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Research article

Intuitionistic fuzzy monotonic DOWA operators

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Abstract: A new measure for intuitionistic fuzzy numbers (IFNs) is proposed to reflect the magnitude of IFNs, and a novel ranking approach for IFNs is presented based on this measure. Furthermore, the theoretical basis of the ranking method is investigated, and several intuitionistic fuzzy monotonic dependent ordered weighted averaging (IFMDOWA) operators are developed, such as the conservative IFMDOWA (COV-IFMDOWA) operator, positive intuitionistic fuzzy monotonic DOWA (POS-IFMDOWA) operator, conservative intuitionistic fuzzy hybrid monotonic dependent order weighted averaging (COV-IFHMDOWA) operator, and positive intuitionistic fuzzy hybrid monotonic dependent order weighted averaging (POS-IFHMDOWA) operator. Finally, a numerical example is given to illustrate the flexibility of our proposed monotonic dependent order weighted averaging operators in a practical decision making process.

Keywords: intuitionistic fuzzy number; strength index; ranking method; MDOWA operator
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1. Introduction

In life, we are usually faced with situations in which we are required to analyze an issue from various perspectives. For instance, in multiple attribute decision-making (MADM) problems, we need to fuse multi-dimensional information into one-dimensional information in order to rank alternatives, and we finish the ranking in one-dimensional space. Therefore, we should first solve the problem of multi-dimensional information fusion. Sometimes, three-way decision-making [1–3] is an effective framework for information fusion. However, information aggregation operators constitute an more powerful tool for solving such problems [4]. The common aggregation operators include weighted averaging (WA) operator, weighted geometric averaging operator and ordered WA (OWA) operator [5]. The OWA operator presented by Yager [6] in 1988 is a widely used aggregation operator in MADM-related fields. The prominent feature of the OWA operator is that it does not weight the attribute values, but the ordered position of the attribute values. A lot of meaningful work concerning OWA operators has been presented in the past 30 years [7–9]. For example, some researchers have focused on discussing the orness measure [7, 10–13]. In addition, some scholars have focused on determining the OWA weights. Some effective methods of obtaining the OWA weights have been proposed [14,15]. For instance, Yager [7] developed a linguistic quantifier guided method to determine the OWA weights. Torra [16] presented an approach to obtain the weights of OWA operators by using sample learning. Ahn [13] derived some methods to obtain the OWA weights by using some functions of dimension n. Sang and Liu [17] employed the method of least squares deviation to obtain the corresponding weights. The most common method of generating the weights associated with OWA operators is the optimization-based method in which the orness level is given in advance [4, 18].

Moreover, some scholars have extended the OWA operator to suit other types of attribute values. For instance, Torra [19] developed the weighted OWA operator in which two weighting vectors are considered, i.e., the one associated with the WA operator and the one corresponding to the OWA operator. Yager [20] investigated the continuous interval number OWA operator in which the inputs are continuous intervals. Chen and Chen [21] introduced the FN-IOWA operators in which the given arguments are fuzzy numbers. Yager [22] presented the centered OWA operators. Xu [23] investigated intuitionistic fuzzy OWA operators. Herrera et al. [9] investigated the linguistic values-based OWA operators in which the aggregated elements are in the form of linguistic values. Xia and Xu [24] developed several hesitant fuzzy OWA operators in which the aggregated attribute values are hesitant fuzzy elements. Alcantud [25] combined multi-agent decision-making with N-soft sets to present novel OWA operators.

For convenience of application, OWA operators are usually divided into two categories: the OWA operators associated with constant weights and the OWA operators associated with changeable weights. The characteristic of the latter is that the weights of the OWA operator are independent of the arguments' ordering, but dependent on the aggregated values. Therefore, Xu [26] called this type the dependent OWA (DOWA) operator. The prominent advantage of the DOWA operator is that it is not necessary to give the weights in advance, since the weights are determined by the aggregated attribute values. Therefore, the DOWA operators can fully reflect the expert's risk preference and attitudinal characters. Xu [26] proposed some DOWA operators and discussed the methods for generating the corresponding weights of the DOWA operators. Xu and Da [27] proposed the uncertain OWA (UOWA) operator in which the criterion values are interval values. Xu [28] further presented several dependent UOWA operators in which the weights of the operators depend on the synthesized interval numbers. Wei [29] presented some dependent uncertain linguistic OWA operators in which the input elements are uncertain linguistic values. Zeng et al. [15] introduced a new kind of DOWA operator, namely, monotonic argument-dependent OWA (MDOWA) operators. The weights of MDOWA operators are generated by a collection of functions which are monotonic with respect to the corresponding variables. The salient characteristic of MDOWA operators is that they can easily grasp the varying trends of the weights through the vector function. Therefore, the MDOWA operators are very flexible in terms of their ability to represent the attitudes of the decision-makers handling complex information.

In real life, fuzzy set theory is a powerful tool for handling imprecise and uncertain information [30, 31]. However, with the progress and development of science and technology, the information that people need to process is becoming more and more complex. Thus, researchers have further developed some new theories and methods to handle the more complex information. As a generalization of fuzzy sets, intuitionistic fuzzy set (IFSs) developed by Atanassov [32], is more appropriate and effective than the ordinary fuzzy set in dealing with uncertain information. Alcantud [33] also proposed complemental fuzzy set theory for a complex world. Ever since Atanassov put forward the concept of the IFS, it has attracted a large amount of interest and attention from the scholars. Some researchers have accomplished a lot of meaningful work focused on this research topic. For example, several scholars investigated the methods of calculating distance between IFSs [34–36]. Moreover, Melo-Pinto et al. [37] presented some new approaches for image-thresholding based on IFSs.

Considering that intuitionistic fuzzy numbers (IFNs) play an important role in the process of handling with intuitionistic fuzzy information, Chen and Tan [38] defined the score function $s(\beta)$ for any IFN β. Feng et al. [39] proposed novel Minkowski weighted score functions for intuitionistic fuzzy values. Moreover, Hong and Choi [40] developed the accuracy function $h(\beta)$ to express the accuracy degree of β. By utilizing the score function and accuracy function, Xu [23] proposed a comparison law to compare any two IFNs. Furthermore, several new methods of ranking IFNs have been developed by researchers [41–43].

To fuse intuitionistic fuzzy information, Xu [23] proposed a series of intuitionistic fuzzy aggregation operators, such as the intuitionistic fuzzy WA (IFWA) operator, intuitionistic fuzzy OWA (IFOWA) operator and so on. To accommodate the intuitionistic fuzzy environment, by employing Einstein operations, Wang and Liu [44] established some novel intuitionistic fuzzy aggregation operators. Further, Zhao and Wei [45] developed several new Einstein hybrid aggregation operators in which the aggregated elements were IFNs. Xu [46] proposed several new power aggregation operators to aggregate IFNs. To solve multi-criteria decision-making problems in which the criteria values are IFNs, Chen et al. [47] presented two new IFN score functions for estimating the degree of suitability of the project; in their method, not only ae the membership degree, non-membership degree and the hesitance degree of an IFN considered, the decision-makers' preference is also taken into account. Focusing on developing new aggregation operators to aggregate IFNs, in this paper we introduce a kind of new measure for IFNs and present a novel ranking method for IFNs. The main idea of the presented ranking method is to compare the magnitudes of IFNs by giving a new measure to express the magnitude of each IFN. Some theoretical basis of the ranking method is investigated. Later, by using the proposed new IFN measure, we develop some monotonic DOWA operators to aggregate IFNs, including conservative intuitionistic fuzzy monotonic DOWA (COV-IFMDOWA) operator and positive intuitionistic fuzzy monotonic DOWA (POS-IFMDOWA) operator.

