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Research article

Pullback attractors for the nonclassical diffusion equations with memory in time-dependent spaces

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Abstract: In this paper, we consider the asymptotic behavior of nonclassical diffusion equations with hereditary memory and time-dependent perturbed parameter on whole space $\mathbb{R}^n$. Under a general assumption on the memory kernel k, the existence and regularity of time-dependent global attractors are proven using a new analytical technique. It is remarkable that the nonlinearity f has no restriction on the upper growth.

Keywords: nonclassical diffusion equation; time-dependent global attractor; memory; unbounded domain
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1. Introduction

This paper concentrates on the following nonclassical diffusion equation:

$$u_t - \varepsilon(t)\Delta u_t - \Delta u - \int_0^\infty k(s)\Delta u(t-s)ds + \lambda u + f(x,u) = g, \ \text{ in } \mathbb{R}^n \times \mathbb{R}^\tau, \tag{1.1}$$

and initial value

$$u(x,t) = u_\tau(x,t), \ \text{ in } \mathbb{R}^n \times (-\infty, \tau], \tag{1.2}$$

where the forcing term $g = g(x) \in L^2(\mathbb{R}^n)(n \geq 3)$ is known, $\lambda > 0$ is a constant and $\mathbb{R}^\tau = [\tau, \infty)$.

To study problem (1.1) with (1.2), we assume that the time-dependent perturbed parameter $\varepsilon(t)$, the nonlinearity f and the hereditary memory $k(s)$ satisfy the following conditions, respectively:

(**H1**) The time-dependent perturbed parameter $\varepsilon(t) \in C^1(\mathbb{R})$ is a decreasing bounded function satisfying

$$\lim_{t\to\infty} \varepsilon(t) = 0, \tag{1.3}$$

and there exists a constant $L > 0$, such that

$$\sup_{t\in\mathbb{R}}(|\varepsilon(t)| + |\varepsilon'(t)|) \leq L. \tag{1.4}$$

(**H2**) The hereditary memory kernel k is a nonnegative summable function having the explicit form

$$k(s) = \int_s^\infty \mu(\sigma)d\sigma,$$

and $\mu(s) \in L^1(\mathbb{R}^+)$ is a decreasing piecewise absolutely continuous in each interval $[0, T]$ with $T > 0$. In particular, $\mu(s)$ is allowed to exhibit (infinitely many) jumps. Moreover, we require that $\mu(0) < \infty$ and

$$k(s) \leq \theta\mu(s) \tag{1.5}$$

holds for some $\theta > 0$ and almost every $s > 0$. As shown in [1, 2], this is completely equivalent to the requirement that

$$\mu(s + r) \leq Me^{-\delta r}\mu(s)$$

for some $M \geq 1$, $\delta > 0$, every $r \geq 0$, and almost every $s > 0$, then we obtain

$$\mu(\infty) = \lim_{s\to\infty} \mu(s) = 0.$$

For convenience, we also assume

$$k(0) = \int_0^\infty \mu(s)ds = 1.$$

(**H3**) The nonlinearity $f \in C^1(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$ satisfies

$$f(x, s)s \geq -\alpha|s|^2 - \varphi(x), \quad \forall (x, s) \in \mathbb{R}^n \times \mathbb{R}, \tag{1.6}$$

and

$$\frac{\partial}{\partial s} f(x, s) \geq -l, \quad \forall (x, s) \in \mathbb{R}^n \times \mathbb{R}, \tag{1.7}$$

where $\varphi(x) \in L^1(\mathbb{R}^n)$, $f_0(x) = f(x, 0) \in L^2(\mathbb{R}^n)$ is given, α, l are two positive constants and $\alpha < \lambda$.

We denote by F the function $F(x, s) = \int_0^s f(x, \upsilon)d\upsilon$, then,

$$F(x, s) \leq f(x, s)s + \frac{l}{2}s^2.$$

Let $f_1(x, s) = f(x, s) + \alpha s$ and $F_1(x, s) = \int_\tau^s f_1(x, s)ds$, then,

$$f_1(x, s) + \varphi(x) \geq 0, \tag{1.8}$$

$$F_1(x, s) - f_0(x)s + \frac{l - \alpha}{2}s^2 \geq 0, \tag{1.9}$$

and

$$F_1(x, s) \leq f_1(x, s)s + \frac{l - \alpha}{2}s^2. \tag{1.10}$$

Remark 1.1. The assumption of the nonlinearity comes from [3], but some constraints have been relaxed. For example, we will not require that $\varphi(x)$ is a nonnegative function and $l < \lambda$. The class of nonlinearities studied in [4–6] have a restriction on the upper growth by which an exponential nonlinearity (e.g., $f(u) = e^u$) does not hold ture.

According to the Dafermos' idea in [7], we also add a new variable η^t to describe the past history of u, which is defined as follows:

$$\eta^t = \eta^t(x, s) := \int_0^s u(x, t - r)dr, \ \ s \in \mathbb{R}^+. \tag{1.11}$$

Let $\eta_t^t = \frac{\partial}{\partial t}\eta^t$, $\eta_s^t = \frac{\partial}{\partial s}\eta^t$, then it gets

$$\eta_t^t = -\eta_s^t + u, \tag{1.12}$$

where $\mathbb{R}^+ = [0, +\infty)$. Historical variable $u_\tau(\cdot, \tau - s)$ of u satisfies

$$\int_0^\infty e^{-\sigma(s-\tau)}\|u_\tau(\tau - s)\|_{H^1(\mathbb{R}^n)}^2 ds \le \mathfrak{R}, \tag{1.13}$$

where $\mathfrak{R} > 0$ and $\sigma \le \delta$ (δ is from (1.5)).

Thus the system (1.1) with (1.2) can be rewritten as follows:

$$\begin{cases} u_t - \varepsilon(t)\Delta u_t - \Delta u - \displaystyle\int_0^\infty \mu(s)\Delta\eta^t(s)ds + (\lambda - \alpha)u + f_1(x, u) = g, \\ \eta_t^t = -\eta_s^t + u, \end{cases} \tag{1.14}$$

and the initial data

$$u(x, \tau) = u_\tau(x, \tau), \ \eta^\tau(x, s) = \int_0^s u_\tau(x, \tau - r)dr. \tag{1.15}$$

From (1.13), it is easy to obtain the following estimate:

$$\int_0^\infty \mu(s)\|\eta^\tau(s)\|_{H^1(\mathbb{R}^n)}^2 ds \le \mathfrak{R}.$$

For Eq (1.1), it is often used to describe some physical phenomena, for example, non-Newtonian flows, soil mechanics and heat conduction theory. Specifically, when we study heat conduction problems in fluid mechanics or solid mechanics, if the influence of viscosity is emphasized, the classic heat-conduction equation is often extended to the following form (see e.g., [8–10]):

$$c\dot{u} - ca\Delta\dot{u} - k\Delta u = 0.$$

However, when we consider polymer and high viscous liquid, etc., some important factors such as the historical influence of u and the disturbance coefficient of viscosity must be included [11], that is, the following evolution equation:

$$u_t - \varepsilon\Delta u_t - \nu\Delta u - \int_0^\infty k(s)\Delta u(t - s)ds + f(u) = g. \tag{1.16}$$

The asymptotic behavior of solutions of Eq (1.16) has been studied by many scholars (see e.g., [3, 12–19] and the references therein). However, the research focused on the nonclassical diffusion equation with constant coefficient and bounded smooth domain early on, see e.g., [20–26]. In [11], the diffusion equation with memory was proposed in the study of heat conduction and relaxation of high viscosity liquids. The convolution term represents the influence of past history on its future evolution and describes more accurately the diffusive process in certain materials, such as high viscosity liquids at low temperatures and polymers. Hence, it is necessary and scientifically significant to study the nonclassical diffusion equation with the time-dependent coefficient (i.e., variable coefficient) and memory. Furthermore, for Eq (1.16), we focus on the nonclassical diffusion equation with memory and a time-dependent perturbed parameter $\varepsilon(t)$ on a bounded domain. For example, for the case of $k(s) = 0$ in (1.1), in [27, 28], the authors proved the existence and regularity of the time-dependent global attractors on time-dependent spaces when the nonlinearity satisfies $|f''(u)| \leq c(1 + |u|)$ and critically exponential growth, respectively. In addition, Wang and Ma studied the existence, regularity and asymptotic structure of the time-dependent global attractors in [29] for this equation when f meets polynomial growth of arbitrary order. In particular, Wang et al. [6] proved the existence of the time-dependent global attractors for the problem with the nonlinearity of critical exponential growth.

More recently, the fourth author of the article and other co-authors considered the autonomous and nonautonomous nonclassical diffusion equation with memory or without memory on bounded domains, see e.g., [4, 5, 19, 30–32]. In these papers, the operator decomposition and contractive functional methods are used to obtain the asymptotic regularity of the solutions and verify asymptotic compactness. It is worth mentioning that Xie et al. [4] have obtained the existence of the time-dependent global attractors on bounded domain for the nonclassical diffusion equation (1.16), lacking instantaneous damping with the nonlinearity that satisfies the polynomial growth of arbitrary order. However, we focus on the unbounded domain. Therefore, we can see that there is few relevant studies for the asymptotic behavior of solutions of Eq (1.1) in time-dependent whole spaces under assumptions **(H1)–(H3)**. This is because there are two major difficulties to obtain the existence of time-dependent global attractors.

(*i*) First, because of the nonlinearity with no restriction on the upper growth, the higher asymptotic regularity of the solutions of Eq (1.1) can not be obtained using the method of [33, 34].

(*ii*) Second, due to the influence of the time-dependent perturbed parameter $\varepsilon(t)$ and the lacking of compact embedding theorem on unbounded domains $\mathbb{R}^n$, it is impossible to directly construct the contractive function to prove the asymptotic compactness for the corresponding process $\{U(t,\tau)\}_{t\geq\tau}$ of Eq (1.1) (see e.g., [29, 30, 35]).

For solving these problems, a new analytical technique combined with the operator decomposition method is used to obtain contractive function, and then the pullback asymptotic compactness for the process $\{U(t,\tau)\}_{t\geq\tau}$ of Eq (1.14) is proved. Furthermore, using this operator decomposition method, the asymptotic regularity of the solutions for Eq (1.14) is also proved. Then, the regularity of time-dependent global attractors for this equation on $\mathcal{H}_t^1 \times L^2_\mu(\mathbb{R}^+; H^1(\mathbb{R}^n))$ ($\mathcal{H}_t^1$ is defined later) are established.

