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*Research article*

# Solutions for Schrödinger equations with variable separated type nonlinear terms

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**Abstract:** In this paper, we consider the following semilinear Schrödinger equation:

$$\begin{cases} -\Delta u + V(x)u = a(x)g(u) & \text{for } x \in \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty, \end{cases}$$

where $a(x) > 0$ for all $\mathbb{R}^N$. Under some different superlinear conditions on $g(u)$, we obtain the existence of solutions for the above problem. In order to regain the compactness of the Sobolev embedding, a competing condition between $a(x)$ and $V(x)$ is introduced.

**Keywords:** Schrödinger equations; compact embedding theorem; variable separated nonlinearities; superlinear
**Mathematics Subject Classification:** 35Q55, 47J30

## 1. Introduction and main results

In this paper, we consider the existence of solutions for the following semilinear Schrödinger equation:

$$\begin{cases} -\Delta u + V(x)u = f(x,u) & \text{for } x \in \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{1.1}$$

Due to its important applications in mathematical physics, Eq (1.1) receives much attention from mathematicians to look for its solutions. For example, (1.1) is also known as the Gross-Pitaevskii equation, which can be simulated in the Bose-Einstein condensate (see [3]). In high dimension, this equation has also been considered by some physicians (see [6]).

In the last two decades, with the development of variational methods and critical points theory, many mathematicians used the variational methods to show the existence and multiplicity of solutions for problem (1.1) and obtained many interesting results [1, 2, 4, 5, 7–12, 15, 17, 19, 20, 22]. Using this method to deal with problem (1.1), one of the difficulties is to get compactness of the embedding from the working space to $L^2(\mathbb{R}^N)$. The periodic and coercive conditions are introduced to regain the compactness. In this paper, we mainly consider the coercive case. The following coercive conditions on $V(x)$ is first introduced by Rabinowitz in [9].

$(V_1)$ $V \in C^1(\mathbb{R}^N, \mathbb{R})$ and there exists a $\overline{V} > 0$ such that $V(x) \geq \overline{V}$ for all $x \in \mathbb{R}^N$;

$(V_2)$ $V(x) \to \infty$ as $|x| \to \infty$.

However, $(V_1)$ and $(V_2)$ are so strong that many functions cannot be involved with them. Then, many mathematicians tried to relax these conditions. For example, in [16], $V$ is required to be of $C$ class and $V(x) \geq -V_0$ with $V_0 > 0$, which generalized condition $(V_1)$. In order to generalize condition $(V_2)$, Bartsch and Wang in [2] introduced the following condition:

$(V_3)$ $\inf_{x\in\mathbb{R}^N} V(x) > 0$ and for every $M > 0$, the set $\Sigma_M = \{x \in \mathbb{R}^N : V(x) < M\}$ has finite Lebesgue measure.

Condition $(V_3)$ has been used by many mathematicians to obtain the existence and multiplicity of solutions for problem (1.1). Under $(V_3)$, $V$ may not have a limit at infinity. In 2000, Sirakov [11] introduced the following condition on $V$ to guarantee the compactness of embedding.

$(V_4)$ For any $r > 0$ and any sequence $\{x_n\} \subset \mathbb{R}^N$ which goes to infinity,

$$\lim_{n\to\infty} \inf_{u\in A_n} \int_{B_n} (|\nabla u|^2 + V(x)u^2)dx = +\infty,$$

where $A_n = \{u \in H_0^1(B_n) | \|u\|_{L^2(B_n)} = 1\}$ and $B_n = B(x_n, r)$ is the open ball with center $x_n$ and the radius $r$.

It has been shown in [11] that condition $(V_4)$ is weaker than $(V_2)$ and $(V_3)$. Moreover, $V$ is allowed to change sign. In [11], $f$ is required to satisfy the following growth condition:

$(AR)$ There exists $\iota > 2$ such that

$$tf(x,t) \geq \iota F(x,t) = \iota \int_0^t f(x,v)dv > 0 \quad \text{for all } \ x \in \mathbb{R}^N \ \text{ and } \ t \in \mathbb{R} \setminus \{0\}.$$

Condition $(AR)$ is a classical condition introduced by Ambrosetti and Rabinowitz, which provides a global growth condition of $f$ at both origin and infinity. $(AR)$ also plays an important role in showing the boundedness of Palais-Smale sequences and the geometrical structure of the corresponding function. However, the $(AR)$ condition is so strict that many functions do not satisfy this condition. By replacing $(AR)$ with the following condition, Wan and Tang [16] obtained existence of solutions for problem (1.1).

$(MC)$ there exists a constant $\theta \geq 1$ such that $\theta\widetilde{F}(x,t) \geq \widetilde{F}(x,st)$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ and $s \in [0,1]$, where $\widetilde{F}(x,t) = f(x,t)t - 2F(x,t)$.

In this paper, we consider a class of variable separated nonlinear functions that has received limited attention from researchers, as mentioned in [11]. Our purpose is to establish the existence of solutions for (1.1) by introducing novel conditions to replace $(AR)$ and $(MC)$. Additionally, we provide examples to highlight the distinctions between our theorems and prior ones. Precisely, we assume that $f$ is a variable separated function defined as follows:

$$f(x,t) = a(x)g(t), \tag{1.2}$$

where $a(x)$ is allowed to go to zero at infinity.

Let $G(t) = \int_0^t g(v)dv$. Now we state our main results.

