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Research article

On generalization of Petryshyn's fixed point theorem and its application to the product of n-nonlinear integral equations

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Abstract: Regarding the Hausdorff measure of noncompactness, we provide and demonstrate a generalization of Petryshyn's fixed point theorem in Banach algebras. Comparing this theorem to Schauder and Darbo's fixed point theorems, we can skip demonstrating closed, convex and compactness properties of the investigated operators. We employ our fixed point theorem to provide the existence findings for the product of n-nonlinear integral equations in the Banach algebra of continuous functions $C(I_a)$, which is a generalization of various types of integral equations in the literature. Lastly, a few specific instances and informative examples are provided. Our findings can successfully be extended to several Banach algebras, including AC, C^1 or BV-spaces.

Keywords: Petryshyn's fixed point theorem (F.P.T.); Measures of noncompactness (M.N.C.); product of n-nonlinear integral equations
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1. Introduction

Different types of integral equations are crucial to the study of economics, biology, mechanics, mathematical physics, control theory, vehicular traffic, population dynamics and other fields (cf. [1,2]).

Recent years have seen some successful attempts to examine the qualitative behavior of solutions for many different types of nonlinear differential or integral equations employing the notion of the measure of noncompactness (M.N.C.) connected to the fixed point approach (F.P.T.) (cf. [3–10]).

Based on this methodology, we first offer and demonstrate a generalization of Petryshyn's F.P.T. connected with the Hausdorff M.N.C., which is a generalization of numerous F.P.T. types, including

Darbo's, Schauder's and traditional Petryshyn's F.P.T.s [11]. The benefit of the proposed F.P.T. is that it enables us to skip demonstrating closed, convex and compactness properties of the investigated operators. These enable us to investigate various varieties of differential and integral equations under a weaker and more general set of presumptions.

Second, we employ the presented F.P.T. to solve the product of n-nonlinear Volterra integral equations, which are a generalization of the classical and quadratic integral equations of the form

$$z(v) = \prod_{i=1}^{n} f_i\left(v, z(\alpha_i(v)), z(\beta_i(v)), \int_0^{\varphi_i(v)} h_i(v, s, z(\gamma_i(s)))ds\right), \qquad v \in I_a = [0, a] \tag{1.1}$$

for $n \geq 2$, in the Banach algebra $C(I_a)$.

In particular, for $n = 2,\ f_i(v, z_1, z_2, z_3) = g_i + z_3,\ h_i = l_i(v-s)z_3(s)$, equation (1.1) yields a Gripenberg equation

$$z(v) = k\left(g_1(v) + \int_0^v l_1(v-s)z(s)\, ds\right)\left(g_2(v) + \int_0^v l_2(\varphi - s)z(s)\, ds\right),$$

that has significant applications in biology (SI models, cf. [12]).

In [13] the authors utilized the F.P.T. approach to establish the existence of $C[a, b]$-solutions of the equation

$$z(v) = \prod_{i=1}^{n}\left(h_i(v) + \int_a^v K_i(v, s, z(s))\, ds\right),\ v \in [a, b].$$

The authors in [14] presented an extension of Darbo F.P.T. in Banach algebra to solve the q-integral equation

$$z(v) = \prod_{i=1}^{n}\left(h_i(v) + \frac{g_i(v, z(v))}{\Gamma_q(\alpha_i)}\int_a^v (v - qs)^{\alpha_i - 1}u_i(s, z(s))\, ds\right),\ v \in [0, 1].$$

A generalization of Darbo F.P.T. was used to investigate the existence results for the equation

$$z(v) = \prod_{i=1}^{n}\left(h_i(v) + \lambda_i \cdot \int_a^b K_i(v, s)f_i(s, z(s))\, ds\right),\ v \in [a, b]$$

in ideal spaces (not be Banach algebras) in [15] see also [16–18].

We focus on applying a generalization of Petryshyn's F.P.T. to solve a general form of product-type integral problems in the Banach algebra $C(I_a)$.

2. Preliminaries

We employ the following symbols in the sequel:

- $\mathbb{E}$: Banach space;
- $\bar{B}_r$: A ball of radius r and center at 0;
- $\partial\bar{B}_r$: Sphere in E with radius $r > 0$ around 0;
- $C(I_a)$: Space of continuous and real-valued functions on $I_a = [0, a]$;
- (F.P.T.): Fixed point theorem;
- (M.N.C.): Measure of noncompactness.

We recall some theorems & definitions that are required for the sequel.

Definition 2.1. *[19] Let $Z \subset \mathbb{E}$ be a bounded set, then*

$$\alpha(Z) = \inf\{\rho > 0 : \ \exists \ \textit{a finite number of sets of diameter} \leq \rho \textit{ that can cover } Z\}$$

is said to be the Kuratowski M.N.C.

Definition 2.2. *[20] Let $Z \subset \mathbb{E}$ be a bounded set, then*

$$\mu(Z) = \inf\{\rho > 0 : Z \textit{ has a finite } \rho\textit{-net in } \mathbb{E}\}$$

is said to be the Hausdorff M.N.C.

Theorem 2.3. *[20] For a bounded set $Z \subset \mathbb{E}$, the M.N.C.s α and μ fulfill*

$$\mu(Z) \leq \alpha(Z) \leq 2\mu(Z).$$

For more information about the properties of the M.N.C. see [11, 20].