To this end, we have organized this paper as follows. In Section 2, some basic notions of OWA operators and IFSs are reviewed. In Section 3, a new IFN measure is proposed, and it can reflect the magnitude of IFN effectively. Based on the proposed measure and score function of IFNs, a new method for ranking IFNs is developed. The reasonability of the proposed ranking method is also investigated. In Section 4, several novel MDOWA operators for aggregating IFNs are developed. In Section 5, an application example is given to illustrate the flexibility of the developed operators. The conclusion is given in the last section.

2. Preliminaries

2.1. OWA operators

In what follows, we briefly introduce some results associated with OWA operators.

Definition 2.1. *[7] Let $t = (t_1, t_2, \cdots, t_m) \in [0, 1]^m$; the OWA operator is defined as follows:*

$$OWA(t_1, t_2, \cdots, t_m) = \sum_{k=1}^{m} w_k t_{\sigma(k)}, \tag{2.1}$$

where $W = (w_1, w_2, \cdots, w_m)$ is the weighting vector that satisfies that $w_k \geq 0, (k = 1, 2, \cdots, m)$, where $\sum_{k=1}^{m} w_k = 1$, $t_{\sigma(k)}$ is the kth largest of the t_k.

Suppose that the associated weighting vector of the OWA operator is $W = (w_1, w_2, \cdots, w_m)$; Yager [7] proposed the orness measure to reflect the decision maker's attitude as follows:

$$orness(W) = \frac{1}{m-1} \sum_{k=1}^{m} (n-k) w_k. \tag{2.2}$$

Yager [7] proposed several special OWA operators, in which the corresponding weights depend on the synthesized values. For convenience, Xu [26] called them DOWA operators. Furthermore

$$DOWA(t_1, t_2, \cdots, t_m) = \sum_{k=1}^{m} w_k(t_{\sigma(1)}, t_{\sigma(2)}, \cdots, t_{\sigma(m)}) t_{\sigma(k)}. \tag{2.3}$$

The weights associated with the DOWA operators are functions of argument elements. For convenience, Zeng et al. [15] called them changeable weights, in which every weight is a function of m-dimensions.

Yager [7] provided a collection of argument-dependent weights, as follows:

$$w_k(t_1, t_2, \cdots, t_m) = \frac{t_{\sigma(k)}^{\alpha}}{\sum_{l=1}^{m} t_{\sigma(l)}^{\alpha}}, \quad k = 1, 2, \cdots, m, \tag{2.4}$$

where $\alpha \in (-\infty, +\infty)$.

To expand the application range of DOWA operators, Zeng et al. [15] developed a kind of special DOWA operators which are able to control the varying trend of the weights easily.

Definition 2.2. *[15] Let $t = (t_1, t_2, \cdots, t_m) \in [0, 1]^m$; suppose that $T_k(t_1, t_2, \cdots, t_m) : [0, 1]^m \rightarrow [0, +\infty)(k = 1, 2, \cdots, m)$ satisfies the following properties:*

(a) Each function $T_k(t_1, t_2, \cdots, t_m)$ is continuous in every variable t_l, $k, l = 1, 2, \cdots, m$;

(b) Each function $T_k(t_1, t_2, \cdots, t_m)$ is monotonically decreasing on t_k $(k = 1, 2, \cdots, m)$;

(c) $T_k(t_{\sigma(1)}, t_{\sigma(2)}, \cdots, t_{\sigma(m)}) \leq T_j(t_{\sigma(1)}, t_{\sigma(2)}, \cdots, t_{\sigma(m)})$ if $k < j$,

where $t_{\sigma(k)}$ is the kth largest of the t_k. Then, we say that $T(t) = (T_1(t), T_2(t), \cdots, T_m(t))$ is a conservative function vector.

Remark 2.1. *We say that $T(t) = T(t_1, t_2, \cdots, t_m)$ is a positive function vector, if it satisfies the above condition (a) in Definition 2.2 and the following properties:*

(b′) $T_k(t_1, t_2, \cdots, t_m)$ is monotonically increasing on t_k $(k = 1, 2, \cdots, m)$;

(c′) $T_k(t_{\sigma(1)}, t_{\sigma(2)}, \cdots, t_{\sigma(m)}) \geq T_j(t_{\sigma(1)}, t_{\sigma(2)}, \cdots, t_{\sigma(m)})$ if $k < j$.

Both conservative and positive function vectors can express the overall influences of the synthesized values. For convenience, we simply call them function vectors.

Let $t = (t_1, t_2, \cdots, t_m)$, and $T(t) = (T_1(t), T_2(t), \cdots, T_m(t))$ be a function vector of dimension m; Zeng et al. [15] developed the changeable weights associated with DOWA operator by using $T(t)$ as follows:

$$w_k^T(t) = \frac{T_k(\sigma(t))}{\sum_{j=1}^{n} T_j(\sigma(t))}, \quad k = 1, 2, \cdots, m, \tag{2.5}$$

where $t_{\sigma(k)}$ is the kth largest of t_k.

A characteristic of the above changeable weighting vector $w(t) = (w_1^T(t), w_2^T(t), \cdots, w_m^T(t))$ is that $w_i^T(t) \leq w_j^T(t)(i < j)$ if $T(t)$ is a conservative function vector, and $w_i^T(t) \geq w_j^T(t)$ $(i < j)$ if $T(t)$ is a positive function vector. Therefore, Zeng et al. [15] proposed the MDOWA operator as follows:

$$MDOWA^T(t_1, t_2, \cdots, t_m) = \sum_{k=1}^{m} w_k^T(t) t_{\sigma(k)} = \frac{\sum_{k=1}^{m} T_k(\sigma(t)) t_{\sigma(k)}}{\sum_{j=1}^{m} T_j(\sigma(t))}, \tag{2.6}$$

where $\sigma(t) = (t_{\sigma(1)}, t_{\sigma(2)}, \cdots, t_{\sigma(m)})$ and $t_{\sigma(k)}$ is the kth largest of t_k.

If $T(t)$ is a conservative function vector, we call the above MDOWA operator a conservative MDOWA operator associated with T, expressed as the COV-MDOWA^T operator. Similarly, if $T(t)$ is a positive function vector, we call the MDOWA operator a positive MDOWA operator associated with T, expressed as the POS-MDOWA^T operator.

