Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(12): 30527–30536.
DOI: 10.3934/math.20231560
Received: 01 September 2023
Revised: 01 November 2023
Accepted: 07 November 2023
Published: 10 November 2023

Research article

Dual Leonardo numbers

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Abstract: This paper introduced the concept of dual Leonardo numbers to generalize the earlier studies in harmony and establish key formulas, including the Binet formula and the generating function. Both were employed to obtain specific elements from the sequence. Moreover, we presented a range of identities that provided deeper insights into the relationships within this numerical family, such as the Cassini and d'Ocagne identities, along with various summation formulas.

Keywords: Leonardo numbers; Fibonacci numbers; dual numbers; identities; summation formulas
Mathematics Subject Classification: 11B39, 11R52

1. Introduction

In this paper, we introduce the novel concept of dual Leonardo numbers, an extension of the Leonardo numbers defined in 2019 [6]. Drawing inspiration from the extensive history and applications of sequences akin to Fibonacci numbers, this new area will have applications in various domains, including the number theory, graph theory and error correction codes, much like the established applications of Fibonacci numbers.

Our contributions encompass the derivation of the Binet formula and the specific generating function tailored to the dual Leonardo numbers. These fundamental equations are crucial for the proofs of the theorems and finding the needed element of the sequence. Moreover, we delve into intricate mathematical identities such as Cassini and d'Ocagne's identities, which offer a profound comprehension of the interrelations within this numerical domain. Additionally, we unveil several enlightening summation formulas for these numbers.

We begin by introducing the general information and fundamental properties of dual numbers. Dual numbers were originally defined by Clifford and have since been extensively studied from various perspectives. They find applications in diverse fields including mathematics, engineering and physics. To start, we offer a concise definition of dual numbers, along with some key attributes that will be utilized in this paper.

The dual numbers are defined as

$$\mathbb{D} = \{(k, l) = k + l\varepsilon \mid k, l \in \mathbb{R} \text{ with } \varepsilon^2 = 0\}.$$

The addition operation is component wise. The multiplication operation uses the property of dual unit as

$$(k + l\varepsilon)(k' + l'\varepsilon) = k^2 + (kl' + lk')\varepsilon.$$

The conjugate of a dual number is defined as

$$\overline{(k, l)} = \overline{k + l\varepsilon} = (k, -l) = k - l\varepsilon.$$

The norm is defined with the help of the conjugate as

$$N(k + l\varepsilon) = (k + l\varepsilon)\overline{(k + l\varepsilon)} = k^2.$$

It is clear that the norm of the dual numbers is degenerate. Because of this property, dual numbers are used to describe the Galilean plane. This non-euclidean geometry is used in Galilean geometry with the following matrix representation [17]. The matrix representation of dual numbers is

$$k + l\varepsilon \to \begin{bmatrix} k & l \\ 0 & k \end{bmatrix}.$$

This mapping is an isomorphism because the matrix addition and multiplication correspond to the operations of addition and multiplication in the dual number system.

The dual number system that we give basics of is used in various areas in geometry [8, 10, 14, 22].

The rest of this paper is structured as follows: In Section 2, we leverage the definition of Leonardo numbers, review pertinent literature, and establish properties that will be utilized throughout the paper. Section 3 is dedicated to the definition and exploration of dual Leonardo numbers, accompanied by a concise overview of previous research. Additionally, we derive significant identities including the Cassini identity, as well as equations involving these numbers. Finally, in Section 4, we provide suggestions for future research and offer a summary of the paper.

2. Leonardo numbers

In 2019, Leonardo numbers were defined by Catarino and Borges [6]. The authors studied the Binet formula and the generating function, which are equations to obtain the desired element of the sequence. Also, they gave some identities and sum formulas, including both Fibonacci and Lucas numbers.

The authors named their study after Leonardo Fibonacci, who was a trader in the city of Pisa. Being a trader, Leonardo was capable of doing calculations in both Roman and Indo-Arabic numeration systems. He decided to write a book about the superiority of the Indo-Arabic system, which was the Liber Abaci. He also added some problems in his book and one of them was the rabbit problem, which generates the Fibonacci sequence [15, 18].

The famous Fibonacci sequence has initial values $f_0 = 0$ and $f_1 = 1$, and for $n \geq 0$ recurrence relation is

$$f_{n+2} = f_{n+1} + f_n. \tag{2.1}$$

This sequence holds a prominent position in number theory, having been the subject of extensive study, as elaborated in [11, 13, 15]. Moreover, the Fibonacci sequence is popular in many different fields, and some of them can be stated in [1,4,19].

