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Research article

More on complex hesitant fuzzy graphs

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Abstract: Correctly determining a company's market worth during an entire year or a certain period presents a difficulty to decision-makers. In the case of the merger of companies, the need performs heavier when both the companies' owners are attracted to establishing a fair price at the optimal time to merge. The effectiveness of representing, connecting and manipulating both uncertainty and periodicity information becomes highly required. Hence, study and nhance some properties and conditions of the algebraic structure of complex hesitant fuzzy graphs. Therefore, the degree of composition between two complex hesitant fuzzy graphs is proposed. Also, the formal definitions of union, joint and complement are presented to be covered in the realm of complex hesitant fuzzy graphs. A real-life application is illustrated to show the relation between vertices and edges in the form of complex hesitant fuzzy graphs.

Keywords: complex hesitant fuzzy graph; composition of two complex hesitant fuzzy graphs; union of two complex hesitant fuzzy graphs; joint of two complex hesitant fuzzy graphs; complement of complex hesitant fuzzy graph
Mathematics Subject Classification: 05C07, 05C76, 94D05

1. Introduction

The hesitant fuzzy set (HFS) was introduced by Torra [1] in 2010. HFS is considered as a generalization of fuzzy set (FS) [2] to overcome the limitation of the appearance of two or more sources of vagueness. In 2018, Carlos et al. [3] studied the decomposition theorems of HFS. Also, they defined two novel extension principles as a function that maps HFS to HFS. The concept of intuitionistic fuzzy sets (IFS) was defined in 1986 [4]. IFS enables the inclusion of some hesitation values. This is demonstrated with membership and non-membership functions belonging to $[0, 1]$. In the literature, IF has several generalizations, such as HFS with several hesitant membership functions, not only two as in IFS. Also, Pythagorean fuzzy sets [5, 6], fermatean fuzzy sets [7] and q-rung orthopair fuzzy sets [8] are a generalization of IFS by extending the domain of values based on raising the power of membership and non-membership functions to squared, cubed or a general number called q. It is necessary to point out that this topic is of significant interest to many scholars. Recently, Alcantud [9] introduced a Semantic Justification of q-Rung orthopair fuzzy sets based on different fuzzy negations. Other classes like bipolar complex fuzzy sets [10], complex intuitionistic fuzzy sets [11] and complex Pythagorean fuzzy sets [12] were also introduced and studied the semantic contents of both IFS and its extensions.

Hesitant fuzzy sets (HFSs) are an impressive method to represent unsure and ambiguous information whose applications can be explored and studied by clustering algorithms and decision-making (DM) problems [13–15]. Many scholars have given its extensions, such as the interval valued hesitant fuzzy set [16], hesitant fuzzy linguistic term set [17], dual hesitant fuzzy set [18] and so on. A Fuzzy graph was presented by Kauffmann [19] at the biggenings of 70's. Also, Rosenfeld in 1975 [20] developed the structure of fuzzy graphs and obtained analogs of many graph theoretical concepts. The hesitancy fuzzy graph was constructed in 2015 by Pathinathan et al. [21] and some basic concepts concerning this structure were added too. Recently, Karaaslan [22] introduced and studied a new hesitant fuzzy graph (HFGr) structure and some concepts, operations and decision-making applications related to it.

Recently, many researchers have proposed the idea of extending the range of membership function to a unit disk, such as complex fuzzy set (CFS) [23], complex intuitionistic fuzzy set (CIFS) [11], complex hesitant fuzzy set (CHFS) [24], complex bipolar fuzzy set (CBFS) [10], complex Pythagorean fuzzy set (CPFS) [12], complex fermatean fuzzy set (CFFS) [25] and others. The significance of this idea lies in its ability to convey and represent uncertainty and periodicity (phase/level/factors) semantics at the same time using one mathematical structure. In addition, this representation does not lose the full meaning of information during the transformation from human knowledge to mathematical formulas and vice versa. In recent times, several scholars have used, applied and implemented CFS, CIFS, CHFS, CBFS, CPFS, CFFS and other complex uncertainty sets in decision-making problems, cluster algorithms, robotics technology and other fields, see [10, 12, 25–31].

It is clear in the literature that most of the above uncertainty tools that are generalized to the complex realm have been applied and combined to the graph area. Tamir et al. [32] extended the fuzzy graph (FGr) to the complex fuzzy graph (CFGr). In 2019, Yaqoob et al. [33] introduced a complex intuitionistic fuzzy graph (CIFGr). In 2023, AbuHijleh [34] defined a complex hesitant fuzzy graph (CHFGr), besides a direct product, tensor product and normal product of two CHFGr. Numerous scholars have employed and generalized fuzzy graphs and complex fuzzy graphs in several fields.

We highlighted some of these studies, such as Akram [35] introducing bipolar fuzzy graphs with an application in social groups. Also, Verma et al. [36] presented and studied some results on Pythagorean fuzzy graphs. Furthermore, Nandhini and Amsaveni [37, 38] defined and studied the notion of bipolar complex Pythagorean fuzzy graphs. Finally, the fields of image processing, multicriteria decision-making, network, data mining, optimization, preference relations and aggregation operators are practically employed in combination with uncertainty and graphs; see [24, 33, 39, 40].

In this paper, the priority is to study more concepts and relations on CHFGr in AbuHijleh [34], highlighting the need to use CHFGr. Unfortunately, the main definition of CHFGr needs to be modified by adding a condition to confine and moderate more results on CHFGr. Also, the notion of order of CHFGr is missing. In addition, some properties and connected operations are missing, such as composition, joint, union of two CHFGr and the complement of CHFGr. Therefore, the purpose of this paper is to cover all these concepts and study the new properties of the new CHFGr.