Definition 2.3. *[15] Suppose that $M(t) = M(t_1, t_2, \cdots, t_m) : [0, 1]^m \to R$ is a function. Let*

$$T_k^M(t) = \frac{\partial M(t)}{\partial t_k}, \quad k = 1, 2, \cdots, m. \tag{2.7}$$

1) If $T^M(t) = (T_1^M(t), T_2^M(t), \cdots, T_m^M(t))$ forms a conservative function vector, then we call $M(t)$ conservative fundamental function;

2) If $T^M(t) = (T_1^M(t), T_2^M(t), \cdots, T_m^M(t))$ forms a positive function vector, we call $M(t)$ a positive fundamental function.

Assume that $M(t)$ is a fundamental function, then, $W^M(t) = (w_1^M(x), w_2^M(t), \cdots, w_m^M(t))$ expresses the weighting vector of the MDOWA operator generated by $T^M(t)$ according to Eq (2.7). MDOWA^M expresses the MDOWA operator which has an associated changeable weighting vector $W^M(t)$.

2.2. IFS

Definition 2.4. *[32] An IFS in a given set Y is denoted by*

$$I = \{\langle y, \mu_I(y), \nu_I(y)\rangle | y \in Y\}, \tag{2.8}$$

where $\mu_I : Y \to [0, 1]$ and $\nu_I : Y \to [0, 1]$ are the membership and non-membership functions, respectively, with the condition that $0 \leq \mu_I(y) + \nu_I(y) \leq 1$. $\pi_I(y) = 1 - \mu_I(y) - \nu_I(y)$ is called the degree of indeterminacy.

For simplicity, Xu [23] simply denoted $\beta = \langle \mu_\beta, \nu_\beta \rangle$ *and called it IFN. Assuming that* $\beta_1 = \langle \mu_{\beta_1}, \nu_{\beta_1} \rangle$ *and* $\beta_2 = \langle \mu_{\beta_2}, \nu_{\beta_2} \rangle$ *are two IFNs, Xu and Yager [48] developed several operations, as follows:*

(1) $\beta_1 \oplus \beta_2 = \langle \mu_{\beta_1} + \mu_{\beta_2} - \mu_{\beta_1}\mu_{\beta_2}, \nu_{\beta_1}\nu_{\beta_2} \rangle$;

(2) $\beta_1 \otimes \beta_2 = \langle \mu_{\beta_1}\mu_{\beta_2}, \nu_{\beta_1} + \nu_{\beta_2} - \nu_{\beta_1}\nu_{\beta_2} \rangle$;

(3) $\gamma\beta = \langle 1 - (1-\mu_\beta)^\gamma, \nu_\beta^\gamma \rangle, \gamma > 0$;

(4) $\beta^\gamma = \langle \mu_\beta^\gamma, 1 - (1-\nu_\beta)^\gamma \rangle, \gamma > 0$.

Using the above operation laws, Xu [23] introduced several intuitionistic fuzzy averaging operators as follows:

Assume that $\beta_1 = \langle \mu_{\beta_1}, \nu_{\beta_1} \rangle, \beta_2 = \langle \mu_{\beta_2}, \nu_{\beta_2} \rangle, \cdots, \beta_m = \langle \mu_{\beta_m}, \nu_{\beta_m} \rangle$ *are a collection of IFNs;*

$$IFWA(\beta_1, \beta_2, \cdots, \beta_m) = \bigoplus_{k=1}^{m} (w_k \beta_k) = \langle 1 - \prod_{k=1}^{m} (1-\mu_{\beta_k})^{w_k}, \prod_{k=1}^{m} (\nu_{\beta_k})^{w_k} \rangle, \tag{2.9}$$

$$IFOWA(\beta_1, \beta_2, \cdots, \beta_m) = \bigoplus_{k=1}^{m} (w_k \beta_{\sigma(k)}) = \langle 1 - \prod_{k=1}^{m} (1-\mu_{\beta_{\sigma(k)}})^{w_k}, \prod_{k=1}^{m} (\nu_{\beta_{\sigma(k)}})^{w_k} \rangle, \tag{2.10}$$

where $w = (w_1, w_2, \cdots, w_m)$ *is the weighting vector that satisfies the conditions that* $w_k \geq 0$*, where* $\sum_{k=1}^{m} w_k = 1$*, and* $\beta_{\sigma(k)}$ *is the kth largest of* β_k.

Ranking IFNs is an important research issue in applications of IFSs. In what follows, we introduce several existing ranking methods for IFNs that have been presented by some researchers.

Method 1: Natural ranking method

The ranking of IFNs was proposed by Atanassov [32], and it is called a natural ranking method of IFNs.

Definition 2.5. *[32] Let* $\beta_1 = \langle \mu_{\beta_1}, \nu_{\beta_1} \rangle$ *and* $\beta_2 = \langle \mu_{\beta_2}, \nu_{\beta_2} \rangle$ *be two IFNs; the natural ranking method of IFNs is defined as follows:*

$\beta_1 \geq \beta_2$ *if and only if* $\mu_{\beta_1} \geq \mu_{\beta_2}$ *and* $\nu_{\beta_1} \leq \nu_{\beta_2}$.

Method 2: Score function based ranking method

To compare any two IFNs, Chen and Tan [38] presented the concept of the score function $s(\beta)$ for IFN β.

Definition 2.6. *[38] For a given IFN* $\beta = \langle \mu_\beta, \nu_\beta \rangle$*, we define the score function of* β *as follows:*

$$s(\beta) = \mu_\beta - \nu_\beta. \tag{2.11}$$

Furthermore, Hong and Choi [40] proposed the accuracy function $h(\beta)$ *for IFN* β *as follows.*

Definition 2.7. *[40] For a given IFN* $\beta = \langle \mu_\beta, \nu_\beta \rangle$*, we define the accuracy function of* β *as follows:*

$$h(\beta) = \mu_\beta + \nu_\beta. \tag{2.12}$$

To compare IFNs β_1 *and* β_2*, by employing* $s(\beta)$ *and* $h(\beta)$*, Xu [23] developed the following comparison laws:*

(1) If $s(\beta_1) > s(\beta_2)$*, then* $\beta_1 > \beta_2$.

(2) If $s(\beta_1) = s(\beta_2)$*, then the following holds:*

(a) If $h(\beta_1) > h(\beta_2)$*, then* $\beta_1 > \beta_2$.

(b) If $h(\beta_1) = h(\beta_2)$*, then* $\beta_1 = \beta_2$.

3. Novel method for ranking IFNs

3.1. Strength index for IFNs

Given an IFN $\alpha = \langle \mu_\alpha, \nu_\alpha \rangle$. It is easy to understand that each IFN can be seen as a point in a rectangular coordinate system. However, it is clear that each IFN can also be seen as a point in the polar coordinate system. Let

$$r_\alpha = \sqrt{\mu_\alpha^2 + \nu_\alpha^2}; \tag{3.1}$$

then, there exists an angle θ_α such that

$$\mu_\alpha = r_\alpha \cos(\theta_\alpha), \tag{3.2}$$

$$\nu_\alpha = r_\alpha \sin(\theta_\alpha), \tag{3.3}$$

where $0 \le \theta_\alpha \le \frac{\pi}{2}$

Hence, for any IFN $\alpha = \langle \mu_\alpha, \nu_\alpha \rangle$, it is associated with a pair of $(r_\alpha, \theta_\alpha)$ which is called the strength vector of α. For convenience, we call r_α the strength of α. Apparently, θ_α determines the direction of r_α. The relationship between (μ_α, ν_α) and $(r_\alpha, \theta_\alpha)$ is denoted by Eqs (3.1)–(3.3).