For conveniences, hereafter let $|\cdot|_p$ be the norm of $L^p(\mathbb{R}^n)$ $(p \geq 1)$. Let $(\cdot,\cdot)$, $(\cdot,\cdot)+(\nabla\cdot,\nabla\cdot) = \langle\cdot,\cdot\rangle_{H^1(\mathbb{R}^n)}$ and $(\cdot,\cdot)+(\nabla\cdot,\nabla\cdot)+(\Delta\cdot,\Delta\cdot) = \langle\cdot,\cdot\rangle_{H^2(\mathbb{R}^n)}$ be the inner product of $L^2(\mathbb{R}^n)$, $H^1(\mathbb{R}^n)$ and $H^2(\mathbb{R}^n)$ respectively.

Let $\|\cdot\|_0, \|\cdot\|_1$ be the norm of $H^1(\mathbb{R}^n)$ and $H^2(\mathbb{R}^n)$ respectively, then,

$$\|x\|_0^2 = |x|_2^2 + |\nabla x|_2^2, \qquad \forall x \in H^1(\mathbb{R}^n);$$

$$\|x\|_1^2 = |x|_2^2 + |\nabla x|_2^2 + |\Delta x|_2^2, \qquad \forall x \in H^2(\mathbb{R}^n).$$

The time-dependent space $\mathcal{H}_t^1 = H^1(\mathbb{R}^n)$ and $\mathcal{H}_t^2 = H^2(\mathbb{R}^n)$ are equipped with the norms:

$$\|\cdot\|_{\mathcal{H}_t^1}^2 = |\cdot|_2^2 + \varepsilon(t)|\nabla\cdot|_2^2 \quad \text{and} \quad \|\cdot\|_{\mathcal{H}_t^2}^2 = \|\cdot\|_0^2 + \varepsilon(t)|\triangle\cdot|_2^2.$$

It is necessary to point out here that $\|\cdot\|_{\mathcal{H}_t^1}^2$ and $|\cdot|_2^2 + \varepsilon(t)\|\cdot\|_0^2$ are equivalent. In fact the following inequality is obvious:

$$\|\cdot\|_{\mathcal{H}_t^1}^2 \le |\cdot|_2^2 + \varepsilon(t)\|\cdot\|_0^2,$$

and

$$\begin{aligned}\|\cdot\|_{\mathcal{H}_t^1}^2 &= \frac{1}{2}|\cdot|_2^2 + \frac{1}{2}|\cdot|_2^2 + \varepsilon(t)|\nabla\cdot|_2^2\\ &\ge \frac{1}{2}|\cdot|_2^2 + \frac{1}{2L}\varepsilon(t)|\cdot|_2^2 + \varepsilon(t)|\nabla\cdot|_2^2\\ &\ge \frac{1}{2(1+L)}\left(|\cdot|_2^2 + \varepsilon(t)\|\cdot\|_0^2\right).\end{aligned}$$

Denote the weight spaces $\mathcal{V}_0 = L_\mu^2(\mathbb{R}^+; L^2(\mathbb{R}^n))$, $\mathcal{V}_1 = L_\mu^2(\mathbb{R}^+; H^1(\mathbb{R}^n))$ and $\mathcal{V}_2 = L_\mu^2(\mathbb{R}^+; H^2(\mathbb{R}^n))$, and their inner products and norms are defined as follows:

$$\langle\psi,\eta\rangle_{\mathcal{V}_0} = \int_0^\infty \mu(s)\,(\psi,\eta)\,ds, \quad |\eta^t|_{\mu,2}^2 = \int_0^\infty \mu(s)|\eta^t(s)|_2^2 ds;$$

$$\langle\psi,\eta\rangle_{\mathcal{V}_1} = \int_0^\infty \mu(s)\,\langle\psi,\eta\rangle_{H^1(\mathbb{R}^n)}\,ds, \quad \left\|\eta^t\right\|_{\mu,0}^2 = \int_0^\infty \mu\,(s)\|\eta^t(s)\|_0^2 ds;$$

and

$$\langle\psi,\eta\rangle_{\mathcal{V}_2} = \int_0^\infty \mu(s)\,\langle\psi,\eta\rangle_{H^2(\mathbb{R}^n)}\,ds, \quad \left\|\eta^t\right\|_{\mu,1}^2 = \int_0^\infty \mu\,(s)\|\eta^t(s)\|_1^2 ds.$$

With the above notation, the phase spaces of Eq (1.14) can be denoted as

$$\mathcal{M}_t := \mathcal{M}_t(\mathbb{R}^n) = \mathcal{H}_t^1 \times \mathcal{V}_1 \ \text{ and } \ \mathcal{M}_t^1 := \mathcal{M}_t^1(\mathbb{R}^n) = \mathcal{H}_t^2 \times \mathcal{V}_2$$

equipped the following norms:

$$\|\cdot\|_{\mathcal{M}_t}^2 = \|\cdot\|_{\mathcal{H}_t^1}^2 + \|\cdot\|_{\mu,0}^2 \ \text{ and } \ \|\cdot\|_{\mathcal{M}_t^1}^2 = \|\cdot\|_{\mathcal{H}_t^2}^2 + \|\cdot\|_{\mu,1}^2$$

respectively.

Particularly, we use $\{\mathcal{M}_t\}_{t\in\mathbb{R}}$ to denote a family of normed time-dependent spaces. Moreover, we introduce some common notations based on processes of time-dependent space(see e.g., [35–38]).

Let $\{\mathcal{M}_t\}_{t\in\mathbb{R}}$ be a family of normed time-dependent space. Note that the ball with radius of R in $\mathcal{M}_t$ is

$$\mathscr{B}_t(R) = \{w \in \mathcal{M}_t : \|w\|_{\mathcal{M}_t} \le R\}.$$

For any given $\varepsilon > 0$, we define the ε neighborhood of set $B \subset \mathcal{M}_t$ as follows:

$$\mathcal{O}_t^\varepsilon(B) = \bigcup_{x\in B}\{y \in \mathcal{M}_t : \|x-y\|_{\mathcal{M}_t} < \varepsilon\} = \bigcup_{x\in B}\{x + \mathscr{B}_t(\varepsilon)\}.$$

Hausdorff semidistance of between two nonempty sets $A, B \subset X_t$ is defined as

$$dist_{\mathcal{M}_t}(A, B) = \sup_{x\in A} \inf_{y\in B} \|x - y\|_{\mathcal{M}_t}.$$

The plan of this paper is as follows. In Section 2, we recall some basic concepts as the time-dependent global attractors and useful results that will be used later. In Section 3, we first prove pullback asymptotic compactness of the process corresponding to problem (1.14) with (1.15) by constructing contractive function, and then we obtain the existence and the regularity of time-dependent global attractors to problem (1.14) with (1.15) in whole space $\mathbb{R}^n$.

2. Preliminaries

In this section, we will recall some basic concepts of time-dependent global attractors and theories of the existence of time-dependent global attractors (see e.g., [35–37]).

Definition 2.1. Let $\{\mathcal{M}_t\}_{t\in\mathbb{R}}$ be a family of normed spaces. A two-parameter family of operators $U(t, \tau) : \mathcal{M}_\tau \to \mathcal{M}_t$ is called a process if it satisfies the following properties:
(i) $U(\tau, \tau) = Id, \tau \in \mathbb{R}$ (identity operator);
(ii) $U(t, s)U(s, \tau) = U(t, \tau),\ \forall t \geq s \geq \tau \in \mathbb{R}$.

Definition 2.2. A family of sets $\tilde{C} = \{C_t : C_t \subset \mathcal{M}_t \ is\ bounded\}_{t\in\mathbb{R}}$ is called uniformly bounded if there exists a constant $R > 0$, such that $C_t \subset \mathscr{B}_t(R)$ for any $t \in \mathbb{R}$.

Definition 2.3. A family of sets $\tilde{B} = \{B_t\}_{t\in\mathbb{R}}$ is called pullback absorbing if $\tilde{B} = \{B_t\}_{t\in\mathbb{R}}$ is uniformly bounded and for all $R > 0$, there exists a constant $t_0 = t_0(t, R) \leq t$ such that $U(t, \tau)\mathscr{B}_\tau(R) \subset B_t$ for any $\tau \leq t_0$.

The process $\{U(t, \tau)\}_{t\geq\tau}$ is called dissipative whenever it enters a pullback absorbing family $\tilde{B}_0 = \{B^0_t\}_{t\in\mathbb{R}}$.

Definition 2.4. A time-dependent absorbing set for the process $U(t, \tau)$ is a uniformly bounded family $\tilde{B} = \{B_t\}_{t\in\mathbb{R}}$ with the following characteristics: For any $R > 0$, there exists $t_0 = t_0(t, R) \geq 0$, such that

$$U(t, \tau)\mathscr{B}_\tau(R) \subset B_t, \quad for\ all\ \tau \leq t - t_0.$$

Definition 2.5. The process $U(t, \tau)$ is called pullback asymptotic compact if for any $t \in \mathbb{R}$, any bounded sequence $\{z_n\}_{n=1}^\infty \subset \mathcal{M}_{\tau_n}$ and $\{\tau_n\}_{n=1}^\infty \subset (-\infty, t]$ with $\tau_n \to -\infty$ as $n \to \infty$, the sequence $\{U(t, \tau_n)z_n\}_{n=1}^\infty$ has a convergent subsequence in $\mathcal{M}_t$.

Definition 2.6. A time-dependent global attractor of the process $U(t, \tau)$ is the smallest family $\tilde{\mathscr{A}} = \{\mathscr{A}_t\}_{t\in\mathbb{R}}$ such that
(i) for every $t \in \mathbb{R}$, $\mathscr{A}_t$ is compact in $\mathcal{M}_t$;
(ii) $\tilde{\mathscr{A}}$ is pullback attracting, namely, $\tilde{\mathscr{A}}$ is uniformly bounded and

$$\lim_{\tau\to-\infty} dist_{\mathcal{M}_t}(U(t, \tau)C_\tau, \mathscr{A}_t) = 0$$

holds for any uniformly bounded family $\tilde{C} = \{C_\tau\}_{\tau\in\mathbb{R}}$ and every fixed $t \in \mathbb{R}$ and $\tau \leq t$.

Remark 2.1. The pullback attracting nature can be equivalently described in the light of pullback absorbing: A (uniformly bounded) family $\mathcal{K} = \{K_t\}_{t\in\mathbb{R}}$ is said to be pullback attracting if for any $\varepsilon > 0$ the family $\{O_t^{\varepsilon}(K_t)\}_{t\in\mathbb{R}}$ is pullback absorbing.