This paper is a continuation of CHFGr [34]; starting with Section 2, where we review the framework with introductory ideas, we continue to investigate CHFGr. We lay out the updated CHFGr notion, establish the CHFGr order and provide a numerical example for illustrating CHFGr in Section 3. The discussion of CHFGr's composition, joint operations, the union of two CHFGr sets and complement is addressed extensively in Section 4. In Section 5, we further investigate the development of the DM issue covered in the study [22]. A comparative analysis is provided in Section 6. Section 7 concludes with comments and a summary of this study's future directions.

2. Preliminaries

Torra [1] defined hesitant fuzzy set (HFS).

Definition 1. *If $\mathbb{X}$ is a fixed set, a HFS $\mathbb{M}$ on $\mathbb{X}$ is given in terms of the function $\eta_{\mathbb{M}}(x)$, such that when applied to $\mathbb{X}$ returns a finite subset of* $[0, 1]$.

In addition, Xia and Xu [41], defined the notion of hesitant fuzzy element (HFE), considered as HFS basic concept,

$$E\ell = \{< x, \eta_{E\ell}(x) > | x \in \mathbb{X}\},$$

where $\eta_{E\ell}(x)$ is a set of some values in $[0, 1]$ and it is hesitant.

In 2020, Javid et al. [40], presented and defined hesitant fuzzy graphs as follows.

Definition 2. *A HFGr of the form $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \Psi, \Phi)$, with Ψ: $\mathcal{V} \to S_g[0, 1]$ and Φ: $\mathcal{E} \to S_g[0, 1]$, where $Sg[0, 1]$ is the family of all finite subsets of* $[0, 1]$. *In addition, Ψ is the membership function of vertex set of HFGr, and Φ is the membership function of edge set of HFGr; $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$.*

Recently, Talafha et al. in 2021 [24], have generalized a HFS to CHFS.

Definition 3. *[24] Let $\mathbb{M}$ be a set, a complex hesitant fuzzy element (CHFE) on $\mathbb{X}$ is in terms of a function that when applied returns a subset of the unit disc in the complex plane. We may express the CHFEs by*

$$\mathbb{M} = \{\langle x, \eta_m(x) = \tau_m(x)e^{2\pi\ i\theta_m(x)}\rangle :\ x \in \mathbb{X}\}, \tag{2.1}$$

where $i = \sqrt{-1}$, $\tau_m(x)$ and $\theta_m(x)$ are a FS of values in [0, 1], denoting the possible amplitude and phase membership degrees of the element $x \in \mathbb{X}$ to the set $\mathbb{M}$. We call

$$\eta_m(x) = \tau_m(x)e^{2\pi\, i\theta_m(x)}$$

CHFE and $\mathbb{M}$ the set of all CHFEs.

To compare between elements of CHFS, Talafha et al. [24] create the following values.

Definition 4. *[24] For a CHFE η, the score of η is*

$$s(\eta) = \frac{1}{2}[s(\eta_\tau) + s(\eta_\theta)] = \frac{1}{2}[\frac{1}{\ell_{\eta_\tau}}\sum_{\tau\in\eta_\tau}\tau + \frac{1}{\ell_{\eta_\theta}}\sum_{\theta\in\eta_\theta}\theta], \tag{2.2}$$

where the number of elements in η_τ and η_θ are ℓ_{η_τ} and ℓ_{η_θ}, respectively.

Definition 5. *[24] For a CHFE η, the deviation degree $\delta(\eta)$ of η is*

$$\delta(\eta) = \frac{1}{2}[(\frac{1}{\ell_{\eta_\tau}}\sum_{\tau\in\eta_\tau}(\tau - s(\eta_\tau))^2)^{\frac{1}{2}} + (\frac{1}{\ell_{\eta_\theta}}\sum_{\theta\in\eta_\theta}(\theta - s(\eta_\theta))^2)^{\frac{1}{2}}]. \tag{2.3}$$

To compare the two elements in CHFS, we have:

Proposition 1. *[24] For two elements η_1 and η_2 in CHFEs, we have:*

- *If $s(\eta_1) < s(\eta_2)$, then $\eta_1 < \eta_2$.*
- *If $s(\eta_1) = s(\eta_2)$ where*
  (1) $\delta(\eta_1) = \delta(\eta_2)$, then $\eta_1 = \eta_2$,
  (2) $\delta(\eta_1) < \delta(\eta_2)$, then $\eta_1 < \eta_2$,
  (3) $\delta(\eta_1) > \delta(\eta_2)$, then $\eta_1 > \eta_2$.

So that, by Proposition 1, $\max\{\eta_1, \eta_2\} = \eta_1 \vee \eta_2$ and $\min\{\eta_1, \eta_2\} = \eta_1 \wedge \eta_2$.

A CHFGr was investigated and defined, by AbuHijleh [34], which can be considered as a combination between HFGr and CHFS.

Definition 6. *[34] A complex hesitant fuzzy graph (CHFGr) is of the form $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \Psi, \Phi)$, with $\Psi: \mathcal{V} \to S_g\{\eta \in \mathbb{C}: |\eta| \le 1\}$ and $\Phi: \mathcal{E} \to S_g\{\eta \in \mathbb{C}: |\eta| \le 1\}$, where $S_g\{\eta \in \mathbb{C}: |\eta| \le 1\}$ is the family of all finite subsets of unit disc. Moreover, the membership functions of the vertex set and edge set of CHFGr are denoted by Ψ and Φ, respectively.*

The addition of two hesitant fuzzy sets defined by Zhu [42].

Definition 7. *Assume that $\mathcal{L} = \{\ell_1, \ell_2, \cdots, \ell_m\}$ and $\mathcal{J} = \{j_1, j_2, \cdots, j_n\}$ be HFS, then*

$$\mathcal{L} \oplus \mathcal{J} = \bigcup_{\ell_i\in\mathcal{L},\ j_k\in\mathcal{J}} \{\ell_i + j_k\}.$$

AbuHijleh [34] defined a vertex degree of CHFGr.