Yager [49] proposed the concept of a Pythagorean fuzzy set, which is a generalization of an IFS. To rank Pythagorean fuzzy values, Yager [49] introduced a measure to represent the magnitude of Pythagorean fuzzy values. Motivated by the idea presented by Yager in Ref. [49], in what follows, we proposed a measure to reflect the magnitude of IFNs.

Definition 3.1. *Let $\alpha = \langle \mu_\alpha, \nu_\alpha \rangle$ be an IFN, in which the strength vector is $(r_\alpha, \theta_\alpha)$. We define the strength index of α as follows:*

$$V(\alpha) = \frac{1}{2} + r_\alpha\left(\frac{1}{2} - \frac{2\theta_\alpha}{\pi}\right). \tag{3.4}$$

3.2. Properties of strength index

In what follows, we investigate the related properties of $V(\alpha)$.

Property 3.1. $0 \le V(\alpha) \le 1$. $V(\alpha) = 0$ *if and only if* $\alpha = \langle 0, 1 \rangle$; $V(\alpha) = 1$ *if and only if* $\alpha = \langle 1, 0 \rangle$.

Proof. Since $0 \le \theta_\alpha \le \frac{\pi}{2}$, and $0 \le r_\alpha \le 1$, $-\frac{1}{2} \le \frac{1}{2} - \frac{2\theta_\alpha}{\pi} \le \frac{1}{2}$. Therefore, we have that $0 \le V(\alpha) \le 1$. If $\alpha = \langle 0, 1 \rangle$, then $r_\alpha = 1$ and $\theta_\alpha = \frac{\pi}{2}$. Hence $V(\alpha) = 0$. On the other hand, if $V(\alpha) = 0$, then $r_\alpha = 1$ and $\theta_\alpha = \frac{\pi}{2}$. By using Eqs (3.2) and (3.3), we have that $\mu_\alpha = 0$ and $\nu_\alpha = 1$.

Similarly, we can prove that $V(\alpha) = 1$ if and only if $\alpha = \langle 1, 0 \rangle$. □

Property 3.2. $V(\alpha) \ge \frac{1}{2}$ *if and only if* $\mu_\alpha \ge \nu_\alpha$; $V(\alpha) \le \frac{1}{2}$ *if and only if* $\mu_\alpha \ge \nu_\alpha$; $V(\alpha) = \frac{1}{2}$ *if and only if* $\mu_\alpha = \nu_\alpha$.

Proof. By using Eq (3.4), the proof is straightforward. □

Lemma 3.1. *Let* $f(x) = \frac{\frac{\pi}{4} - x}{\cos(x)} (0 \le x \le \frac{\pi}{4})$*; then,* $f(x)$ *is strictly monotonically decreasing with respect to* x.

Proof. $f'(x) = \frac{-\cos(x)+(\frac{\pi}{4}-x)\sin(x)}{\cos^2(x)}$. Let $g(x) = -\cos(x) + (\frac{\pi}{4} - x)\sin(x)(0 \le x \le \frac{\pi}{4})$; then, $g'(x) = (\frac{\pi}{4} - x)\cos(x) \ge 0(0 \le x \le \frac{\pi}{4})$. Hence, $g(x)$ is monotonically increasing with respect to x. Since $g(\frac{\pi}{4}) = -\frac{\sqrt{2}}{2} < 0$, $g(x) < 0$. Therefore, $f'(x) < 0$. Thus, $f(x)$ is strictly monotonically decreasing with respect to x.

This completes the proof of Lemma 3.1. □

Theorem 3.1. *Let $\alpha = \langle\mu_\alpha, \nu_\alpha\rangle$ and $\beta = \langle\mu_\beta, \nu_\beta\rangle$ be two IFNs; if $\mu_\alpha \ge \mu_\beta$ and $\nu_\alpha \le \nu_\beta$, then $V(\alpha) \ge V(\beta)$.*

Proof. Assume that $\alpha = \langle\mu_\alpha, \nu_\alpha\rangle$ and $\beta = \langle\mu_\beta, \nu_\beta\rangle$, where $\mu_\alpha = r_\alpha\cos(\theta_\alpha)$, $\nu_\alpha = r_\alpha\sin(\theta_\alpha)$ and $\mu_\beta = r_\beta\cos(\theta_\beta)$, $\nu_\beta = r_\beta\sin(\theta_\beta)$. If $\mu_\alpha \ge \mu_\beta$ and $\nu_\alpha \le \nu_\beta$, then $\theta_\alpha \le \theta_\beta$. By utilizing Eq (3.4), we have that $V(\alpha) = \frac{1}{2} + \frac{2r_\alpha}{\pi}(\frac{\pi}{4} - \theta_\alpha)$ and $V(\beta) = \frac{1}{2} + \frac{2r_\beta}{\pi}(\frac{\pi}{4} - \theta_\beta)$. Hence,

$$V(\alpha) - V(\beta) = \frac{2r_\alpha}{\pi}(\frac{\pi}{4} - \theta_\alpha) - \frac{2r_\beta}{\pi}(\frac{\pi}{4} - \theta_\beta). \tag{3.5}$$

According to the situations of the values of θ_α and θ_β, the proof can be divided into the following three cases:

Case 1. $\theta_\alpha \le \frac{\pi}{4} \le \theta_\beta$.

By using Eq (3.5), we have that $V(\alpha) - V(\beta) \ge 0$; then, $V(\alpha) \ge V(\beta)$.

Case 2. $\theta_\alpha \le \theta_\beta \le \frac{\pi}{4}$.

If $r_\alpha \ge r_\beta$, by using Eq (3.5), we have that $V(\alpha) - V(\beta) \ge 0$; then, $V(\alpha) \ge V(\beta)$.

