac{\Gamma(\alpha+1)}{2(\rho-\sigma)^{\alpha}}\left[\mathfrak{I}_{\sigma+}^{\alpha}F(\rho) + \mathfrak{I}_{\rho-}^{\alpha}F(\sigma)\right]\right| \\ \leq\; & \frac{\rho-\sigma}{2}\left[\Upsilon_1(\alpha) + \Upsilon_3(\alpha)\right](M-m), \end{aligned}$$

where Υ_1 and Υ_3 are defined as in (2.1).

Proof. By Lemma 2.1, we can easily write

$$\frac{1}{3}\left[2F(\sigma) - F\left(\frac{\sigma+\rho}{2}\right) + 2F(\rho)\right] - \frac{\Gamma(\alpha+1)}{2(\rho-\sigma)^{\alpha}}\left[\mathfrak{I}_{\sigma+}^{\alpha}F(\rho) + \mathfrak{I}_{\rho-}^{\alpha}F(\sigma)\right] \tag{4.1}$$

$$
\begin{aligned}
&= \frac{\rho-\sigma}{2}\left[\int_0^{\frac{1}{2}}\left(\eta^\alpha-\frac{2}{3}\right)\left[F'(\eta\rho+(1-\eta)\sigma)-\frac{m+M}{2}\right]d\eta\right. \\
&\quad+\int_{\frac{1}{2}}^1\left(\eta^\alpha-\frac{1}{3}\right)\left[F'(\eta\rho+(1-\eta)\sigma)-\frac{m+M}{2}\right]d\eta \\
&\quad+\int_0^{\frac{1}{2}}\left(\eta^\alpha-\frac{2}{3}\right)\left[\frac{m+M}{2}-F'(\eta\sigma+(1-\eta)\rho)\right]d\eta \\
&\quad\left.\int_{\frac{1}{2}}^1\left(\eta^\alpha-\frac{1}{3}\right)\left[\frac{m+M}{2}-F'(\eta\sigma+(1-\eta)\rho)\right]d\eta\right].
\end{aligned}
$$

By using the properties of modulus in (4.1), we obtain

$$
\begin{aligned}
&\left|\frac{1}{3}\left[2F(\sigma)-F\left(\frac{\sigma+\rho}{2}\right)+2F(\rho)\right]-\frac{\Gamma(\alpha+1)}{2(\rho-\sigma)^\alpha}\left[\Im_{\sigma+}^\alpha F(\rho)+\Im_{\rho-}^\alpha F(\sigma)\right]\right| \\
&\le \frac{\rho-\sigma}{2}\left[\int_0^{\frac{1}{2}}\left|\eta^\alpha-\frac{2}{3}\right|\left|F'(\eta\rho+(1-\eta)\sigma)-\frac{m+M}{2}\right|d\eta\right. \\
&\quad+\int_{\frac{1}{2}}^1\left|\eta^\alpha-\frac{1}{3}\right|\left|F'(\eta\rho+(1-\eta)\sigma)-\frac{m+M}{2}\right|d\eta \\
&\quad+\int_0^{\frac{1}{2}}\left|\eta^\alpha-\frac{2}{3}\right|\left|\frac{m+M}{2}-F'(\eta\sigma+(1-\eta)\rho)\right|d\eta \\
&\quad\left.\int_{\frac{1}{2}}^1\left|\eta^\alpha-\frac{1}{3}\right|\left|\frac{m+M}{2}-F'(\eta\sigma+(1-\eta)\rho)\right|d\eta\right].
\end{aligned}
$$

From the assumption $m \le F'(\eta) \le M$ for $\eta \in [\sigma,\rho]$, we have

$$
\left|F'(\eta\rho+(1-\eta)\sigma)-\frac{m+M}{2}\right| \le \frac{M-m}{2}, \tag{4.2}
$$

and

$$
\left|\frac{m+M}{2}-F'(\eta\sigma+(1-\eta)\rho)\right| \le \frac{M-m}{2}. \tag{4.3}
$$

By the inequalities (4.2) and (4.3), we get

$$
\begin{aligned}
&\left|\frac{1}{3}\left[2F(\sigma)-F\left(\frac{\sigma+\rho}{2}\right)+2F(\rho)\right]-\frac{\Gamma(\alpha+1)}{2(\rho-\sigma)^\alpha}\left[\Im_{\sigma+}^\alpha F(\rho)+\Im_{\rho-}^\alpha F(\sigma)\right]\right| \\
&\le \frac{\rho-\sigma}{2}\left[\int_0^{\frac{1}{2}}\left|\eta^\alpha-\frac{2}{3}\right|d\eta+\int_{\frac{1}{2}}^1\left|\eta^\alpha-\frac{1}{3}\right|d\eta\right](M-m) \\
&= \frac{\rho-\sigma}{2}\left[\Upsilon_1(\alpha)+\Upsilon_3(\alpha)\right](M-m).
\end{aligned}
$$