Definition 8. *For a CHFGr $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \Psi, \Phi)$, the vertex degree v_i is*

$$deg(v_i) = \bigoplus_{e_{i,j}\in\mathcal{E}} \Phi(e_{i,j}) = \sum_j \eta_{i,j} = (\bigcup_{j_1\neq j_2}(\eta_{r_{i,j_1}} \oplus \eta_{r_{i,j_2}}))exp(2\pi i(\bigcup_{j_1\neq j_2}(\eta_{\theta_{i,j_1}} \oplus \eta_{\theta_{i,j_2}}))),$$

where $e_{i,j}$ is an edge incident to v_i and v_j and $\Phi(e_{i,j}) = \eta_{i,j} = \eta_{r_{i,j}}exp(2\pi i\eta_{\theta_{i,j}})$.

3. More on complex hesitance fuzzy graphs

In this section, Definition 9 is considered a modification of the CHFGr notion presented in [34], by adding a condition. Also, we introduce the CHFGr order as Definition 10 in the present study.

Definition 9. *A complex hesitant fuzzy graph (CHFGr) is of the form* $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \Psi, \Phi)$*, with* Ψ*:* $\mathcal{V} \to S_g\{\eta \in \mathbb{C}: |\eta| \leq 1\}$ *and* Φ*:* $\mathcal{E} \to S_g\{\eta \in \mathbb{C}: |\eta| \leq 1\}$*, where* $S_g\{\eta \in \mathbb{C}: |\eta| \leq 1\}$ *is the family of all finite subsets of unit disc. Moreover, the membership functions of vertex set and edge set of CHFGr are denoted by* Ψ *and* Φ*, respectively, where* $\Phi(\nu\omega) \leq \min\{\Psi(\nu), \Psi(\omega)\}$.

Note that, the condition, $\Phi(\nu\omega) \leq \min\{\Psi(\nu), \Psi(\omega)\}$, is as an improvement of the Definition 6.

Definition 10. *For a CHFGr* $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \Psi, \Phi)$*, the order of CHFGr* $\mathcal{G}$ *is*

$$O(\mathcal{G}) = \bigoplus_{\nu_i \in \mathcal{V}} \Psi(\nu_i) = \sum_i \eta_i = (\bigcup_{i_1 \neq i_2} (\eta_{r_{i_1}} \oplus \eta_{r_{i_2}})) exp(2\pi i (\bigcup_{i_1 \neq i_2} (\eta_{\theta_{i_1}} \oplus \eta_{\theta_{i_2}}))),$$

for vertex ν_i*, we have* $\Psi(\nu_i) = \eta_i = \eta_{r_i} exp(2\pi i \eta_{\theta_i})$.

Let's find valuations of vertices and edges within a graph, as well as provide an example of how to determine the vertex degree and CHFGr order, as seen in the example that follows.

Example 1. *A. For CHFGr* $\mathcal{H}$ *and* $\mathcal{F}$*, see Figure 1, the score values (and deviation degree if necessary) are given below:*

(1) $s(\eta_{\nu_1}) = \frac{1}{2}(\frac{0.5+0.7}{2} + \frac{0.4+0.6}{2}) = 0.55$*,* $s(\eta_{\nu_2}) = 0.65$ *and* $s(\eta_{\nu_1\nu_2}) = 0.375$.

(2) $s(\eta_{\omega_1}) = 0.4$*,* $s(\eta_{\omega_2}) = 0.475$*,* $s(\eta_{\omega_3}) = 0.725$*,* $s(\eta_{\omega_1\omega_2}) = 0.4$ *and* $s(\eta_{\omega_2\omega_3}) = 0.45$*. Here in graph* $\mathcal{F}$*, we found degree of deviation for* η_{ω_1}*,* $\eta_{\omega_1\omega_2}$*, to compare among all values of vertices and edges;* $\delta(\eta_{\omega_1}) = 0.2$ *and* $\delta(\eta_{\omega_1\omega_2}) = 0.05$.

B. An example of vertex degree from CHFGr $\mathcal{F}$*:*

$$deg_{\mathcal{F}}(\omega_2) = \Phi_{\mathcal{F}}(\omega_1\omega_2) + \Phi_{\mathcal{F}}(\omega_2\omega_3) = \{0.6, 0.7, 0.8, 0.9\}e^{\{0.9,1\}2\pi i}.$$

C. An order of CHFGr $\mathcal{H}$*:*

$$O(\mathcal{H}) = \Psi(\omega_1) \oplus \Psi(\omega_2) = \{1.1, 1.3\}e^{2\pi i\{1,1.2,1.4\}}.$$

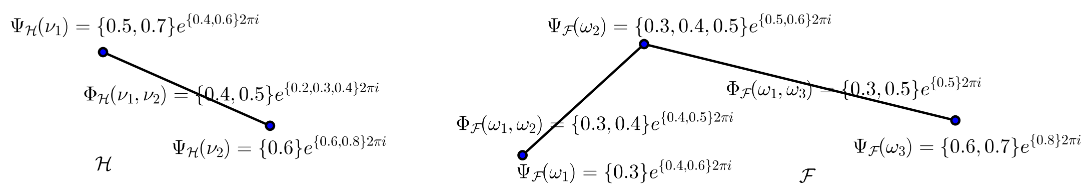

Figure 1. CHFGr $\mathcal{H}$ and $\mathcal{F}$.

4. Operations on complex hesitant fuzzy graphs

AbuHijelh [34] defined the normal product, cartesian product and tensor product of two CHFGr and discussed some of their properties, while this paper defines the composition, union and joint of two CHFGrs and the complement of CHFGr. Also, some properties of the presented notions are investigated and examined.