Theorem 2.1. *A time-dependent global attractor $\tilde{\mathscr{A}}$ exists and it is unique if and only if the process $U(t,\tau)$ is asymptotic compact, i.e., the set*

$$\mathbb{K} = \{\mathcal{K} = \{K_t\}_{t\in\mathbb{R}} : K_t \subset \mathcal{M}_t \text{ is compact, } \mathbb{K} \text{ is pullback attracting}\}$$

is non-empty.

It can be seen from Definition 2.6 that the time-dependent global attractor is not necessarily invariant. This is mainly because that the process is not required to meet some continuity. If the process $U(t,\tau)$ satisfies the appropriate continuity, then the invariance of time-dependent global attractor $\mathscr{A}$ can be obtained.

Definition 2.7. We say that $\tilde{\mathscr{A}} = \{\mathscr{A}_t\}_{t\in\mathbb{R}}$ is invariant if

$$U(t,\tau)\mathscr{A}_\tau = \mathscr{A}_t, \quad t \geq \tau \in \mathbb{R}.$$

Lemma 2.1. *If the time-dependent global attractor $\tilde{\mathscr{A}}$ exists and the process $U(t,\tau)$ is a strongly continuous process, then $\tilde{\mathscr{A}}$ is invariant.*

Next, we will state the definitions of contractive function and contractive process, which will be used to obtain asymptotic compactness of a family of process $\{U(t,\tau)\}_{t\geq\tau}$ (see e.g., [23,35,39–43]).

Definition 2.8. Let $\{\mathcal{M}_t\}_{t\in\mathbb{R}}$ be a family of Banach spaces and $\tilde{B} = \{B_t \subset \mathcal{M}_t\}_{t\in\mathbb{R}}$ be a family of uniformly bounded subset. We call function $\varphi(\cdot,\cdot)$, defined on $\mathcal{M}_\tau \times \mathcal{M}_\tau$, to be a contractive function on $B_\tau \times B_\tau$ if for any sequence $\{z_n\}_{n=1}^{\infty} \subset B_\tau$, there exists a subsequence $\{z_{n_k}\}_{k=1}^{\infty} \subset \{z_n\}_{n=1}^{\infty}$ such that

$$\lim_{k\to\infty}\lim_{l\to\infty} \varphi_\tau^t(z_{n_k}, z_{n_l}) = 0, \quad \forall\, t \geq \tau \in \mathbb{R}.$$

We use $\mathfrak{E}(B_\tau)$ to denote the set all contractive function on $B_\tau \times B_\tau$.

Definition 2.9. Let $U(t,\tau)$ be a process on $\{\mathcal{M}_t\}_{t\in\mathbb{R}}$ and have a pullback bounded absorbing set $\tilde{B} = \{B_t\}_{t\in\mathbb{R}}$. $U(t,\tau)$ is called $\mathcal{M}_t$-contractive process if for any given $\varepsilon > 0$, there exist $T = T(\varepsilon)$ and $\varphi_T^t(\cdot,\cdot) \in \mathfrak{E}(B_T)$ such that

$$\|U(t,T)z_1 - U(t,T)z_2\|_{\mathcal{M}_t} \leq \varepsilon + \varphi_T^t(z_1,z_2), \quad \forall\, z_i \in B_T\ (i = 1,2),$$

where φ_T^t depends on T.

Next, we will give the method to prove the existence of time-dependent global attractors for evolution equations, which will be used in the later discussion.

Theorem 2.2. [35] *Let $\{\mathcal{M}_t\}_{t\in\mathbb{R}}$ be a family of Banach spaces, then $U(t,\tau)$ has a time-dependent global attractor in $\{\mathcal{M}_t\}_{t\in\mathbb{R}}$, if the following conditions hold:*
(i) $U(t,\tau)$ has a pullback absorbing set $\tilde{B} = \{B_t\}_{t\in\mathbb{R}}$ in $\{\mathcal{M}_t\}_{t\in\mathbb{R}}$;
(ii) $U(t,\tau)$ is a $\mathcal{M}_t$-contractive process.

Lemma 2.2. *[44] Let $X \subset\subset H \subset Y$ be Banach spaces, with X reflexive. Suppose that $\{u_n\}_{n=0}^{\infty}$ is a sequence, uniformly bounded in $L^2(\tau,T;X)$ and du_n/dt is uniformly bounded in $L^p(\tau,T;Y)$, for some $p > 1$. Then, there is a subsequence of $\{u_n\}_{n=0}^{\infty}$ that converges strongly in $L^2(\tau,T;H)$.*

3. Time-dependent global attractors in $\{\mathcal{M}_t\}_{t\in\mathbb{R}}$

In this section, we shall consider the existence of time-dependent global attractors in $\{\mathcal{M}_t\}_{t\in\mathbb{R}}$. For this purpose, we have to first discuss the well-posedness for Eq (1.14) with (1.15).

3.1. The well-posedness of equation

The well-posedness for Eq (1.14) with (1.15) can be obtained by using Faedo-Galerkin method (see e.g., [38,45]). To this end, we first give the definition of weak solution.

Definition 3.1. For any $R > 0$ and $T > \tau$, let $I = [\tau, T]$, then the function $z = (u, \eta^t)$ defined on $\mathbb{R}^n \times I$ is called a weak solution of the problem (1.14) with initial value $z_\tau = (u_\tau, \eta^\tau) \in \mathscr{B}_\tau(R) \subset \mathcal{M}_\tau$ if

$$u \in C(I; \mathcal{H}_t^1),\ u_t \in L^2(I; \mathcal{H}_t^1),\ \eta^t \in C(I; L^2_\mu(\mathbb{R}^+; H^1(\mathbb{R}^n))),$$
$$\eta^t_t + \eta^t_s \in L^\infty(I; L^2_\mu(\mathbb{R}^+; H^1(\mathbb{R}^n))) \cap L^2(I; L^2_\mu(\mathbb{R}^+; H^1(\mathbb{R}^n))).$$

Furthermore, the following identity hold:

$$\begin{cases} (u_t, \omega) + \varepsilon(t)\langle u_t, \omega\rangle_{H^1} + \langle u, \omega\rangle_{H^1} + \langle \eta^t, \omega\rangle_{\mathcal{V}_1} + \langle f(u), \omega\rangle = (g, \omega), \\ \langle \eta^t_t + \eta^t_s, \varphi\rangle_{\mathcal{V}_1} = \langle u, \varphi\rangle_{\mathcal{V}_1}, \end{cases} \tag{3.1}$$

holds for any $(\omega, \varphi) \in C^\infty(I, \mathcal{H}_t^1) \times C^\infty(I, L^2_\mu(\mathbb{R}^+; H^1(\mathbb{R}^n)))$ and *a.e.* $t \in I$.

Lemma 3.1. *For any $R > 0$, $T > \tau$ and $z_\tau = (u_\tau, \eta^\tau) \in \mathscr{B}_\tau(R) \subset \mathcal{M}_\tau$, then the problem* (1.14) *with* (1.15) *has unique weak solution*

$$z(t) = (u(x, t), \eta^t) \in C([\tau, T]; \mathcal{H}_t^1 \times \mathcal{V}),$$

which continuously depends on the initial data in $\mathcal{M}_\tau$, i.e., there exists a constant $\kappa > 0$ not related to t such that the process $U(t, \tau)$ is Lipschitz continuous

$$\|U(t,\tau)z^1_\tau - U(t,\tau)z^2_\tau\|_{\mathcal{M}_t} \le Ce^{\kappa(T-\tau)}\|z^1_\tau - z^2_\tau\|_{\mathcal{M}_\tau}, \quad \forall t \in [\tau, T]. \tag{3.2}$$

By Lemma 3.1, we may define the process of solutions on time-dependent space $\{\mathcal{M}_t\}_{t\in\mathbb{R}}$:

$$U(t, \tau) : \mathcal{M}_\tau \to \mathcal{M}_t,\ U(t, \tau)z_\tau = z(t), \quad \forall t \ge \tau. \tag{3.3}$$

In addition, it's easy to obtain that the process $U(t, \tau)$ is a strongly continuous process on the time-dependent phase space $\{\mathcal{M}_t\}_{t\in\mathbb{R}}$.

3.2. Time-dependent absorbing sets

In the following discussion, let C mean any positive constant and $Q(\cdot)$ be a monotonically increasing function on $[0, \infty)$ which may be different from line to line even in the same line.

We always assume that: The assumptions **(H1)–(H3)** are true. Furthermore, $z(t) = (u(t), \eta^t)$ is a sufficiently regular solution of (1.14) with (1.15).

Lemma 3.2. *For any $R > 0$ and $z_\tau = (u_\tau, \eta^\tau) \in \mathscr{B}_\tau(R) \subset \mathcal{M}_\tau$. Then, there exist positive two constants δ_1 and k_1, such that*

$$|u|_2^2 + \varepsilon(t)|\nabla u|_2^2 + \|\eta^t\|^2_{\mu,0} \le Q(R)e^{-\delta_1(t-\tau)} + k_1$$

holds for any $\tau \le t$.

Proof. Multiplying the first equation of (1.14) by u in $L^2(\mathbb{R}^n)$ and the second one by η^t in $\mathcal{V}_0$, then we obtain

$$\frac{1}{2}\frac{d}{dt}\left(|u|_2^2+\varepsilon(t)|\nabla u|_2^2+|\nabla\eta^t|_{\mu,2}^2\right)-\frac{1}{2}\varepsilon'(t)|\nabla u|_2^2-\frac{1}{2}\int_0^\infty\mu'(s)|\nabla\eta^t(s)|_2^2ds$$
$$+|\nabla u|_2^2+\int_{\mathbb{R}^n}f_1(x,u)u+\frac{\lambda-\alpha}{2}|u|_2^2\leq\frac{1}{2(\lambda-\alpha)}|g|_2^2, \tag{3.4}$$

and

$$\frac{1}{2}\frac{d}{dt}|\eta^t|_{\mu,2}^2-\frac{1}{2}\int_0^\infty\mu'(s)|\eta^t(s)|_2^2ds\leq 2|u|_2^2+\frac{1}{8}|\eta^t(s)|_{\mu,2}^2. \tag{3.5}$$

Furthermore, we also get

$$\frac{1}{2}\frac{d}{dt}\int_0^\infty k(s)|\eta^t(s)|_2^2ds+\frac{3}{8}|\eta^t|_{\mu,2}^2\leq 2\theta^2|u|_2^2, \tag{3.6}$$

and

$$\frac{1}{2}\frac{d}{dt}\int_0^\infty k(s)(\nabla\eta^t(s),\nabla\eta^t(s))ds+\frac{1}{4}|\nabla\eta^t|_{\mu,2}^2\leq\theta^2|\nabla u|_2^2. \tag{3.7}$$

