Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(12): 30511–30526.
DOI:10.3934/math.20231559
Received: 21 September 2023
Revised: 14 October 2023
Accepted: 5 November 2023
Published: 10 November 2023

Research article

Counting rational points of two classes of algebraic varieties over finite fields

Guangyan Zhu[1], **Shiyuan Qiang**[2,*] **and Mao Li**[3]

[1] School of Teacher Education, Hubei Minzu University, Enshi 445000, China
[2] Mathematical College, Sichuan University, Chengdu 610064, China
[3] School of Mathematics and Statistics, Southwest University, Chongqing 400715, China

* **Correspondence:** Email: syqiang23@163.com.

Abstract: Let p stand for an odd prime and let $\eta \in \mathbb{Z}^+$ (the set of positive integers). Let $\mathbb{F}_q$ denote the finite field having $q = p^\eta$ elements and $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$. In this paper, when the determinants of exponent matrices are coprime to $q-1$, we use the Smith normal form of exponent matrices to derive exact formulas for the numbers of rational points on the affine varieties over $\mathbb{F}_q$ defined by

$$\begin{cases} a_1 x_1^{d_{11}} \ldots x_n^{d_{1n}} + \ldots + a_s x_1^{d_{s1}} \ldots x_n^{d_{sn}} = b_1, \\ a_{s+1} x_1^{d_{s+1,1}} \ldots x_n^{d_{s+1,n}} + \ldots + a_{s+t} x_1^{d_{s+t,1}} \ldots x_n^{d_{s+t,n}} = b_2 \end{cases}$$

and

$$\begin{cases} c_1 x_1^{e_{11}} \ldots x_m^{e_{1m}} + \ldots + c_r x_1^{e_{r1}} \ldots x_m^{e_{rm}} = l_1, \\ c_{r+1} x_1^{e_{r+1,1}} \ldots x_m^{e_{r+1,m}} + \ldots + c_{r+k} x_1^{e_{r+k,1}} \ldots x_m^{e_{r+k,m}} = l_2, \\ c_{r+k+1} x_1^{e_{r+k+1,1}} \ldots x_m^{e_{r+k+1,m}} + \ldots + c_{r+k+w} x_1^{e_{r+k+w,1}} \ldots x_m^{e_{r+k+w,m}} = l_3, \end{cases}$$

respectively, where $d_{ij}, e_{i'j'} \in \mathbb{Z}^+, a_i, c_{i'} \in \mathbb{F}_q^*, i = 1, \ldots, s+t, j = 1, \ldots, n, i' = 1, \ldots, r+k+w, j' = 1, \ldots, m$, and $b_1, b_2, l_1, l_2, l_3 \in \mathbb{F}_q$. These formulas extend the theorems obtained by Q. Sun in 1997. Our results also give a partial answer to an open question posed by S.N. Hu, S.F. Hong and W. Zhao [The number of rational points of a family of hypersurfaces over finite fields, *J. Number Theory* **156** (2015), 135–153].

Keywords: finite field; algebraic variety; rational point; prime number; Smith normal form
Mathematics Subject Classification: Primary 11T06, Secondary 11T71

1. Introduction

Throughout this paper, p will always denote an odd prime, $\mathbb{Z}^+$ and $\mathbb{F}_q$ denote the set of positive integers and the finite field having $q = p^\eta$ elements, respectively, where $\eta \in \mathbb{Z}^+$. Then $\mathbb{F}_q^* := \mathbb{F}_q \setminus \{0\}$

forms a group under the multiplicative operation. For any finite set $\mathcal{S}$, $|\mathcal{S}|$ means its cardinality. Let $\lambda, n \in \mathbb{Z}^+$ and $\langle\lambda\rangle$ be the set of the first λ positive integers. Let $x_1, ..., x_{n-1}$ and x_n be n indeterminates in $\mathbb{F}_q$, and for brevity, let $\mathbf{x} = (x_1, ..., x_n)$. Let $f_1(\mathbf{x}), ..., f_\lambda(\mathbf{x})$ be the system of n-variable polynomials over $\mathbb{F}_q$, and we denote by $V(f_1, ..., f_\lambda) = V(f_1(\mathbf{x}), ..., f_\lambda(\mathbf{x}))$ the affine variety determined by the vanishing of these polynomials. Define

$$N(V) = \left|\{\mathbf{x} = (x_1, ..., x_n) \in \mathbb{F}_q^n : f_1(\mathbf{x}) = ... = f_\lambda(\mathbf{x}) = 0\}\right|.$$

When $\lambda = 1$, one writes $N(V) = N(f)$. Finding an accurate formula for $N(V)$ is a common and significant subject. However, such a problem is hard in general. In the past 70 years, many mathematicians were devoted to this subject and made much vital progress (see [1–11, 13–27]).

In 1997, the number $N(f)$ of rational points over $\mathbb{F}_q$ on the following affine hypersurface

$$f = a_1 x_1^{e_{11}} ... x_n^{e_{1n}} + ... + a_s x_1^{e_{s1}} ... x_n^{e_{sn}} - b = 0,\ e_{ij} \in \mathbb{Z}^+, a_i \in \mathbb{F}_q^*, b \in \mathbb{F}_q, i \in \langle s\rangle, j \in \langle n\rangle$$

was investigated by Sun [18]. Besides, the accurate formula for the number $N(f)$ of rational points was found in [18]:

$$N(f) = \begin{cases} q^n - (q-1)^n + \dfrac{q-1}{q} A(n-1) & \text{if } b = 0, \\ \dfrac{1}{q} A(n) & \text{otherwise} \end{cases}$$

provided $s = n$ and $\gcd(\det(e_{ij}), q-1) = 1$, where $A(\jmath) := (q-1)^{\jmath} - (-1)^{\jmath},\ \forall \jmath \in \mathbb{Z}^+$. Eight years later, the result of [18] was successfully extended by Wang and Sun [21]. Actually, they attained a formula for the number of $(x_1, ...x_{n_2}) \in \mathbb{F}_q^{n_2}$ on the following hypersurface

$$a_1 x_1^{d_{11}} ... x_{n_1}^{d_{1n_1}} + ... + a_{n_1} x_1^{d_{n_1,1}} ... x_{n_1}^{d_{n_1,n_1}} + a_{n_1+1} x_1^{d_{n_1+1,1}} ... x_{n_2}^{d_{n_1+1,n_2}} + ... + a_{n_2} x_1^{d_{n_2,1}} ... x_{n_2}^{d_{n_2,n_2}} = b$$

with $d_{ij} \in \mathbb{Z}^+, a_i \in \mathbb{F}_q^*, 1 \le i, j \le n_2$.