4.1. Composition of two complex hesitant fuzzy graphs

Definition 11. *The composition graph $\mathcal{G}$ of two CHFGr*

$$\mathcal{H} = (\mathcal{V}_{\mathcal{H}},\ \mathcal{E}_{\mathcal{H}},\ \Psi_{\mathcal{H}},\ \Phi_{\mathcal{H}})$$

and

$$\mathcal{F} = (\mathcal{V}_{\mathcal{F}},\ \mathcal{E}_{\mathcal{F}},\ \Psi_{\mathcal{F}},\ \Phi_{\mathcal{F}}),$$

defines as CHFGr

$$\mathcal{G} = \mathcal{H} \circ \mathcal{F} = (\mathcal{V},\ \mathcal{E},\ \Psi_{\mathcal{H}} \circ \Psi_{\mathcal{F}},\ \Phi_{\mathcal{H}} \circ \Phi_{\mathcal{F}}),$$

where

$$\mathcal{V} = \mathcal{V}_{\mathcal{H}} \circ \mathcal{V}_{\mathcal{F}}$$

and

$$\mathcal{E} = \{((\nu_1, \omega_1), (\nu_2, \omega_2)) : \nu_1 = \nu_2, (\omega_1, \omega_2) \in \mathcal{E}_{\mathcal{F}}$$

or

$$\omega_1 = \omega_2,\ (\nu_1, \nu_2) \in \mathcal{E}_{\mathcal{H}}$$

or

$$\omega_1 \neq \omega_2,\ (\nu_1, \nu_2) \in \mathcal{E}_{\mathcal{H}}\}$$

with:

(1) $(\Psi_{\mathcal{H}} \circ \Psi_{\mathcal{F}})(\nu, \omega) = \Psi_{\mathcal{H}}(\nu) \wedge \Psi_{\mathcal{F}}(\omega)$.

(2) $(\Phi_{\mathcal{H}} \circ \Phi_{\mathcal{F}})((\nu_1, \omega_1), (\nu_2, \omega_2)) = \begin{cases} \Psi_{\mathcal{H}}(\nu_1) \wedge \Phi_{\mathcal{F}}(\omega_1, \omega_2) : & \nu_1 = \nu_2,\ (\omega_1, \omega_2) \in \mathcal{E}_{\mathcal{F}}, \\ \Phi_{\mathcal{H}}(\nu_1, \nu_2) \wedge \Psi_{\mathcal{F}}(\omega_1) : & \omega_1 = \omega_2,\ (\nu_1, \nu_2) \in \mathcal{E}_{\mathcal{H}}, \\ \Phi_{\mathcal{H}}(\nu_1, \nu_2) \wedge \Psi_{\mathcal{F}}(\omega_1) \wedge \Psi_{\mathcal{F}}(\omega_2) : & \omega_1 \neq \omega_2,\ (\nu_1, \nu_2) \in \mathcal{E}_{\mathcal{H}}. \end{cases}$

Figure 2 is an example of composition graph $\mathcal{G} = (\mathcal{H} \circ \mathcal{F})$, according to previous definition, where $\mathcal{H}$ and $\mathcal{F}$ are in the Figure 1. Hence, using score values (and deviation degree if necessary), from Example 1, the labellings evaluated for this composition graph. Moreover, the labellings in the Figure 2 will assures the following proposition.

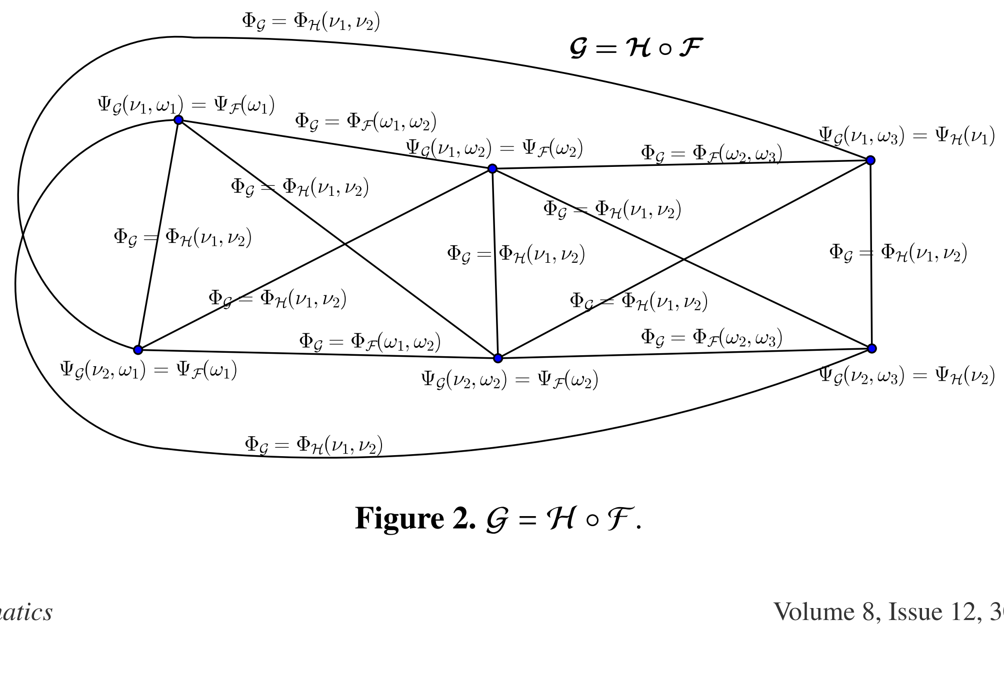

Figure 2. $\mathcal{G} = \mathcal{H} \circ \mathcal{F}$.