From (3.5) and (3.6), one gets

$$\frac{1}{2}\frac{d}{dt}\left(|\eta^t|_{\mu,2}^2+|\eta^t|_{k,2}^2\right)+\frac{1}{4}|\eta^t|_{\mu,2}^2\leq 2(1+\theta^2)|u|_2^2. \tag{3.8}$$

Taking $\sigma=\frac{\lambda-\alpha}{8(1+\theta^2)}$ and let

$$E(t)=\frac{1}{2}\left[|u|_2^2+\varepsilon(t)|\nabla u|_2^2+|\nabla\eta^t|_{\mu,2}^2+\sigma\left(|\eta^t|_{\mu,2}^2+|\eta^t|_{k,2}^2\right)+\frac{1}{2\theta^2}|\nabla\eta^t|_{k,2}^2\right];$$
$$H(t)=-\frac{1}{2}\varepsilon'(t)|\nabla u|_2^2-\frac{1}{2}\int_0^\infty\mu'(s)|\nabla\eta^t(s)|_2^2ds+\frac{1}{2}|\nabla u|_2^2+\frac{1}{8\theta^2}|\nabla\eta^t|_{\mu,2}^2$$
$$+\int_{\mathbb{R}^n}f_1(x,u)u+\frac{\lambda-\alpha}{4}|u|_2^2+\frac{\sigma}{4}|\eta^t|_{\mu,2}^2.$$

Then combining (3.4) and (3.5), it follows that

$$\frac{d}{dt}E(t)+H(t)\leq\frac{1}{2(\lambda-\alpha)}|g|_2^2. \tag{3.9}$$

Furthermore, let

$$a_0=\max\{1+\frac{1}{2\theta},\sigma(1+\theta)\},\ \text{ and }\ \widetilde{a}_0=\min\{1,\sigma\}.$$

Then we get

$$E(t)\leq a_0\left(|u|_2^2+\varepsilon(t)|\nabla u|_2^2+\|\eta^t\|_{\mu,0}^2\right), \tag{3.10}$$

$$E(t)\geq\widetilde{a}_0\left(|u|_2^2+\varepsilon(t)|\nabla u|_2^2+\|\eta^t\|_{\mu,0}^2\right), \tag{3.11}$$

and

$$
\begin{aligned}
H(t) \geq & \frac{1}{2}|\nabla u|_2^2 + \frac{1}{8\theta^2}|\nabla \eta^t|_{\mu,2}^2 + \int_{\mathbb{R}^n} f_1(x,u)u + \frac{\lambda-\alpha}{4}|u|_2^2 + \frac{\sigma}{4}|\eta^t|_{\mu,2}^2 \\
\geq & \frac{\lambda-\alpha}{4}|u|_2^2 + \frac{\varepsilon(t)}{2L}|\nabla u|_2^2 + \frac{1}{8\theta^2}|\nabla\eta^t|_{\mu,2}^2 + \frac{\sigma}{4}|\eta^t|_{\mu,2}^2 + \int_{\mathbb{R}^n}(f_1(x,u)u + \varphi(x)) - |\varphi|_1 \\
\geq & \delta_1 a_0 \left(|u|_2^2 + \varepsilon(t)|\nabla u|_2^2 + \|\eta^t\|_{\mu,0}^2\right) + \int_{\mathbb{R}^n}(f_1(x,u)u + \varphi(x)) - |\varphi|_1 \\
\geq & \delta_1 E(t) - |\varphi|_1, \qquad (3.12)
\end{aligned}
$$

where $\delta_1 = \frac{1}{2a_0}\min\{\frac{1}{L}, \frac{1}{4\theta^2}, \frac{\sigma}{2}, \frac{\lambda-\alpha}{2}\}$.

Then from (3.9) and (3.12) we get

$$
\frac{d}{dt}E(t) + \delta_1 E(t) \leq \frac{1}{2(\lambda-\alpha)}|g|_2^2 + |\varphi|_1.
$$

Applying Grönwall lemma, we have

$$
E(t) \leq E(\tau)e^{-\delta_1(t-\tau)} + \frac{1}{2\delta_1(\lambda-\alpha)}|g|_2^2 + \frac{1}{\delta_1}|\varphi|_1.
$$

By (3.11), it follows that

$$
|u|_2^2 + \varepsilon(t)|\nabla u|_2^2 + \|\eta^t\|_{\mu,0}^2 \leq Q(R)e^{-\delta_1(t-\tau)} + \frac{1}{\widetilde{a}_0}\left(\frac{1}{2\delta_1(\lambda-\alpha)}|g|_2^2 + \frac{1}{\delta_1}|\varphi|_1\right).
$$

Let

$$
k_1 = \frac{1}{\widetilde{a}_0}\left(\frac{1}{2\delta_1(\lambda-\alpha)}|g|_2^2 + \frac{1}{\delta_1}|\varphi|_1\right).
$$

Then the proof is complete.

Corollary 3.1. *For any given $R \in \mathbb{R}^+$, let $z_\tau = (u_\tau, \eta^\tau) \in \mathscr{B}_\tau(R) \subset \mathcal{M}_\tau$, then the process $U(t,\tau)$ corresponding to Eq (1.14) with (1.15) possesses a time-dependent bounded absorbing set $\tilde{B}_0 = \{B_t^0 = \mathscr{B}_t(\rho_0)\}_{t\in\mathbb{R}}$, that is there exists $t_0 = t_0(t,R) < t$, such that*

$$
U(t,\tau)\mathscr{B}_\tau(R) \subset B_t^0, \quad \forall\, \tau \leq t_0.
$$

In fact, let $t_0 = t_0(t,R) = t - \frac{1}{\delta_1}\ln\frac{Q(R)}{k_1} < t$ and $\rho_0 = 2k_1$, then the conclusion can be directly obtained from Lemma 3.2.

For brevity, later in this article, let $\tilde{B}^0$ be the bounded uniformly absorbing set obtained in Lemma 3.2, i.e.,

$$
\tilde{B}^0 = \{B_t : z = (u,\eta^t) \in B_t,\ \|z\|_{\mathcal{M}_t}^2 = |u|_2^2 + \varepsilon(t)|\nabla u|_2^2 + \|\eta^t\|_{\mu,0}^2 \leq \rho_0\}_{t\in\mathbb{R}}. \qquad (3.13)
$$

From Lemma 3.2, it follows that:

Corollary 3.2. *For any $R > 0$ and $z_\tau = (u_\tau, \eta^\tau) \in \mathscr{B}_\tau(R) \subset \mathcal{M}_\tau$, then there exists $\mathcal{K}_0 = \mathcal{K}_0(R)$ such that*

$$
|u(t)|_2^2 + \varepsilon(t)|\nabla u(t)|_2^2 + \|\eta^t\|_{\mu,0}^2 \leq \mathcal{K}_0
$$

holds for all $t - \tau \geq 0$.

Corollary 3.3. *For any $R > 0$ and $z_\tau = (u_\tau, \eta^\tau) \in \mathscr{B}_\tau(R) \subset \mathcal{M}_\tau$, then there exists $\rho_1 = \rho_1(k_1) > 0$, such that the following estimate:*

$$\int_t^{t+1} \left(|u(s)|_2^2 + |\nabla u(s)|_2^2 + \int_{\mathbb{R}^n} f_1(x, u(s))u(s) \right) ds \leq Q(R)e^{-\delta_1(t-\tau)} + \rho_1$$

holds for any $\tau \leq t$.

Proof. By (3.4), it's easy to obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(|u|_2^2 + \varepsilon(t)|\nabla u|_2^2 + |\nabla \eta^t|_{\mu,2}^2\right) + \frac{\lambda - \alpha}{2}|u|_2^2 + |\nabla u|_2^2 \\ &+ \int_{\mathbb{R}^n} (f_1(x, u)u + \varphi(x)) \leq \frac{1}{2(\lambda - \alpha)}|g|_2^2 + |\varphi|_1. \end{aligned} \tag{3.14}$$

Let $b_1 = \min\{1, \frac{\lambda-\alpha}{2}\}$ and integrating (3.14) on $[t, t+1]$, we get

$$\begin{aligned} &\int_t^{t+1} \left(|u(s)|_2^2 + |\nabla u(s)|_2^2 + \int_{\mathbb{R}^n} (f_1(x, u(s))u(s) + \varphi(x)) \right) ds \\ \leq &\frac{1}{b_1}\left(\frac{1}{2(\lambda - \beta_1)})|g|_2^2 + |\varphi|_1 + \frac{1}{2}\left(|u(t)|_2^2 + \varepsilon(t)|\nabla u(t)|_2^2 + |\nabla \eta^t|_{\mu,2}^2\right) \right) \\ \leq &Q(R)e^{-\delta_1(t-\tau)} + \frac{1}{b_1}\left(\frac{1}{2(\lambda - \beta_1)})|g|_2^2 + |\varphi|_1 \right) + \frac{1}{2b_1}k_1. \end{aligned}$$

Let $\rho_1 = \frac{1}{b_1}\left(\frac{1}{2(\lambda-\beta_1)})|g|_2^2 + |\varphi|_1\right) + \frac{1}{2b_1}k_1 + |\varphi|_1$ and $\tau \leq t$, then the proof is complete.

Lemma 3.3. *For any $R > 0$ and $z_\tau = (u_\tau, \eta^\tau) \in \mathscr{B}_\tau(R) \subset \mathcal{M}_\tau$. Then, there exist positive two constants δ_2 and k_2, such that*

$$|u(t)|_2^2 + |\nabla u(t)|_2^2 + \int_{\mathbb{R}^n} F_1(x, u) \leq Q(R)e^{-\delta_2(t-\tau)} + k_2$$

holds for any $\tau < t$.