**Theorem 1.1.** *Suppose that (1.2) and the following conditions hold:*

$(g_1)$ *There exists $V_0 > 0$ such that $V(x) \geq V_0$ for all $x \in \mathbb{R}^N$;*

$(g_2)$ *$a(x) \in L^\infty_{loc}(\mathbb{R}^N)$ and $a(x) > 0$ for all $x \in \mathbb{R}^N$;*

$(g_3)$ *$\frac{a(x)}{V(x)} \to 0$ as $|x| \to \infty$;*

$(g_4)$ *There exist $\nu > 2$ and $d_1, \rho_\infty > 0$ such that*

$$g(t)t - \nu G(t) \geq -d_1 t^2 \quad \text{for all} \quad |t| \geq \rho_\infty;$$

$(g_5)$ *There exists $d_2 > 0$ such that $G(t) \geq -d_2 t^2$ for all $t \in \mathbb{R}$;*

$(g_6)$ *$g(t) = o(|t|)$ as $t \to 0$;*

$(g_7)$ *$G(t)/t^2 \to +\infty$ as $|t| \to \infty$;*

$(g_8)$ *There exist $\beta > 1$ and $d_3 > 0$ such that*

$$a(x) \leq d_3(V(x)^{\frac{1}{\beta}} + 1) \quad \text{for all} \quad x \in \mathbb{R}^N.$$

$(g_9)$ *There exist $\zeta \in (2, \beta^*)$ and $d_4 > 0$ such that*

$$|g(t)| \leq d_4(|t| + |t|^{\zeta - 1}) \quad \text{for all} \quad t \in \mathbb{R},$$

*where $\beta^* = \frac{2N}{N-2} - \frac{4}{\beta(N-2)}$ if $N \geq 3$, $\beta^* = +\infty$ if $N = 1, 2$.*

*Then, problem (1.1) possesses at least one nontrivial solution.*

**Remark 1.1.** *Condition $(g_3)$ is a mixed condition and the function $a(x) = \frac{1}{1+|x|^2}$ is allowed in Theorem 1.1 if $V(x) = 1$, which means $a(x)$ can vanish at infinity. In some most recent paper, the authors also considered the vanishing cases. In 2020, Toon and Ubilla [13] obtianed the existence of positive solution for Schrödinger equation, where $V(x)$ is required to be vanishing at infinity and $(g_3)$ is also needed. By strengthening $(g_3)$ with*

$(g_3')$ *For any $\delta \in (0, 1], \omega(x) := a(x)V^{-\delta}(x) > 0$ satisfies $\omega(x) \to 0$ almost everywhere (a.e.) as $|x| \to \infty$.*

*Toon and Ubilla [14] obtained solutions for a class of Hamiltonian systems of Schrödinger equations. However, in our theorem, we remove the vanishing property of $V$ and only need the competition condition $(g_3)$. In another paper, Wu, Li and Lin [18] introduced a new coercive condition on $V$ to obtain the existence of (1.1) with asymptotically linear nonlinearities. Our theorems can not involved in above results since, besides $(g_3)$, we also introduced some new superlinear conditions. In the following remark, we give some examples to show the differences.*

**Remark 1.2.** *As we know, there are many superlinear condition on $f$, which are weaker than the $(AR)$ condition. However, in most papers, the following condition is required:*

$(SQ)$ *$\widetilde{G}(t) \triangleq g(t)t - 2G(t) \geq 0$ for any $t \in \mathbb{R}$.*

*In Theorem 1.1, we drop this condition.*

**Theorem 1.2.** *Suppose that (1.2), $(g_1)$–$(g_3)$, $(g_6)$–$(g_9)$, $(SQ)$ and the following conditions hold:*

$(g_{10})$ *There exist constants $d_5, l_\infty > 0$ and $\kappa > \frac{\beta^*}{\beta^*-2}$ such that*

$$\widetilde{G}(t) \geq d_5 \left( \frac{|G(t)|}{t^2} \right)^\kappa \quad \text{for all} \quad |t| \geq l_\infty.$$

*Then, problem (1.1) possesses at least one nontrivial solution.*

**Remark 1.3.** *Condition* $(g_{10})$ *is introduced by Ding and Luan [4], which is used by many mathematicians to obtain the existence and multiplicity of solutions for problem (1.1).*

**Theorem 1.3.** *Suppose that (1.2),* $(g_1)$–$(g_3)$*,* $(g_6)$–$(g_9)$*,* $(SQ)$ *and the following condition holds:*
$(g_{11})$ *there exist constants* $\mu > \beta^*$, $\lambda_0 \in (0,1)$, $d_6$, $d_7 > 0$, $s \in [2, \beta^*)$ *and* $r_\infty > 0$ *such that*

$$\frac{1-\lambda^2}{2} g(t)t + G(\lambda t) - G(t) \geq -d_6 \lambda^\mu |t|^{\beta^*} - d_7 \lambda^s t^s, \quad \forall \lambda \in [0, \lambda_0],\ |t| \geq r_\infty.$$

*Then, problem (1.1) possesses at least one nontrivial solution.*

**Remark 1.4.** *When* $d_6 = d_7 = 0$ *and* $r_\infty = 0$*,* $(g_{11})$ *goes back to the condition introduced by Tang in [12]. As the author said in [12],* $(g_{11})$ *unifies the* $(AR)$ *and the following weak Nehari type condition:*
$(WN)$ $t \mapsto g(t)/|t|$ *is increasing on* $(-\infty, 0) \cup (0, \infty)$.
*By an easy computation, we see that* $(g_{11})$ *is weaker than* $(g_4)$.