If $r_\alpha < r_\beta$, suppose that $\gamma = \langle\mu_\gamma, \nu_\gamma\rangle$ is an IFN, where $\mu_\gamma = \mu_\beta$ and $\nu_\gamma = \nu_\alpha$. Let $r_\gamma = \sqrt{\mu_\gamma^2 + \nu_\gamma^2}$, $\mu_\gamma = r_\gamma\cos(\theta_\gamma)$, $\nu_\gamma = r_\gamma\sin(\theta_\gamma)$. Then, $\theta_\alpha < \theta_\gamma < \theta_\beta$ and $r_\gamma < r_\alpha < r_\beta$. Thus, we have that $V(\alpha) \ge V(\gamma)$. In addition, $r_\gamma(\frac{\pi}{4} - \theta_\gamma) = \frac{\mu_\gamma}{\cos(\theta_\gamma)}(\frac{\pi}{4} - \theta_\gamma)$ and $r_\beta(\frac{\pi}{4} - \theta_\beta) = \frac{\mu_\beta}{\cos(\theta_\beta)}(\frac{\pi}{4} - \theta_\beta)$. Since $\theta_\gamma < \theta_\beta$ and $\mu_\beta = \mu_\gamma$, by Lemma 1, we have $\frac{\mu_\gamma}{\cos(\theta_\gamma)}(\frac{\pi}{4} - \theta_\gamma) > \frac{\mu_\beta}{\cos(\theta_\beta)}(\frac{\pi}{4} - \theta_\beta)$, that is $r_\gamma(\frac{\pi}{4} - \theta_\gamma) > r_\beta(\frac{\pi}{4} - \theta_\beta)$. Hence, we obtain $V(\gamma) \ge V(\beta)$. Therefore, we have that $V(\alpha) \ge V(\beta)$.

Case 3. $\frac{\pi}{4} \le \theta_\alpha \le \theta_\beta$.

It can be proved analogously to Case 2. Thus, we have completed the proof of this theorem.

Theorem 3.1 denotes that the strength index $V(\alpha)$ of α can be used to compare the magnitude of α. □

3.3. Strength index-based ranking method for IFNs

In this subsection, based on the strength index $V(\beta)$ and the score function $s(\beta)$ for IFNs, we present a novel approach for ranking IFNs as follows:

For any two IFNs $\beta_1 =< \mu_{\beta_1}, \nu_{\beta_1} >$ and $\beta_2 =< \mu_{\beta_2}, \nu_{\beta_2} >$, the following holds:

(1) If $V(\beta_1) > V(\beta_2)$, then $\beta_1 > \beta_2$.

(2) If $V(\beta_1) = V(\beta_2)$, then we have the following:

(a) If $s(\beta_1) > s(\beta_2)$, then $\beta_1 > \beta_2$;

(b) If $s(\beta_1) = s(\beta_2)$, then $\beta_1 = \beta_2$.

Example 3.1. *Let $\alpha_1 = \langle 0.5, 0.4\rangle$ and $\alpha_2 = \langle 0.4, 0.3\rangle$. By applying Eq (3.4), we obtain that $V(\alpha_1) =$ 0.5451 and $V(\alpha_2) =$ 0.5452. Then $V(\alpha_2) > V(\alpha_1)$. By utilizing our proposed ranking approach, we have that $\alpha_2 > \alpha_1$.*

Example 3.2. *Let* $\alpha = \langle 0.4\cos(\frac{\pi}{10}), 0.4\sin(\frac{\pi}{10})\rangle$ *and* $\beta = \langle 0.3\cos(\frac{\pi}{20}), 0.3\sin(\frac{\pi}{20})\rangle$ *be two IFNs. By using Eq (3.4), we have that* $V(\alpha) = 0.62$*,* $V(\beta) = 0.62$*. And, given Eq (2.11), we have that* $s(\alpha) = 0.2568$ *and* $s(\beta) = 0.2494$*. Since* $s(\alpha) > s(\beta)$*, then* $\alpha > \beta$*.*

Example 3.3. *Let* $\alpha_3 =< 0.4, 0.01 >$ *and* $\beta_3 =< 0.5, 0.1 >$ *be two IFNs. By using Eq (2.11), we have that* $s(\alpha_3) = 0.39$ *and* $s(\beta_3) = 0.4$*. By using Eq (3.4), we have* $V(\alpha_3) = 0.6937, V(\beta_3) = 0.6908$*. When we use the score function, we have that* $s(\alpha_3) < s(\beta_3)$*. When we use the strength index, we have that* $V(\alpha_3) > V(\beta_3)$*.*

4. Intuitionistic fuzzy monotonic dependent OWA operators

4.1. Intuitionistic fuzzy monotonic dependent OWA operators

Let $C_1, C_2, \ldots, C_m$ be m criteria and $G = (\beta_1, \beta_2, \ldots, \beta_m)$ be the corresponding criterion values, where $\beta_k = \langle \mu_{\beta_k}, \nu_{\beta_k}\rangle$ is an IFN expressing the value of the criterion $C_k (k = 1, 2, \ldots, m)$. $V(G) = (V(\beta_1), V(\beta_2), \ldots, V(\beta_m))$ is the strength index vector for G. Suppose that $T(t) = (T_1(t), T_2(t), \ldots, T_m(t))$ be m-dimensional conservative(positive) function vector satisfying the conditions in Definition 2.2 and $t \in [0, 1]^m$. We give the definition of an intuitionistic fuzzy conservative (positive) function vector as follows.

Definition 4.1. *1) Suppose that* $T(t) = (T_1(t), T_2(t), \ldots, T_m(t))$ *is an m-dimensional conservative function vector satisfying the conditions in Definition 2.2 and* $t \in [0, 1]^m$*. Let* $G = (\beta_1, \beta_2, \ldots, \beta_m)$ *be an IFN vector and* $V(G) = (V(\beta_1), V(\beta_2), \ldots, V(\beta_m))$ *be the strength index vector for G. We state that* $T(G) = (T_1(V(G)), T_2(V(G)), \ldots, T_m(V(G)))$ *is an intuitionistic fuzzy conservative function vector.*

2) Suppose that $T(t) = (T_1(t), T_2(t), \ldots, T_m(t))$ *is an m-dimensional positive function vector satisfying the conditions in Definition 2.2 and* $t \in [0, 1]^m$*. Let* $G = (\beta_1, \beta_2, \ldots, \beta_m)$ *be IFN vector, and* $V(G) = (V(\beta_1), V(\beta_2), \ldots, V(\beta_m))$ *be the strength index vector for G. We state that* $T(G) = (T_1(V(G)), T_2(V(G)), \ldots, T_m(V(G)))$ *is an intuitionistic fuzzy positive function vector.*

The changeable weights of MDOWA operators generated by $T(G)$ *are as follows:*

$$w_k^T(G) = \frac{T_k(\sigma(V(G)))}{\sum_{j=1}^m T_j(\sigma(V(G)))}, \quad k = 1, 2, \cdots, m \tag{4.1}$$

where $\sigma(V(G)) = (V(\beta_{\sigma(1)}), V(\beta_{\sigma(2)}), \cdots, V(\beta_{\sigma(m)}))$ *is a permutation of* $(V(\beta_1), V(\beta_2), \ldots, V(\beta_m))$*, such that* $V(\beta_{\sigma(k-1)}) \geq V(\beta_{\sigma(k)})$ *for all* $k = 2, 3, \cdots, m$*.*

Furthermore, we establish the intuitionistic fuzzy monotonic dependent OWA (IFMDOWA) operator as follows:

$$\text{IFMDOWA}(\beta_1, \beta_2, \cdots, \beta_m) = \bigoplus_{k=1}^m (w_k^T(G)\beta_{\sigma(k)}) = \langle 1 - \prod_{k=1}^m (1 - \mu_{\beta_{\sigma(k)}})^{w_k^T(G)}, \prod_{k=1}^m (\nu_{\beta_{\sigma(k)}})^{w_k^T(G)}\rangle \tag{4.2}$$

where $\beta_{\sigma(k)}$ is the kth largest of β_k.