In 2015, Hu, Hong and Zhao [9] gave a uniform generalization to the results of [20, 21]. Actually, they used the Smith normal form to deduce an accurate formula for $N(f)$ of $(x_1, ...x_{n_t}) \in \mathbb{F}_q^{n_t}$ on the hypersurface over $\mathbb{F}_q$ defined by

$$f := f(x_1, ..., x_{n_t}) = \sum_{j=0}^{t-1} \sum_{i=1}^{r_{j+1}-r_j} a_{r_j+i} x_1^{e_{r_j+i,1}} ... x_{n_{j+1}}^{e_{r_j+i,n_{j+1}}} - b, \tag{1.1}$$

where the integers $t > 0$, $0 = r_0 < r_1 < r_2 < ... < r_t$, $1 \le n_1 < n_2 < ... < n_t$, $b \in \mathbb{F}_q$, $a_i \in \mathbb{F}_q^*$ and $e_{ij} \in \mathbb{Z}^+$, $i \in \langle r_t\rangle$, $j \in \langle n_t\rangle$. Under some restrictions on f, a little bit simple formula about the number of rational points on the hypersurface (1.1) was given in [13]. One notices that the result of [9] was extended by Hu and Zhao [11] from the hypersurface case to certain algebraic variety case.

An open problem was raised at the end of [9, Section 3]. For the case of the variety consisting of two hypersurfaces, Hu, Qin and Zhao [10] and Zhu and Hong [27] obtained some partial answers to this problem. In other words, Hu, Qin and Zhao [10] gave an explicit formula for $N(V(f_1, f_2))$, where

$$\begin{cases} f_1 := \sum\limits_{i=1}^{r_1} a_{1i} x_1^{e_{i1}^{(1)}} ... x_{n_1}^{e_{i,n_1}^{(1)}} + \sum\limits_{i=r_1+1}^{r_2} a_{1i} x_1^{e_{i1}^{(1)}} ... x_{n_2}^{e_{i,n_2}^{(1)}} - b_1 \\ f_2 := \sum\limits_{i'=1}^{r_3} a_{2i'} x_1^{e_{i'1}^{(2)}} ... x_{n_3}^{e_{i,n_3}^{(2)}} + \sum\limits_{i'=r_3+1}^{r_4} a_{2i'} x_1^{e_{i'1}^{(2)}} ... x_{n_4}^{e_{i',n_4}^{(2)}} - b_2 \end{cases}$$

with $1 \leq r_1 < r_2, 1 \leq r_3 < r_4, 1 \leq n_1 < n_2, 1 \leq n_3 < n_4$, $e^{(1)}_{i,j}$, $e^{(2)}_{i',j'} \in \mathbb{Z}^+$, $b_1, b_2 \in \mathbb{F}_q$, and $a_{1i}, a_{2i'} \in \mathbb{F}_q^*$, $i \in \langle r_2 \rangle$, $i' \in \langle r_4 \rangle$, $j \in \langle n_2 \rangle$, $j' \in \langle n_4 \rangle$. Zhu and Hong [27] used and developed the techniques in [9] to get an exact formula for the number of rational points on $V = V(f_1, f_2)$ over $\mathbb{F}_q$ with

$$\begin{cases} f_1 := f_1(x_1, ..., x_{n_t}) = \sum_{i=1}^{r} a_i^{(1)} x_1^{e_{i1}^{(1)}} ... x_n^{e_{in}^{(1)}} - b_1, \\ f_2 := f_2(x_1, ..., x_{n_t}) = \sum_{j'=0}^{t-1} \sum_{i'=1}^{r_{j'+1}-r_{j'}} a_{r_{j'}+i'}^{(2)} x_1^{e^{(2)}_{r_{j'}+i',1}} ... x_{n_{j'+1}}^{e^{(2)}_{r_{j'}+i',n_{j'+1}}} - b_2, \end{cases} \tag{1.2}$$

where $b_i \in \mathbb{F}_q, i = 1, 2$, $t \in \mathbb{Z}^+$, $0 = n_0 < n_1 < n_2 < ... < n_t$, $n_{k-1} < n \leq n_k$ for some $k \in \langle t \rangle$, $0 = r_0 < r_1 < r_2 < ... < r_t$, $a_i^{(1)}, a_{i'}^{(2)} \in \mathbb{F}_q^*$, $i \in \langle r \rangle, i' \in \langle r_t \rangle$, $e^{(1)}_{i,j}$, $e^{(2)}_{i',j'} \in \mathbb{Z}^+$, $j \in \langle n \rangle$, $j' \in \langle n_t \rangle$.

Inspired by the works of [9, 18, 21, 27], we consider in this paper the question of counting rational points on the variety $V(f_1, f_2)$ with

$$\begin{cases} f_1 := a_1 x_1^{d_{11}} ... x_n^{d_{1n}} + ... + a_s x_1^{d_{s1}} ... x_n^{d_{sn}} - b_1, \\ f_2 := a_{s+1} x_1^{d_{s+1,1}} ... x_n^{d_{s+1,n}} + ... + a_{s+t} x_1^{d_{s+t,1}} ... x_n^{d_{s+t,n}} - b_2, \end{cases} \tag{1.3}$$

and the variety $V(f_1, f_2, f_3)$ with

$$\begin{cases} f_1 := c_1 x_1^{e_{11}} ... x_m^{e_{1m}} + ... + c_r x_1^{e_{r1}} ... x_m^{e_{rm}} - l_1, \\ f_2 := c_{r+1} x_1^{e_{r+1,1}} ... x_m^{e_{r+1,m}} + ... + c_{r+k} x_1^{e_{r+k,1}} ... x_m^{e_{r+k,m}} - l_2, \\ f_3 := c_{r+k+1} x_1^{e_{r+k+1,1}} ... x_m^{e_{r+k+1,m}} + ... + c_{r+k+w} x_1^{e_{r+k+w,1}} ... x_m^{e_{r+k+w,m}} - l_3, \end{cases} \tag{1.4}$$

where $d_{ij}, e_{i'j'} \in \mathbb{Z}^+, a_i, c_{i'} \in \mathbb{F}_q^*, i \in \langle s+t \rangle, j \in \langle n \rangle, i' \in \langle r+k+w \rangle, j' \in \langle m \rangle$, and $b_1, b_2, l_1, l_2, l_3 \in \mathbb{F}_q$. Let

$$E_1 = \begin{pmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ \vdots & \vdots & & \vdots \\ d_{s1} & d_{s2} & \cdots & d_{sn} \\ d_{s+1,1} & d_{s+1,2} & \cdots & d_{s+1,n} \\ \vdots & \vdots & & \vdots \\ d_{s+t,1} & d_{s+t,2} & \cdots & d_{s+t,n} \end{pmatrix} \tag{1.5}$$

with $d_{ij}, i \in \langle s + t \rangle, j \in \langle n \rangle$ being given as in (1.3), and let

$$E_2 = \begin{pmatrix} e_{11} & e_{12} & \cdots & e_{1m} \\ \vdots & \vdots & & \vdots \\ e_{r1} & e_{r2} & \cdots & e_{rm} \\ e_{r+1,1} & e_{r+1,2} & \cdots & e_{r+1,m} \\ \vdots & \vdots & & \vdots \\ e_{r+k,1} & e_{r+k,2} & \cdots & e_{r+k,m} \\ e_{r+k+1,1} & e_{r+k+1,2} & \cdots & e_{r+k+1,m} \\ \vdots & \vdots & & \vdots \\ e_{r+k+w,1} & e_{r+k+w,2} & \cdots & e_{r+k+w,m} \end{pmatrix} \tag{1.6}$$

with $e_{i'j'}, i' \in \langle r+k+w \rangle, j' \in \langle m \rangle$ being given as in (1.4).