**Remark 1.5.** *From* $(g_1)$–$(g_3)$*, there exists* $A > 0$ *such that*

$$a(x) \leq AV(x) \quad \text{for all} \quad x \in \mathbb{R}^{\mathbb{N}}.$$

**Remark 1.6.** *There are examples satisfying conditions of Theorems 1.3, but not* $(g_4)$ *or* $(g_{10})$*. Setting* $2 < p < 2^*$, $0 < \epsilon < p-2$*, consider*

$$G(t) = |t|^p + a(p-2)|t|^{p-\epsilon} \sin^2(|t|^\epsilon/\epsilon). \tag{1.3}$$

*For any* $\gamma > 2$*, let* $\max\left\{0, \frac{p-\gamma}{p-2}\right\} < a < 1$ *and* $t_n = \left(\epsilon\left(n\pi + \frac{3\pi}{4}\right)\right)^{1/\epsilon}$*, then,*

$$\begin{aligned}
&\frac{g(t_n)t_n - \gamma G(t_n)}{t_n^2} \\
= \ & \frac{1}{t_n^2}\left[(p-\gamma)|t_n|^p + a(p-2)(p-\gamma-\epsilon)|t_n|^{p-\epsilon}\sin^2(|t_n|^\epsilon/\epsilon) + a(p-2)|t_n|^p \sin(2|t_n|^\epsilon/\epsilon)\right] \\
= \ & |t_n|^{p-2}\left[(p-\gamma) - a(p-2) + \frac{a(p-2)(p-\gamma-\epsilon)}{2|t_n|^\epsilon}\right] \\
\leq \ & \frac{1}{2}\left[(p-\gamma) - a(p-2)\right]|t_n|^{p-2} \\
\to \ & -\infty \ \ \text{as} \ \ n \to \infty.
\end{aligned}$$

*Hence, (1.3) does not satisfy* $(g_4)$*. Moreover, for* $|t|$ *large enough and any* $\kappa > 1$*, we have*

$$\begin{aligned}
&\left(\frac{1}{2}g(t)t - G(t)\right)\left(\frac{t^2}{|G(t)|}\right)^\kappa \\
\leq \ & 2^{\kappa-1}(p-2)|t|^{p-\kappa(p-2)}\left[(1 + a\sin(2|t|^\epsilon/\epsilon)) + \frac{a(p-2-\epsilon)\sin^2(|t|^\epsilon/\epsilon)}{|t|^\epsilon}\right] \\
\leq \ & 2^\kappa (1+a)(p-2)|t|^{p-\kappa(p-2)}.
\end{aligned}$$

*If* $p > \frac{2N}{N-2} - \frac{4}{\beta(N-2)} = \beta^*$ *and* $\kappa > \frac{\beta^*}{\beta^*-2}$*, we can deduce that* $p - \kappa(p-2) < 0$*. Then, we can not find* $d_5 > 0$ *such that* $(g_{10})$ *holds. Next, we show that (1.3) satisfies* $(g_{11})$*. Obviously,*

$$
\begin{aligned}
& \frac{\lambda^2}{2} g(t)t - G(\lambda t) \\
= & \left[\frac{\lambda^2}{2} p - \lambda^p\right] |t|^p + a(p-2)\left[\frac{\lambda^2}{2}(p-\epsilon)\sin^2(|t|^\epsilon/\epsilon) - \lambda^{p-\epsilon}\sin^2(|\lambda t|^\epsilon/\epsilon)\right]|t|^{p-\epsilon} \\
& + \frac{\lambda^2 a(p-2)}{2}|t|^p \sin(2|t|^\epsilon/\epsilon) \\
\leq & \frac{\lambda^2}{2}\left(p + a(p-2)(p-\epsilon) + a(p-2)\right)|t|^p,
\end{aligned}
$$

*and for all $t \in \mathbb{R}$,*

$$
\begin{aligned}
\frac{1}{2} g(t)t - G(t) &= \frac{p-2}{2}|t|^p \left[(1 + a\sin(2|t|^\epsilon/\epsilon)) + \frac{a(p-2-\epsilon)\sin^2(|t|^\epsilon/\epsilon)}{|t|^\epsilon}\right] \\
&\geq \frac{p-2}{2}|t|^p \left[1 + a\sin(2|t|^\epsilon/\epsilon)\right] \\
&\geq \frac{1}{4}(1-a)(p-2)|t|^p,
\end{aligned}
$$

*which implies*

$$
\frac{1-\lambda^2}{2} g(t)t + G(\lambda t) - G(t) \geq \frac{1}{8}(1-a)(p-2)|t|^p
$$

*for $\lambda$ small enough. Hence, we see $(g_{11})$ is fulfilled with $d_6 = d_7 = 1$.*

## 2. Preliminaries

Set

$$
E := \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) dx < \infty \right\}
$$

with the inner product

$$
\langle u, v \rangle = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V(x)uv) dx
$$

and the norm $\|u\|^2 = \langle u, u \rangle$. Then $E$ is a Hilbert space. For any $2 \leq p \leq 2^*$, we denote

$$
\|u\|_p = \left( \int_{\mathbb{R}^N} |u|^p dx \right)^{1/p},
$$

where $2^* = \frac{2N}{N-2}$ if $N \geq 3$, $2^* = +\infty$ if $N = 1, 2$. Since we have $(g_1)$, the embedding theorem shows that $E \hookrightarrow L^p(\mathbb{R}^N)$ continuously for $p \in [2, 2^*]$, which implies that there exists a constant $C_p > 0$ such that

$$
\|u\|_p \leq C_p \|u\| \tag{2.1}
$$

for all $u \in E$. For any $b(x) \geq 0$ and $2 \leq q \leq 2^*$, let $L^q_b(\mathbb{R}^N, \mathbb{R})$ be a weighted space of measure functions under the norm as follow:

$$
\|u\|_{L^q_b} = \left( \int_{\mathbb{R}^N} b(x)|u|^q dx \right)^{1/q}. \tag{2.2}
$$

**Lemma 2.1.** *Under assumptions* $(g_1)$, $(g_3)$ *and* $(g_8)$, *the embedding* $E \hookrightarrow L_a^q(\mathbb{R}^N, \mathbb{R})$ *is continuous for all* $q \in [2, \beta^*]$ *and compact for all* $q \in [2, \beta^*)$.