For conveniences, we call the above IFMDOWA operator an intuitionistic fuzzy MDOWA operator according to T, represented by IFMDOWA^T simply. Especially, if T is an intuitionistic fuzzy conservative function vector, then we call the above IFMDOWA operator a conservative IFMDOWA operator according to T, represented by CON-IFMDOWA^T. If T is an intuitionistic fuzzy positive function vector, then we call the above IFMDOWA operator a positive IFMDOWA operator according to T, denoted by POS-IFMDOWA^T.

4.2. Intuitionistic fuzzy hybrid monotonic dependent OWA operators

Note that IFMDOWA operators only weight the sorted positions of the IFN, not the IFN itself. To overcome this limitation, in what follows, we propose an intuitionistic fuzzy hybrid MDOWA (IFHMDOWA) operator, which not only weights the aggregated IFN, but it also weights its sorted position.

Let $C_1, C_2, \ldots, C_m$ be m criteria and w=$(w_1, w_2, \ldots, w_m)$ be the weighting vector for all criteria, which satisfies that $\sum_{k=1}^{m} w_k = 1$ and $w_k \in [0,1]$. Suppose that $G = (\beta_1, \beta_2, \ldots, \beta_m)$ denotes the corresponding criteria values, where $\beta_k = \langle \mu_{\beta_k}, \nu_{\beta_k} \rangle$ is an IFN expressing the value of the criterion C_k $(k = 1, 2, \ldots, m)$. $w \odot G = (w_1\beta_1, w_2\beta_2, \ldots, w_m\beta_m)$ is the weighted criterion value vector. $V(w \odot G) = (V(w_1\beta_1), V(w_2\beta_2), \ldots, V(w_m\beta_m))$ is the strength index vector for $w \odot G$. Suppose that $T(t) = (T_1(t), T_2(t), \ldots, T_m(t))$ is an m-dimensional conservative (positive) function vector satisfying the conditions in Definition 2.2 and $t \in [0,1]^m$. We give the definition of an intuitionistic fuzzy hybrid conservative (positive) function vector as follows.

Definition 4.2. *Let $G = (\beta_1, \beta_2, \ldots, \beta_m)$ be an IFN vector, and w=$(w_1, w_2, \ldots, w_m)$ be the weighting vector, which satisfies that $\sum_{k=1}^{m} w_k = 1$ and $w_k \in [0,1]$. $w \odot G = (w_1\beta_1, w_2\beta_2, \ldots, w_m\beta_m)$ is the weighted IFN vector. $V(w \odot G) = (V(w_1\beta_1), V(w_2\beta_2), \ldots, V(w_m\beta_m))$ is the strength index vector for $w \odot G$.*

1) Suppose that $T(t) = (T_1(t), T_2(t), \ldots, T_m(t))$ is an m-dimensional conservative function vector satisfying the conditions in Definition 2.2 and $t \in [0,1]^m$. We call $T(w \odot G) = (T_1(V(w \odot G)), T_2(V(w \odot G)), \ldots, T_m(V(w \odot G)))$ as intuitionistic fuzzy hybrid conservative function vector.

2) Suppose that $T(t) = (T_1(t), T_2(t), \ldots, T_m(t))$ be m-dimensional positive function vector satisfying the conditions in Definition 2.2 and $t \in [0,1]^m$. We call $T(w \odot G) = (T_1(V(w \odot G)), T_2(V(w \odot G)), \ldots, T_m(V(w \odot G)))$ an intuitionistic fuzzy hybrid positive function vector.

Furthermore, we propose the IFHMDOWA operator as follows:

$$\text{IFHMDOWA}(\beta_1, \beta_2, \cdots, \beta_m) = \bigoplus_{k=1}^{m} (w_k^T(w \odot G)\tilde{\beta}_{\sigma(k)}) = \langle 1 - \prod_{k=1}^{m} (1 - \mu_{\tilde{\beta}_{\sigma(k)}})^{w_k^T(w \odot G)}, \prod_{k=1}^{m} (\nu_{\tilde{\beta}_{\sigma(k)}})^{w_k^T(w \odot G)} \rangle, \tag{4.3}$$

where $\tilde{\beta}_k = w_k\beta_k, (k = 1, 2, \cdots, k)$ and $\tilde{\beta}_{\sigma(k)}$ is the kth largest of $\tilde{\beta}_k$.

For convenience, we call the above IFHMDOWA operator an intuitionistic fuzzy hybrid MDOWA operator according to T, represented by IFHMDOWA^T simply. Especially, if T is an intuitionistic fuzzy conservative function vector, then we call the above IFHMDOWA operator a conservative IFHMDOWA operator according to T, represented by COV-IFHMDOWA^T. If T is an intuitionistic fuzzy positive function vector, then we call the above IFHMDOWA operator as positive intuitionistic fuzzy hybrid MDOWA operator according to T, represented by POS-IFHMDOWA^T.

Here, the changeable weights of IFHMDOWA operators generated by $T(w \odot G)$ are as follows:

$$w_k^T(w \odot G) = \frac{T_k(\sigma(V(w \odot G)))}{\sum_{j=1}^{m} T_j(\sigma(V(w \odot G))}, \quad k = 1, 2, \cdots, m, \tag{4.4}$$

where $\sigma(V(w \odot G)) = (V(w_{\sigma(1)}\beta_{\sigma(1)}), V(w_{\sigma(2)}\beta_{\sigma(2)}), \cdots, V(w_{\sigma(m)}\beta_{\sigma(m)}))$ is a permutation of $(V(w_1\beta_1), V(w_2\beta_2), \ldots, V(w_m\beta_m))$, such that $V(w_{\sigma(k-1)}\beta_{\sigma(k-1)}) \geq V(w_{\sigma(k)}\beta_{\sigma(k)})$ for all $k = 2, 3, \cdots, m$.

Example for changeable weights. To illustrate details of computing changeable weights, we add this example to improve readability. Assume that there are four criteria and the corresponding weighting vector is given as $W = (0.2, 0.1, 0.3, 0.4)$; one sample is $G = \{< 0.4, 0.5 >, < 0.5, 0.4 >, < 0.2, 0.7 >, < 0.2, 0.5 >\}$. Then, we can obtain that $w \odot G = \{< 0.0971, 0.8706 >, < 0.0670, 0.9124 >, < 0.0647, 0.8985 >, < 0.0854, 0.7579 >\}$ through Definition 2.4. Further, we can obtain the results of $V(w \odot G) = \{0.3265, 0.3793, 0.1780, 0.2475\}$ through Definition 2.4 and Eq (3.4). And we apply descending order for this vector and obtain the result $\sigma(V(w \odot G)) = \{0.3793, 0.3265, 0.2475, 0.1780\}$. We use the conservative fundamental function $T(x) = \sum_{i=1}^{4} \frac{e^{\lambda x_i}}{\lambda} (\lambda < 0)$. Let $\lambda = -0.5$; we can obtain the final changeable weights $w_k^T(w \odot G) = \{0.2444, 0.2380, 0.2632, 0.2543\}$ through Eq (4.4).