Proof. We now multiply the first equation of (1.14) by u_t in $L^2(\mathbb{R}^n)$, then we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[(\lambda - \alpha)|u|_2^2 + |\nabla u|_2^2 + 2\int_{\mathbb{R}^n} F_1(x, u) + 2\int_0^\infty \mu(s)\left(\nabla u, \nabla \eta^t(s)\right) ds\right] + \frac{1}{2}|u_t|_2^2 + \frac{\varepsilon(t)}{2}|\nabla u_t|_2^2 \\ \leq &2|\nabla u|_2^2 - \frac{\mu^2(0)}{4}\int_0^\infty \mu'(s)|\nabla \eta^t(s)|_2^2 ds + \frac{1}{2}|g|_2^2. \end{aligned} \tag{3.15}$$

Setting

$$\begin{aligned} \mathcal{F}(u) &= \int_{\mathbb{R}^n} \left(F_1(x, u) - f_0(x)u + \frac{l - \alpha}{2}|u|^2 \right) \geq 0, \\ N(t) &= \frac{1}{2}\left[(\lambda - \alpha)|u|_2^2 + |\nabla u|_2^2 + 2\int_{\mathbb{R}^n} F_1(x, u) + 2\int_0^\infty \mu(s)\left(\nabla u, \nabla \eta^t(s)\right) ds\right]. \end{aligned}$$

From Lemma 3.2 and Corollary 3.2, we get that for any $t > \tau$, there exists $t^\star \in (\tau, t]$ such that

$$|u(t^\star)|_2^2 + |\nabla u(t^\star)|_2^2 + |\nabla \eta^{t^\star}|_{\mu,2}^2 + \int_{\mathbb{R}^n} (f_1(x, u(t^\star))u(t^\star) + \varphi(x)) \leq Q(R).$$

Furthermore, associating with (1.10), it is easy to obtain the following inequality:

$$|u(t^\star)|_2^2 + |\nabla u(t^\star)|_2^2 + |\nabla \eta^{t^\star}|_{\mu,2}^2 + \mathcal{F}(u(t^\star)) \leq Q(R) \tag{3.16}$$

holds for some $t^\star \in (\tau, t]$. Let

$$B_\beta(t) = \frac{1}{2}\left(|u(t)|_2^2 + \varepsilon(t)|\nabla u(t)|_2^2 + |\nabla \eta^t|_{\mu,2}^2\right) + \beta N(t) + \frac{\beta}{2\theta^2}\int_0^\infty k(s)|\nabla \eta^t(s)|_2^2 ds,$$

where

$$\beta = \frac{1}{2}\min\{\frac{1}{1 + |\lambda - l|}, \frac{1}{4}, \frac{2}{\mu^2(0)}, \sqrt{4|\alpha - l|^2 + 2(\lambda - \alpha)} - 2|\alpha - l|\} < \frac{1}{4}.$$

Then, $1 - (1 + |\lambda - l|)\beta > \frac{1}{2}, 1 - 2\beta > \frac{1}{2}$. Let

$$\begin{aligned} a_1 &= \frac{1}{2}\min\{1 - (1 + |\lambda - l|)\beta, 2\beta, 1 - 2\beta\}, \\ \tilde{a}_1 &= \frac{1}{2}\max\{1 + (1 + |\lambda - l|)\beta, 2\beta, L + 2\beta, 1 + \beta + \frac{\beta}{\theta}\}. \end{aligned}$$

Therefore, we get

$$\begin{aligned} B_\beta(t) &\geq \frac{1}{2}\left(1 - (1 + |\lambda - l|\beta)\right)|u(t)|_2^2 + \frac{\beta}{4}|\nabla u(t)|_2^2 + \frac{1}{2}(1 - 2\beta)|\nabla \eta^t|_{\mu,2}^2 + \beta\mathcal{F}(u) - \frac{\beta}{2}|f_0|_2^2 \\ &\geq a_1\left(|u(t)|_2^2 + |\nabla u(t)|_2^2 + |\nabla \eta^t|_{\mu,2}^2 + \mathcal{F}(u)\right) - \frac{\beta}{2}|f_0|_2^2 \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} B_\beta(t) \leq& \frac{1}{2}\left(1 + (1 + |\lambda - l|\beta)\right)|u(t)|_2^2 + \frac{1}{2}(L + 2\beta)|\nabla u(t)|_2^2 + \frac{1}{2}(1 + \beta + \frac{\beta}{\theta})|\nabla \eta^t|_{\mu,2}^2 + \beta\mathcal{F}(u) + \frac{\beta}{2}|f_0|_2^2 \\ \leq& \tilde{a}_1\left(|u(t)|_2^2 + |\nabla u(t)|_2^2 + |\nabla \eta^t|_{\mu,2}^2 + \mathcal{F}(u)\right) + \frac{1}{4}|f_0|_2^2. \end{aligned} \tag{3.18}$$

Combining with (3.4) and (3.15), it follows that

$$\begin{aligned} &\frac{d}{dt}B_\beta(t) + \frac{1}{2}|u_t|_2^2 + \frac{\varepsilon(t)}{2}|\nabla u_t|_2^2 - \frac{1}{2}\varepsilon'(t)|\nabla u|_2^2 - \frac{1}{2}\left(1 - \frac{\mu^2(0)\beta}{2}\right)\int_0^\infty \mu'(s)|\nabla \eta^t(s)|_2^2 ds \\ &+(1 - 3\beta)|\nabla u|_2^2 + \frac{\lambda - \alpha}{2}|u|_2^2 + \frac{\beta}{4\theta^2}|\nabla \eta^t|_{\mu,2}^2 + \int_{\mathbb{R}^n} f_1(x, u)u \leq C|g|_2^2. \end{aligned} \tag{3.19}$$

Let

$$\begin{aligned} \tilde{B}_\beta(t) =& \frac{1}{2}|u_t|_2^2 + \frac{\varepsilon(t)}{2}|\nabla u_t|_2^2 - \frac{1}{2}\varepsilon'(t)|\nabla u|_2^2 - \frac{1}{2}\left(1 - \frac{\mu^2(0)\beta}{2}\right)\int_0^\infty \mu'(s)|\nabla \eta^t(s)|_2^2 ds \\ &+ (1 - 3\beta)|\nabla u|_2^2 + \frac{\lambda - \alpha}{2}|u|_2^2 + \frac{\beta}{4\theta^2}|\nabla \eta^t|_{\mu,2}^2 + \int_{\mathbb{R}^n} f_1(x, u)u. \end{aligned}$$

Then,

$$
\begin{aligned}
\tilde{B}_\beta(t) \geq &(1-3\beta)|\nabla u|_2^2 + \frac{\lambda-\alpha}{2}|u|_2^2 + \frac{\beta}{4\theta^2}|\nabla\eta^t|_{\mu,2}^2 + \int_{\mathbb{R}^n} f_1(x,u)u \\
\geq &(1-3\beta)|\nabla u|_2^2 + \frac{\lambda-\alpha}{2}|u|_2^2 + \frac{\beta}{4\theta^2}|\nabla\eta^t|_{\mu,2}^2 + \beta\int_{\mathbb{R}^n} (f_1(x,u)u + \varphi(x)) - |\varphi|_1 \\
\geq &(1-3\beta)|\nabla u|_2^2 + \frac{\lambda-\alpha}{2}|u|_2^2 + \frac{\beta}{4\theta^2}|\nabla\eta^t|_{\mu,2}^2 + \beta\mathcal{F}(u) + \beta\int_{\mathbb{R}^n} \Big(f_0(x)u + (\alpha-l)u^2 + \varphi(x)\Big) - |\varphi|_1 \\
\geq &(1-3\beta)|\nabla u|_2^2 + \left(\frac{\lambda-\alpha}{2} - |\alpha-l|\beta - \frac{\beta^2}{4}\right)|u|_2^2 + \frac{\beta}{4\theta^2}|\nabla\eta^t|_{\mu,2}^2 + \beta\mathcal{F}(u) - C(|f_0|_2^2 + |\varphi|_1).
\end{aligned}
$$

Then, there is constant $\delta_2 > 0$, such that

$$
\tilde{B}_\beta(t) \geq \delta_2 B_\beta(t) - C(|f_0|_2^2 + |\varphi|_1). \tag{3.20}
$$

Thus, we can rewrite (3.17) as follows:

$$
\frac{d}{dt}B_\beta(t) + \delta_2 B_\beta(t) \leq C(|g|_2^2 + |f_0|_2^2 + |\varphi|_1).
$$

Applying Grönwall lemma, we have

$$
\begin{aligned}
&B_\beta(t) \leq B_\beta(t^\star)e^{-\delta_2(t-t^\star)} + C(|\varphi|_1 + |g|_2^2 + |f_0|_2^2) \\
\leq &\tilde{a}_1\Big(|u(t^\star)|_2^2 + |\nabla u(t^\star)|_2^2 + |\nabla\eta^{t^\star}|_{\mu,2}^2 + \mathcal{F}(u(t^\star))\Big)e^{-\delta_2(t-\tau)} + C(|\varphi|_1 + |g|_2^2 + |f_0|_2^2) \\
\leq &Q(R)e^{-\delta_2(t-\tau)} + C(|\varphi|_1 + |g|_2^2 + |f_0|_2^2)
\end{aligned} \tag{3.21}
$$

holds for any $\tau < t$. Then the proof is complete.

3.3. Time-dependent global attractors

In subsection, we will prove the existence of time-dependent global attractors in $\{\mathcal{M}_t\}_{t\in\mathbb{R}}$ through the process $U(t,\tau)$ defined by (3.3). In order to prove Theorem 3.1, we first give the following lemmas.

Lemma 3.4. *For any $R > 0$ and $z_\tau = (u_\tau, \eta^\tau) \in \mathscr{B}_\tau(R) \subset \mathcal{M}_\tau$, then there exists a positive constant $\mathcal{K}_1 = \mathcal{K}_1(R)$, such that*

$$
\int_t^{t+1} (|u_t(s)|_2^2 + \varepsilon(s)|\nabla u_t(s)|_2^2)ds \leq \mathcal{K}_1
$$

holds for any $t \geq \tau$.

Proof. From (3.19) and the value of β, one gets

$$
\frac{d}{dt}B_\beta(t) + \frac{1}{2}|u_t|_2^2 + \frac{\varepsilon(t)}{2}|\nabla u_t|_2^2 \leq C(|\varphi|_1 + |g|_2^2 + |f_0|_2^2). \tag{3.22}
$$

Integrating t from t to $t+1$ on both sides of (3.22), and organizing it to obtain

$$
\begin{aligned}
&\int_t^{t+1}\left(|u_t(s)|_2^2+\varepsilon(s)|\nabla u_t(s)|_2^2\right)ds \le B_\beta(t)+C(|\varphi|_1+|g|_2^2+|f_0|_2^2)\\
\le& Q(R)e^{-\delta_2(t-\tau)}+C(|\varphi|_1+|g|_2^2+|f_0|_2^2)+a_1\beta|f_0|_2^2\\
\le& Q(R)+C(|\varphi|_1+|g|_2^2+|f_0|_2^2)+a_1\beta|f_0|_2^2
\end{aligned}
$$

holds for any $\tau < t$. Let $\mathcal{K}_1 = Q(R) + C(|\varphi|_1 + |g|_2^2 + |f_0|_2^2) + a_1\beta|f_0|_2^2$, then the proof is complete.

Lemma 3.5. *Let B be any bounded subset $\mathcal{M}_\tau$ and $z_\tau \in B$. Then for any $\varepsilon > 0$, there exist positive constants K_1 large enough and $t_1 \le t$, such that*

$$
\int_{B_k^c}\left(|u|^2+\varepsilon(t)|\nabla u|^2\right)+\int_0^\infty \mu(s)\int_{B_k^c}|\nabla\eta^t(s)|^2 ds \le C\varepsilon
$$

holds for every $\tau \le t_1 \le t$ and $k \ge K_1$, where $B_k^c = \{x \in \mathbb{R}^n : |x| \ge k\}$.