*Proof.* Since $2 < \beta^* \leq 2^*$, by $(g_8)$ and (2.1), for any $u \in L_a^q(\mathbb{R}^N, \mathbb{R})$ with $q \in [2, \beta^*]$, we obtain

$$\begin{aligned} \|u\|_{L_a^q}^q &= \int_{\mathbb{R}^N} a(x)|u|^q dx \\ &\leq d_3 \left( \int_{\mathbb{R}^N} V(x)^{\frac{1}{\beta}} |u|^q dx + \int_{\mathbb{R}^N} |u|^q dx \right) \\ &\leq d_3 \left( \int_{\mathbb{R}^N} V(x) u^2 dx \right)^{\frac{1}{\beta}} \left( \int_{\mathbb{R}^N} |u|^{\frac{\beta q - 2}{\beta - 1}} dx \right)^{\frac{\beta - 1}{\beta}} + d_3 C_q^q \|u\|^q \\ &\leq d_3 \left( C_{\frac{\beta q - 2}{\beta - 1}}^{\frac{\beta q - 2}{\beta}} + C_q^q \right) \|u\|^q, \end{aligned}$$

which implies that the embedding is continuous. Moreover, there exists a constant $K_p > 0$ such that

$$\|u\|_{L_a^q} \leq K_p \|u\| \tag{2.3}$$

for all $u \in E$ and $q \in [2, \beta^*]$. Next, we prove the compactness of the embedding. Let $\{u_k\} \subset E$ be a sequence such that $u_k \rightharpoonup u$ in $E$. Subsequently, we show that $u_k \to u$ in $L_a^q(\mathbb{R}^N, \mathbb{R})$ for all $q \in [2, \beta^*)$. By Banach-Steinhaus theorem, there exists $M_1 > 0$ such that

$$\sup_{k \in \mathbb{N}} \|u_k\| \leq M_1 \quad \text{and} \quad \|u\| \leq M_1. \tag{2.4}$$

It follows from $(g_3)$ that for any $\varepsilon > 0$, there exists $T > 0$ such that

$$a(x) \leq \varepsilon V(x) \tag{2.5}$$

for all $|x| \geq T$. We can deduce from (2.4) and (2.5) that

$$\begin{aligned} \int_{|x| \geq T} a(x)|u_k - u|^2 dx &\leq \varepsilon \int_{|x| \geq T} V(x)|u_k - u|^2 dx \leq 2\varepsilon \int_{|x| \geq T} V(x)(u_k^2 + u^2) dx \\ &\leq 2\varepsilon(\|u_k\|^2 + \|u\|^2) \leq 4M_1 \varepsilon. \end{aligned} \tag{2.6}$$

for all $k \in \mathbb{N}$. Moreover, by Sobolev's theorem, there exists $k_0 > 0$ such that

$$\int_{|x| \leq T} a(x)|u_k - u|^2 dx \leq \varepsilon \tag{2.7}$$

for all $k \geq k_0$. From (2.6) and (2.7), we obtain $u_k \to u$ in $L_a^2(\mathbb{R}^N, \mathbb{R})$ as $k \to \infty$, which shows that the embedding from $E$ to $L_a^2(\mathbb{R}^N, \mathbb{R})$ is compact. By the Gagliardo-Nirenberg inequality the embedding from $E$ to $L_a^q(\mathbb{R}^N, \mathbb{R})$ is also compact for $q \in (2, \beta^*)$. □

The corresponding functional of (1.1) is defined on $E$ by

$$\begin{aligned} I(u) &= \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x) u^2) dx - \int_{\mathbb{R}^N} F(x, u) dx \\ &= \frac{1}{2} \|u\|^2 - \int_{\mathbb{R}^N} a(x) G(u) dx. \end{aligned} \tag{2.8}$$

**Lemma 2.2.** *Suppose that* $(g_1)$, $(g_3)$, $(g_5)$, $(g_6)$, $(g_8)$ *and* $(g_9)$ *hold, then the functional I is well defined and of* $C^1$ *class with*

$$\langle I'(u), v\rangle = \langle u, v\rangle - \langle \psi'(u), v\rangle, \tag{2.9}$$

*for all* $v \in E$, *where* $\psi(u) = \int_{\mathbb{R}^N} F(x, u)dx$. *Moreover, the critical points of I in E are solutions for problem (1.1).*

*Proof.* First, we show $I$ is well defined. By $(g_6)$, for any $\varepsilon > 0$, there exists $\sigma > 0$ such that

$$|g(t)| \le \varepsilon|t|, \qquad |t| \le \sigma. \tag{2.10}$$

We can deduce from (2.10), $(g_6)$ and $(g_9)$, for any $\varepsilon > 0$, there exists $M_\varepsilon > 0$ such that

$$|g(t)| \le \varepsilon|t| + M_\varepsilon|t|^{\zeta-1}, \qquad \forall t \in \mathbb{R}, \tag{2.11}$$

and

$$|G(t)| \le \varepsilon t^2 + M_\varepsilon|t|^{\zeta}, \qquad \forall t \in \mathbb{R}. \tag{2.12}$$

By (2.3) and (2.12), we have

$$\begin{aligned}\int_{\mathbb{R}^N} |F(x, u)|dx &= \int_{\mathbb{R}^N} a(x)|G(u)|dx \\ &\le \varepsilon \int_{\mathbb{R}^N} a(x)u^2 dx + M_\varepsilon \int_{\mathbb{R}^N} a(x)|u|^{\zeta} dx \\ &\le \varepsilon K_2^2\|u\|^2 + M_\varepsilon K_\zeta^\zeta \|u\|^\zeta \\ &< \infty,\end{aligned}$$

which means that $I$ is well defined. It is standard to see that $I$ is $C^1$ on $E$ and (2.9) holds. □

From Lemma 2.2, we can obtain

$$\langle I'(u), u\rangle = \|u\|^2 - \int_{\mathbb{R}^N} f(x, u)udx. \tag{2.13}$$

For the reader's convenience, we state the classical Mountain Pass Theorem as follow.