Proposition 2. *The composition of two CHFGrs is CHFGr.*

Proof. We have three cases:

(1) If $((\nu,\omega_1),(\nu,\omega_2)) \in \mathcal{E}_{\mathcal{G}}$, then,

$$\Phi_{\mathcal{G}}((\nu,\omega_1),(\nu,\omega_2)) = \Psi_{\mathcal{H}}(\nu) \wedge \Phi_{\mathcal{F}}(\omega_1,\omega_2) \leq \Psi_{\mathcal{H}}(\nu) \wedge \Psi_{\mathcal{F}}(\omega_1) \wedge \Psi_{\mathcal{F}}(\omega_2),$$

where $\mathcal{F}$ is CHFGr. However,

$$\Psi_{\mathcal{G}}(\nu,\omega_1) \wedge \Psi_{\mathcal{G}}(\nu,\omega_2) = \Psi_{\mathcal{H}}(\nu) \wedge \Psi_{\mathcal{F}}(\omega_1) \wedge \Psi_{\mathcal{F}}(\omega_2),$$

see Definition 11. Hence,

$$\Phi_{\mathcal{G}}((\nu,\omega_1),(\nu,\omega_2)) \leq \Psi_{\mathcal{G}}(\nu,\omega_1) \wedge \Psi_{\mathcal{G}}(\nu,\omega_2).$$

(2) If $((\nu_1,\omega),(\nu_2,\omega)) \in \mathcal{E}_{\mathcal{G}}$, the result follows by similar argument.

(3) If $((\nu_1,\omega_1),(\nu_2,\omega_2)) \in \mathcal{E}_{\mathcal{G}}$ and $\nu_1 \neq \nu_2,\ \omega_1 \neq \omega_2$, then

$$\begin{aligned}\Phi_{\mathcal{G}}((\nu_1,\omega_1),(\nu_2,\omega_2)) &= \Phi_{\mathcal{H}}(\nu_1,\nu_2) \wedge \Psi_{\mathcal{F}}(\omega_1) \wedge \Psi_{\mathcal{F}}(\omega_2)\\ &\leq \Psi_{\mathcal{H}}(\nu_1) \wedge \Psi_{\mathcal{H}}(\nu_2) \wedge \Psi_{\mathcal{F}}(\omega_1) \wedge \Psi_{\mathcal{F}}(\omega_2)\\ &= \Psi_{\mathcal{G}}(\nu_1,\omega_1) \wedge \Psi_{\mathcal{G}}(\nu_2,\omega_2),\end{aligned}$$

see Definition 11. Hence result follows. □

The degree of vertices of composition graph in CHFGr, can be specified when we add some constrains, see the following theorem.

Theorem 1. *Let*

$$\mathcal{G} = \mathcal{H} \circ \mathcal{F} = (\mathcal{V},\ \mathcal{E},\ \Psi_{\mathcal{H}} \circ \Psi_{\mathcal{F}},\ \Phi_{\mathcal{H}} \circ \Phi_{\mathcal{F}})$$

defined a composition of two CHFGr

$$\mathcal{H} = (\mathcal{V}_{\mathcal{H}},\ \mathcal{E}_{\mathcal{H}},\ \Psi_{\mathcal{H}},\ \Phi_{\mathcal{H}}) \ \text{and}\ \mathcal{F} = (\mathcal{V}_{\mathcal{F}},\ \mathcal{E}_{\mathcal{F}},\ \Psi_{\mathcal{F}},\ \Phi_{\mathcal{F}}).$$

If $\max\{\Phi_{\mathcal{H}}\} \leq \min\{\Psi_{\mathcal{F}}\}$ *and* $\max\{\Phi_{\mathcal{F}}\} \leq \min\{\Psi_{\mathcal{H}}\}$*, then,*

$$deg_{\mathcal{G}}(\nu_1,\omega_1) = |\mathcal{V}_{\mathcal{F}}| deg_{\mathcal{H}}(\nu_1) \bigoplus deg_{\mathcal{F}}(\omega_1).$$

Proof. By Definition 8, the vertex degree of (ν_1,ω_1) in the composition graph is

$$\begin{aligned}deg_{\mathcal{G}}(\nu_1,\omega_1) &= \bigoplus_{((\nu_1,\omega_1),(\nu_2,\omega_2))\in\mathcal{E}} (\Phi_{\mathcal{H}} \circ \Phi_{\mathcal{F}})((\nu_1,\omega_1),(\nu_2,\omega_2))\\ &= \left\{ \bigoplus_{\nu_1=\nu_2,\ (\omega_1,\omega_2)\in\mathcal{E}_{\mathcal{F}}} \Psi_{\mathcal{H}}(\nu_1) \wedge \Phi_{\mathcal{F}}(\omega_1,\omega_2) \right\}\\ &\quad \oplus \left\{ \bigoplus_{\omega_1=\omega_2,\ (\nu_1,\nu_2)\in\mathcal{E}_{\mathcal{H}}} \Phi_{\mathcal{H}}(\nu_1,\nu_2) \wedge \Psi_{\mathcal{F}}(\omega_1) \right\}\\ &\quad \oplus \left\{ \bigoplus_{\omega_1\neq\omega_2,\ (\nu_1,\nu_2)\in\mathcal{E}_{\mathcal{H}}} \Phi_{\mathcal{H}}(\nu_1,\nu_2) \wedge \Psi_{\mathcal{F}}(\omega_1) \wedge \Psi_{\mathcal{F}}(\omega_2) \right\},\end{aligned}$$

by Definition 11. However, $\max\{\Phi_{\mathcal{H}}\} \leq \min\{\Psi_{\mathcal{F}}\}$ and $\max\{\Phi_{\mathcal{F}}\} \leq \min\{\Psi_{\mathcal{H}}\}$, so that