Proof. Let $\theta(\cdot) \in C^\infty(\mathbb{R}^+)$ satisfies

$$
0 \le \theta(s) \le 1, \quad 0 \le \theta'(s) \le b, \ \forall s \in \mathbb{R}^+,
$$

and

$$
\theta(s) = 0, \ \forall s \in [0,1), \ \ \theta(s) = 1, \ \forall s \in [2,\infty),
$$

where $b > 0$ is a constant. Setting $\theta_k = \theta(\frac{2|x|^2}{k^2})$ and multiplying the first equation of (1.14) by $\theta_k^2 u$ in $L^2(\mathbb{R}^n)$, then we obtain

$$
\begin{aligned}
&\frac{d}{dt}E_k(t)-\frac{\varepsilon'(t)}{2}|\theta_k\nabla u|_2^2-\frac{1}{2}\int_0^\infty \mu'(s)|\theta_k\nabla\eta^t(s)|_2^2 ds+|\theta_k\nabla u|_2^2+\frac{\lambda-\alpha}{2}|\theta_k u|_2^2\\
\le&-\frac{8}{k^2}\int_0^\infty \mu(s)\int_{\mathbb{R}^n}\theta_k\theta_k' u(x\cdot\nabla\eta^t(s))ds-\frac{8\varepsilon(t)}{k^2}\int_{\mathbb{R}^n}\theta_k\theta_k' u(x\cdot\nabla u_t)\\
&-\frac{8}{k^2}\int_{\mathbb{R}^n}\theta_k\theta_k' u(x\cdot\nabla u)+\frac{1}{2(\lambda-\alpha)}|\theta_k g|_2^2-\int_{\mathbb{R}^n}\theta_k^2 f_1(x,u)u, \qquad (3.23)
\end{aligned}
$$

where $E_k(t) = \frac{1}{2}\left[|\theta_k u|_2^2 + \varepsilon(t)|\theta_k\nabla u|_2^2 + |\theta_k\nabla\eta^t|_{\mu,2}^2\right]$. Similar to (3.7), we define the function $N_k(t)$ as follows:

$$
N_k(t) = \int_0^\infty k(s)\left(\theta_k\nabla\eta^t, \theta_k\nabla\eta^t\right)ds. \qquad (3.24)
$$

Then it is follows that

$$
\frac{\mathrm{d}}{\mathrm{d}t}N_k(t)+\frac{1}{2}|\theta_k\nabla\eta^t|_2^2 \le 2\theta^2|\theta_k\nabla u|_2^2. \qquad (3.25)
$$

We can define a functional with a undetermined coefficient κ as follows:

$$
H_k(t) = E_k(t) + \frac{\kappa}{2}N_k(t). \qquad (3.26)
$$

Let $0 < \kappa < \frac{1}{2\theta^2}$ be sufficiently small, then it follows that

$$
H_k(t) \ge \frac{1}{2}\left(|\theta_k u|_2^2+\varepsilon(t)|\theta_k\nabla u|_2^2+|\theta_k\nabla\eta^t|_{\mu,2}^2\right), \qquad (3.27)
$$

and

$$H_k(t) \le \frac{1+\theta\kappa}{2}\left(|\theta_k u|_2^2 + \varepsilon(t)|\theta_k \nabla u|_2^2 + |\theta_k \nabla \eta^t|_{\mu,2}^2\right). \tag{3.28}$$

Combining with (3.23), (3.25) and (3.26), we obtain

$$\begin{aligned}
&\frac{d}{dt}H_k(t) + \frac{\lambda-\alpha}{2}|\theta_k u|_2^2 + \frac{1}{2}(1-\theta^2\kappa)|\theta_k \nabla u|_2^2 + \frac{\kappa}{4}|\theta_k \nabla \eta^t|_{\mu,2}^2 \\
\le& -\frac{8}{k^2}\int_0^\infty \mu(s)\int_{\mathbb{R}^n} \theta_k\theta_k' u(x\cdot\nabla\eta^t(s))ds - \frac{8\varepsilon(t)}{k^2}\int_{\mathbb{R}^n}\theta_k\theta_k' u(x\cdot\nabla u_t) \\
&+ \frac{64}{k^4}\int_{\mathbb{R}^n}(\theta_k')^2|u|^2|x|^2 + \frac{1}{2(\lambda-\alpha)}|\theta_k g|_2^2 - \int_{\mathbb{R}^n}\theta_k^2 f_1(x,u)u.
\end{aligned} \tag{3.29}$$

From (1.9), there is

$$\begin{aligned}
-\int_{\mathbb{R}^n}\theta_k^2 f_1(x,u)u =& -\int_{\mathbb{R}^n}\theta_k^2\left(f_1(x,u)u + \varphi(x)\right) + \int_{\mathbb{R}^n}\theta_k^2\varphi(x) \\
\le& -\int_{\mathbb{R}^n}\theta_k^2\left(f_1(x,u)u + \varphi(x)\right) + |\theta_k\varphi|_1 \\
\le& |\theta_k\varphi|_1.
\end{aligned}$$

Let $\delta_3 = \min\{\lambda-\alpha, \frac{1-\theta^2\kappa}{L}, \frac{\kappa}{2}\}$, then we have that

$$\begin{aligned}
&\frac{d}{dt}H_k(t) + \frac{\delta_3}{2}\left(|\theta_k u|_2^2 + \varepsilon(t)|\theta_k \nabla u|_2^2 + |\theta_k \nabla \eta^t|_{\mu,2}^2\right) \\
\le& -\frac{8}{k^2}\int_0^\infty \mu(s)\int_{\mathbb{R}^n} \theta_k\theta_k' u(x\cdot\nabla\eta^t(s))ds - \frac{8\varepsilon(t)}{k^2}\int_{\mathbb{R}^n}\theta_k\theta_k' u(x\cdot\nabla u_t) \\
&- \frac{8}{k^2}\int_{\mathbb{R}^n}\theta_k\theta_k' u(x\cdot\nabla u) + \frac{1}{2(\lambda-\alpha)}|\theta_k g|_2^2 + |\theta_k\varphi|_1.
\end{aligned} \tag{3.30}$$

Next, we will estimate each item on the right side of (3.30). According to the definition of θ, it can be seen that

$$\theta_k' = 0, \quad |x| < \frac{k}{2} \text{ or } |x| > k.$$

Therefore,

$$\begin{aligned}
\frac{64}{k^4}\int_{\mathbb{R}^n}(\theta_k')^2|u|^2|x|^2 =& \frac{64}{k^4}\int_{\frac{k}{2}\le|x|\le k}(\theta_k')2|u|^2|x|^2 \\
\le& \frac{64}{k^2}\int_{\frac{k}{2}\le|x|\le k} b^2|u|^2 \\
\le& \frac{C}{k^2}|u|_2^2.
\end{aligned}$$

Similarly, we also can obtain the following estimates:

$$-\frac{8}{k^2}\int_0^\infty \mu(s)\int_{\mathbb{R}^n}\theta_k\theta_k' u(x\cdot\nabla\eta^t(s))ds \le \frac{C}{k}\left(|u|_2^2 + |\nabla\eta^t|_{\mu,2}^2\right);$$

$$-\frac{8\varepsilon(t)}{k^2}\int_{\mathbb{R}^n}\theta_k\theta_k' u(x\cdot\nabla u_t(s))\le\frac{C}{k}\left(|u|_2^2+\varepsilon(t)|\nabla u_t|_2^2\right).$$

Combining with (3.28), then (3.30) can be rewritten as follows:

$$\begin{aligned}&\frac{d}{dt}H_k(t)+\delta_2 H_k(t)\\ \le&\frac{C}{k}\left(|u|_2^2+|\nabla\eta^t|_{\mu,2}^2+\varepsilon(t)|\nabla u_t|_2^2\right)+C\left(|\theta_k g|_2^2+|\theta_k\varphi|_1\right).\end{aligned} \tag{3.31}$$

By Grönwall lemma, it follows that

$$\begin{aligned}&H_k(t)\\ \le&H_k(\tau)e^{-\delta_3(t-\tau)}+C\left(|\theta_k g|_2^2+|\theta_k f_0|_2^2+|\theta_k\varphi|_1\right)+\frac{C}{k}e^{-\delta_3 t}\int_\tau^t e^{\delta_2 s}\left(|u(s)|_2^2+|\nabla\eta^s|_{\mu,2}^2+\varepsilon(s)|\nabla u_t(s)|_2^2\right)ds.\end{aligned}$$

Combining with Lemma 3.3, we have

$$H_k(t)\le Q(R)e^{-\delta_3(t-\tau)}+C\left(|\theta_k g|_2^2+|\theta_k f_0|_2^2+|\theta_k\varphi|_1\right)+\frac{C}{k}\mathcal{K}_0,$$

and for any $\varepsilon>0$, there exists $K_1\ge 0$ large enough, such that for every $k\ge K_1$,

$$\begin{aligned}&\left(|\theta_k g|_2^2+|\theta_k f_0|_2^2+|\theta_k\varphi|_1\right)=\int_{B_k^c}\left(|\theta_k g|^2+|\theta_k f_0|^2+|\theta_k\varphi|\right)<C\varepsilon;\\ &\frac{C}{k}\mathcal{K}_0<C\varepsilon,\end{aligned}$$

and let $t_1=t-\frac{1}{\delta_3}\ln\frac{Q(R)}{\varepsilon}$, for every $\tau\le t_1\le t$, it follows that

$$Q(R)e^{-\delta_3(t-\tau)}<C\varepsilon.$$

By (3.27), then,

$$|\theta_k u|_2^2+\varepsilon(t)|\theta_k\nabla u|_2^2+|\theta_k\nabla\eta^t|_{\mu,2}^2+\varepsilon(t)\mathcal{F}_1(\theta_k u)\le C\varepsilon$$

holds for every $\tau\le t_1\le t$ and $k\ge K_1$. So we have that

$$\int_{B_k^c}\left(|u|^2+\varepsilon(t)|\nabla u|^2\right)+\int_0^\infty\mu(s)\int_{B_k^c}|\nabla\eta^t(s)|^2 ds\le C\varepsilon.$$