**Lemma 2.3.** *(Mountain Pass Theorem, see [10], Theorem 2.2) Let E be a real Banach space and* $I : \mathbb{R} \to \mathbb{R}^N$ *be a* $C^1$*-smooth functional and satisfy the* $(C)$ *condition that is,* $(u_j)$ *has a convergent subsequence in* $W^{1,2}(\mathbb{R}, \mathbb{R}^N)$ *whenever* $\{I(u_j)\}$ *is bounded and* $\|I'(u_j)\|(1 + \|u_j\|) \to 0$ *as* $n \to \infty$. *If*

*(i)* $I(0) = 0$;

*(ii) There exist constants* $\varrho, \alpha > 0$ *such that* $I_{|\partial B_\varrho(0)} \ge \alpha$;

*(iii) There exists* $e \in E \setminus \bar{B}_\varrho(0)$ *such that* $I(e) \le 0$,

*where* $B_\varrho(0)$ *is an open ball in E of radius* $\varrho$ *centred at 0, then I possesses a critical value* $c \ge \alpha$ *given by*

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} I(g(s)),$$

*where*

$$\Gamma = \{g \in C([0, 1], E) : g(0) = 0, \;\; g(1) = e\}.$$

## 3. Proofs of the main results

In this section, we use the Mountain Pass Theorem to show the existence of critical points of $I$ which help us to prove Theorems 1.1–1.3. In Lemma 3.1, we show that the $(C)$ condition is fulfilled for $I$ under the conditions of Theorem 1.1. In the *Step 1* and *Step 2*, we show $I$ satisfies the conditions $(i)$–$(iii)$ in the Mountain Pass Theorem. In Lemmas 3.3 and 3.4, we show that $I$ satisfies the $(C)$ condition under the conditions of Theorems 1.2 and 1.3 respectively.

**Lemma 3.1.** *Suppose that (1.2) and* $(g_1)$–$(g_9)$ *hold, then I satisfies the* $(C)$ *condition.*

*Proof.* Assume that $\{u_n\} \subset E$ being a sequence such that $\{I(u_n)\}$ is bounded and $\|I'(u_n)\|(1 + \|u_n\|) \to 0$ as $n \to \infty$. Then, there exists a constant $M_2 > 0$ such that

$$|I(u_n)| \leq M_2, \quad \|I'(u_n)\|(1 + \|u_n\|) \leq M_2. \tag{3.1}$$

Now we prove that $\{u_n\}$ is bounded in $E$. Arguing in an indirect way, we assume that $\|u_n\| \to +\infty$ as $n \to \infty$. Set $z_n = \frac{u_n}{\|u_n\|}$, then $\|z_n\| = 1$, which implies that there exists a subsequence of $\{z_n\}$, still denoted by $\{z_n\}$, such that $z_n \rightharpoonup z_0$ in $E$. By (2.8) and (3.1), we get

$$\left|\int_{\mathbb{R}^N} \frac{F(x, u_n)}{\|u_n\|^2} dx - \frac{1}{2}\right| = \left|-\frac{I(u_n)}{\|u_n\|^2}\right| \leq \frac{M_2}{\|u_n\|^2}, \tag{3.2}$$

which implies that

$$\int_{\mathbb{R}^N} \frac{F(x, u_n)}{\|u_n\|^2} dx \to \frac{1}{2} \quad \text{as} \quad n \to \infty. \tag{3.3}$$

The following discussion is divided into two cases.

*Case* 1: $z_0 \not\equiv 0$. Let $\Omega = \{x \in \mathbb{R}^N |\, |z_0(x)| > 0\}$. Then we can see that $meas(\Omega) > 0$, where $meas$ denotes the Lebesgue measure. Then there exists $J > 0$ such that $meas(\Lambda) > 0$, where $\Lambda = \Omega \bigcap \Upsilon_J(0)$ and $\Upsilon_r(\bar{x}) = \{x \in \mathbb{R}^N : |x - \bar{x}| \leq r\}$. Since $\|u_n\| \to +\infty$ as $n \to \infty$ and $|u_n| = |z_n| \cdot \|u_n\|$, then we have $|u_n| \to +\infty$ as $n \to \infty$ for a.e. $x \in \Lambda$. Let $a_1 = \inf_{x \in \Upsilon_J(0)} a(x) > 0$. By (1.2), $(g_5)$, $(g_7)$, (3.1), Remark 1.5 and Fatou's lemma, we can obtain

$$\begin{aligned}
\liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{F(x, u_n)}{\|u_n\|^2} dx &= \liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{a(x)G(u_n)}{\|u_n\|^2} dx \\
&= \liminf_{n\to\infty} \int_{\Lambda} \frac{a(x)G(u_n)}{\|u_n\|^2} dx + \liminf_{n\to\infty} \int_{\mathbb{R}^N\setminus\Lambda} \frac{a(x)G(u_n)}{\|u_n\|^2} dx \\
&\geq a_1 \liminf_{n\to\infty} \int_{\Lambda} \frac{G(u_n)}{|u_n|^2} |z_n|^2 dx - d_2 \limsup_{n\to\infty} \int_{\mathbb{R}^N\setminus\Lambda} a(x)|z_n|^2 dx \\
&\geq a_1 \liminf_{n\to\infty} \int_{\Lambda} \frac{G(u_n)}{|u_n|^2} |z_n|^2 dx - d_2 A \limsup_{n\to\infty} \int_{\mathbb{R}^N\setminus\Lambda} V(x)|z_n|^2 dx \\
&\geq a_1 \liminf_{n\to\infty} \int_{\Lambda} \frac{G(u_n)}{|u_n|^2} |z_n|^2 dx - d_2 A \\
&= +\infty,
\end{aligned}$$

which contradicts (3.3). So $\|u_n\|$ is bounded in this case.