The space $C[0, a]$ yields to a Banach space under the norm $\|z\| = \sup\{|z(v)| : v \in I_a\}$ and we shall write the modulus of continuity of a function $z \in C(I_a)$ as

$$\omega(z, \rho) = \sup\{|z(v) - z(s)| : \ |v - s| \leq \rho\}.$$

Theorem 2.4. *[20] For a bounded set $Z \subset C(I_a)$, the M.N.C. in $C(I_a)$ is denoted by*

$$\mu(Z) = \lim_{\rho \to 0} \sup_{z \in Z} \omega(z, \rho). \tag{2.1}$$

Definition 2.5. *[21] Let $P : \mathbb{E} \to \mathbb{E}$ be a continuous map. P is said to be a contraction map if for all $Z \subset C(I_a)$ be bounded, $P(Z)$ be bounded and*

$$\alpha(PZ) \leq k\alpha(Z), \quad 0 < k < 1.$$

Moreover, P is said to be condensing (densifying) map if

$$\alpha(PZ) < \alpha(Z).$$

Note that a contraction map yields condensing (densifying) but not vice versa.

Remark 2.6. *In $C(I_a)$, the M.N.C. μ fulfills condition (m) (cf. [22]) and its generalization for a finite sequence of bounded sets $\{N_i\}_{i=1,\ldots,n}$, $n \geq 2$ (cf. [14]) i.e.*

$$\mu\Big(\prod_{i=1}^{n} N_i\Big) \leq \sum_{i=1}^{n} \prod_{j=1, j\neq i}^{n} \|N_j\| \cdot \mu(N_i).$$

3. Main results

In order to solve Eq (1.1), we first give a fixed point $z \in \bar{B}_r$ of the problem

$$z = Pz = \prod_{i=1}^{n} P_i z, \tag{3.1}$$

where $P_i : \bar{B}_r \to \mathbb{E},\ i = 1, \cdots, n,\ n \geq 2$ are known operators.

Definition 2.5 should be rewritten in view of the M.N.C. μ in $C(I_a)$.

Definition 3.1. *The operator $P : C(I_a) \to C(I_a)$ is said to be a contraction map if for all $Z \subset C(I_a)$ be bounded set, $P(Z)$ be bounded set and*

$$\mu(PZ) \leq 2k\mu(Z),\ \ 0 < k < \frac{1}{2}.$$

Moreover, P is said to be condensing (densifying) map if

$$\mu(PZ) < \mu(Z).$$

Proof. Since Z and $P(Z)$ are bounded sets in $C(I_a)$ and by using Theorem 2.3, we have

$$\begin{aligned} \mu(PZ) &\leq \alpha(PZ) \leq k\alpha(Z) \leq 2k\mu(Z) \\ &\Rightarrow \mu(PZ) \leq 2k\mu(Z). \end{aligned}$$

The above inequality with $0 < k < \frac{1}{2}$ finishes the proof. □

Note that a contraction map related to the M.N.C. μ yields condensing (densifying) with $0 < k < \frac{1}{2}$ but not vice versa.

The following Proposition can be presented and proven by us.

Proposition 3.2. *Suppose that the operators $P_i : \bar{B}_r \to \mathbb{E},\ i = 1, \cdots n$ and that:*

(B1) *P_i are continuous on $\bar{B}_r,\ \ i = 1, \cdots n$.*

(B2) *There exist $k_i > 0$ such that P_i fulfill:*

$$\mu(P_i(Z)) \leq k_i\, \mu(Z),\ i = 1, \cdots n$$

for arbitrary bounded $Z \subset \mathbb{E}$,

(B3) $\mathbb{K} = \sum_{i=1}^{n} k_i \prod_{j=1, j\neq i}^{n} \|P_j \bar{B}_r\| < \frac{1}{2}$,

(B4) *$P(z) = kz$, for some $z \in \partial \bar{B}_r$ then $k \leq 1$,*

then the set $Fix(P)$ of fixed points of P in $\bar{B}_r$ is nonempty.

Proof. Let $\emptyset \neq Z \subset \bar{B}_r$. By utilizing the above assumptions, we obtain

$$\begin{aligned}\mu(PZ) = \mu\Big(\prod_{i=1}^{n} P_i Z\Big) &\leq \sum_{i=1}^{n} \prod_{j=1, j\neq i}^{n} \|P_j Z\| \cdot \mu(P_i Z)\\ &\leq \sum_{i=1}^{n} k_i \prod_{j=1, j\neq i}^{n} \|P_j Z\| \cdot \mu(Z)\\ &\leq \Big(\sum_{i=1}^{n} k_i \prod_{j=1, j\neq i}^{n} \|P_j \bar{B}_r\|\Big) \cdot \mu(Z)\\ &= \mathbb{K} \cdot \mu(Z).\end{aligned}$$

By using Petryshyn's F.P.T., we have finished. □

Remark 3.3.
- *If $n = 1$, Proposition 3.2 reduces to classical Petryshyn's F.P.T. [11], which is a generalization of classical Darbo and Schauder F.P.Ts.*
- *If $n = 2$, Proposition 3.2 reduces to the F.P.T. presented in [11, 21, 21], which is a generalization of the results presented in [22].*
- *If $n \geq 2$, Proposition 3.2 is a general form of the F.P.T. presented in [14, 15].*

Now, we will apply Proposition 3.2 to check the solvability of Eq. (1.1) under the assumptions:

(A1) Assume that $\alpha_i, \beta_i, \gamma_i : I_a \to I_a$ and $\varphi_i : I_a \to R^+$ are continuous s.t. $\varphi_i(v) \leq B$, for $i = 1, \cdots, n$ and $B \geq 0$, $v \in I_a$.