From [12], it guarantees the existences of unimodular matrices U_2 and V_2 with the property

$$U_2 E_2 V_2 = \begin{pmatrix} D_2 & 0 \\ 0 & 0 \end{pmatrix}, \tag{1.7}$$

where

$$D_2 := \operatorname{diag}(g_1^{(E_2)}, ..., g_{v'}^{(E_2)})$$

with $g_1^{(E_2)}, ..., g_{v'}^{(E_2)} \in \mathbb{Z}^+$ and $g_1^{(E_2)}|...|g_{v'}^{(E_2)}$. The diagonal matrix on the right side of (1.7) is called *Smith normal form* of E_2, and abbreviated as SNF(E_2). That is,

$$\mathrm{SNF}(E_2) = \begin{pmatrix} D_2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Fix $\alpha \in \mathbb{F}_q^*$ as a primitive element of $\mathbb{F}_q$, then for any $\beta \in \mathbb{F}_q^*$, one can find a unique integer $\gamma \in [1, q-1]$ with $\beta = \alpha^\gamma$, and such an integer γ is said to be the *index* of β on the basis α. We write $\mathrm{ind}_\alpha \beta := \gamma$.

Consider the variety defined by

$$\begin{cases} \sum\limits_{i=1}^{r} c_i \upsilon_i = l_1, \\ \sum\limits_{i=r+1}^{r+k} c_i \upsilon_i = l_2, \\ \sum\limits_{i=r+k+1}^{r+k+w} c_i \upsilon_i = l_3, \end{cases} \tag{1.8}$$

where $c_{i'}, l_1, l_2, l_3, i' \in \langle r+k+w \rangle$ are given as in (1.4). Let $\mathcal{N}$ denote the number of rational points $(\upsilon_1, ..., \upsilon_{r+k+w}) \in (\mathbb{F}_q^*)^{r+k+w}$ on (1.8) satisfying

$$\begin{cases} \gcd(q-1, g_j^{(E_2)}) | h_j^{(E_2)} \text{ for } j \in \langle v' \rangle \\ (q-1) | h_j^{(E_2)} \text{ for } j \in \langle r+k+w \rangle \setminus \langle v' \rangle, \end{cases} \tag{1.9}$$

where

$$(h_1^{(E_2)}, ..., h_{r+k+w}^{(E_2)})^{\mathrm{T}} := U_2(\mathrm{ind}_\alpha(\upsilon_1), ..., \mathrm{ind}_\alpha(\upsilon_{r+k+w}))^{\mathrm{T}}.$$

We can now state our main results.

Theorem 1.1. *Let V be the variety defined by* (1.3). *If $s+t=n$ and* $\gcd(q-1, \det(E_1)) = 1$, *then*

$$N(V) = \begin{cases} q^n - (q-1)^n + \dfrac{(q-1)^2}{q^2} A(s-1)A(t-1) & \text{if } b_1 = b_2 = 0, \\ \dfrac{q-1}{q^2} A(s)A(t-1) & \text{if } b_1 \neq 0, b_2 = 0, \\ \dfrac{q-1}{q^2} A(s-1)A(t) & \text{if } b_1 = 0, b_2 \neq 0, \\ \dfrac{1}{q^2} A(s)A(t) & \text{if } b_1 \neq 0, b_2 \neq 0. \end{cases} \tag{1.10}$$

Theorem 1.2. *Let V be the variety defined by* (1.4). *Then*

$$N(V) = \begin{cases} q^m - (q-1)^m + \mathscr{N}R & \text{if } l_1 = l_2 = l_3 = 0, \\ \mathscr{N}R & \text{otherwise}, \end{cases} \tag{1.11}$$

where $R := (q-1)^{m-v'} \prod_{j=1}^{v'} \gcd(q-1, g_j^{(E_2)})$.

From Theorem 1.2, one can derive the third main result of this paper as follows.

Theorem 1.3. *Let V denote the affine variety defined by* (1.4). *If* $r+k+w = m$ *and* $\gcd(q-1, \det(E_2)) = 1$*, then*

$$N(V) = \begin{cases} q^m - (q-1)^m + \frac{(q-1)^3}{q^3} A(r-1)A(k-1)A(w-1) & \text{if } l_l = l_2 = l_3 = 0, \\ \frac{(q-1)^2}{q^3} A(r-1)A(k-1)A(w) & \text{if } l_l = l_2 = 0, l_3 \neq 0, \\ \frac{(q-1)^2}{q^3} A(r-1)A(k)A(w-1) & \text{if } l_l = l_3 = 0, l_2 \neq 0, \\ \frac{(q-1)^2}{q^3} A(r)A(k-1)A(w-1) & \text{if } l_1 \neq 0, l_2 = l_3 = 0, \\ \frac{q-1}{q^3} A(r)A(k)A(w-1) & \text{if } l_l \neq 0, l_2 \neq 0, l_3 = 0, \\ \frac{q-1}{q^3} A(r)A(k-1)A(w) & \text{if } l_l \neq 0, l_2 = 0, l_3 \neq 0, \\ \frac{q-1}{q^3} A(r-1)A(k)A(w) & \text{if } l_1 = 0, l_2 \neq 0, l_3 \neq 0, \\ \frac{1}{q^3} A(r)A(k)A(w) & \text{if } l_1 \neq 0, l_2 \neq 0, l_3 \neq 0. \end{cases} \tag{1.12}$$

Obviously, Theorems 1.1 to 1.3 also give a partial answer to the open problem proposed at the end of [9, Section 3].

In Section 2, in order to prove Theorems 1.1 to 1.3, we give several auxiliary results. Then in Section 3, one presents the details of the proofs of Theorems 1.1 to 1.3. Finally, four examples are provided in Section 4.

2. Auxiliary results

In this section, we present several preliminary results which are needed in the proofs of Theorems 1.1 to 1.3. We begin with a result due to Zhu and Hong [27].