*Case* 2: $z_0 \equiv 0$. Set

$$\widehat{G}(t) = g(t)t - \nu G(t),$$

where $\nu$ is defined in ($g_4$). From (2.11) and (2.12), we can deduce that there exits $M_3 > 0$ such that

$$|\widehat{G}(t)| \leq M_3(t^2 + |t|^{\zeta}), \qquad \forall t \in \mathbb{R}. \tag{3.4}$$

It follows from (3.1), ($g_4$), (3.4) and Lemma 2.1 that

$$\begin{aligned}
o(1) &= \frac{\nu M_2 + M_2}{\|u_n\|^2} \\
&\geq \frac{\nu I(u_n) - \langle I'(u_n), u_n\rangle}{\|u_n\|^2} \\
&\geq \left(\frac{\nu}{2} - 1\right) + \frac{1}{\|u_n\|^2}\int_{\mathbb{R}^N} a(x)\widehat{G}(u_n)dx \\
&\geq \left(\frac{\nu}{2} - 1\right) + \frac{1}{\|u_n\|^2}\left(\int_{|u_n|\leq\rho_\infty} a(x)\widehat{G}(u_n)dx + \int_{|u_n|>\rho_\infty} a(x)\widehat{G}(u_n)dx\right) \\
&\geq \left(\frac{\nu}{2} - 1\right) - \frac{M_3}{\|u_n\|^2}\left(\int_{|u_n|\leq\rho_\infty} a(x)|u_n|^2dx + \int_{|u_n|\leq\rho_\infty} a(x)|u_n|^{\zeta}dx\right) - \frac{d_1}{\|u_n\|^2}\int_{|u_n|>\rho_\infty} a(x)|u_n|^2dx \\
&\geq \left(\frac{\nu}{2} - 1\right) - M_3\left(1 + \rho_\infty^{\zeta-2}\right)\int_{\mathbb{R}^N} a(x)|z_n|^2dx - d_1\int_{\mathbb{R}^N} a(x)|z_n|^2dx \\
&\to \left(\frac{\nu}{2} - 1\right) \quad \text{as } n \to \infty,
\end{aligned}$$

which is a contradiction. Hence, $\|u_n\|$ is still bounded in this case, which implies that $\{u_n\}$ is bounded in $E$. The following proof is similar to *Step 3* of the main proof in [16]. □

Subsequently, we show that $I$ possesses the Mountain Pass geometric structure under the conditions of Theorem 1.1. The proof is divided into two steps.

*Step 1.* We show that there exist constants $\varrho_1, \alpha_1 > 0$ such that $I\mid_{\partial B_{\varrho_1}(0)} \geq \alpha_1$. For $\varepsilon = \frac{1}{4A}$, it follows from ($g_8$), Remark 1.5 and (2.12) that

$$\begin{aligned}
I(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x,u)dx \\
&= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} a(x)G(u)dx \\
&\geq \frac{1}{2}\|u\|^2 - \varepsilon\int_{\mathbb{R}^N} a(x)u^2dx - M_\varepsilon\int_{\mathbb{R}^N} a(x)|u|^{\zeta}dx \\
&\geq \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_{\mathbb{R}^N} V(x)u^2dx - M_\varepsilon d_3\left(\int_{\mathbb{R}^N} V(x)^{\frac{1}{\beta}}|u|^{\zeta}dx + \int_{\mathbb{R}^N} |u|^{\zeta}dx\right) \\
&\geq \frac{1}{4}\|u\|^2 - M_\varepsilon d_3\left(\int_{\mathbb{R}^N} V(x)u^2dx\right)^{\frac{1}{\beta}}\left(\int_{\mathbb{R}^N} |u|^{\frac{\beta\zeta-2}{\beta-1}}dx\right)^{\frac{\beta-1}{\beta}} - M_\varepsilon d_3 C_\zeta^{\zeta}\|u\|^{\zeta} \\
&\geq \frac{1}{4}\|u\|^2 - M_\varepsilon d_3\left(C_{\frac{\beta\zeta-2}{\beta-1}}^{\frac{\beta\zeta-2}{\beta}} + C_\zeta^{\zeta}\right)\|u\|^{\zeta}.
\end{aligned}$$

It is easy to see that there exist positive constants $\varrho_1$ and $\alpha_1$ such that $I|_{\partial B_{\varrho_1}} \geq \alpha_1$. We finish the proof of this step.

*Step 2.* Now, we prove that there exists $\bar{e} \in E$ such that $\|\bar{e}\| > \varrho_1$ and $I(\bar{e}) \leq 0$. Set $e_0 \in C_0^\infty(\Upsilon_1(0), \mathbb{R})$ such that $\|e_0\| = 1$. Let $a_2 = \inf_{t\in\Upsilon_1(0)} a(x) > 0$ and $a_3 = \sup_{t\in\Upsilon_1(0)} a(x) > 0$. For $M_4 > \left(2a_2 \int_{\Upsilon_1(0)} |e_0|^2 dx\right)^{-1}$, it follows from $(g_7)$ that there exists $Q > 0$ such that

$$G(t) \geq M_4 t^2 \tag{3.5}$$

for all $|t| > Q$. Then, we can deduce from $(g_5)$ and (3.5) that

$$G(t) \geq M_4(t^2 - Q^2) - d_2 Q^2 \tag{3.6}$$

for all $t \in \mathbb{R}$. By (2.8) and (3.6), for every $\eta \in \mathbb{R}^+$, we have

$$\begin{aligned} I(\eta e_0) &= \frac{\eta^2}{2}\|e_0\|^2 - \int_{\mathbb{R}^N} a(x)G(\eta e_0)dx \\ &\leq \frac{\eta^2}{2} - \int_{\Upsilon_1(0)} a(x)[M_4(|\eta e_0|^2 - Q^2) - d_2 Q^2]dx \\ &\leq \left(\frac{1}{2} - M_4 a_2 \int_{\Upsilon_1(0)} |e_0|^2 dx\right)\eta^2 + a_3(M_4 + d_2)Q^2 meas\Upsilon_1(0), \end{aligned}$$

which implies that

$$I(\eta e_0) \to -\infty \quad \text{as} \quad \eta \to +\infty.$$

Hence, there exists $\eta_1 > 0$ such that $I(\eta_1 e_0) < 0$ and $\|\eta_1 e_0\| > \varrho_1$, which finish the proof of this step.