(A2) The functions $h_i \in C(I_a \times [0, B] \times \mathbb{R}, \mathbb{R})$ and $f_i \in C(I_a \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, where there exist constants $k_i > 0$, s.t.

$$|f_i(v, \Omega_1, \Omega_2, \Omega_3) - f_i(v, \bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)| \leq k_i(|\Omega_1 - \bar{\Omega}_1| + |\Omega_2 - \bar{\Omega}_2| + |\Omega_3 - \bar{\Omega}_3|),\ i = 1, \cdots, n.$$

(A3) There exists $M_i \geq 0$ and $r_0 \geq 0$ such that

$$\sup\left\{\Big|\prod_{i=1}^{n} f_i(v, \Omega_1, \Omega_2, \Omega_3)\Big| : v \in I_a,\ |\Omega_j| \leq r_0,\ j = 1, 2,\ |\Omega_3| \leq \prod_{i=1}^{n} BM_i,\ i = 1, \cdots, n\right\} \leq r_0,$$

where

$$M_i = \sup\{|h_i(v, s, z)|;\ \forall\ v \in I_a,\ s \in [0, B], z \in [-r_0, r_0]\}.$$

(A4) $\mathbb{K} = \sum_{i=1}^{n} 2k_i \prod_{j=1, j\neq i}^{n} \|f_j\| < \frac{1}{2}$.

Theorem 3.4. *Under the tacit assumption (A1)–(A4) above, Eq.* (1.1) *has at least one solution in $C(I_a)$.*

Proof. First, let us define the operators $P_i : B_{r_0} \to C(I_a)$, as follows

$$Pz(v) = \prod_{i=1}^{n} P_i z(v) = \prod_{i=1}^{n} f_i\left(v, z(\alpha_i(v)), z(\beta_i(v)), \int_0^{\varphi_i(v)} h_i(v, s, z(\gamma_i(s)))ds\right),\ \ v \in I_a. \tag{3.2}$$

Next, we will divide the proof into some steps according to Proposition 3.2. **Step 1.** The operator P is well defined on $C(I_a)$. Obviously from assumptions (A1) and (A2), we have $P : C(I_a) \to C(I_a)$.

Step 2. We will demonstrate that the operators $P,\ P_i, i = 1, \cdots, n$ are continuous on the ball B_{r_0}.

Take arbitrary $z, y \in B_{r_0}$ and $\varepsilon > 0$ s.t. $\|z - y\| \le \varepsilon$, then for $v \in I_a$, we obtain

$$\begin{aligned}
& \Big|(P_iz)(v) - (P_iy)(v)\Big| \\
= & \left|f_i\Big(v, z(\alpha_i(v)), z(\beta_i(v)), \int_0^{\varphi_i(v)} h_i(v, s, z(\gamma_i(s)))ds\Big) - f_i\Big(v, y(\alpha_i(v)), y(\beta_i(v)), \int_0^{\varphi_i(v)} h_i(v, s, y(\gamma_i(s)))ds\Big)\right| \\
\le & \left|f_i\Big(v, z(\alpha_i(v)), z(\beta_i(v)), \int_0^{\varphi_i(v)} h_i(v, s, z(\gamma_i(s)))ds\Big) - f_i\Big(v, y(\alpha_i(v)), z(\beta_i(v)), \int_0^{\varphi_i(v)} h_i(v, s, z(\gamma_i(s)))ds\Big)\right| \\
+ & \left|f_i\Big(v, y(\alpha_i(v)), z(\beta_i(v)), \int_0^{\varphi_i(v)} h_i(v, s, z(\gamma_i(s)))ds\Big) - f_i\Big(v, y(\alpha_i(v)), y(\beta_i(v)), \int_0^{\varphi_i(v)} h_i(v, s, z(\gamma_i(s)))ds\Big)\right| \\
+ & \left|f_i\Big(v, y(\alpha_i(v)), y(\beta_i(v)), \int_0^{\varphi_i(v)} h_i(v, s, z(\gamma_i(s)))ds\Big) - f_i\Big(v, y(\alpha_i(v)), y(\beta_i(v)), \int_0^{\varphi_i(v)} h_i(v, s, y(\gamma_i(s)))ds\Big)\right| \\
\le & \ k_i\Big|z(\alpha_i(v)) - y(\alpha_i(v))\Big| + k_i\Big|z(\beta_i(v)) - y(\beta_i(v))\Big| + k_i\int_0^{\varphi_i(v)} \Big|h_i(v, s, z(\gamma_i(s))) - h_i(v, s, y(\gamma_1(s)))\Big|ds \\
\le & \ 2k_i\|z - y\| + k_iB \cdot \omega(h_i, \varepsilon),
\end{aligned}$$

where $\omega(h_i, \varepsilon) = \sup\{|h_i(v, s, z) - h_i(v, s, y)| : v \in I_a, s \in [0, B], z, y \in [-r_0, r_0], \|z - y\| \le \varepsilon\}$.