Lemma 2.1. [27, Lemma 2.6] *Let* $c_{ij} \in \mathbb{F}_q^*$, $i \in \langle m \rangle$, $j \in \langle \hbar_i \rangle$, $c_1, ..., c_m \in \mathbb{F}_q$. *Let* $N(c_1, ..., c_m)$ *stand for the number of* $(u_{11}, ..., u_{1\hbar_1}, ..., u_{m1}, ..., u_{m\hbar_m}) \in (\mathbb{F}_q^*)^{\hbar_1 + ... + \hbar_m}$ *such that*

$$\begin{cases} c_{11}u_{11} + ... + c_{1\hbar_1}u_{1\hbar_1} = c_1 \\ \vdots \\ c_{m1}u_{m1} + ... + c_{m\hbar_m}u_{m\hbar_m} = c_m. \end{cases}$$

Then

$$N(c_1, ..., c_m) = \frac{(q-1)^{|\{1\le i\le m: c_i=0\}|}}{q^m} \prod_{\substack{i=1\\ c_i=0}}^{m} A(k_i - 1) \prod_{\substack{i=1\\ c_i\neq 0}}^{m} A(k_i).$$

Lemma 2.2. *Let $a_i \in \mathbb{F}_q^*$, $i \in \langle s+t\rangle$, $b_1, b_2 \in \mathbb{F}_q$. Let $N(b_1, b_2)$ denote the number of $(u_1, ..., u_{s+t}) \in (\mathbb{F}_q^*)^{s+t}$ with*

$$\begin{cases} \sum_{i=1}^{s} a_i u_i = b_1, \\ \sum_{i=s+1}^{s+t} a_i u_i = b_2. \end{cases} \tag{2.1}$$

Then

$$N(b_1, b_2) = \begin{cases} \frac{(q-1)^2}{q^2} A(s-1)A(t-1) & \text{if } b_1 = b_2 = 0, \\ \frac{q-1}{q^2} A(s)A(t-1) & \text{if } b_1 \neq 0, b_2 = 0, \\ \frac{q-1}{q^2} A(s-1)A(t) & \text{if } b_1 = 0, b_2 \neq 0, \\ \frac{1}{q^2} A(s)A(t) & \text{if } b_1 \neq 0, b_2 \neq 0. \end{cases} \tag{2.2}$$

Proof. The result follows immediately from Lemma 2.1. □

Lemma 2.3. *Let $c_i \in \mathbb{F}_q^*$ for all $i \in \langle r+k+w\rangle$ and let $l_1, l_2, l_3 \in \mathbb{F}_q$. Let $N(l_1, l_2, l_3)$ denote the number of $(v_1, ..., v_{r+k+w}) \in (\mathbb{F}_q^*)^{r+k+w}$ satisfying (1.8). Then*

$$N(l_1, l_2, l_3) = \begin{cases} \frac{(q-1)^3}{q^3} A(r-1)A(k-1)A(w-1) & \text{if } l_l = l_2 = l_3 = 0, \\ \frac{(q-1)^2}{q^3} A(r-1)A(k-1)A(w) & \text{if } l_l = l_2 = 0, l_3 \neq 0, \\ \frac{(q-1)^2}{q^3} A(r-1)A(k)A(w-1) & \text{if } l_l = l_3 = 0, l_2 \neq 0, \\ \frac{(q-1)^2}{q^3} A(r)A(k-1)A(w-1) & \text{if } l_1 \neq 0, l_2 = l_3 = 0, \\ \frac{q-1}{q^3} A(r)A(k)A(w-1) & \text{if } l_l \neq 0, l_2 \neq 0, l_3 = 0, \\ \frac{q-1}{q^3} A(r)A(k-1)A(w) & \text{if } l_l \neq 0, l_2 = 0, l_3 \neq 0, \\ \frac{q-1}{q^3} A(r-1)A(k)A(w) & \text{if } l_1 = 0, l_2 \neq 0, l_3 \neq 0, \\ \frac{1}{q^3} A(r)A(k)A(w) & \text{if } l_1 \neq 0, l_2 \neq 0, l_3 \neq 0. \end{cases} \tag{2.3}$$

Proof. This follows immediately from Lemma 2.1. □

Reference [12] tells us that by using elementary transformation, we can readily find unimodular matrices U_1 and V_1 with the property

$$U_1 E_1 V_1 = \begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.4}$$

where E_1 is given as in (1.5),

$$D_1 := \mathrm{diag}(g_1^{(E_1)}, ..., g_v^{(E_1)})$$

with $g_1^{(E_1)}, ..., g_v^{(E_1)} \in \mathbb{Z}^+$ and $g_1^{(E_1)}|...|g_v^{(E_1)}$. Let $\mathscr{M}$ represent the number of $(u_1, ..., u_{s+t}) \in (\mathbb{F}_q^*)^{s+t}$ on (2.1) under the following additional restrictions:

$$\begin{cases} \gcd(q-1, g_j^{(E_1)})|h_j^{(E_1)} \text{ for } j \in \langle v\rangle \\ (q-1)|h_j^{(E_1)} \text{ for } j \in \langle s+t\rangle \setminus \langle v\rangle, \end{cases} \tag{2.5}$$

where

$$(h_1^{(E_1)}, ..., h_{s+t}^{(E_1)})^{\mathrm{T}} := U_1(\mathrm{ind}_\alpha(u_1), ..., \mathrm{ind}_\alpha(u_{s+t}))^{\mathrm{T}}.$$

As a special case of [27, Theorem 1.2], one has the following result.

Lemma 2.4. *Let V be the variety* (1.3). *Then*

$$N(V) = \begin{cases} q^n - (q-1)^n + \mathscr{M}(q-1)^{n-v} \prod\limits_{j=1}^{v} \gcd(q-1, g_j^{(E_1)}) & \text{if } b_1 = b_2 = 0, \\ \mathscr{M}(q-1)^{n-v} \prod\limits_{j=1}^{v} \gcd(q-1, g_j^{(E_1)}) & \text{otherwise.} \end{cases}$$

Let $g_{ij}, b_i (i \in \langle \ell\rangle, j \in \langle u\rangle)$ and a be integers. Let $Y = (y_1, ..., y_u)^{\mathrm{T}}$ and $\mathscr{B} = (b_1, ..., b_\ell)^{\mathrm{T}}$. Then one forms an $\ell \times u$ matrix $\mathscr{G} = (g_{ij})$ and the following system of congruences:

$$\mathscr{G}Y \equiv \mathscr{B} \pmod{a}. \tag{2.6}$$

From [12], one can use elementary transformation of matrices to find unimodular matrices $\mathcal{U}$ and $\mathcal{V}$ with the property

$$\mathcal{U}\mathscr{G}\mathcal{V} = \mathrm{SNF}(\mathscr{G}) = \begin{pmatrix} \mathscr{D} & 0 \\ 0 & 0 \end{pmatrix},$$

where $\mathscr{D} := \mathrm{diag}(d_1, ..., d_\tau)$ with $d_i \in \mathbb{Z}^+, i \in \langle\tau\rangle$ and $d_i|d_{i+1}, i \in \langle \tau - 1\rangle$.

Lemma 2.5. [9, Lemma 2.3] *Let $\mathscr{B}' = (b_1', ..., b_\ell')^{\mathrm{T}} = \mathcal{U}\mathscr{B}$, then a necessary and sufficient condition for the system* (2.6) *of linear congruences to have a solution is* $\gcd(a, d_i)|b_i'$ *for all $i \in \langle\tau\rangle$ and $a|b_i'$ for all $i \in \langle\ell\rangle \setminus \langle\tau\rangle$. In addition, the number of solutions $(y_1, ..., y_u)^{\mathrm{T}}$ of* (2.6) *equals* $a^{u-\tau} \prod_{i=1}^{\tau} \gcd(a, d_i)$.