To recall the definition of the authors in [6], we restate some fundamental equations and give some of them without proofs.

First, we present the definition of the Leonardo sequence from [6].

Definition 2.1. For $n \geq 2$, the nth Leonardo number is defined by

$$Le_n = Le_{n-1} + Le_{n-2} + 1 \tag{2.2}$$

with initial values $Le_0 = Le_1 = 1$.

Also, using Eq (2.2), the following relation can be written

$$Le_{n+1} = 2Le_n - Le_{n-2}, \tag{2.3}$$

where $n \geq 2$.

Second, the Binet formula for nth Leonardo numbers is

$$Le_n = \frac{\alpha(2\alpha^n - 1) - \beta(2\beta^n - 1)}{\alpha - \beta}, \tag{2.4}$$

where

$$\alpha = \frac{1+\sqrt{5}}{2},\ \beta = \frac{1-\sqrt{5}}{2} \text{ and } n \geq 0.$$

Third, the generating function for the Leonardo numbers is

$$g(t) = \frac{1 - t + t^2}{1 - 2t + t^3} \tag{2.5}$$

for $1 - 2t + t^3$ is not equal to zero.

In the following identities for Leonardo numbers, we encounter both Fibonacci and Lucas numbers. Because of their similarities in definition, Leonardo numbers are closely linked with the Fibonacci and Lucas numbers. For $n \geq 0$,

$$Le_n = 2F_{n+1} - 1, \tag{2.6}$$

$$Le_n = 2\frac{L_n + L_{n+2}}{5} - 1 \tag{2.7}$$

and

$$Le_n = L_{n+2} - F_{n+2} - 1. \tag{2.8}$$

For proofs of these equations and more, see [2, 6].

We present a corollary for the Leonardo numbers, which we employ in the following section.

Corollary 2.2. *For nonnegative integer n, we have*

$$Le_n Le_{n+1} = -Le_{n+2} + 4\sum_{i=1}^{n+1} f_i^2. \tag{2.9}$$

Note that, Eq (2.9) can be rewritten using the Fibonacci identities as

$$Le_n Le_{n+1} + Le_{n+2} = 4 f_{n+1} f_{n+2}. \tag{2.10}$$

In the next section, we define the dual Leonardo numbers and we give some fundamental equations and identities.

3. Dual Leonardo numbers

In the literature, the authors studied number sequences in different types of dual structures. One of the studies is [16], which defines dual Fibonacci quaternions. Another author provided the Horadam sequence, which is a generalization of the Fibonacci sequence in dual bicomplex quaternion structure [9]. Also, authors searched for different types of sequences like third-order Jacobsthal sequence on the dual quaternion settings [7]. Moreover, some studies use octonions [12, 20], which defines the generalization of Fibonacci numbers in the dual eight-dimensional octonion algebra and studies Fibonacci and Lucas numbers, respectively. Furthermore, researchers investigated related polynomials with the sequences [5, 21]. There are many papers concerning the dual structure and Fibonacci-like sequences, but we confine ourselves.

In the following definition, we use the definition of Leonardo numbers and expand it to the dual structure.

Definition 3.1. For $n \geq 0$, the nth dual Leonardo number DLe_n is defined as

$$DLe_n = Le_n + \varepsilon Le_{n+1}. \tag{3.1}$$

Using this definition, we can obtain the recurrence relation of dual Leonardo numbers for $n \geq 2$ as

$$DLe_n = (Le_{n-1} + Le_{n-2} + 1) + \varepsilon(Le_n + Le_{n-1} + 1) = DLe_{n-1} + DLe_{n-2} + \omega, \tag{3.2}$$

where $\omega = 1 + \varepsilon$.

In addition, using the recurrence relation we have

$$DLe_{n+1} = 2DLe_n - DLe_{n-2}, \tag{3.3}$$

where $n \geq 2$.

The Binet formula is used to find the nth element of the sequence. In the next theorem, we give the Binet formula for dual Leonardo numbers.

Theorem 3.2. *For $n > 1$, the Binet formula for the nth dual Leonardo number DLe_n is*

$$DLe_n = \frac{2\underline{\alpha}\alpha^{n+1} - 2\underline{\beta}\beta^{n+1}}{\alpha - \beta} - \omega, \tag{3.4}$$

where α and β are roots of characteristic equation, $\underline{\alpha} = 1 + \varepsilon\alpha$ and $\underline{\beta} = 1 + \varepsilon\beta$.