*Proof of Theorem 1.1.* It is known that the Mountain Pass Theorem still holds when the usual $(PS)$ condition is replaced by condition $(C)$. From the above proofs and Lemma 2.3 under $(C)$ condition, $I$ possesses a critical value $c \geq \alpha_1$ and a critical point $u_0$ such that $I(u_0) = c$, which means problem (1.1) has at least one nontrivial solution.

*Proof of Theorem 1.2.* In Theorem 1.2, we show the existence of solutions for problem (1.1) under growth condition $(g_{10})$. Similarly, we rewrite only the proof of Lemma 3.1 and the following proof is similar to that of Theorem 1.1.

**Lemma 3.2.** *Suppose that $(g_6)$ and $(SQ)$ hold, then $G(t) \geq 0$ for all $t \in \mathbb{R}$.*

*Proof.* The proof of this lemma is similar to that of Lemma 2.2 in [21]. □

**Lemma 3.3.** *Suppose that (1.2), $(g_1)$–$(g_3)$, $(g_6)$–$(g_{10})$ and $(SQ)$ hold, then $I$ satisfies the $(C)$ condition.*

*Proof.* Assume that $\{u_n\} \subset E$ being a sequence such that $\{I(u_n)\}$ is bounded and $\|I'(u_n)\|(1 + \|u_n\|) \to 0$ as $n \to \infty$. Then, there exists a constant $M_5 > 0$ such that

$$|I(u_n)| \leq M_5, \quad \|I'(u_n)\|(1 + \|u_n\|) \leq M_5. \tag{3.7}$$

Now we prove that $\{u_n\}$ is bounded in $E$. Arguing in an indirect way, we assume that $\|u_n\| \to +\infty$ as $n \to \infty$. Set $w_n = \frac{u_n}{\|u_n\|}$. Then $\|w_n\| = 1$ and there exists a subsequence of $\{w_n\}$, still denoted by $\{w_n\}$, such that $w_n \rightharpoonup w_0$ in $E$. By Lemma 2.1, we have

$$w_n \to w_0 \ \text{ in } L_a^q(\mathbb{R}^N) \quad \text{for any} \ \ q \in [2, \beta^*). \tag{3.8}$$

Similar to Lemma 3.1, we can obtain (3.3). The following proof is divided into two cases.

*Case 1:* $w_0 \neq 0$. The proof is similar to Lemma 3.1.

*Case 2:* $w_0 \equiv 0$. By (2.8), (2.9) and (3.7), we obtain

$$\begin{aligned} 2M_5 &\geq 2I(u_n) + \|I'(u_n)\|(1+\|u_n\|) \\ &\geq 2I(u_n) - \langle I'(u_n), u_n\rangle \\ &\geq \int_{\mathbb{R}^N} a(x)\widetilde{G}(u_n)dx. \end{aligned} \tag{3.9}$$

On one hand, by (2.12) and Lemma 2.1, we can deduce that

$$\int_{|u_n|<l_\infty} \frac{F(x,u_n)}{\|u_n\|^2}dx \leq \left(\varepsilon + M_\varepsilon l_\infty^{\zeta-2}\right)\int_{\mathbb{R}^N} a(x)|w_n|^2 dx \to 0 \quad \text{as} \quad n\to\infty. \tag{3.10}$$

On the other hand, it follows from $(g_{10})$, $(SQ)$ and Lemma 2.1 that

$$\begin{aligned} \int_{|u_n|\geq l_\infty} \frac{F(x,u_n)}{\|u_n\|^2}dx &= \int_{|u_n|\geq l_\infty}\left(a^{\frac{1}{\kappa}}(x)\frac{G(u_n)}{u_n^2}\right)\left(a^{\frac{\kappa-1}{\kappa}}(x)w_n^2\right)dx \\ &\leq \left(\int_{|u_n|\geq l_\infty} a(x)\left(\frac{|G(u_n)|}{u_n^2}\right)^\kappa dx\right)^{\frac{1}{\kappa}}\left(\int_{|u_n|\geq l_\infty} a(x)|w_n|^{\frac{2\kappa}{\kappa-1}}dx\right)^{\frac{\kappa-1}{\kappa}} \\ &\leq d_5^{\frac{1}{\kappa}}\left(\int_{|u_n|\geq l_\infty} a(x)\widetilde{G}(u_n)dx\right)^{\frac{1}{\kappa}}\left(\int_{|u_n|\geq l_\infty} a(x)|w_n|^{\frac{2\kappa}{\kappa-1}}dx\right)^{\frac{\kappa-1}{\kappa}} \\ &\leq d_5^{\frac{1}{\kappa}}\left(\int_{\mathbb{R}^N} a(x)\widetilde{G}(u_n)dx\right)^{\frac{1}{\kappa}}\left(\int_{\mathbb{R}^N} a(x)|w_n|^{\frac{2\kappa}{\kappa-1}}dx\right)^{\frac{\kappa-1}{\kappa}} \\ &\to 0 \quad \text{as} \quad n\to\infty. \end{aligned} \tag{3.11}$$

It follows from (3.10) and (3.11) that

$$\int_{\mathbb{R}^N} \frac{F(x,u_n)}{\|u_n\|^2}dx < \frac{1}{4}$$

for $n$ large enough, which contradicts (3.3). Then we can see that $\|u_n\|$ is bounded in $E$. The following proof is similar to $Step$ 3 of the main proof in [16]. □

*Proof of Theorem 1.3.* In Theorem 1.3, we replace condition $(g_4)$ by condition $(g_{11})$. Condition $(g_4)$ is only used in the proof of the boundedness of $(C)$ sequence. Hence, we rewrite only the proof of Lemma 3.1 and the following proof is similar to that of Theorem 1.1.