From assumption (A2), the functions $h_i = h_i(v, s, z)$ are uniformly continuous on $[0, a] \times [0, B] \times \mathbb{R}$, we indicate that $\omega(h_i, \varepsilon) \to 0$ as $\varepsilon \to 0$. Thus, the operators $P_i, i = 1, \cdots, n$ are continuous on B_{r_0} and consequently, the operator $P = \prod_{i=1}^n P_i$ is continuous on B_{r_0}.

Step 3. We will demonstrate that the operator P fulfills the densifying condition in view of μ.

Take arbitrary $\rho > 0$ and $z \in M \subset C(I_a)$ is bounded set and for $v_1, v_2 \in I_a$ s.t. $v_1 \le v_2$ with $v_2 - v_1 \le \rho$, we obtain

$$\begin{aligned}
\Big|(P_iz)(v_2) - (P_iz)(v_1)\Big| = & \left|f_i\Big(v_2, z(\alpha_i(v_2)), z(\beta_i(v_2)), \int_0^{\varphi_i(v_2)} h_i(v_2, s, z(\gamma_i(s)))ds\Big)\right. \\
& \left. - f_i\Big(v_1, z(\alpha_i(v_1)), z(\beta_i(v_1)), \int_0^{\varphi_i(v_1)} h_i(v_1, s, z(\gamma_i(s)))ds\Big)\right| \\
\le & \left|f_i\Big(v_2, z(\alpha_i(v_2)), z(\beta_i(v_2)), \int_0^{\varphi_i(v_2)} h_i(v_2, s, z(\gamma_i(s)))ds\Big)\right. \\
& \left. - f_i\Big(v_2, z(\alpha_i(v_2)), z(\beta_i(v_2)), \int_0^{\varphi_i(v_1)} h_i(v_1, s, z(\gamma_i(s)))ds\Big)\right| \\
+ & \left|f_i\Big(v_2, z(\alpha_i(v_2)), z(\beta_i(v_2)), \int_0^{\varphi_i(v_1)} h_i(t_i, s, z(\gamma_i(s)))ds\Big)\right. \\
& \left. - f_i\Big(v_2, z(\alpha_i(v_2)), z(\beta_i(v_1)), \int_0^{\varphi_i(v_1)} h_i(v_1, s, z(\gamma_i(s)))ds\Big)\right| \\
+ & \left|f_i\Big(v_2, z(\alpha_i(v_2)), z(\beta_i(v_1)), \int_0^{\varphi_i(v_1)} h_i(v_1, s, z(\gamma_i(s)))ds\Big)\right. \\
& \left. - f_i\Big(v_2, z(\alpha_i(v_1)), z(\beta_i(v_1)), \int_0^{\varphi_i(v_1)} h_i(v_1, s, z(\gamma_i(s)))ds\Big)\right|
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \left| f_i\Big(v_2, z(\alpha_i(v_1)), z(\beta_i(v_1)), \int_0^{\varphi_i(v_1)} h_i(v_1, s, z(\gamma_i(s)))ds\Big) \right. \\
&\quad \left. -f_i\Big(v_1, z(\alpha_i(v_1)), z(\beta_i(v_1)), \int_0^{\varphi_i(v_1)} h_i(v_1, s, z(\gamma_i(s)))ds\Big) \right| \\
&\leq \quad k_i \left| \int_0^{\varphi_i(v_2)} h_i(v_2, s, z(\gamma_i(s)))ds - \int_0^{\varphi_i(v_1)} h_i(v_1, s, z(\gamma_i(s)))ds \right| \\
&\quad +k_i \Big| z(\beta_i(v_2)) - z(\beta_i(v_1)) \Big| + k_i \Big| z(\alpha_i(v_2)) - z(\alpha_i(v_1)) \Big| + \omega^i_{f_i}(I_a, \rho) \\
&\leq \quad k_i \left| \int_0^{\varphi_i(v_1)} \omega^i_{h_i}(I_a, \rho)ds + \int_{\varphi_i(v_1)}^{\varphi_i(v_2)} h_i(v_2, s, z(\gamma_i(s)))ds \right| \\
&\quad +k_i\omega(z, \omega(\beta_i, \rho)) + k_i\omega(z, \omega(\alpha_i, \rho)) + \omega^i_{f_i}(I_a, \rho),
\end{aligned}
$$

where

$$\omega^i_{h_i}(I_a, \rho) = \sup\{|h_i(v, s, z) - h_i(\bar{v}, s, z)| : |v - \bar{v}| \leq \rho, v \in I_a,\ s \in [0, B],\ z \in [-r_0, r_0]\},$$

$$\omega^i_{f_i}(I_a, \rho) = \sup\{|f_i(v, z_1, z_2, z_3) - f_i(\bar{v}, z_1, z_2, z_3)| : |v - \bar{v}| \leq \rho, v \in I_a,\ z_1, z_2 \in [-r_0, r_0],\ z_3 \in [-BM_i, BM_i]\}$$

and

$$M_i = \sup\{|h_i(v, s, z)| : \quad v \in I_a, \quad s \in [0, B], \quad z \in [-r_0, r_0]\}.$$