Proof. Using Eq (3.1) and the Binet of the Leonardo numbers, we get

$$DLe_n = \frac{\alpha(2\alpha^n - 1) - \beta(2\beta^n - 1)}{\alpha - \beta} + \varepsilon\left(\frac{\alpha(2\alpha^{n+1} - 1) - \beta(2\beta^{n+1} - 1)}{\alpha - \beta}\right).$$

Making necessary calculations, we have

$$DLe_n = \frac{2\alpha^{n+1}(1 + \varepsilon\alpha) - 2\beta^{n+1}(1 + \varepsilon\beta) + (1 + \varepsilon)(\beta - \alpha)}{\alpha - \beta},$$

which is equal to (3.4) with

$$\underline{\alpha} = 1 + \varepsilon\alpha, \quad \underline{\beta} = 1 + \varepsilon\beta \text{ and } \omega = 1 + \varepsilon.$$

□

In the following theorem, we give the generating function for dual Leonardo numbers.

Theorem 3.3. *The generating function for dual Leonardo numbers is*

$$g(t) = \frac{DLe_0 - t(1-\varepsilon) + t^2(1-\varepsilon)}{1 - 2t + t^3}, \tag{3.5}$$

where $1 - 2t + t^3$ *is not equal to zero.*

Proof. To prove the theorem, we write $g(t)$ as

$$g(t) = DLe_0 t^0 + DLe_1 t^1 + DLe_2 t^2 + \cdots + DLe_n t^n + \cdots .$$

Now, making necessary calculations, we obtain $2tg(t)$ and $-t^3 g(t)$ as

$$2tg(t) = \sum_{n=0}^{\infty} 2DLe_n t^{n+1} \quad \text{and} \quad t^3 g(t) = \sum_{n=0}^{\infty} DLe_n t^{n+3}.$$

Utilizing the above terms and Eq (3.3), we get

$$(1 - 2t + t^3)g(t) = DLe_0 - t(1-\varepsilon) + t^2(1-\varepsilon).$$

After dividing both sides with $(1 - 2t + t^3)$, we get the desired result as

$$g(t) = \frac{DLe_0 - t(1-\varepsilon) + t^2(1-\varepsilon)}{1 - 2t + t^3}.$$

□

Our Binet and generating formulas are compatible with the earlier result in [3]. Similar to the Binet formula, the generating function is also used to find the *n*th element of the sequence, which is why they are so important and generally provided in studies. Another famous equation is the Cassini identity. For the next theorem, we give this identity.

Theorem 3.4. *The Cassini identity for dual Leonardo numbers is*

$$(DLe_n)^2 - DLe_{n-1}DLe_{n+1} = Le_{n-1} - Le_{n-2} + 4(-1)^n + (1 + Le_{n-1} + 4(-1)^2)\varepsilon, \tag{3.6}$$

where Le_n *is the nth Leonardo number.*

Proof. We start the proof via explicitly writing the identity

$$LHS = (Le_n + Le_{n+1}\varepsilon)^2 - (Le_{n-1} + Le_n\varepsilon)(Le_{n+1} + Le_{n+2}\varepsilon).$$

By separating reals and duals, we get the following equation

$$LHS = {Le_n}^2 - Le_{n-1}Le_{n+1} + (2Le_nLe_{n+1} - Le_{n-1}Le_{n+2} - Le_nLe_{n+1})\varepsilon.$$

Calculating both real and dual parts of the equation separately, we can rewrite the equation as

$$(DLe_n)^2 - DLe_{n-1} - DLe_{n+1} = Le_{n-1} - Le_{n-2} + 4(-1)^n + (1 + Le_{n-1} + 4(-1)^2)\varepsilon,$$

which completes the proof. □

Now, we present another identity in the next theorem known as the d'Ocagne's identity for the dual Leonardo numbers.

Theorem 3.5. *The d'Ocagne's identity for the dual Leonardo numbers is*

$$DLe_m DLe_{n+1} - DLe_{m+1} DLe_n = [2(-1)^{n+1}(Le_{m-n-1} + 1) + Le_{m-1} - Le_{n-1}](1+\varepsilon) + (Le_m - Le_n)\varepsilon, \quad (3.7)$$

where $m > n$ and $n > 0$.

Proof. In order to prove Eq (3.7), we use the definition of the dual Leonardo numbers (3.1)

$$LHS = (Le_m + Le_{m+1}\varepsilon)(Le_{n+1} + Le_{n+2}\varepsilon) - (Le_{m+1} + Le_{m+2}\varepsilon)(Le_n + Le_{n+1}\varepsilon).$$

After calculations and collocating the Leonardo numbers, we get the following equation

$$LHS = Le_m Le_{n+1} - Le_{m+1} Le_n + (Le_m Le_{n+2} - Le_{m+2} Le_n)\varepsilon.$$

Using properties of the numbers, we obtain

$$LHS = [2(-1)^{n+1}(Le_{m-n-1} + 1) + Le_{m-1} - Le_{n-1}](1+\varepsilon) + (Le_m - Le_n)\varepsilon,$$

which is the desired result. □

In the next theorem, we give the sum of squares of the iterative dual Leonardo numbers.