Lemma 2.6. *Let r, k and w be positive integers. Then*

$$\sum_{\substack{(\upsilon_1,...,\upsilon_{r+k+w})\in(\mathbb{F}_q^*)^{r+k+w}\\ (1.8)\text{ holds}}} \left|\left\{(x_1,...,x_m)\in(\mathbb{F}_q^*)^m : x_1^{e_{i1}}...x_m^{e_{im}}=\upsilon_i, i\in\langle r+k+w\rangle\right\}\right|$$
$$=\mathcal{N}(q-1)^{m-v'}\prod_{j=1}^{v'}\gcd(q-1,g_j^{(E_2)}). \tag{2.7}$$

Proof. First of all, for any given $(\upsilon_1,...,\upsilon_{r+k+w})\in(\mathbb{F}_q^*)^{r+k+w}$ satisfying (1.8), we have the system of congruences:

$$\sum_{j=1}^{m} e_{ij}\text{ind}_\alpha(x_i)\equiv\text{ind}_\alpha(\upsilon_i) \pmod{q-1}, i\in\langle r+k+w\rangle, \tag{2.8}$$

then

$$\begin{aligned}&\left|\{(x_1,...,x_m)\in(\mathbb{F}_q^*)^m : x_1^{e_{i1}}...x_m^{e_{im}}=\upsilon_i,\ i\in\langle r+k+m\rangle\}\right|\\ =&\left|\{(x_1,...,x_m)\in(\mathbb{F}_q^*)^m : \alpha^{\sum_{j=1}^m e_{ij}\text{ind}_\alpha(x_i)}=\alpha^{\text{ind}_\alpha(\upsilon_i)},\ i\in\langle r+k+m\rangle\}\right|\\ =&\left|\{(x_1,...,x_m)\in(\mathbb{F}_q^*)^m : (2.8)\text{ holds}\}\right|.\end{aligned}$$

However, Lemma 2.5 tells us that the necessary and sufficient condition for (2.8) to have a solution is that (1.9) holds. In addition, if (2.8) has a solution, then the number of the m-tuples $(\text{ind}_\alpha(x_1),...,\text{ind}_\alpha(x_m))\in\langle q-1\rangle^m$ satisfying (2.8) is equal to

$$(q-1)^{m-v'}\prod_{j=1}^{v'}\gcd(q-1,g_j^{(E_2)}).$$

Namely, if (1.9) is satisfied, then

$$\begin{aligned}&\left|\left\{(x_1,...,x_m)\in(\mathbb{F}_q^*)^m : x_1^{e_{i1}}...x_m^{e_{im}}=\upsilon_i, i\in\langle r+k+w\rangle\right\}\right|\\ =&(q-1)^{m-v'}\prod_{j=1}^{v'}\gcd(q-1,g_j^{(E_2)}).\end{aligned}$$

Thus, the left hand side of (2.7) is equal to

$$((q-1)^{m-v'}\prod_{j=1}^{v'}\gcd(q-1,g_j^{(E_2)}))\times\sum_{\substack{(\upsilon_1,...,\upsilon_{r+k+w})\in(\mathbb{F}_q^*)^{r+k+w}\\ (1.8)\text{ and }(1.9)\text{ hold}}}1. \tag{2.9}$$

However,

$$\mathcal{N}=\sum_{(\upsilon_1,...,\upsilon_{r+k+w})\in(\mathbb{F}_q^*)^{r+k+w}\text{ such that (1.8) and (1.9) hold}}1. \tag{2.10}$$

Hence putting (2.10) into (2.9) gives us the wanted result (2.7). □

3. Proofs of Theorems 1.1 to 1.3

In this section, we present the proofs of Theorems 1.1 to 1.3. We begin with the proof of Theorem 1.1.

Proof of Theorem 1.1. Taking determinants on both sides of (2.4), we can deduce that

$$\det(U_1)\det(E_1)\det(V_1) = g_1^{(E_1)}...g_n^{(E_1)}.$$

Since $\det(U_1) = \pm 1$ and $\det(V_1) = \pm 1$, the condition

$$\gcd(q-1, \det(E_1)) = 1$$

implies that

$$\gcd(q-1, g_j^{(E_1)}) = 1 \text{ for all } j \in \langle n \rangle.$$

So (2.5) holds.

Further, by Lemma 2.2, one has

$$\mathscr{M} = \sum_{(u_1,...,u_{s+t})\in(\mathbb{F}_q^*)^{s+t} \text{ such that (2.1) and (2.5) hold}} 1 = N(b_1, b_2) \tag{3.1}$$

with $N(b_1, b_2)$ being given as in (2.2). It follows from Lemma 2.4 that

$$N(V) = \begin{cases} q^n - (q-1)^n + \mathscr{M} & \text{if } b_1 = b_2 = 0, \\ \mathscr{M} & \text{otherwise.} \end{cases} \tag{3.2}$$

Thus, putting (3.1) and (3.2) together gives the expected result (1.10).

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. It is clear that

$$N(V) = \sum_{\substack{(\upsilon_1,...,\upsilon_{r+k+w})\in(\mathbb{F}_q)^{r+k+w} \\ \text{(1.8) holds}}} \left|\left\{(x_1,...,x_m) \in \mathbb{F}_q^m : x_1^{e_{i1}}...x_m^{e_{im}} = \upsilon_i, i \in \langle r+k+w\rangle\right\}\right|. \tag{3.3}$$

One defines the set $T(l_1, l_2, l_3)$ of $\mathbb{F}_q$-rational points as follows:

$$T(l_1, l_2, l_3) := \{(\upsilon_1, ..., \upsilon_{r+k+w}) \in \mathbb{F}_q^{r+k+w} : \text{(1.8) holds}\}. \tag{3.4}$$

Substituting (3.4) into (3.3) yields

$$N(V) = \sum_{(\upsilon_1,...,\upsilon_{r+k+w})\in T(l_1,l_2,l_3)} \left|\left\{(x_1,...,x_m) \in \mathbb{F}_q^m : x_1^{e_{i1}}...x_m^{e_{im}} = \upsilon_i, i \in \langle r+k+w\rangle\right\}\right|. \tag{3.5}$$

Define the set $T(0)$ by $T(0) := \emptyset$ if l_1, l_2 and l_3 are not all zero, and if $l_1 = l_2 = l_3 = 0$, then $T(0)$ consists of the zero vector of dimension $r+k+w$. For any integer ρ with $1 \le \rho \le r+k+w$, one defines the set $T(\rho)$ to be the subset of $T(l_1, l_2, l_3)$ consisting of $(\upsilon_1, ..., \upsilon_{r+k+w}) \in \mathbb{F}_q^{r+k+w}$ with exactly ρ nonzero components. Noticing that $\vartheta_1, ..., \vartheta_{r+k+w}$ are simultaneously zero, or simultaneously nonzero, one has $T(\rho) = \emptyset$ when $0 < \rho < r + k + w$. Hence,

$$T(l_1, l_2, l_3) = \bigcup_{\rho=0}^{r+k+w} T(\rho) = T(0) \cup T(r + k + w). \tag{3.6}$$