This completes the proof. □

Corollary 4.1. *If we take $\alpha = 1$ in Theorem 4.1, then we get the following inequality*

$$
\left|\frac{1}{3}\left[2F(\sigma)-F\left(\frac{\sigma+\rho}{2}\right)+2F(\rho)\right]-\frac{1}{\rho-\sigma}\int_\sigma^\rho F(\eta)d\eta\right| \le \frac{5(\rho-\sigma)}{24}(M-m),
$$

which is given in [29, Corollary 2].

Corollary 4.2. *Under assumptions of Theorem 4.1, if there exists $M \in \mathbb{R}^+$ such that $|F'(\eta)| \leq M$ for all $\eta \in [\sigma, \rho]$, then we have*

$$\begin{aligned} &\left|\frac{1}{3}\left[2F(\sigma) - F\left(\frac{\sigma+\rho}{2}\right) + 2F(\rho)\right] - \frac{\Gamma(\alpha+1)}{2(\rho-\sigma)^{\alpha}}\left[\mathfrak{I}^{\alpha}_{\sigma+}F(\rho) + \mathfrak{I}^{\alpha}_{\rho-}F(\sigma)\right]\right| \\ \leq\ & M(\rho-\sigma)\left[\Upsilon_1(\alpha) + \Upsilon_3(\alpha)\right]. \end{aligned}$$

Remark 4.1. If we choose $\alpha = 1$ in Corollary 4.2, then we get the inequality

$$\left|\frac{1}{3}\left[2F(\sigma) - F\left(\frac{\sigma+\rho}{2}\right) + 2F(\rho)\right] - \frac{1}{\rho-\sigma}\int_{\sigma}^{\rho} F(\eta)d\eta\right| \leq \frac{5}{12}(\rho-\sigma)M,$$

which is given by Alomari and Liu in [39].

5. Fractional Milne type inequality for Lipschitzian functions

In this section, we give some fractional Milne type inequalities for Lipschitzian functions.

Theorem 5.1. *Suppose that the assumptions of Lemma 2.1 hold. If F' is an L-Lipschitzian function on $[\sigma, \rho]$, then we have the the following Milne type inequality for Riemann-Liouville fractional integrals*

$$\begin{aligned} &\left|\frac{1}{3}\left[2F(\sigma) - F\left(\frac{\sigma+\rho}{2}\right) + 2F(\rho)\right] - \frac{\Gamma(\alpha+1)}{2(\rho-\sigma)^{\alpha}}\left[\mathfrak{I}^{\alpha}_{\sigma+}F(\rho) + \mathfrak{I}^{\alpha}_{\rho-}F(\sigma)\right]\right| \\ \leq\ & \frac{(\rho-\sigma)^2}{2}\left[\Upsilon_1(\alpha) - 2\Upsilon_2(\alpha) + 2\Upsilon_4(\alpha) - \Upsilon_3(\alpha)\right]L, \end{aligned}$$

where Υ_1-Υ_4 are defined as in (2.1).

Proof. We can rewite Lemma 2.1 as

$$\begin{aligned} &\frac{1}{3}\left[2F(\sigma) - F\left(\frac{\sigma+\rho}{2}\right) + 2F(\rho)\right] - \frac{\Gamma(\alpha+1)}{2(\rho-\sigma)^{\alpha}}\left[\mathfrak{I}^{\alpha}_{\sigma+}F(\rho) + \mathfrak{I}^{\alpha}_{\rho-}F(\sigma)\right] \\ =\ & \frac{\rho-\sigma}{2}\left[\int_0^{\frac{1}{2}}\left(\eta^{\alpha} - \frac{2}{3}\right)\left[F'(\eta\rho + (1-\eta)\sigma) - F'(\eta\sigma + (1-\eta)\rho)\right]d\eta\right. \\ &\left. + \int_{\frac{1}{2}}^{1}\left(\eta^{\alpha} - \frac{1}{3}\right)\left[F'(\eta\rho + (1-\eta)\sigma) - F'(\eta\sigma + (1-\eta)\rho)\right]d\eta\right]. \end{aligned}$$