Theorem 3.6. *For $n \geq 0$, the following identity holds true:*

$$DLe_{n+1}^2 + DLe_n^2 = 2(Le_{2n+2} - Le_{n+2} + 1) + 2\varepsilon(4f_{n+2}(f_{n+2} + 2f_{n+1}) - (Le_{n+4} - 1)), \quad (3.8)$$

where f_n is the nth Fibonacci number.

Proof. To prove the theorem, we start with using the explicit form of the DLe_n

$$DLe_{n+1}^2 + DLe_n^2 = (Le_{n+1} + \varepsilon Le_{n+2})^2 + (Le_n + \varepsilon Le_{n+1})^2.$$

After the multiplication, the equation becomes

$$LHS = Le_{n+1}^2 + Le_n^2 + 2\varepsilon(Le_{n+2}Le_{n+1} + Le_{n+1}Le_n).$$

After using necessary identities we get

$$LHS = 2(Le_{2n+2} - Le_{n+2} + 1) + 2\varepsilon(4f_{n+2}(f_{n+2} + 2f_{n+1}) - (Le_{n+4} - 1)).$$

□

In the Theorem 3.7, we obtain an identity that includes the sum of squares in a more generalized way.

Theorem 3.7. *For the dual Leonardo numbers where m and n are nonnegative integers with $m \geq n$, we have*

$$DLe_{m+n}^2 + DLe_{m-n}^2 = 2\Big(2f_{2m+2}f_{2n} - Le_{m+n} + Le_{m-n} + (Le_{m+n}Le_{m+n+1} - Le_{m-n}Le_{m-n+1})\varepsilon\Big). \quad (3.9)$$

Proof. In order to prove the theorem, we employ the definition of the dual Leonardo numbers and the definition of the Leonardo numbers using the Fibonacci numbers.

$$\begin{aligned} DLe_{m+n}^2 + DLe_{m-n}^2 &= (Le_{m+n} + Le_{m+n+1}\varepsilon)^2 + (Le_{m-n} + Le_{m-n+1}\varepsilon)^2 \\ &= (2f_{m+n+1} - 1 + (2f_{m+n+2} - 1)\varepsilon)^2 + (2f_{m-n+1} - 1 + (2f_{m-n+2} - 1)\varepsilon)^2. \end{aligned}$$

Making calculations and employing the necessary identities we get the desired result:

$$DLe_{m+n}^2 + DLe_{m-n}^2 = 2\Big(2f_{2m+2}f_{2n} - Le_{m+n} + Le_{m-n} + (Le_{m+n}Le_{m+n+1} - Le_{m-n}Le_{m-n+1})\varepsilon\Big).$$

□

Note that by using corollary (2.2) or Eq (2.10), we can get different versions of this result, but this one is very compact.

Theorem 3.8. *For $n, m > 2$, the following identity holds true:*

$$DLe_{m+1}DLe_{n+1} - DLe_{m-1}DLe_{n-1} = 2Le_{m+n+1} - Le_m - Le_n + \varepsilon(4Le_{m+n+2} - Le_{m+2} - Le_{n+2} + 2). \quad (3.10)$$

Proof. For the proof, we start with the definition of dual Leonardo numbers and we have

$$LHS = (Le_{m+1} + \varepsilon Le_{m+2})(Le_{n+1} + \varepsilon Le_{n+2}) - (Le_{m-1} + \varepsilon Le_m)(Le_{n-1} + \varepsilon Le_n).$$

After separating the duals and the reals, we need to deal with the following equations:

$$LHS = (Le_{m+1}Le_{n+1} - Le_{m-1}Le_{n-1}) + \varepsilon(Le_{m+1}Le_{n+2} - Le_{m-1}Le_n) + \varepsilon(Le_{m+2}Le_{n+1} - Le_mLe_{n-1}).$$

Using the Leonardo number identities and some additional computation, we get the desired result as

$$DLe_{m+1}DLe_{n+1} - DLe_{m-1}DLe_{m-1} = 2Le_{m+n+1} - Le_m - Le_n + \varepsilon(4Le_{m+n+2} - Le_{m+2} - Le_{n+2} + 2).$$

□

In the next theorem we present another identity before giving summation formulas, including the dual Leonardo numbers.