Now, applying Lemma 2.6, we have

$$\begin{aligned}
&\sum_{(\vartheta_1,...,\vartheta_{r+k+w})\in T(r+k+w)} \left|\left\{(x_1, ..., x_m) \in \mathbb{F}_q^m : x_1^{e_{i1}}...x_m^{e_{im}} = \vartheta_i, i \in \langle r + k + w\rangle\right\}\right| \\
=& \sum_{\substack{(\vartheta_1, ..., \vartheta_{r+k+w}) \in (\mathbb{F}_q^*)^{r+k+w} \\ (1.8) \text{ holds}}} \left|\left\{(x_1, ..., x_m) \in \mathbb{F}_q^m : x_1^{e_{i1}}...x_m^{e_{im}} = \vartheta_i, i \in \langle r + k + w\rangle\right\}\right| \\
=& \sum_{\substack{(\vartheta_1, ..., \vartheta_{r+k+w}) \in (\mathbb{F}_q^*)^{r+k+w} \\ (1.8) \text{ holds}}} \left|\left\{(x_1, ..., x_m) \in (\mathbb{F}_q^*)^m : x_1^{e_{i1}}...x_m^{e_{im}} = \vartheta_i, i \in \langle r + k + w\rangle\right\}\right| \\
&(\text{since } (\vartheta_1, ..., \vartheta_{r+k+w}) \in (\mathbb{F}_q^*)^{r+k+w} \text{ implying } (x_1, ..., x_m) \in (\mathbb{F}_q^*)^m) \\
=& \mathcal{N}(q-1)^{m-v'} \prod_{j=1}^{v'} \gcd(q-1, g_j^{(E_2)}).
\end{aligned} \tag{3.7}$$

It readily follows that if at least one of l_1, l_2 and l_3 is nonzero, then $T(0) = \emptyset$, and so by (3.5) to (3.7), one has

$$\begin{aligned}
N(V) =& \sum_{(\vartheta_1,...,\vartheta_{r+k+w})\in T(0)\cup T(r+k+w)} |\{(x_1, ..., x_m) \in \mathbb{F}_q^m : x_1^{e_{i1}}...x_m^{e_{im}} = \vartheta_i, i \in \langle r + k + w\rangle\}| \\
=& \sum_{(\vartheta_1,...,\vartheta_{r+k+w})\in T(r+k+w)} |\{(x_1, ..., x_m) \in \mathbb{F}_q^m : x_1^{e_{i1}}...x_m^{e_{im}} = \vartheta_i, i \in \langle r + k + w\rangle\}| \\
=& \mathcal{N}(q-1)^{m-v'} \prod_{j=1}^{v'} \gcd(q-1, g_j^{(E_2)}).
\end{aligned}$$

If $l_1 = l_2 = l_3 = 0$, then by using (3.5) to (3.7), we derive that

$$\begin{aligned}
N(V) =& \sum_{(\vartheta_1,...,\vartheta_{r+k+w})\in T(0)\cup T(r+k+w)} |\{(x_1, ..., x_m) \in \mathbb{F}_q^m : x_1^{e_{i1}}...x_m^{e_{im}} = \vartheta_i, i \in \langle r + k + w\rangle\}| \\
=& \sum_{(\vartheta_1,...,\vartheta_{r+k+w})\in T(0)} |\{(x_1, ..., x_m) \in \mathbb{F}_q^m : x_1^{e_{i1}}...x_m^{e_{im}} = \vartheta_i, i \in \langle r + k + w\rangle\}| \\
&+ \sum_{(\vartheta_1,...,\vartheta_{r+k+w})\in T(r+k+w)} |\{(x_1, ..., x_m) \in \mathbb{F}_q^m : x_1^{e_{i1}}...x_m^{e_{im}} = \vartheta_i, i \in \langle r + k + w\rangle\}| \\
=& |\{(x_1, ..., x_m) \in \mathbb{F}_q^m : x_1^{e_{i1}}...x_m^{e_{im}} = 0, i \in \langle r + k + w\rangle\}| \\
&+ \mathcal{N}(q-1)^{m-v'} \prod_{i=1}^{v'} \gcd(q-1, g_i^{(E_2)}) \\
=& |\{(x_1, ..., x_m) \in \mathbb{F}_q^m : x_1^{e_{11}}...x_m^{e_{1m}} = 0\}| + \mathcal{N}(q-1)^{m-v'} \prod_{i=1}^{v'} \gcd(q-1, g_i^{(E_2)})
\end{aligned}$$

$$
\begin{aligned}
=&|\{(x_1,...,x_m)\in\mathbb{F}_q^m : x_1...x_m=0\}| + \mathscr{N}(q-1)^{m-v'}\prod_{i=1}^{v'}\gcd(q-1,g_i^{(E_2)})\\
=&\sum_{j=1}^{m}\binom{m}{j}(q-1)^{m-j} + \mathscr{N}(q-1)^{m-v'}\prod_{i=1}^{v'}\gcd(q-1,g_i^{(E_2)})\\
=&q^m-(q-1)^m + \mathscr{N}(q-1)^{m-v'}\prod_{i=1}^{v'}\gcd(q-1,g_i^{(E_2)})
\end{aligned}
$$

as expected.

This finishes the proof of Theorem 1.2. □

Proof of Theorem 1.3. Taking determinants on both sides of (1.7), one has

$$\det(U_2)\det(E_2)\det(V_2) = g_1^{(E_2)}...g_m^{(E_2)}.$$

Because $\det(U_2)=\pm1$ and $\det(V_2)=\pm1$, the condition $\gcd(q-1,\det(E_2))=1$ guarantees that

$$\gcd(q-1,g_i^{(E_2)})=1 \text{ for all } i\in\langle m\rangle.$$

This ensures that (1.9) is satisfied.

Noting that

$$\mathscr{N} = \sum_{\substack{(v_1,...,v_{r+k+w})\in(\mathbb{F}_q^*)^{r+k+w} \text{ such that (1.8) and (1.9) hold}}} 1 = N(l_1,l_2,l_3) \tag{3.8}$$

with $N(l_1,l_2,l_3)$ being given as in (2.3), it follows from (1.11) that

$$N(V)=\begin{cases} q^m-(q-1)^m+\mathscr{N} & \text{if } l_1=l_2=l_3,\\ \mathscr{N} & \text{otherwise.}\end{cases} \tag{3.9}$$

Therefore, by the identities (3.8) and (3.9), the desired result (1.12) follows immediately.

This concludes the proof of Theorem 1.3. □

4. Examples

In this section, we give four examples to demonstrate the validity of Theorems 1.1 to 1.3.