From the above relations we get

$$|(P_i z)(v) - (P_i y)(v)| \leq k_i B\omega^i_{h_i}(I_a, \rho) + k_i M_i \omega(\varphi_i, \rho) + k_i\omega(z, \omega(\beta_i, \rho)) + k_i\omega(z, \omega(\alpha_i, \rho)) + \omega^i_{f_i}(I_a, \rho).$$

Let $\rho \to 0$, we get

$$\omega(P_i z, \rho) \leq 2k_i\omega(z, \rho).$$

This yields the following estimation:

$$\mu(P_i M) \leq 2k_i\mu(M).$$

Therefore,

$$\mu(PM) = \mu\Big(\prod_{i=1}^n P_i M\Big) \leq \mathbb{K} = \left(\sum_{i=1}^n 2k_i \prod_{j=1, j\neq i}^n \|f_j\|\right)\mu(M).$$

From assumption (A4), we get P is a condensing map with $\mathbb{K} < \frac{1}{2}$.

Step 4. We will demonstrate assumption (B4) of Proposition 3.2.
Suppose $z \in \partial\bar{B}_{r_0}$. If $Tz = kz$ then we get $kr_0 = k\|z\| = \|Pz\|$ and by (H3) we have

$$|Pz(v)| = \left|\prod_{i=1}^n P_i z(v)\right| = \left|\prod_{i=1}^n f_i\left(v, z(\alpha_i(v)), z(\beta_i(v)), \int_0^{\varphi_i(v)} h_i(v, s, z(\gamma_i(s)))ds\right)\right| \leq r_0,$$

for all $v \in I_a$, hence $\|Pz\| \leq r_0$, so this shows $k \leq 1$.

Step 5. The proof is completed when Proposition 3.2 is applied. □

4. Applications and examples

To demonstrate the value of our results, we provide a few examples and instances of integral equations.

- If $n = 2$, $f_1(v, \Omega_1, \Omega_2, \Omega_3) = f(v, \Omega_1) + p(v, \Omega_1, \Omega_3)$, $\alpha_1(v) = \varphi_1(v) = v$, $f_2(v, \Omega_1, \Omega_2, \Omega_3) = q(v, \Omega_1, \Omega_3)$, $\varphi_2(v) = a$, then we have

$$z(v) = \Big(f(v, z(v)) + p(v, z(\beta_1(v)), \int_0^v h_1(v, s, z(\gamma_1(s)))ds\Big) \times q\Big(v, z(\alpha_2(v)), \int_0^a h_2(v, s, z(\gamma_2(s)))ds\Big),$$

which was inspected in [23].

- For $n = 2$, $f_i(v, \Omega_1, \Omega_2, \Omega_3) = p_i(v, \Omega_1, \Omega_3)$, $\gamma_1(v) = \gamma_2(v) = \varphi_1(v) = v$, $\varphi_2(v) = a$, we have

$$z(v) = p_1\Big(v, z(\alpha_1(v)), \int_0^v h_1(v, s, z(s)ds\Big) \times p_2\Big(v, z(\alpha_2(v)), \int_0^a h_2(v, s, z(s))ds\Big),$$

which was inspected in [24, 25].

- If $n = 2$, $f_I(v, \Omega_1, \Omega_2, \Omega_3) = p_i(v, \Omega_1, \Omega_3)$, $\varphi_2(v) = 1$ then we get

$$z(v) = p_1\Big(v, z(\alpha_1(v)), \int_0^{\varphi_1(v)} h_1(v, s, z(\gamma_1(s)))ds\Big) \times p_2\Big(v, z(\alpha_2(v)), \int_0^1 h_2(v, s, z(\gamma_2(s)))ds\Big),$$

which was inspected in [26, 27].

- If $n = 2$, $f_1(v, \Omega_1, \Omega_2, \Omega_{3_1}) = a(v) \cdot \Omega_{3_1}$, $f_2(v, \Omega_1, \Omega_2, \Omega_{3_2}) = \Omega_{3_1} \cdot \Omega_{3_2}$, $\alpha_i(v) = \varphi_1(v) = \gamma_i(v) = \gamma_2(v) = v, \varphi_2(v) = a$, then we get

$$z(v) = a(v) \int_0^a h_2(v, s, z(s))ds + \Big(\int_0^v h_1(v, s, z(s))ds\Big)\Big(\int_0^a h_2(v, s, z(s))ds\Big),$$

which was inspected in [28].