Theorem 3.9. *For nonnegative integer n, the following identity holds true:*

$$DLe_nDLe_{n+3} - DLe_{n+1}DLe_{n+2} = 4(-1)^n - (Le_n + 1) - \Big(Le_n + Le_{n+4} - 2Le_{n+2} - (-1)^n3\Big)\varepsilon. \quad (3.11)$$

Proof. For the proof of the theorem, we first use the expanded form of the dual Leonardo numbers

$$(Le_n + \varepsilon Le_{n+1})(Le_{n+3} + \varepsilon Le_{n+4}) - (Le_{n+1} + \varepsilon Le_{n+2})(Le_{n+2} + \varepsilon Le_{n+3}).$$

After calculations, we study the real and the dual parts separately. The real part is

$$Le_nLe_{n+3} - Le_{n+1}Le_{n+2} = 4(f_{n+1}f_{n+4} - f_{n+2}f_{n+3}) - 2f_{n+1}.$$

The dual part is

$$Le_nLe_{n+4} - Le_{n+2}^2.$$

Second, we employ the Fibonacci identities for the real part.

Third, we use a special case of the Catalan identity for the Leonardo numbers [6]. Making the adjustments leads us to the result

$$4(-1)^n - (Le_n + 1) - \Big(Le_n + Le_{n+4} - 2Le_{n+2} - (-1)^n3\Big)\varepsilon.$$

□

In the next theorem, we provide some summation formulas for dual Leonardo numbers.

Theorem 3.10. *For the dual Leonardo numbers, we have the following summation formulas with n being the positive integer:*

$$\sum_{j=0}^{n} DLe_j = DLe_{n+2} - (n+2)\omega - 2\varepsilon, \tag{3.12}$$

$$\sum_{j=0}^{n} DLe_{2j} = DLe_{2n+1} - n\omega - 2\varepsilon, \tag{3.13}$$

$$\sum_{j=0}^{n} DLe_{2j+1} = DLe_{2n+2} - (n+2)\omega. \tag{3.14}$$

Proof. For abbreviation, we present the proof of Eq (3.12). The other proofs are similar, so we omit them. Using the definition of the dual Leonardo numbers (3.1), we can write the formula as

$$\sum_{j=0}^{n} DLe_j = \sum_{j=0}^{n} (Le_j + \varepsilon Le_{j+1}) = \sum_{j=0}^{n} Le_j + \varepsilon \sum_{j=0}^{n} Le_{j+1}.$$

Calculating the summation formula for the Leonardo numbers and rearranging duals and reals of the equation, we can rewrite the expression as

$$\sum_{j=0}^{n} DLe_j = DLe_{n+2} - (n+2)\omega - 2\varepsilon,$$

which completes the proof. □

In the following theorem, we present the sum of squared Leonardo numbers, which outputs a result that contains the Fibonacci and the Leonardo numbers.

Theorem 3.11. *For $n \geq 1$, the following equation holds true:*

$$\sum_{j=1}^{n} (DLe_j)^2 = (Le_n - 1)(Le_{n+1}) + (n+1) + 2\sum_{j=1}^{n} \Big(4\sum_{i=1}^{j+1} f_i^2 - Le_{i+2}\Big)\varepsilon. \tag{3.15}$$

Proof. First, we use the definition of the dual Leonardo numbers and obtain real and dual parts of the equation as follows:

$$\sum_{j=1}^{n} (DLe_j)^2 = \sum_{j=1}^{n} (Le_j)^2 + 2\sum_{j=1}^{n} Le_n Le_{n+1}.$$

Second, for the real part, we use the definition of the Leonardo numbers and get the desired result. Third, for the dual part we employ corollary (2.2) and obtain

$$\sum_{j=1}^{n} (DLe_j)^2 = (Le_n - 1)(Le_{n+1}) + (n+1) + 2\sum_{j=1}^{n} \Big(4\sum_{i=1}^{j+1} f_i^2 - Le_{i+2}\Big)\varepsilon,$$

which ends the proof. □

4. Conclusions

In this study, we established the concept of dual Leonardo numbers, which is an extension of the Leonardo numbers. We provided fundamental equations for this sequence, including the Binet formula and the generating function, both of which are employed to obtain specific elements from the sequence. Moreover, we offered a range of identities, such as the Cassini and d'Ocagne's identities. Furthermore, we introduced multiple summation formulas for the dual Leonardo numbers. For future research, graph theoretical applications and generalizations of the dual Leonardo numbers can be studied.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares that he has no conflict of interest.

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