**Lemma 3.4.** *Suppose that (1.2), $(g_1)$–$(g_3)$, $(g_6)$–$(g_9)$, $(g_{11})$ and $(SQ)$ hold, then $I$ satisfies the $(C)$ condition.*

*Proof.* Assume that $\{u_n\}\subset E$ being a sequence such that $\{I(u_n)\}$ is bounded and $\|I'(u_n)\|(1+\|u_n\|)\to 0$ as $n\to\infty$. Then there exists a constant $M_6>0$ such that

$$|I(u_n)| \leq M_6, \qquad \|I'(u_n)\|(1+\|u_n\|) \leq M_6. \tag{3.12}$$

Now, we prove that $\{u_n\}$ is bounded in $E$. Arguing in an indirect way, we assume $\|u_n\| \to +\infty$ as $n \to \infty$. Set $w_n = \frac{u_n}{\|u_n\|}$. Similar to Lemma 3.3, we have (3.8). The following discussion is divided into two cases.

*Case 1:* $w_0 \not\equiv 0$. The proof is similar to Case 1 in Lemma 3.1.

*Case 2:* $w_0 \equiv 0$. Let $R = (2M_6 + 2)^{1/2}$. By (2.12), one can obtain

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |F(x, Rw_n)| dx \le \limsup_{n\to\infty} \int_{\mathbb{R}^N} a(x)(\varepsilon R^2 w_n^2 + M_\varepsilon |Rw_n|^\zeta) dx = 0. \tag{3.13}$$

Set $\lambda_n = \frac{R}{\|u_n\|}$. It follows from (3.12), (2.11), (2.12), $(g_{11})$, $(SQ)$, (3.13), (2.3) and (3.8) that

$$\begin{aligned}
M_6 &\ge I(u_n) \\
&= I(\lambda_n u_n) + \frac{1-\lambda_n^2}{2}\|u_n\|^2 + \int_{\mathbb{R}^N} (F(x,\lambda_n u_n) - F(x,u_n))\, dx \\
&= I(\lambda_n u_n) + \frac{1-\lambda_n^2}{2}\langle I'(u_n), u_n\rangle + \int_{\mathbb{R}^N} a(x)\left(\frac{1-\lambda_n^2}{2} g(u_n)u_n + G(\lambda_n u_n) - G(u_n)\right) dx \\
&= I(Rw_n) + \frac{1}{2}\left(1 - \frac{R^2}{\|u_n\|^2}\right)\langle I'(u_n), u_n\rangle + \int_{|u_n|\le r_\infty} a(x)\left(\frac{1-\lambda_n^2}{2} g(u_n)u_n + G(\lambda_n u_n) - G(u_n)\right) dx \\
&\quad + \int_{|u_n|\ge r_\infty} a(x)\left(\frac{1-\lambda_n^2}{2} g(u_n)u_n + G(\lambda_n u_n) - G(u_n)\right) dx \\
&\ge \frac{R^2}{2} - \int_{\mathbb{R}^N} F(x, Rw_n) dx + \int_{|u_n|\le r_\infty} a(x)\left(-\frac{\lambda_n^2}{2} g(u_n)u_n + G(\lambda_n u_n)\right) dx \\
&\quad -d_6 \int_{|u_n|\ge r_\infty} a(x)\lambda_n^\mu |u_n|^{\beta^*} dx - d_7 \int_{|u_n|\ge r_\infty} a(x)\lambda_n^s |u_n|^s dx + o(1) \\
&\ge \frac{R^2}{2} - M_7 \int_{|u_n|\le r_\infty} a(x)\left(\lambda_n^2|u_n|^2 + \lambda_n^2|u_n|^\zeta + \lambda_n^\zeta |u_n|^\zeta\right) dx \\
&\quad -d_6 \frac{R^\mu}{\|u_n\|^{\mu-\beta^*}} \int_{|u_n|\ge r_\infty} a(x)|w_n|^{\beta^*} dx - d_7 R^s \int_{|u_n|\ge r_\infty} a(x)|w_n|^s dx + o(1) \\
&\ge \frac{R^2}{2} - M_7 \int_{|u_n|\le r_\infty} a(x)\left(R^2|w_n|^2 + r_\infty^{\zeta-2}R^2|w_n|^2 + R^\zeta |w_n|^\zeta\right) dx - d_6 \frac{R^\mu K_{\beta^*}^{\beta^*}}{\|u_n\|^{\mu-\beta^*}} + o(1) \\
&= \frac{R^2}{2} + o(1) = M_6 + 1 + o(1)
\end{aligned}$$

for some $M_7 > 0$, which is a contradiction. Hence, $\|u_n\|$ is still bounded in this case, which implies that $\{u_n\}$ is bounded in $E$. The following proof is similar to *Step 3* of the main proof in [16]. □

## 4. Conclusions

In this paper, we obtain a compact embedding theorem by using a new competition condition on the potentials which involve the vanishing cases. Then, we show the existence of solutions for Schrödinger equations with different superlinear conditions via the Mountain Pass Theorem. Some examples are given to show the difference between our theorems and the results in previous works.

## Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare that they have no conflicts of interest.

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