In order to obtain the asymptotic regularity estimates later, we decompose the solution $U(t,\tau)z_\tau=(u(t),\eta^t)$ into the following sum:

$$U(t,\tau)z_\tau=U_1(t,\tau)z_\tau+K(t,\tau)z_\tau, \tag{3.32}$$

where $U_1(t,\tau)z_\tau=(v(t),\xi^t)$ and $K(t,\tau)z_\tau=(\omega(t),\zeta^t)$ solve the following equations respectively:

$$\begin{cases}v_t-\varepsilon(t)\Delta v_t-\Delta v-\int_0^\infty\mu(s)\Delta\xi^t(s)ds+f(x,u)-f(x,\omega)+(\lambda+l)v=0,\\ \xi_t^t=-\xi_s^t+v,\end{cases} \tag{3.33}$$

where l is a constant from (1.5). It has an initial data

$$v(x,\tau) = u_\tau(x),\ \xi^\tau(x,s) = \int_0^s u_\tau(x,\tau-\theta)d\theta, \quad (x,s) \in \mathbb{R}^n \times \mathbb{R}^+. \tag{3.34}$$

And

$$\begin{cases} \omega_t - \varepsilon(t)\Delta\omega_t - \Delta\omega - \int_0^\infty \mu(s)\Delta\zeta^t(s)ds + f(x,\omega) + (\lambda+l)\omega = lu + g, \\ \zeta_t^t = -\zeta_s^t + \omega, \end{cases} \tag{3.35}$$

with the initial value conditions

$$\omega(x,\tau) = 0,\ \zeta^\tau(x,s) = 0, \quad (x,s) \in \mathbb{R}^n \times \mathbb{R}^+. \tag{3.36}$$

Lemma 3.6. *Assume that $K(t,\tau)z_\tau = (\omega(t),\zeta^t)$ is the solution of Eq* (3.35) *with* (3.36)*. Then there exist positive constants $k_i(i=3,4)$, such that*

$$|\omega(t)|_2^2 + |\nabla\omega(t)|_2^2 + \|\zeta^t\|_{\mu,0}^2 + \mathcal{F}(\omega(t)) \le k_3$$

and

$$\int_t^{t+1} \left(|\omega_t(s)|_2^2 + \varepsilon(s)|\nabla\omega_t(s)|_2^2\right) ds \le k_4$$

hold for any $t-\tau>0$.

Proof. This proof can imitate the proof of Lemma 3.2, Corollary 3.3 and Lemma 3.4 word by word, it should be noted that (3.17), (3.21) and $\|(\omega(\tau),\zeta^\tau)\|_{\mathcal{M}_\tau} = 0$. Then hence is omitted.

Lemma 3.7. *Assume that $U_1(t,\tau)z_\tau = (v(t),\xi^t)$ is the solution of Eq (3.33) with (3.34). Then,*

$$\lim_{\tau\to-\infty}\left(\|U_1(t,\tau)z_\tau\|_{\mathcal{M}_t}^2\right) = 0$$

holds for every $t \in \mathbb{R}^\tau$ fixed.

Proof. Multiplying the first equation of (3.33) by $v(t)$ and integrating in $L^2(\mathbb{R}^n)$ and the second one by ξ^t in $\mathcal{V}$, we have

$$\frac{1}{2}\frac{d}{dt}\left(|v|_2^2 + \varepsilon(t)|\nabla v|_2^2 + |\nabla\xi^t|_{\mu,2}^2\right) + \frac{1}{2}\int_0^\infty -\mu'(s)|\nabla\xi^t(s)|_2^2 ds - \varepsilon'(t)|\nabla v|_2^2 + |\nabla v|_2^2 + \lambda|v|_2^2 \le 0, \tag{3.37}$$

and

$$\frac{1}{2}\frac{d}{dt}\int_0^\infty \mu(s)|\xi^t(s)|_2^2 ds - \frac{1}{2}\int_0^\infty \mu'(s)|\xi^t(s)|_2^2 ds \le |v|_2|\xi^t|_{\mu,2}. \tag{3.38}$$

Furthermore,

$$\frac{1}{2}\frac{d}{dt}\int_0^\infty k(s)|\xi^t(s)|_2^2 ds + \frac{1}{4}|\xi^t(s)|_{\mu,2}^2 \le \theta^2|v|_2^2. \tag{3.39}$$

Similar to (3.7), we introduce the functional

$$N_1(t) = \int_0^\infty k(s)(\nabla\xi^t(s), \nabla\xi^t(s))ds.$$

Then,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}N_1(t) + \frac{1}{4}|\nabla\xi(t)|^2_{\mu,2} \le \theta^2|\nabla v(t)|_2^2. \tag{3.40}$$

Setting up

$$\gamma = \frac{1}{2(2+\theta^2)}. \tag{3.41}$$

Furthermore, we define the energy-like functional

$$D_\gamma(t,\tau) = \frac{1}{2}\left(|v|_2^2 + \varepsilon(t)|\nabla v|_2^2 + |\nabla\xi^t|^2_{\mu,2} + \gamma|\xi^t|^2_{\mu,2} + \gamma\int_0^\infty k(s)|\xi^t(s)|_2^2 ds + \gamma N_1(t)\right). \tag{3.42}$$

Then we get

$$D_\gamma(t,\tau) \le a_3\left(|v|_2^2 + \varepsilon(t)|\nabla v|_2^2 + \|\xi^t\|^2_{\mu,0}\right) = a_3\|U_1(t,\tau)z_\tau\|^2_{\mathcal{M}_t}, \tag{3.43}$$

and

$$D_\gamma(t,\tau) \ge \tilde{a}_3\left(|v|_2^2 + \varepsilon(t)|\nabla v|_2^2 + \|\xi^t\|^2_{\mu,0}\right) = \tilde{a}_3\|U_1(t,\tau)z_\tau\|^2_{\mathcal{M}_t}, \tag{3.44}$$

where $a_3 = \frac{1+\gamma\theta}{2}$ and $\tilde{a}_3 = \frac{\gamma}{2(1+\gamma)} > 0$.

Therefore, it follows that

$$\begin{aligned} &\frac{d}{dt}D_\gamma(t,\tau) - \frac{1}{2}\int_0^\infty \mu'(s)|\nabla\xi^t(s)|_2^2 ds - \varepsilon'(t)|\nabla v|_2^2 + \frac{\gamma}{4}|\nabla\xi^t|^2_{\mu,2} + \frac{\gamma}{8}|\xi^t|^2_{\mu,2} \\ &+ (\lambda - 2\gamma - \gamma\theta^2)|v|_2^2 + \frac{1-\gamma\theta^2}{L}\varepsilon(t)|\nabla v|_2^2 \le 0. \end{aligned} \tag{3.45}$$

By (3.41), we have

$$\lambda - (2+\theta^2)\gamma = \frac{1}{2},\ 1 - \gamma\theta^2 > \frac{1}{2}.$$

Let $\delta_4 = \frac{1}{4a_3}\min\{\frac{\gamma}{8}, \lambda - 2\gamma - \gamma\theta^2, \frac{1-\gamma\theta^2}{L}\}$, then (3.45) can be rewritten as

$$\frac{d}{dt}D_\gamma(t,\tau) + \delta_4 D_\gamma(t,\tau) \le 0. \tag{3.46}$$

Applying Grönwall lemma, we get that

$$D_\gamma(t,\tau) \le D_\gamma(\tau,\tau)e^{-\delta_4(t-\tau)}.$$

Then, combining with (3.43) and (3.44), we have

$$\lim_{\tau\to-\infty}\|U_1(t,\tau)z_\tau\|^2_{\mathcal{M}_t} = 0$$

is true for any $t \in \mathbb{R}^\tau$.

Lemma 3.8. *There exists a positive constant $\mathcal{K}_3 = \mathcal{K}_3(R)$ which depends on t and τ, such that*

$$|\nabla\omega(t)|_2^2 + \varepsilon(t)|\triangle\omega(t)|_2^2 + \|\zeta^t\|^2_{\mu,1} \le \mathcal{K}_3 \tag{3.47}$$

holds for any $\tau \le t \in \mathbb{R}$.

Proof. Acting on the first equation of (3.35) by $-\Delta\omega(t)$ on $L^2(\mathbb{R}^n)$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(|\nabla\omega|_2^2+\varepsilon(t)|\Delta\omega|_2^2+|\Delta\zeta^t|_{\mu,2}^2\right)-\frac{\varepsilon'(t)}{2}|\Delta\omega|_2^2-\frac{1}{2}\int_0^\infty\mu'(s)|\Delta\zeta^t(s)|_2^2ds\\&+\lambda|\nabla\omega|_2^2+\frac{1}{2}|\Delta\omega|_2^2\le l^2|u|_2^2+|g|_2^2.\end{aligned}\tag{3.48}$$

Similar to (3.7), we introduce the functional

$$N_2(t)=\int_0^\infty k(s)(\Delta\zeta^t(s),\Delta\zeta^t(s))ds.$$

Then,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}N_2(t)+\frac{1}{4}|\Delta\zeta^t|_{\mu,2}^2\le\theta^2|\Delta\omega|_2^2.\tag{3.49}$$

Therefore, from (3.48) and (3.49), it is that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(|\nabla\omega|_2^2+\varepsilon(t)|\Delta\omega|_2^2+|\Delta\zeta^t|_{\mu,2}^2+\frac{1}{4\theta^2}N_2(t)\right)-\frac{\varepsilon'(t)}{2}|\Delta\omega|_2^2-\frac{1}{2}\int_0^\infty\mu'(s)|\Delta\zeta^t(s)|_2^2ds\\&+\frac{1}{16\theta^2}|\Delta\zeta^t|_{\mu,2}^2+\lambda|\nabla\omega|_2^2+\frac{1}{4}|\Delta\omega|_2^2\le l^2|u|_2^2+|g|_2^2.\end{aligned}\tag{3.50}$$

By Corollary 3.2 and the Grönwall lemma, we have

$$|\nabla\omega(t)|_2^2+\varepsilon(t)|\Delta\omega(t)|_2^2+|\Delta\zeta^t|_{\mu,2}^2\le C\left(\mathcal{K}_0+|g|_2^2\right)$$

holds for any $\tau\le t\in\mathbb{R}$. Let

$$\mathcal{K}_3=C(\mathcal{K}_0+|g|_2^2),$$

then it follows that

$$|\nabla\omega(t)|_2^2+\varepsilon(t)|\Delta\omega(s)|_2^2+|\Delta\zeta^t|_{\mu,2}^2\le\mathcal{K}_3$$

holds for any $\tau\le t\in\mathbb{R}$.

Next, we will verify the existence and regularity of the pullback global attractors $\tilde{\mathscr{A}}$ for Eq (1.1).