5. Application example

In what follows, we present a numerical example to illustrate the application of the developed operators.

Example 5.1. *(adapted from [45]) An investment firm decides to invest some money profitably. There are five potential alternatives in which to invest the sum of money. Y_1 is a real estate, Y_2 is a furniture industry, Y_3 is an electronic commerce firm, Y_4 is a communications firm, Y_5 is an air-conditioning company. Four criteria C_i $(i = 1, 2, 3, 4)$ are taken into account to evaluate the alternatives: C_1 is risk avoidance; C_2 is productivity; C_3 is technological innovation capability; C_4 is the environmental protection. The criterion weighting vector is given as $W = (0.2, 0.1, 0.3, 0.4)$. The decision group provides its evaluations to evaluate the five potential alternatives by applying IFNs over the aforementioned four criteria. The evaluations are listed in Table 1.*

By utilizing Eqs (3.1)–(3.3), we obtain the corresponding strength vectors for the IFNs which are listed in Table 2. Furthermore, by utilizing Eq (3.4), we obtain the corresponding strength index values $V(\alpha)$ for the IFNs. The results are listed in Table 3.

Table 1. The evaluations of IFNs provided by the decision-making group.

	C_1	C_2	C_3	C_4
Y_1	$\langle 0.4, 0.5 \rangle$	$\langle 0.5, 0.4 \rangle$	$\langle 0.2, 0.7 \rangle$	$\langle 0.2, 0.5 \rangle$
Y_2	$\langle 0.6, 0.4 \rangle$	$\langle 0.6, 0.3 \rangle$	$\langle 0.6, 0.3 \rangle$	$\langle 0.3.0.6 \rangle$
Y_3	$\langle 0.5, 0.5 \rangle$	$\langle 0.4, 0.5 \rangle$	$\langle 0.4, 0.4 \rangle$	$\langle 0.5, 0.4 \rangle$
Y_4	$\langle 0.7, 0.2 \rangle$	$\langle 0.5, 0.4 \rangle$	$\langle 0.2, 0.5 \rangle$	$\langle 0.3, 0.7 \rangle$
Y_5	$\langle 0.5, 0.3 \rangle$	$\langle 0.3, 0.4 \rangle$	$\langle 0.6, 0.2 \rangle$	$\langle 0.4, 0.4 \rangle$

Table 2. The corresponding strength vectors $(r_\alpha, \theta_\alpha)$ for the IFNs.

	C_1	C_2	C_3	C_4
Y_1	(0.6403,0.8961)	(0.6403,0.6747)	(0.7280,1.2925)	(0.5385, 1.1903)
Y_2	(0.7211,0.5880)	(0.6708,0.4636)	(0.6708,0.4636)	(0.6708,1.1071)
Y_3	(0.7071,0.7854)	(0.6403,0.8961)	(0.5657,0.7854)	(0.6403,0.6747)
Y_4	(0.7280,0.2783)	(0.6403,0.6747)	(0.5385,1.1903)	(0.7616,1.1659)
Y_5	(0.5831,0.5404)	(0.5000, 0.9273)	(0.6325, 0.3218)	(0.5657,0.7854)

Table 3. The corresponding strength indexes $V(\alpha)$ for the IFNs.

	C_1	C_2	C_3	C_4
Y_1	0.4549	0.5451	0.2650	0.3612
Y_2	0.5906	0.6374	0.6374	0.3626
Y_3	0.5000	0.4549	0.5000	0.5451
Y_4	0.7350	0.5451	0.3612	0.3155
Y_5	0.5909	0.4548	0.6867	0.5000

Suppose that the decision makers emphasize the balance among the criteria; then, we should use the intuitionistic fuzzy conservative function vector to generate the weights associated with the DOWA operators. In this paper, we use the conservative fundamental function $M(x) = \sum_{i=1}^{4} \frac{e^{\lambda x_i}}{\lambda}(\lambda < 0)$ to generate the conservative function vector. If the criterion weights are not taken into account, then we utilize the COV-IFMDOWA operator to aggregate the IFNs. To illustrate the effectiveness of the developed operators, we have employed several different COV-IFMDOWA operators to aggregate IFNs by selecting different values of the parameter λ. The ranking results are listed in Table 4. For convenience, the strength index values for V (IFMDOWA(Y_k))$(k = 1, 2, 3, 4, 5)$ are simply denoted as $V(Y_k)(k = 1, 2, 3, 4, 5)$. For example, if $\lambda = -0.5$, by utilizing Eq (4.2), we obtain the corresponding aggregation values IFMDOWA$(Y_1) = \langle 0.3315, 0.5195\rangle$, IFMDOWA$(Y_2) = \langle 0.5334, 0.3896\rangle$, IFMDOWA$(Y_3) = \langle 0.4517, 0.4478\rangle$, IFMDOWA$(Y_4) = \langle 0.4454, 0.4242\rangle$ and IFMDOWA$(Y_5) = \langle 0.4567, 0.3169\rangle$. By employing Eq (3.4), we have the following $V(Y_1) = 0.4147$, $V(Y_2) = 0.5650$, $V(Y_3) = 0.5018$, $V(Y_4) = 0.5095$, $V(Y_5) = 0.5633$. Since $V(Y_2) > V(Y_5) > V(Y_4) > V(Y_3) > V(Y_1)$, we have that $Y_2 \succ Y_5 \succ Y_4 \succ Y_3 \succ Y_1$. Thus, the most desirable alternative is Y_2.

Table 4. Ranking results applying IFMDOWA operators with different values of λ.

	$V(Y_1)$	$V(Y_2)$	$V(Y_3)$	$V(Y_4)$	$V(Y_5)$	Ranking results
$\lambda = -5.5$	0.3622	0.4896	0.4967	0.4000	0.5268	$Y_5 \succ Y_3 \succ Y_2 \succ Y_4 \succ Y_1$
$\lambda = -4.5$	0.3721	0.5062	0.4977	0.4159	0.5333	$Y_5 \succ Y_2 \succ Y_3 \succ Y_4 \succ Y_1$
$\lambda = -3.5$	0.3825	0.5225	0.4987	0.4350	0.5403	$Y_5 \succ Y_2 \succ Y_3 \succ Y_4 \succ Y_1$
$\lambda = -2.5$	0.3931	0.5380	0.4997	0.4572	0.5477	$Y_5 \succ Y_2 \succ Y_3 \succ Y_4 \succ Y_1$
$\lambda = -1.5$	0.4039	0.5522	0.5008	0.4823	0.5553	$Y_5 \succ Y_2 \succ Y_3 \succ Y_4 \succ Y_1$
$\lambda = -0.5$	0.4147	0.5650	0.5018	0.5095	0.5633	$Y_2 \succ Y_5 \succ Y_4 \succ Y_3 \succ Y_1$