Example 4.1. *Consider the integral equation in* $C[0, 1]$

$$\begin{aligned} z(v) = &\left(\frac{v^2}{15(1+v^2)} \sin(|z(v)|) + \frac{1}{2} \ln(1 + |z(\sqrt{v})|) + \frac{1}{4} \int_0^{\sqrt{v}} \frac{s \sin(z(\sqrt{s}))}{1 + s + e^v} ds\right) \\ &\times \left(\frac{e^{-v}(z(v) + 2z(1-v))}{6 + v} + \frac{1}{8 + v} \int_0^{\frac{1}{2}v} \frac{v(1 + \arctan(\frac{z(s^2)}{1+z(s^2)}))}{2 + s} ds\right) \\ &\times \left(\frac{v^4 e^{-v} z(\frac{1}{2}v)}{3} + \frac{1}{2 + \ln(1+s) + e^v} \int_0^{v^3} \frac{s e^{-2t} z(s)}{2 + |\cos(z(s))|} ds\right) \qquad v \in [0, 1]. \end{aligned} \tag{4.1}$$

Equation (4.1) *is a particular form of Eq* (1.1) *such that:*

$$\alpha_i(v) = v, i = 1, 2,\ \alpha_3(v) = \frac{v}{2}v,\ \beta_1(v) = \sqrt{v}, \beta_2(v) = 1 - v, \beta_3(v) = v,$$
$$\gamma_1(v) = \sqrt{v}, \gamma_2(v) = v^2, \gamma_3 = v \quad \varphi_1(v) = \sqrt{v}, \varphi_2(v) = \frac{v}{2}, \varphi^3(v) = v^3,$$

- $f_1(v, z(\alpha_1(v)), z(\beta_1(v)), W_1) = \frac{v^2}{15(1+v^2)}\sin(|z(v)|) + \frac{1}{2}\ln(1+|z(\sqrt{v})|) + \frac{1}{4}W_1, \quad W_1 = \int_0^{\sqrt{v}} \frac{s\sin(z(\sqrt{s}))}{1+s+e^v}ds,$
- $f_2(v, z(\alpha_2(v)), z(\beta_2(v)), W_2) = \frac{e^{-v}(z(v)+2z(1-v))}{6+v} + \frac{1}{8+v}W_2, \quad W_2 = \int_0^{\frac{1}{2}v} \frac{v\left(1+\arctan(\frac{z(s^2)}{1+z(s^2)})\right)}{2+s}ds,$
- $f_3(v, z(\alpha_2(v)), z(\beta_2(v)), W_3) = \frac{v^4e^{-v}z(\frac{1}{2}v)}{3} + \frac{1}{2+\ln(1+s)+e^v}W_3, \quad W_3 = \int_0^{v^3} \frac{se^{-2t}z(s)}{2+|\cos(z(s))|}ds.$

It can be seen that

$$|f_1(v,\Omega_1,\Omega_2,\Omega_3) - f_1(v,\bar{\Omega}_1,\bar{\Omega}_2,\bar{\Omega}_3)| \le \frac{1}{2}(|\Omega_1-\bar{\Omega}_1| + |\Omega_2-\bar{\Omega}_2| + |\Omega_3 -\bar{}_3|),$$

$$|f_2(v,\Omega_1,\Omega_2,\Omega_3) - f_2(v,\bar{\Omega}_1,\bar{\Omega}_2,\bar{\Omega}_3)| \le \frac{1}{3}(|\Omega_1-\bar{\Omega}_1| + |\Omega_2-\bar{\Omega}_2| + |\Omega_3-\bar{\Omega}_3|),$$

$$|f_3(v,\Omega_1,\Omega_2,\Omega_3) - f_3(v,\bar{\Omega}_1,\bar{\Omega}_2,\bar{\Omega}_3)| \le \frac{1}{3}(|\Omega_1-\bar{\Omega}_1| + |\Omega_2-\bar{\Omega}_2| + |\Omega_3-\bar{\Omega}_3|).$$

So we can choose

$$k_1 = \frac{1}{2},\ k_2 = \frac{1}{3},\ k_3 = \frac{1}{3}$$

and so the conditions (A1) and (A2) hold. Moreover, for $\|z\| \le r_0$, *we get*

$$\begin{aligned}|z(v)| &\le \left|f_1(v, z(\alpha_1(v)), z(\beta_1(v)), W_1)\right| \cdot \left|f_2(v, z(\alpha_2(v)), z(\beta_2(v)), W_2)\right| \cdot \left|f_3(v, z(\alpha_3(v)), z(\beta_3(v)), W_3)\right| \\ &\le \left(\frac{1}{15} + \frac{1}{2}r_0 + \frac{1}{4}\right)\left(\frac{1}{2}r_0 + \frac{(1+r_0)}{16}\right)\left(\frac{1}{3}r_0 + \frac{1}{4}r_0\right) \le r_0.\end{aligned}$$

This shows that $r_0 \le 2.1104$. *Also, for* $r_0 \in [0, 0.64368] \subset [0, 2.1104]$ *we have*

$$\mathbb{K} = 2k_1(\|f_2\|\cdot\|f_3\|) + 2k_2(\|f_1\|\cdot\|f_3\|) + 2k_3(\|f_1\|\cdot\|f_2\|) < \frac{1}{2}.$$

Therefore, assumptions (A1)–(A4) be fulfilled and Theorem 3.4 indicates the solution of (4.1) in $C[0,1]$.