$$deg_{\mathcal{G}}(\nu_1, \omega_1) = \left\{ \bigoplus_{\nu_1=\nu_2,\ (\omega_1,\omega_2)\in\mathcal{E}_{\mathcal{F}}} \Phi_{\mathcal{F}}(\omega_1, \omega_2) \right\} \oplus \left\{ \bigoplus_{\omega_1=\omega_2,\ (\nu_1,\nu_2)\in\mathcal{E}_{\mathcal{H}}} \Phi_{\mathcal{H}}(\nu_1, \nu_2) \right\} \oplus \left\{ \bigoplus_{\omega_1\neq\omega_2,\ (\nu_1,\nu_2)\in\mathcal{E}_{\mathcal{H}}} \Phi_{\mathcal{H}}(\nu_1, \nu_2) \right\}.$$

Then

$$deg_{\mathcal{G}}(\nu_1, \omega_1) = deg_{\mathcal{F}}(\omega_1) \bigoplus deg_{\mathcal{H}}(\nu_1) \bigoplus |\mathcal{V}_{\mathcal{F}} - \{\omega_1\}| deg_{\mathcal{H}}(\nu_1),$$

hence

$$deg_{\mathcal{G}}(\nu_1, \omega_1) = deg_{\mathcal{F}}(\omega_1) \bigoplus |\mathcal{V}_{\mathcal{F}}| deg_{\mathcal{H}}(\nu_1).$$

□

4.2. Union of two complex hesitant fuzzy graphs

Definition 12. *The union graph is*

$$\mathcal{G} = \mathcal{H} \cup \mathcal{F} = (\mathcal{V},\ \mathcal{E},\ \Psi_{\mathcal{H}\cup\mathcal{F}},\ \Phi_{\mathcal{H}\cup\mathcal{F}})$$

of

$$\mathcal{H} = (\mathcal{V}_{\mathcal{H}},\ \mathcal{E}_{\mathcal{H}},\ \Psi_{\mathcal{H}},\ \Phi_{\mathcal{H}})$$

and

$$\mathcal{F} = (\mathcal{V}_{\mathcal{F}},\ \mathcal{E}_{\mathcal{F}},\ \Psi_{\mathcal{F}},\ \Phi_{\mathcal{F}}),$$

where:
(1)

$$(\Psi_{\mathcal{H}\cup\mathcal{F}})(\nu) = \left\{ \begin{array}{ll} \Psi_{\mathcal{H}}(\nu) & : \nu \in \mathcal{V}_{\mathcal{H}} \\ \Psi_{\mathcal{F}}(\nu) & : \nu \in \mathcal{V}_{\mathcal{F}} \\ \max\{\Psi_{\mathcal{H}}(\nu), \Psi_{\mathcal{F}}(\nu)\} & : \nu \in \mathcal{V}_{\mathcal{H}} \cap \mathcal{V}_{\mathcal{F}} \end{array} \right\},$$

(2)

$$(\Phi_{\mathcal{H}\cup\mathcal{F}})(\nu\omega) = \left\{ \begin{array}{ll} \Phi_{\mathcal{H}}(\nu\omega) & : \nu\omega \in \mathcal{E}_{\mathcal{H}} \\ \Phi_{\mathcal{F}}(\nu\omega) & : \nu\omega \in \mathcal{E}_{\mathcal{F}} \\ \max\{\Phi_{\mathcal{H}}(\nu\omega), \Phi_{\mathcal{F}}(\nu\omega)\} & : \nu\omega \in \mathcal{E}_{\mathcal{H}} \cap \mathcal{E}_{\mathcal{F}} \end{array} \right\}.$$

So that, for $\mathcal{G} = \mathcal{H} \cup \mathcal{F}$, if $\mathcal{H} \cap \mathcal{F} = \emptyset$, then it will be disconnected graph with two components. Note that, a general example is given in the Figure 3, where $\mathcal{H} \cap \mathcal{F} \neq \emptyset$.

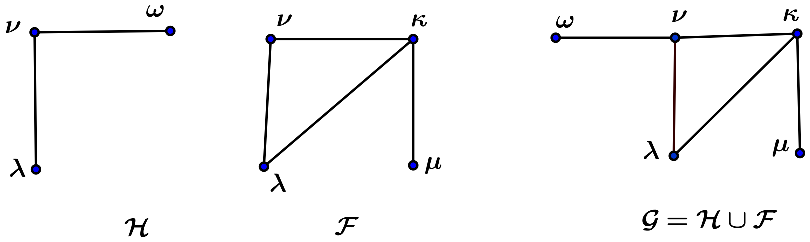

Figure 3. $\mathcal{G} = \mathcal{H} \cup \mathcal{F}$.

Moreover, it is an easy consequence that $\mathcal{G}$ is CHFGr.

Proposition 3. *The union of two CHFGrs is CHFGr.*

Lemma 1. *For CHFGr $\mathcal{G} = \mathcal{H} \cup \mathcal{F}$ and by Definition 12, the vertex degree of $\mathcal{G}$ is*

$$deg_{\mathcal{H}\cup\mathcal{F}}(\nu) = \left\{ \begin{array}{ll} deg_{\mathcal{H}}(\nu) & : \nu \in \mathcal{H} \\ deg_{\mathcal{F}}(\nu) & : \nu \in \mathcal{F} \\ \Phi_{\mathcal{H}}(\nu\omega) + \Phi_{\mathcal{F}}(\nu\kappa) + \max\{\Phi_{\mathcal{H}}(\nu\lambda), \Phi_{\mathcal{F}}(\nu\lambda)\} & : \nu\lambda \in \mathcal{E}_{\mathcal{H}} \cap \mathcal{E}_{\mathcal{F}}, \\ & \nu\omega \in \mathcal{E}_{\mathcal{H}},\ \nu\kappa \in \mathcal{E}_{\mathcal{F}} \end{array} \right\},$$

where the proof is straight forward, see Figure 3 as an example.