From the aforementioned results, it is easy to see that different parameter values for the fundamental function can provide different ranking results. In other words, the attitudinal character of decision-makers can affect the decision results. When $\lambda = -1.5, -2.5, -3.5, -4.5$, the ranking results obtained by the proposed IFMDOWA operator are the same as the results provided by Zhao and Wei's method [45]. To further investigate the influence of the parameter values of λ, we can calculate the orness measures of the changeable weights associated with the IFMDOWA operator. For each given value of λ, every alternative is associated with different changeable weights. The orness measures associated with parameter values of λ are listed in Table 5. It can be easily seen that the smaller the value of λ, the smaller of orness measure. Therefore, a smaller value of λ indicates a more conservative decision-maker and greater closeness of the IFMDOWA operator to the minimum operator. Moreover, the aforementioned results also indicate that, as long as the experts correctly describe their attitude by selecting an appropriate value of λ, the developed IFMDOWA operators can present more accurate and robust decision-making results. Therefore, the proposed operators are very efficient and flexible because the decision makers have more choices according to their interests and actual needs.

Table 5. Orness measures for changeable weights associated with different values of λ.

	$orness(Y_1)$	$orness(Y_2)$	$orness(Y_3)$	$orness(Y_4)$	$orness(Y_5)$
$\lambda = -5.5$	0.2994	0.2680	0.4383	0.2534	0.3892
$\lambda = -4.5$	0.3323	0.3112	0.4494	0.2843	0.4069
$\lambda = -3.5$	0.3672	0.3554	0.4606	0.3213	0.4257
$\lambda = -2.5$	0.4039	0.3993	0.4718	0.3647	0.4457
$\lambda = -1.5$	0.4418	0.4417	0.4831	0.4146	0.4666
$\lambda = -0.5$	0.4805	0.4814	0.4944	0.4704	0.4886

If the criterion weights $w = (0.2, 0.3, 0.1, 0.15, 0.25)$ are taken into account, then we can utilize the COV-IFHMDOWA operator to aggregate the IFNs. We can also use the conservative fundamental function $M(x) = \sum_{i=1}^{5} \frac{e^{\lambda x_i}}{\lambda}(\lambda < 0)$ to generate the conservative function vector. We selected different values of λ to obtain different fundamental functions and aggregate the IFNs by utilizing Eq (4.3). The ranking results for the alternatives obtained by using the COV-IFHMDOWA operator with different values of λ are listed in Table 6.

Table 6. Ranking results obtained by applying IFHMDOWA operators with different values of λ.

	$V(Y_1)$	$V(Y_2)$	$V(Y_3)$	$V(Y_4)$	$V(Y_5)$	Ranking results
$\lambda = -6.5$	0.1120	0.1677	0.1387	0.1367	0.1548	$Y_2 \succ Y_5 \succ Y_3 \succ Y_4 \succ Y_1$
$\lambda = -5.5$	0.1131	0.1709	0.1447	0.1398	0.1641	$Y_2 \succ Y_5 \succ Y_3 \succ Y_4 \succ Y_1$
$\lambda = -4.5$	0.1142	0.1744	0.1510	0.1431	0.1737	$Y_2 \succ Y_5 \succ Y_3 \succ Y_4 \succ Y_1$
$\lambda = -3.5$	0.1153	0.1780	0.1575	0.1465	0.1834	$Y_5 \succ Y_2 \succ Y_3 \succ Y_4 \succ Y_1$
$\lambda = -2.5$	0.1165	0.1817	0.1641	0.1502	0.1932	$Y_5 \succ Y_2 \succ Y_3 \succ Y_4 \succ Y_1$
$\lambda = -1.5$	0.1177	0.1856	0.1709	0.1541	0.2029	$Y_5 \succ Y_2 \succ Y_3 \succ Y_4 \succ Y_1$
$\lambda = -0.5$	0.1189	0.1897	0.1778	0.1582	0.2123	$Y_5 \succ Y_2 \succ Y_3 \succ Y_4 \succ Y_1$

We can easily see that the ranking result obtained by the COV-IFHMDOWA operator is different from the ranking order obtained by using the COV-IFMDOWA operator when we select the same value of λ. The difference just indicates impact of the criterion weights on the decision-making results.

If we use an IFWA operator to aggregate the IFNs, by utilizing Eq (2.9), we have the following: IFWA(Y_1) $= \langle 0.2794, 0.5409\rangle$, IFWA($Y_2$) $= \langle 0.4996, 0.4193\rangle$, IFWA($Y_3$) $= \langle 0.4622, 0.4277\rangle$, IFWA($Y_4$) $= \langle 0.4053, 0.4657\rangle$, IFWA($Y_5$) $= \langle 0.4798, 0.3067\rangle$. By Eq (3.4), we have the following: V(IFWA(Y_1)) $= 0.3804$, V(IFWA(Y_2)) $= 0.5362$, V(IFWA(Y_3)) $= 0.5155$, V(IFWA(Y_4)) $= 0.4728$, V(IFWA(Y_5)) $= 0.5785$. Thus, the final result is $Y_5 \succ Y_2 \succ Y_3 \succ Y_4 \succ Y_1$. We can easily see that when $\lambda = -0.5, -1.5, -2.5, -3.5$, the result obtained by the proposed IFHMDOWA operator is the same as the results obtained by using the IFWA operator and Zhao and Wei's method [45]. However, if $\lambda = -4.5, -5.5, -6.5$, the results obtained by using the IFHMDOWA operator are different from the results obtained by using the IFWA operator and Zhao and Wei's method [45]. This indicates that the IFHMDOWA operator has more flexibility since it provides the decision-makers with more options by selecting different values of λ. Thus, the decision-maker's attitudinal character can be fully expressed in the decision making process.

Using the novel proposed aggregation operators, we performed comparison experiments with IFWA aggregation operators. And, we adjusted different λ values to obtain different ranking results. The POS-IFMDOWA and COV-IFHMDOWA aggregation operators have been shown to provide more flexibility and effectiveness.

6. Conclusions

IFSs constitute a useful tool for modeling the ambiguous and uncertain information in decision-making-related fields. Ranking IFNs plays an essential role in intuitionistic fuzzy decision making problems. To solve this problem, we have presented a new measure to reflect the magnitude of IFNs. By applying the presented measure of IFN, we proposed a new ranking method for IFNs. Furthermore, by utilizing the presented measure and the new ranking method for IFNs, we have developed several new intuitionistic fuzzy DOWA operators, such as IFMDOWA operators and IFHMDOWA operators, for the aggregation of intuitionistic fuzzy informations. The weights associated with the developed operators can be generated and adjusted dynamically and automatically. Therefore, the developed DOWA operators can reflect the decision maker's attitudinal character and risk preference more flexibly than the ordinary OWA operator, as well as have some potential applications.

In future research, we will further investigate the methodology for selecting appropriate parameter values for the fundamental function in the intuitionistic fuzzy decision making process.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

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