Example 4.2. *Consider the integral equation in* $C[0,1]$

$$\begin{aligned}z(v) = &\left(\frac{1}{2}ve^{-v} + \frac{\cos(z(v^3))}{8} + \frac{1}{8}\sin(\frac{z(\sqrt{v})}{2+v}) + \frac{1}{8+v^2}\int_0^v \frac{(\sqrt{1+2|z(\sqrt{s})|}+ts^2)\cos(s)}{4+3\sqrt{s}}ds\right) \\ &\times\left(\frac{1}{9}v\cos(z(v)) + \frac{tz(\sqrt{v})}{9(1+z(\sqrt{v}))} + \frac{1}{9(e^v+3t^4)}\int_0^{\sqrt{v}} \frac{(1+\cos(\sqrt{s}))(\sqrt{1+2|z(\sqrt{s})|})}{1+st\ln(1+s)}ds\right).\end{aligned} \tag{4.2}$$

- *Here* $\alpha_1(v) = v^3,\ \alpha_2(v) = \sqrt{v}, \quad \beta_1(v) = \beta_2(v) = \sqrt{v}, \gamma_1(v) = \gamma_2(v) = \sqrt{v}, , \varphi_1(v) = \sqrt{v}, \varphi_2(v) = \sqrt{v},$
- $f_1(v, z(\alpha_1(v)), z(\beta_1(v)), W_1) = \frac{1}{2}ve^{-v} + \frac{\cos(z(v^3))}{8} + \frac{1}{8}\sin(\frac{z(\sqrt{v})}{2+v}) + \frac{1}{8+v^2}W_1,$
- $f_2(v, z(\alpha_2(v)), z(\beta_2(v)), W_2) = \frac{1}{9}v\cos(z(v)) + \frac{tz(\sqrt{v})}{9(1+z(\sqrt{v}))} + \frac{1}{9(e^v+3t^4)}W_2,$
- $W_1 = \int_0^v \frac{(\sqrt{1+2|z(\sqrt{s})|}+ts^2)\cos(s)}{4+3\sqrt{s}}ds, \quad W_2 = \int_0^{\sqrt{v}} \frac{(1+\cos(\sqrt{s}))(\sqrt{1+2|z(\sqrt{s})|})}{1+st\ln(1+s)}ds.$

It can be seen that

$$|f_1(v,\Omega_1,\Omega_2,\Omega_3)-f_1(v,\bar{\Omega}_1,\bar{\Omega}_2,\bar{\Omega}_3)|\leq\frac{1}{8}(|\Omega_1-\bar{\Omega}_1|+|\Omega_2-\bar{\Omega}_2|+|\Omega_3-\bar{\Omega}_3|),$$

$$|f_2(v,\Omega_1,\Omega_2,\Omega_3)-f_2(v,\bar{\Omega}_1,\bar{\Omega}_2,\bar{\Omega}_3)|\leq\frac{1}{9}(|\Omega_1-\bar{\Omega}_1|+|\Omega_2-\bar{\Omega}_2|+|\Omega_3-\bar{\Omega}_3|).$$

So we can choose

$$k_1=\frac{1}{8},\ k_2=\frac{1}{9}$$

and so the conditions (A1) and (A2) hold. Moreover, for $\|z\|\leq r_0$, *we get*

$$\begin{aligned}|z(v)|&\leq\left|f_1(v,z(\alpha_1(v)),z(\beta_1(v)),W_1)\right|\cdot\left|f_2(v,z(\alpha_2(v)),z(\beta_2(v)),W_2)\right|\\&\leq\left(\frac{1}{2}+\frac{1}{8}+\frac{1}{8}+\frac{1}{32}(\sqrt{1+2r_0})\right)\left(\frac{1}{9}+\frac{1}{9}+\frac{2}{9}(\sqrt{1+2r_0})\right)\leq r_0\end{aligned}$$

This shows that $r_0\geq 0.41410$. *Also, for* $r_0\in[0.41410,9.3765]$ *we have*

$$\mathbb{K}=2k_1\|f_2\|+2k_2\|f_1\|<\frac{1}{2}.$$

Therefore, assumptions (A1)–(A4) be fulfilled and Theorem 3.4 indicates the solution of (4.1) in $C[0,1]$.

5. Conclusions and perspective

In this article, a generalization of Petryshyn F.P.T. and the MNC idea were used to analyze the solutions for products of n-nonlinear integral equations in the Banach algebra $C(I_a)$. The presented F.P.T. is a generalization of Darbo, Schauder and the classical Petryshyn F.P.T. Examples are provided to demonstrate the usefulness of our findings. The upcoming work in this field will consider different Banach algebras, including AC, C^1 or BV spaces.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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References