4.3. The joint graph of two complex hesitant fuzzy graphs

Definition 13. *The joint graph is*

$$\mathcal{G} = \mathcal{H} + \mathcal{F} = (\mathcal{V},\ \mathcal{E},\ \Psi_{\mathcal{H}+\mathcal{F}},\ \Phi_{\mathcal{H}+\mathcal{F}})$$

of

$$\mathcal{H} = (\mathcal{V}_{\mathcal{H}},\ \mathcal{E}_{\mathcal{H}},\ \Psi_{\mathcal{H}},\ \Phi_{\mathcal{H}})$$

and

$$mathcalF = (\mathcal{V}_{\mathcal{F}},\ \mathcal{E}_{\mathcal{F}},\ \Psi_{\mathcal{F}},\ \Phi_{\mathcal{F}}),$$

where:
(1)

$$(\Psi_{\mathcal{H}+\mathcal{F}})(\nu) = \left\{ \begin{array}{ll} \Psi_{\mathcal{H}}(\nu) & : \nu \in \mathcal{V}_{\mathcal{H}} \\ \Psi_{\mathcal{F}}(\nu) & : \nu \in \mathcal{V}_{\mathcal{F}} \end{array} \right\},$$

(2)

$$(\Phi_{\mathcal{H}+\mathcal{F}})(\nu\omega) = \left\{ \begin{array}{ll} \Phi_{\mathcal{H}}(\nu\omega) & : \nu\omega \in \mathcal{E}_{\mathcal{H}} \\ \Phi_{\mathcal{F}}(\nu\omega) & : \nu\omega \in \mathcal{E}_{\mathcal{F}} \\ \Psi_{\mathcal{H}}(\nu) \wedge \Psi_{\mathcal{F}}(\omega) & : \nu \in \mathcal{V}_{\mathcal{H}} \textit{ and } \omega \in \mathcal{V}_{\mathcal{F}} \end{array} \right\}.$$

By considering CHFGr $\mathcal{H}$ and $\mathcal{F}$ in the Figure 1, the joint graph $\mathcal{G}$ is in the Figure 4; where we look to scores of vertices in the Example 1 and that for third case of labellings edges.

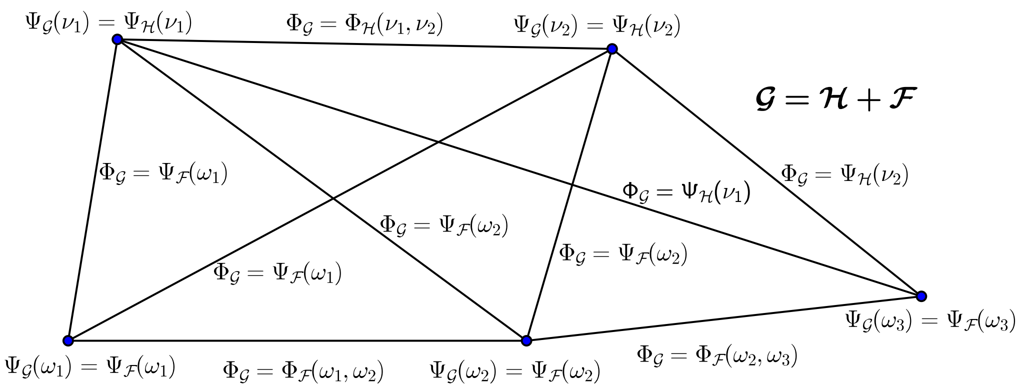

Figure 4. $\mathcal{G} = \mathcal{H} + \mathcal{F}$.

Proposition 4. *The joint of two CHFGrs is CHFGr.*

Proof. It is straight forward to show that $(\Phi_{\mathcal{H}+\mathcal{F}})(\nu\omega) \leq \Psi_{\mathcal{H}}(\nu) \wedge \Psi_{\mathcal{F}}(\omega)$. □

Lemma 2. *Let $\mathcal{G}$ be a joint graph of two CHFGr $\mathcal{H}$ and $\mathcal{F}$, then the vertex degree of $\mathcal{G}$ equal*

$$deg_{\mathcal{H}+\mathcal{F}}(\nu) = \left\{ \begin{array}{ll} deg_{\mathcal{H}}(\nu) + \sum_{\omega \in \mathcal{V}_{\mathcal{F}}} \min\{\Psi_{\mathcal{H}}(\nu), \Psi_{\mathcal{F}}(\omega)\} & : \nu \in \mathcal{H} \\ deg_{\mathcal{F}}(\nu) + \sum_{\omega \in \mathcal{V}_{\mathcal{H}}} \min\{\Psi_{\mathcal{F}}(\nu), \Psi_{\mathcal{H}}(\omega)\} & : \nu \in \mathcal{F} \end{array} \right\},$$

where the proof of previous lemma is straight forward, by Definition 13, and see Figure 4 as an example.

4.4. The complement graph of complex hesitant fuzzy graphs

Definition 14. *The complement graph of the CHFGr $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \Psi, \Phi)$, is denoted by $\bar{\mathcal{G}}$ where $\mathcal{V}_{\bar{\mathcal{G}}} = \mathcal{V}$ and $\mathcal{E}_{\bar{\mathcal{G}}} = \{\nu\omega : \nu\omega \notin E$ and $\nu, \omega \in \mathcal{V}\}$. Moreover,*

(1) $$\Psi_{\bar{\mathcal{G}}} = \Psi_{\mathcal{G}},$$

(2)

$$(\Phi_{\bar{\mathcal{G}}})(\nu\omega) = \left\{ \begin{array}{ll} \Psi(\nu) \wedge \Psi(\omega) & : \nu\omega \notin \mathcal{E} \\ 0 & : \nu\omega \in \mathcal{E} \end{array} \right\}.$$

As an example of $\mathcal{G}$ and the complement of it, see Figure 5.