Example 4.1. *We calculate the number $N(V)$ of rational points on the variety*

$$\begin{cases} f_1(x_1,...,x_5) = x_1x_2^2x_3^3x_4^4x_5^5 + x_1^2x_2^2x_3^4x_4^5x_5 - 2 = 0,\\ f_2(x_1,...,x_5) = x_1^3x_2^4x_3^2x_4^3x_5 + x_1^2x_2^5x_3^4x_4^2x_5^2 + x_1^2x_2^3x_3x_4^2x_5^2 = 0\end{cases}$$

over $\mathbb{F}_{11}$.

Clearly, we have

$$b_1 = 2, b_2 = 0, q = 11, q - 1 = 10, s = 2, t = 3, n = 5,$$

and

$$E_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 2 & 4 & 5 & 1 \\ 3 & 4 & 2 & 3 & 1 \\ 2 & 5 & 4 & 2 & 2 \\ 2 & 3 & 1 & 2 & 2 \end{pmatrix}.$$

Since $\det(E_1) = 9$, one derives that $\gcd(q-1, \det(E_1)) = 1$. By Theorem 1.1, we can calculate and obtain that

$$N(V) = \frac{1}{11^2}((11-1)^2 - (-1)^2)((11-1)^3 + (-1)^3 \cdot (11-1)) = 810.$$

Example 4.2. *We compute the number $N(V)$ of rational points on the variety*

$$\begin{cases} f_1(x_1, ..., x_6) = x_1 x_2^2 x_3^3 x_4^4 x_5^5 x_6 + x_1^2 x_2^2 x_3^4 x_4^5 x_5 x_6 = 1, \\ f_2(x_1, ..., x_6) = x_1^3 x_2^4 x_3^2 x_4^3 x_5 x_6^2 + x_1^2 x_2^5 x_3^4 x_4^2 x_5^2 x_6 = 2, \\ f_3(x_1, ..., x_6) = x_1 x_2^2 x_3 x_4^2 x_5^3 x_6 + x_1^2 x_2^3 x_3 x_4^2 x_5^2 x_6 = 0 \end{cases} \tag{4.1}$$

over $\mathbb{F}_7$.

Evidently, we have

$$l_1 = l_2 = l_3 = 0, q = 7, m = 6, r = k = w = 2$$

and

$$E_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 1 \\ 2 & 2 & 4 & 5 & 1 & 1 \\ 3 & 4 & 2 & 3 & 1 & 2 \\ 2 & 5 & 4 & 2 & 2 & 1 \\ 1 & 2 & 1 & 2 & 3 & 1 \\ 2 & 3 & 1 & 2 & 2 & 1 \end{pmatrix}.$$

Hence, $\det(E_2) = -22$, one observes that $\gcd(q-1, \det(E_2)) \neq 1$.

By using Maple, we can find two unimodular matrices

$$U_2 = \begin{pmatrix} 1 & 1 & -5 & 0 & -7 & 10 \\ 3 & -6 & 18 & -1 & 9 & -26 \\ 5 & -6 & 16 & -1 & 3 & -21 \\ -5 & 6 & -15 & 1 & -2 & 19 \\ 4 & -5 & 13 & -1 & 3 & -17 \\ 7 & -10 & 28 & -2 & 9 & -38 \end{pmatrix}$$

and

$$V_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 0 & -15 \\ 0 & 0 & 1 & 0 & 0 & -12 \\ 0 & 0 & 0 & 1 & 0 & 11 \\ 0 & 0 & 0 & 0 & 1 & -10 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

such that

$$U_2E_2V_2 = \mathrm{SNF}(E_2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 22 \end{pmatrix}.$$

Thus,

$$g_1^{(E_2)} = g_2^{(E_2)} = g_3^{(E_2)} = g_4^{(E_2)} = g_5^{(E_2)} = 1, g_6^{(E_2)} = 22 \text{ and } v' = 6.$$

Still using Maple, we compute and get that the number $\mathscr{N}$ of vectors $(\upsilon_1, ..., \upsilon_6) \in (\mathbb{F}_q^*)^6$ with

$$\begin{cases} \upsilon_1 + \upsilon_2 = 0 \\ \upsilon_3 + \upsilon_4 = 0 \\ \upsilon_5 + \upsilon_6 = 0 \end{cases}$$

under the extra restriction (1.9) is equal to 108. Thus, by Theorem 1.2, we have

$$N(V) = 7^6 - 6^6 + 108 \times 2 = 71209.$$

Example 4.3. *We compute the number $N(V)$ of rational points on the variety*

$$\begin{cases} f_1(x_1, ..., x_6) = x_1x_2^2x_3^3x_4^4x_5^5x_6^4 + x_1^{11}x_2^5x_3^4x_4^5x_5x_6^4 = 0, \\ f_2(x_1, ..., x_6) = x_1^3x_2^4x_3^2x_4^3x_5x_6^3 + x_1^7x_2^3x_3^5x_4^2x_5x_6^4 = 0, \\ f_3(x_1, ..., x_6) = x_1^2x_2^6x_3^3x_4^2x_5^2x_6^3 + x_1^8x_2^2x_3^{11}x_4^5x_5^3x_6^5 = 0 \end{cases}$$

over $\mathbb{F}_{13}$.

Obviously, we have

$$l_1 = l_2 = l_3 = 0, q = 13, q - 1 = 12, m = 6, r = k = w = 2$$

and

$$E_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 4 \\ 11 & 5 & 4 & 5 & 1 & 4 \\ 3 & 4 & 2 & 3 & 1 & 3 \\ 7 & 3 & 5 & 2 & 1 & 4 \\ 2 & 6 & 3 & 2 & 2 & 3 \\ 8 & 2 & 11 & 5 & 3 & 5 \end{pmatrix}.$$

Since $\det(E_2) = 4387 = 41 \times 107$, we deduce that $\gcd(q - 1, \det(E_2)) = 1$. By Theorem 1.3, we compute and obtain that

$$N(V) = 13^6 - 12^6 + \frac{1}{13^3}(12^2 + 12)(12^2 + 12)(12^2 + 12) = 1842553.$$

Example 4.4. *We calculate the number $N(V)$ of rational points on the variety*

$$\begin{cases} f_1(x_1,...,x_7) = x_1x_2^2x_3^3x_4^4x_5^5x_6^4x_7^5 + x_1^2x_2^5x_3^4x_4^5x_5x_6^4x_7^4 - 2 = 0, \\ f_2(x_1,...,x_7) = x_1^3x_2^4x_3^2x_4^3x_5x_6^3x_7^2 + x_1^2x_2^3x_3^5x_4^2x_5x_6^4x_7^5 + x_1^2x_2^6x_3^3x_4^2x_5^2x_6^3x_7^5 = 0, \\ f_3(x_1,...,x_7) = x_1^2x_2^2x_3^3x_4^3x_5^5x_6^3x_7^3 + x_1^2x_2^2x_3^4x_4^5x_5^3x_6^5x_7^3 = 0 \end{cases}$$

over $\mathbb{F}_{11}$.