Since F' is L-Lipschitzian function, we obtain

$$\begin{aligned} &\left|\frac{1}{3}\left[2F(\sigma) - F\left(\frac{\sigma+\rho}{2}\right) + 2F(\rho)\right] - \frac{\Gamma(\alpha+1)}{2(\rho-\sigma)^{\alpha}}\left[\mathfrak{I}^{\alpha}_{\sigma+}F(\rho) + \mathfrak{I}^{\alpha}_{\rho-}F(\sigma)\right]\right| \\ \leq\ & \frac{\rho-\sigma}{2}\left[\int_0^{\frac{1}{2}}\left|\eta^{\alpha} - \frac{2}{3}\right|\left|F'(\eta\rho + (1-\eta)\sigma) - F'(\eta\sigma + (1-\eta)\rho)\right|d\eta\right. \\ &\left. + \int_{\frac{1}{2}}^{1}\left|\eta^{\alpha} - \frac{1}{3}\right|\left|F'(\eta\rho + (1-\eta)\sigma) - F'(\eta\sigma + (1-\eta)\rho)\right|d\eta\right] \\ \leq\ & \frac{\rho-\sigma}{2}\left[\int_0^{\frac{1}{2}}\left|\eta^{\alpha} - \frac{2}{3}\right|L(1-2\eta)(\rho-\sigma)\,d\eta\right. \end{aligned}$$

$$+\int_{\frac{1}{2}}^{1}\left|\eta^{\alpha}-\frac{1}{3}\right|L(2\eta-1)(\rho-\sigma)\,d\eta\Bigg]$$
$$=\frac{(\rho-\sigma)^{2}}{2}\left[\Upsilon_{1}(\alpha)-2\Upsilon_{2}(\alpha)+2\Upsilon_{4}(\alpha)-\Upsilon_{3}(\alpha)\right]L.$$

□

Remark 5.1. If we take $\alpha = 1$ in Theorem 5.1, then we get the inequality

$$\left|\frac{1}{3}\left[2F(\sigma)-F\left(\frac{\sigma+\rho}{2}\right)+2F(\rho)\right]-\frac{1}{\rho-\sigma}\int_{\sigma}^{\rho}F(\eta)d\eta\right|\leq\frac{(\rho-\sigma)^{2}}{8}L,$$

which is given in [29, Corollary 4].

6. Fractional Milne type inequality for functions of bounded variation

In this section, we prove a fractional Milne type inequality for function of bounded variation.

Theorem 6.1. *Let $F:[\sigma,\rho]\to\mathbb{R}$ be a function of bounded variation on $[\sigma,\rho]$. Then we have the following Milne type inequality for Riemann-Liouville fractional integrals*

$$\left|\frac{1}{3}\left[2F(\sigma)-F\left(\frac{\sigma+\rho}{2}\right)+2F(\rho)\right]-\frac{\Gamma(\alpha+1)}{2(\rho-\sigma)^{\alpha}}\left[\mathfrak{J}_{\sigma+}^{\alpha}F(\rho)+\mathfrak{J}_{\rho-}^{\alpha}F(\sigma)\right]\right|\leq\frac{2}{3}\bigvee_{\sigma}^{\rho}(F),\tag{6.1}$$

where $\bigvee_{\sigma}^{\rho}(F)$ denotes the total variation of F on $[\sigma,\rho]$.

Proof. Define the mappings $K_{\alpha}(\kappa)$ by,

$$K_{\alpha}(\kappa)=\begin{cases}(\kappa-\sigma)^{\alpha}-\frac{2(\rho-\sigma)^{\alpha}}{3}, & \sigma\leq\kappa\leq\frac{\sigma+\rho}{2}\\[2ex] (\kappa-\sigma)^{\alpha}-\frac{(\rho-\sigma)^{\alpha}}{3} & \frac{\sigma+\rho}{2}<\kappa\leq\rho,\end{cases}$$

and

$$L_{\alpha}(\kappa)=\begin{cases}\frac{(\rho-\sigma)^{\alpha}}{3}-(\rho-\kappa)^{\alpha}, & \sigma\leq\kappa\leq\frac{\sigma+\rho}{2}\\[2ex] \frac{2(\rho-\sigma)^{\alpha}}{3}-(\rho-\kappa)^{\alpha} & \frac{\sigma+\rho}{2}<\kappa\leq\rho.\end{cases}$$