Theorem 3.1. *The family of process $U(t,\tau)$ for Eq* (1.14) *with initial conditions* (1.15) *is $\mathcal{M}_t$-contractive process on $B_T\in\tilde{B}^0$ (from (3.13)).*

Proof. Let $z_i(t)=(u_i(t),\xi_i^t)=U(t,\tau)z_\tau^i$ $(i=1,2)$ be the solutions to Eq (1.14) with the parameter $\varepsilon(t)$ and initial data $z_\tau^i\in B_\tau\in\tilde{B}^0$ $(i=1,2)$ ($\tilde{B}^0$ is from (3.13)) respectively.

By (3.32), there is

$$z_i(t)=U(t,\tau)z_\tau^i=U_1(t,\tau)z_\tau^i+K(t)z_\tau^i=(v_i(t),\xi_i^t)+(\omega_i(t),\zeta_i^t).$$

It gets

$$\begin{aligned}&\|U(t,\tau)z_\tau^1-U(t,\tau)z_\tau^2\|_{\mathcal{M}_t}^2\\\le&2\|U_1(t,\tau)z_\tau^1-U_1(t,\tau)z_\tau^2\|_{\mathcal{M}_t}^2+2\left(|\omega_1(t)-\omega_2(t)|_2^2+\varepsilon(t)|\nabla\omega_1(t)-\nabla\omega_2(t)|_2^2+\|\zeta_1^t-\zeta_2^t\|_{\mu,0}^2\right),\end{aligned}\tag{3.51}$$

and

$$\lim_{\tau\to-\infty}\|U_1(t,\tau)z_\tau^1-U_1(t,\tau)z_\tau^2\|_{\mathcal{M}_t}^2\leq 2\lim_{\tau\to-\infty}\left(\|U_1(t,\tau)z_\tau^1\|_{\mathcal{M}_t}^2+\|U_1(t,\tau)z_\tau^2\|_{\mathcal{M}_t}^2\right)=0.$$

Then for any $\varepsilon>0$, there is $T=T(\varepsilon)\leq t$ such that

$$2\|U(t,T)z_T^1-U(t,T)z_T^2\|_{\mathcal{M}_t}^2<\varepsilon \tag{3.52}$$

holds for any fixed $t\geq T$.

Let $(\varpi(t),\varsigma^t)=(\omega_1(t)-\omega_2(t),\zeta_1^t-\zeta_2^t)$ be the solution of the following system:

$$\begin{cases}\varpi_t-\varepsilon(t)\Delta\varpi_t-\Delta\varpi-\int_0^\infty\mu(s)\Delta\varsigma^t(s)ds+f(x,\omega_1)-f(x,\omega_2)+(\lambda+l)\varpi=l(u_1-u_2),\\ \varsigma_t^t=-\varsigma_s^t+\varpi.\end{cases}$$

It has an initial value conditions

$$\varpi(x,\tau)=0,\ \varsigma^\tau(x,s)=0,\quad (x,s)\in\mathbb{R}^n\times\mathbb{R}^+.$$

Next, we also define the functional

$$\Gamma(t)=\frac{1}{2}\left(|\varpi(t)|_2^2+\varepsilon(t)|\nabla\varpi(t)|_2^2+|\nabla\varsigma^t|_{\mu,2}^2+\frac{1}{2\theta^2}|\nabla\varsigma^t|_{k,2}^2+\frac{\lambda}{2(1+\theta^2)}\left(|\varsigma^t|_{\mu,2}^2+|\varsigma^t|_{k,2}^2\right)\right).$$

Then,

$$\begin{aligned}\Gamma(t)&\leq a_4\left(|\varpi(t)|_2^2+\varepsilon(t)|\nabla\varpi(t)|_2^2+\|\varsigma^t\|_{\mu,0}^2\right),\\ \Gamma(t)&\geq \tilde{a}_4\left(|\varpi(t)|_2^2+\varepsilon(t)|\nabla\varpi(t)|_2^2+\|\varsigma^t\|_{\mu,0}^2\right),\end{aligned}$$

where $a_4=\frac{1}{2}\max\{1+\frac{1}{2\theta},\frac{\lambda(1+\theta)}{2(1+\theta^2)}\}$ and $\tilde{a}_4=\frac{1}{2}\min\{1,\frac{\lambda}{2(1+\theta^2)}\}$.

Similar to (3.5)–(3.7), we get

$$\frac{1}{2}\frac{d}{dt}\left(|\varsigma^t|_{\mu,2}^2+|\varsigma^t|_{k,2}^2\right)+\frac{1}{4}|\varsigma^t|_{\mu,2}^2\leq 2(1+\theta^2)|\varpi|_2^2, \tag{3.53}$$

and

$$\frac{1}{2}\frac{d}{dt}\int_0^\infty k(s)(\nabla\varsigma^t(s),\nabla\varsigma^t(s))+\frac{1}{4}|\nabla\varsigma^t|_{\mu,2}^2\leq\theta^2|\nabla\varpi|_2^2.$$

Then it yields

$$\begin{aligned}&\frac{d}{dt}\Gamma(t)-\frac{1}{2}\int_0^\infty\mu'(s)|\nabla\varsigma^t(s)|_2^2ds-\frac{\varepsilon'(t)}{2}|\nabla\varpi|_2^2+\frac{1}{2}|\nabla\varpi(t)|_2^2\\ &+\frac{\lambda}{4}|\varpi(t)|_2^2+\frac{1}{8\theta^2}|\nabla\varsigma^t|_{\mu,2}^2+\frac{\lambda}{16(1+\theta^2)}|\varsigma^t|_{\mu,2}^2\\ \leq&\frac{l^2}{\lambda}\left(|u_1(t)-u_2(t)|_0^2\right).\end{aligned} \tag{3.54}$$

Let $\delta_5 = \frac{1}{2a_4}\min\{\frac{1}{L}, \frac{1}{4\theta^2}, \frac{\lambda}{8(1+\theta^2)}\}$, then it follows that

$$\frac{d}{dt}\Gamma(t) + \delta_5\Gamma(t) \le \frac{l^2}{\lambda}|u_1(t) - u_2(t)|_0^2.$$

So, we get

$$|\varpi(t)|_2^2 + \varepsilon(t)|\nabla\varpi(t)|_2^2 + \|\varsigma^t\|_{\mu,0}^2 \le \frac{l^2}{\tilde{a}_4\lambda}e^{-\delta_5 t}\int_T^t e^{\delta_5 s}|u_1(s) - u_2(s)|_0^2 ds. \tag{3.55}$$

And

$$\begin{aligned}
&e^{-\delta_5 t}\int_T^t e^{\delta_5 s}|u_1(s) - u_2(s)|_0^2 ds \\
\le &\int_T^t \int_{B_k} |u_1(s) - u_2(s)|^2 ds + e^{-\delta_5 t}\int_T^t e^{\delta_5 s}\int_{B_k^c} |u_1(s) - u_2(s)|^2 ds \\
\le &\int_T^t \int_{B_k} |u_1(s) - u_2(s)|^2 ds + C\varepsilon.
\end{aligned}$$

Then setting

$$\varphi_T^t(z_1, z_2) = C\int_T^t \int_{B_k} \left(|u_1(s) - u_2(s)|_2^2\right) ds. \tag{3.56}$$

Combining with Lemmas 3.3 and 3.4. Applying Lemma 2.2, then the sequences $\{u_n(s)\}_{n=1}^{\infty}$ is relatively compact in $L^2(T, t; L^2(B_k))$. In other words, for any sequences $\{z_n(T) = (u_n(T), \eta_n^T)\} \subset B_T \in \tilde{B}^0$, $\{z_n(t) = (u_n(t), \eta_n^t)\}$ is the solution of Eq (1.14) with the initial data $\{z_n(T) = (u_n(T), \eta_n^T)\}$ respectively. Then there exist subsequence $\{z_{n_k}\} \subset \{z_n\}$ satisfying

$$\lim_{k\to\infty}\lim_{l\to\infty}\varphi_T^t(z_{n_k}, z_{n_l}) = 0.$$

It follows that $\varphi_T^t \in \mathfrak{C}(B_T)$. Substituting (3.56) and (3.52) into (3.51), one gets

$$\|U(t,T)x - U(t,T)y\|_{\mathcal{M}_t}^2 \le C\varepsilon + \varphi_T^t(x, y).$$

By Definitions 2.8 and 2.9, then φ_T^t is contractive function in B_T. Therefore, it's easy to obtain that the process $U(t,\tau)$ is $\mathcal{M}_t$-contractive process on $B_T \in \tilde{B}^0$ (from (3.13)).

As the end of this article, we will deduce the main conclusion as the following theorem:

Theorem 3.2. *The process $U(t,\tau)$ defined by* (3.3) *possesses a time-dependent global attractor $\tilde{\mathscr{A}}$ in $\{\mathcal{M}_t\}_{t\in\mathbb{R}}$, and $\tilde{\mathscr{A}}$ is non-empty, compact, invariant in $\{\mathcal{M}_t\}_{t\in\mathbb{R}}$ and pullback attracting in $\{\mathcal{M}_t\}_{t\in\mathbb{R}}$. Furthermore,*

$$\tilde{\mathscr{A}} \subset \{\mathcal{M}_t^1\}_{t\in\mathbb{R}}.$$

Proof. Thanks to Lemma 3.2 and Theorem 3.1, it's easy to get the existence of time-dependent global attractor $\tilde{\mathscr{A}}$ for the process $U(t,\tau)$ defined by (3.3) in time-dependent spaces $\{\mathcal{M}_t\}_{t\in\mathbb{R}}$. According to Lemmas 3.7 and 3.8, the pullback asymptotic regularity of the solutions of Eq (1.1) is proved, and the regularity of the time-dependent global attractor $\tilde{\mathscr{A}}$ is obtained. By Lemma 2.1 and (3.2), it follows that the invariance of time-dependent global attractor $\tilde{\mathscr{A}}$.

4. Conclusions

We conclude the existence, uniqueness and regularity of time-dependent global attractors on whole space. The findings of this study can be considered as a supplement to our previous works, such as [4, 5]. We overcome some essential difficulties for studying this kind of problem, including that the compact embedding is no longer valid under the case of unbounded domain and the nonlinear term fulfills the supcritical growth as well as the memory kernel satisfying more general assumptions. However, our results show that the method of operator decomposition that was proposed in [46] is available for dealing with the case of unbounded domain like (1.1).

Unfortunately, we fail to consider the existence of time-dependent global attractors for Eq (1.1) which lacks instantaneous damping on whole space, and further study the upper-semicontinuity of time-dependent global attractors between two kinds of equations. Future studies shall consider such issues using the ideas of the paper and [4, 5], i.e., the asymptotic behavior of solutions for nonautonomous and autonomous equations (1.1) lacking instantaneous damping on whole space.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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