It is clear that

$$l_1 = 2, l_2 = l_3 = 0, q = 11, q-1 = 10, m = 7, r = 2, k = 3, w = 2$$

and

$$E_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 4 & 5 \\ 2 & 5 & 4 & 5 & 1 & 4 & 4 \\ 3 & 4 & 2 & 3 & 1 & 3 & 2 \\ 2 & 3 & 5 & 2 & 1 & 4 & 5 \\ 2 & 6 & 3 & 2 & 2 & 3 & 5 \\ 2 & 2 & 3 & 3 & 5 & 3 & 3 \\ 2 & 2 & 4 & 5 & 3 & 5 & 3 \end{pmatrix}.$$

Thus, $\det(E_2) = 957$ which infers that $\gcd(q-1, \det(E_2)) = 1$. Therefore, by employing Theorem 1.3, we compute and obtain that

$$N(V) = \frac{1}{11^3}((11-1)^2 - (-1)^2)((11-1)^3 + (-1)^3(11-1))((11-1)^2 + (-1)^2(11-1)) = 8100.$$

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The research of M. Li was supported partially by National Science Foundation of China (Grant No. 12071375).

Conflict of interest

The authors declare that there is no conflict of interest in this paper.

References

1. J. Ax, Zeros of polynomials over finite fields, *Amer. J. Math.*, **86** (1964), 255–261. https://doi.org/10.2307/2373163
2. W. Cao, Q. Sun, A reduction for counting the number of zeros of general diagonal equation over finite fields, *Finite Fields Appl.* **12** (2006), 681–692. https://doi.org/10.1016/j.ffa.2005.07.001

3. W. Cao, Q. Sun, On a class of equations with special degrees over finite fields, *Acta Arith.*, **130** (2007), 195–202. https://doi.org/10.4064/aa130-2-8

4. L. Carlitz, Pairs of quadratic equations in a finite field, *Amer. J. Math.*, **76** (1954), 137–154. https://doi.org/10.2307/2372405

5. S. Chowla, J. Cowles, M. Cowles, On the number of zeros of diagonal cubic forms, *J. Number Theory*, **9** (1977), 502–506. https://doi.org/10.1016/0022-314X(77)90010-5

6. Y. L. Feng, S. F. Hong, Improvements of p-adic estimates of exponential sums, *Proc. Amer. Math. Soc.*, **150** (2022), 3687–3698. https://doi.org/10.1090/proc/15995

7. S. F. Hong, L-functions of twisted diagonal exponential sums over finite fields, *Proc. Amer. Math. Soc.*, **135** (2007), 3099–3108. https://doi.org/10.1090/s0002-9939-07-08873-9

8. S. F. Hong, C. X. Zhu, On the number of zeros of diagonal cubic forms over finite fields, *Forum Math.*, **33** (2021), 697–708. https://doi.org/10.1515/forum-2020-0354

9. S. N. Hu, S. F. Hong, W. Zhao, The number of rational points of a family of hypersurfaces over finite fields, *J. Number Theory*, **156** (2015), 135–153. https://doi.org/10.1016/j.jnt.2015.04.006

10. S. N. Hu, X. E. Qin, J. Y. Zhao, Counting rational points of an algebraic variety over finite fields, *Results Math.*, **74** (2019), 37. https://doi.org/10.1007/s00025-019-0962-6

11. S. N. Hu, J. Y. Zhao, The number of rational points of a family of algebraic variety over finite fields, *Algebra Colloq.*, **24** (2017), 705–720. https://doi.org/10.1142/S1005386717000475

12. L. K. Hua, *Introduction to Number Theory,* Springer-Verlag, Berlin-Heidelberg, 1982. https://mathscinet.ams.org/mathscinet/article?mr=665428

13. H. Huang, W. Gao, W. Cao, Remarks on the number of rational points on a class of hypersurfaces over finite fields, *Algebra Colloq.*, **25** (2018), 533–540. https://doi.org/10.1142/S1005386718000366

14. L. K. Hua, H. S. Vandiver, Characters over certain types of rings with applications to the theory of equations in a finite field, *Proc. Nat. Acad. Sci.*, **35** (1949), 94–99. https://doi.org/10.1073/pnas.35.2.94

15. G. Myerson, On the number of zeros of diagonal cubic forms, *J. Number Theory*, **11** (1979), 95–99. https://doi.org/10.1016/0022-314X(79)90023-4

16. D. R. Richman, Some remarks on the number of solutions to the equation $f(x_1) + ... + f(x_n) = 0$, *Stud. Appl. Math.*, **71** (1984), 263–266. https://doi.org/10.1002/sapm1984713263

17. Q. Sun, On diagonal equations over finite fields, *Finite Fields Appl.*, **3** (1997), 175–179. https://doi.org/10.1006/ffta.1996.0173

18. Q. Sun, On the formula of the number of solutions of some equations over finite fields, *Chinese Ann. Math. Ser. A*, **18** (1997), 403–408.

19. D. Q. Wan, Zeros of diagonal equations over finite fields, *Proc. Amer. Math. Soc.*, **103** (1988), 1049–1052. https://doi.org/10.1090/s0002-9939-1988-0954981-2

20. W. S. Wang, Q. Sun, The number of solutions of certain equations over a finite field, *Finite Fields Appl.*, **11** (2005), 182–192. https://doi.org/10.1016/j.ffa.2004.06.004

21. W. S. Wang, Q. Sun, An explicit formula of solution of some special equations over a finite field, *Chinese Ann. Math. Ser. A*, **26** (2005), 391–396.

22. A. Weil, On some exponential sums, *Proc. Natl. Acad. Sci. USA*, **34** (1948), 204–207. https://doi.org/10.1073/pnas.34.5.204

23. J. Wolfmann, The number of solutions of certain diagonal equations over finite fields, *J. Number Theory*, **42** (1992), 247–257. https://doi.org/10.1016/0022-314X(92)90091-3

24. J. Wolfmann, New results on diagonal equations over finite fields from cyclic codes, in: *Finite Fields: Theory, Applications, and Algorithms* (Las Vegas, NV, 1993), Contemp. Math., **168**, Amer. Math. Soc., Providence, RI, (1994), 387–395. https://dx.doi.org/10.1090/conm/168

25. J. Y. Zhao, Y. L. Feng, S. F. Hong, C. X. Zhu, On the number of zeros of diagonal quartic forms over finite fields, *Forum Math.*, **34** (2022), 385–405. https://doi.org/10.1515/forum-2021-0196

26. C. X. Zhu, Y. L. Feng, S. F. Hong, J. Y. Zhao, On the number of zeros to the equation $f(x_1) + ... + f(x_n) = a$ over finite fields, *Finite Fields Appl.*, **78** (2021), 101922. https://doi.org/10.1016/j.ffa.2021.101922

27. G. Y. Zhu, S. A. Hong, On the number of rational points of certain algebraic varieties over finite fields, *Forum Math.*, **35** (2023), 1511–1532. https://doi.org/10.1515/forum-2022-0324

AIMS Press