Integrating by parts, we get

$$\int_{\sigma}^{\rho}K_{\alpha}(\kappa)dF(\kappa)\tag{6.2}$$
$$=\int_{\sigma}^{\frac{\sigma+\rho}{2}}\left((\kappa-\sigma)^{\alpha}-\frac{2(\rho-\sigma)^{\alpha}}{3}\right)dF(\kappa)+\int_{\frac{\sigma+\rho}{2}}^{\rho}\left((\kappa-\sigma)^{\alpha}-\frac{(\rho-\sigma)^{\alpha}}{3}\right)dF(\kappa)$$

$$
\begin{aligned}
&= \left((\kappa-\sigma)^{\alpha}-\frac{2(\rho-\sigma)^{\alpha}}{3}\right)F(\kappa)\Big|_{\sigma}^{\frac{\sigma+\rho}{2}} - \alpha\int_{\sigma}^{\frac{\sigma+\rho}{2}}(\kappa-\sigma)^{\alpha-1}F(\kappa)d\kappa \\
&\quad + \left((\kappa-\sigma)^{\alpha}-\frac{(\rho-\sigma)^{\alpha}}{3}\right)F(\kappa)\Big|_{\frac{\sigma+\rho}{2}}^{\rho} - \alpha\int_{\frac{\sigma+\rho}{2}}^{\rho}(\kappa-\sigma)^{\alpha-1}F(\kappa)d\kappa \\
&= \left(\frac{(\rho-\sigma)^{\alpha}}{2^{\alpha}}-\frac{2(\rho-\sigma)^{\alpha}}{3}\right)F\left(\frac{\sigma+\rho}{2}\right)+\frac{2(\rho-\sigma)^{\alpha}}{3}F(\sigma) \\
&\quad +\frac{2(\rho-\sigma)^{\alpha}}{3}F(\rho)-\left(\frac{(\rho-\sigma)^{\alpha}}{2^{\alpha}}-\frac{(\rho-\sigma)^{\alpha}}{3}\right)F\left(\frac{\sigma+\rho}{2}\right)-\alpha\int_{\sigma}^{\rho}(\kappa-\sigma)^{\alpha-1}F(\kappa)d\kappa \\
&= \frac{(\rho-\sigma)^{\alpha}}{3}\left[2F(\sigma)-F\left(\frac{\sigma+\rho}{2}\right)+2F(\rho)\right]-\Gamma(\alpha+1)\,\mathfrak{J}_{\rho-}^{\alpha}F(\sigma).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\int_{\sigma}^{\rho} L_{\alpha}(\kappa)dF(\kappa) \\
&= \frac{(\rho-\sigma)^{\alpha}}{3}\left[2F(\sigma)-F\left(\frac{\sigma+\rho}{2}\right)+2F(\rho)\right]-\Gamma(\alpha+1)\,\mathfrak{J}_{\sigma+}^{\alpha}F(\rho).
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
&\frac{1}{3}\left[2F(\sigma)-F\left(\frac{\sigma+\rho}{2}\right)+2F(\rho)\right]-\frac{\Gamma(\alpha+1)}{2(\rho-\sigma)^{\alpha}}\left[\mathfrak{J}_{\sigma+}^{\alpha}F(\rho)+\mathfrak{J}_{\rho-}^{\alpha}F(\sigma)\right] \\
&= \frac{1}{2(\rho-\sigma)^{\alpha}}\int_{\sigma}^{\rho}\left[K_{\alpha}(\kappa)+L_{\alpha}(\kappa)\right]dF(\kappa).
\end{aligned}
$$

It is well known that if $g, F : [\sigma,\rho] \to \mathbb{R}$ are such that g is continuous on $[\sigma,\rho]$ and F is of bounded variation on $[\sigma,\rho]$, then $\int_{\sigma}^{\rho} g(\eta)dF(\eta)$ exist and

$$
\left|\int_{\sigma}^{\rho} g(\eta)dF(\eta)\right| \le \sup_{\eta\in[\sigma,\rho]}|g(\eta)|\bigvee_{\sigma}^{\rho}(F). \tag{6.3}
$$

On the other hand, using (6.3), we get

$$
\begin{aligned}
&\left|\frac{1}{3}\left[2F(\sigma)-F\left(\frac{\sigma+\rho}{2}\right)+2F(\rho)\right]-\frac{\Gamma(\alpha+1)}{2(\rho-\sigma)^{\alpha}}\left[\mathfrak{J}_{\sigma+}^{\alpha}F(\rho)+\mathfrak{J}_{\rho-}^{\alpha}F(\sigma)\right]\right| \\
&\le \frac{1}{2(\rho-\sigma)^{\alpha}}\left[\left|\int_{\sigma}^{\rho}K_{\alpha}(\kappa)dF(\kappa)\right|+\left|\int_{\sigma}^{\rho}L_{\alpha}(\kappa)dF(\kappa)\right|\right]
\end{aligned}
$$