1. A. Ben Amar, A. Jeribi, M. Mnif, Some fixed point theorems and application to biological model, *Numer. Funct. Anal. Optim.*, **29** (2008), 1–23. https://doi.org/10.1080/01630560701749482
2. C. Corduneanu, *Integral Equations and Applications,* Cambridge University Press, New York, 1990.
3. R. P. Agarwal, N. Hussain, M. A. Taoudi, Fixed point theorems in ordered Banach spaces and applications to nonlinear integral equations, *Abstr. Appl. Anal.*, **2012**, ID 245872. https://doi.org/10.1155/2012/245872
4. A. Aghajani, J. Banaś, Y. Jalilian, Existence of solutions for a class of nonlinear Volterra singular integral equations, *Comput. Math. Appl.*, **62** (2011), 1215–1227. https://doi.org/10.1016/j.camwa.2011.03.049
5. A. Alsaadi, M. Cichoń, M. Metwali, Integrable solutions for Gripenberg-type equations with m-product of fractional operators and applications to initial value problems, *Mathematics*, **10** (2022), 1172. https://doi.org/10.3390/math10071172
6. J. Banaś, Measures of noncompactness in the study of solutions of nonlinear differential and integral equations, *Cent. Eur. J. Math.*, **10** (2012), 2003–2011. https://doi.org/10.2478/s11533-012-0120-9
7. Y. Guo, M. Chen, X. B. Shu, F. Xu, The existence and Hyers-Ulam stability of solution for almost periodical fractional stochastic differential equation with fBm, *Stoch. Anal. Appl.*, **39** (2021), 643–666. https://doi.org/10.1080/07362994.2020.1824677
8. L. Shu, X. B. Shu, J. Mao, Approximate controllability and existence of mild solutions for Riemann-Liouville fractional stochastic evolution equations with nonlocal conditions of order $1 < \alpha < 2$, *Fract. Calc. Appl. Anal.*, **22** (2019), 1086–1112. https://doi.org/10.1515/fca-2019-0057
9. M. Metwali, Solvability in weighted L_1-spaces for the m-product of integral equations and model of the dynamics of the capillary rise, *J. Math. Anal. Appl.*, **515** (2022), 126461. https://doi.org/10.1016/j.jmaa.2022.126461
10. İ. Özdemir, Ü. Çakan, B. İlhan, On the existence of the solutions for some nonlinear Volterra integral equations, *Abstr. Appl. Anal.*, **5** (2013), ID 698234. https://doi.org/10.1155/2013/698234
11. W. V. Petryshyn, Structure of the fixed points sets of k-set-contractions, *Arch. Rational Mech. Anal.*, **40** (1971), 312–328. https://doi.org/10.1007/BF00252680
12. G. Gripenberg, On some epidemic models, *Quart. Appl. Math.*, **39** (1981), 317–327. https://doi.org/10.1090/qam/636238
13. I. M. Olaru, Generalization of an integral equation related to some epidemic models, *Carpathian J. Math.*, **26** (2010), 92–96.
14. M. Jleli, B. Samet, Solvability of a q-fractional integral equation arising in the study of an epidemic model, *Adv. Difference Equ.*, **21** (2017). https://doi.org/10.1186/s13662-017-1076-7
15. M. Metwali, On a fixed point theorems and applications to product of n-nonlinear integral operators in ideal spaces, *Fixed Point Theory*, **23** (2022), 557–572. https://doi.org/10.24193/fpt-ro.2022.2.09

16. M. Metwali, K. Cichoń, Solvability of the product of n-integral equations in Orlicz spaces, *Rend. Circ. Mat. Palermo,* **II** (2023). https://doi.org/10.1007/s12215-023-00916-1
17. M. Metwali, V. N. Mishra, On the measure of noncompactness in $L_p(\mathbb{R}^+)$ and applications to a product of n-integral equations, *Turkish J. Math.*, **47** (2023), 372–386. https://doi.org/10.55730/1300-0098.3365
18. E. Brestovanská, M. Medved, Fixed point theorems of the Banach and Krasnosel'skii type for mappings on m-tuple Cartesian product of Banach algebras and systems of generalized Gripenberg's equations, *Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math.*, **51** (2012), 27–39.
19. K. Kuratowski, Sur les espaces completes, *Fund. Math.*, **15** (1930), 301–309.
20. J. Banaś, K. Goebel, *Measures of Noncompactness in Banach Spaces,* Lect. Notes in Math. 60, M. Dekker, New York, Basel, 1980.
21. M. Kazemi, R. Ezzati, A. Deep, On the solvability of non-linear fractional integral equations of product type, *J. Pseudo-Differ. Oper. Appl.*, **14** (2023) 39. https://doi.org/10.1007/s11868-023-00532-8
22. J. Banaś, M. Lecko, Fixed points of the product of operators in Banach algebra, *Panamer. Math. J.*, **12** (2002), 101–109.
23. Deepmala, H. K. Pathak, A study on some problems on existence of solutions for nonlinear functional-integral equations, *Acta Math. Sci.*, **33** (2013), 1305–1313. https://doi.org/10.1016/S0252-9602(13)60083-1
24. K. Maleknejad, R.Mollapourasl, K. Nouri, Study on existence of solutions for some nonlinear functional-integral equations, *Nonlinear Anal.*, **69** (2008), 2582–2588. https://doi.org/10.1016/j.na.2007.08.040
25. J. Banaś, K. Sadarangani, Solutions of some functional-integral equations in Banach algebra, *Math. Comput. Modelling*, **38** (2003), 245–250, 2003. https://doi.org/10.1016/S0895-7177(03)90084-7
26. J. Caballero, A. B. Mingarelli, K. Sadarangani, Existence of solutions of an integral equation of Chandrasekhar type in the theory of radiative transfer, Electron, *Electron. J. Differential Equations*, **57** (2006), 1–11.
27. İ. Özdemir, B. İlhan, Ü. Çakan, On the solutions of a class of nonlinear integral equations in Banach algebra of the continuous functions and some examples, *An. Univ. Vest. Time. Ser. Mat. Inform*, **2014** (2014), 121–140. https://doi.org/10.2478/awutm-2014-0008
28. B. C. Dhage, On α-condensing mappings in Banach algebras, *Math. Student*, **63** (1994), 146–152.

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