$$\leq \frac{1}{2(\rho-\sigma)^{\alpha}}\left[\left|\int_{\sigma}^{\frac{\sigma+\rho}{2}}\left((\kappa-\sigma)^{\alpha}-\frac{2(\rho-\sigma)^{\alpha}}{3}\right)dF(\kappa)\right|+\left|\int_{\frac{\sigma+\rho}{2}}^{\rho}\left((\kappa-\sigma)^{\alpha}-\frac{(\rho-\sigma)^{\alpha}}{3}\right)dF(\kappa)\right|\right.$$

$$\left.+\left|\int_{\sigma}^{\frac{\sigma+\rho}{2}}\left(\frac{(\rho-\sigma)^{\alpha}}{3}-(\rho-\kappa)^{\alpha}\right)dF(\kappa)\right|+\left|\int_{\frac{\sigma+\rho}{2}}^{\rho}\left(\frac{2(\rho-\sigma)^{\alpha}}{3}-(\rho-\kappa)^{\alpha}\right)dF(\kappa)\right|\right]$$

$$\leq \frac{1}{2(\rho-\sigma)^{\alpha}}\left[\sup_{\kappa\in\left[\sigma,\frac{\sigma+\rho}{2}\right]}\left|(\kappa-\sigma)^{\alpha}-\frac{2(\rho-\sigma)^{\alpha}}{3}\right|\bigvee_{\sigma}^{\frac{\sigma+\rho}{2}}(F)+\sup_{\kappa\in\left[\frac{\sigma+\rho}{2},\rho\right]}\left|(\kappa-\sigma)^{\alpha}-\frac{(\rho-\sigma)^{\alpha}}{3}\right|\bigvee_{\frac{\sigma+\rho}{2}}^{\rho}(F)\right.$$

$$\left.+\sup_{\kappa\in\left[\sigma,\frac{\sigma+\rho}{2}\right]}\left|\frac{(\rho-\sigma)^{\alpha}}{3}-(\rho-\kappa)^{\alpha}\right|\bigvee_{\sigma}^{\frac{\sigma+\rho}{2}}(F)+\sup_{\kappa\in\left[\frac{\sigma+\rho}{2},\rho\right]}\left|\frac{2(\rho-\sigma)^{\alpha}}{3}-(\rho-\kappa)^{\alpha}\right|\bigvee_{\frac{\sigma+\rho}{2}}^{\rho}(F)\right]$$

$$= \frac{1}{2(\rho-\sigma)^{\alpha}}\left[\frac{2(\rho-\sigma)^{\alpha}}{3}\bigvee_{\sigma}^{\frac{\sigma+\rho}{2}}(F)+\frac{2(\rho-\sigma)^{\alpha}}{3}\bigvee_{\sigma}^{\frac{\sigma+\rho}{2}}(F)+\frac{2(\rho-\sigma)^{\alpha}}{3}\bigvee_{\sigma}^{\frac{\sigma+\rho}{2}}(F)+\frac{2(\rho-\sigma)^{\alpha}}{3}\bigvee_{\sigma}^{\frac{\sigma+\rho}{2}}(F)\right]$$

$$= \frac{2}{3}\bigvee_{\sigma}^{\rho}(F).$$

This completes the proof. □

Remark 6.1. If we take $\alpha = 1$ in Theorem 6.1, then we get the inequality

$$\left|\frac{1}{3}\left[2F(\sigma)-F\left(\frac{\sigma+\rho}{2}\right)+2F(\rho)\right]-\frac{1}{\rho-\sigma}\int_{\sigma}^{\rho}F(\eta)d\eta\right|\leq\frac{2}{3}\bigvee_{\sigma}^{\rho}(F)$$

which is given by Alomari and Liu in [39].

7. Conclusions

In this study, we have obtained new variations of Milne-type inequalities in the case of Riemann-Liouville fractional integrals. By utilizing important function classes such as convexity, bounded functions, Lipschitzian functions and functions of bounded variation, we first established a crucial identity for differentiable functions. Leveraging this identity, we derived novel versions of fractional Milne inequalities. The results obtained provide extended and enhanced versions of Milne-type inequalities in the context of Riemann-Liouville fractional integrals. Moreover, the obtained findings highlight the significant role of such fractional integral inequalities in analysis and applications. The implications of the obtained results in other mathematical problems also present an interesting avenue for future investigations. The obtained results not only contribute to the advancement of fractional integral theory and integral inequalities but also indicate potential applications in various mathematical problems. Further extensions and in-depth exploration of the findings in other mathematical domains can be pursued in future research.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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