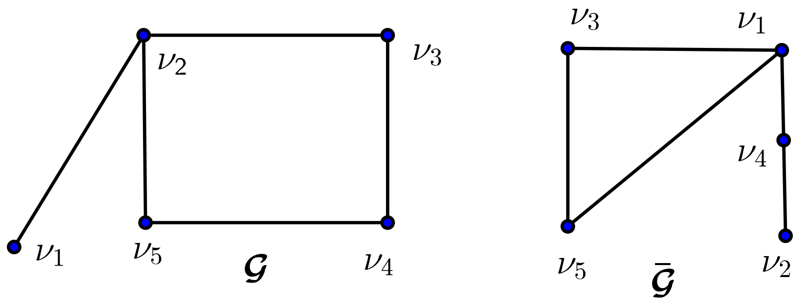

Figure 5. $\mathcal{G}$ and $\bar{\mathcal{G}}$.

In addition, as a consequence of the Definition 14, we have the following results.

Proposition 5. *The complement of CHFGr is CHFGr.*

Lemma 3. *Let $\mathcal{G}$ be a CHFGr, then the vertex degree of $\bar{\mathcal{G}}$ is*

$$deg_{\bar{\mathcal{G}}}(\nu) = \sum_{\omega \in \mathcal{V}_{\mathcal{G}},\ \nu\omega \notin \mathcal{E}_{\mathcal{G}}} \min\{\Psi_{\mathcal{G}}(\nu), \Psi_{\mathcal{G}}(\omega)\}.$$

Note that, the proof is straightforward for the previous proposition and lemma.

5. Practical problem

Tangible and intangible assets represent the market worth of any company. Moreover, the market worth of companies is considered an important indicator that expresses the capability of a future merger between two companies. Several reasons drive numerous researchers to find out and evaluate the fair market worth of a company. These reasons are to buy or sell a company, merge, divide, or liquidate two companies or more. In the case of the merger of companies, the need performs heavier when both the companies' owners are attracted to establishing a fair price. To accomplish an ideal merger of companies with distinct types of CHFGr, we need to study many factors. One of these factors is the connections or relations of each merged company to other companies. Also, it is important to show if there is a joint relation between company X and company Y (merger company to other companies). The presented graphs help the company owners to decide on the merger depending on the market worth besides identifying the optimal time to merge.

First, consider that a network has a set of companies (vertices) besides the mutual collaboration between them (edges). Moreover, for each company (vertex), the amplitude term is the performance of the company's market worth, and the phase term is the time needed to reach that performance. In addition, for each mutual collaboration between companies (edges), the amplitude term is the market worth of companies' joint ventures, and the phase term is the appropriate period for established to have such a market value. Hence, we use the CHFGr model.

Assume that there are four companies $\{v_1, v_2, v_3, v_4\}$. We want to get a measurement of the value of each company and the collaboration between companies at a specific time. The model of CHFGr that represents this problem is shown in Figure 6.

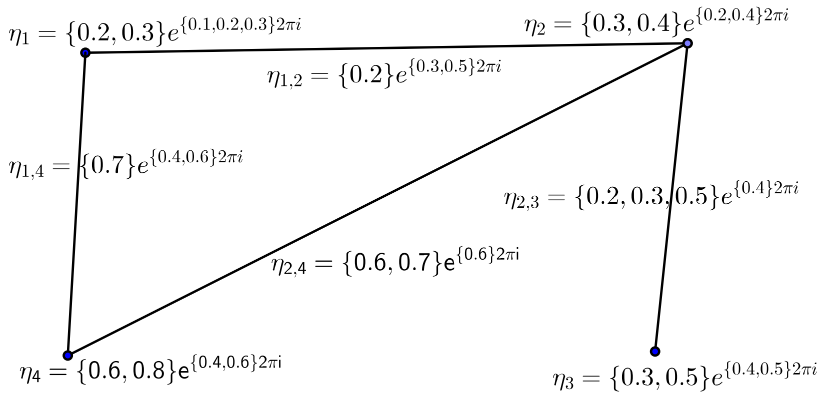

Figure 6. CHFGr.

Here, v_1 has labelling value is $\eta_1 = \{0.2, 0.3\}e^{\{0.1,0.2,0.3\}2\pi i}$ and it represents the value of company v_1. The set $\{0.2, 0.3\}$ represents the performance of the market-worth of the company, and the set $\{0.1, 0.2, 0.3\}$ represents the time needed to reach that performance. Similarly, $\eta_{1,4} = \{0.7\}e^{\{0.4,0.6\}2\pi i}$ represents the collaboration between companies v_1 and v_4. Furthermore, the set $\{0.7\}$ represents the market-worth of companies' joint ventures, and the set $\{0.4, 0.6\}$ represents the appropriate period established to have such a market value. Then apply previous concepts to other vertices and edges in the CHFGr model.

Then, the priority is to define/evaluate the weaknesses and strengths of the collaboration between companies, it is compulsory/needed, to give a suitable decision by decision-makers (DMs). The score (and deviation degree if necessary) for each company (vertex) and between mutual companies (edges) is (are) documented, as follows:

(1) Scores of vertices are: $s(\eta_1) = 0.225$, $s(\eta_2) = 0.325$, $s(\eta_3) = 0.425$ and $s(\eta_4) = 0.6$. The finding explains company v_4 is the best one and company v_1 is the worst one among all.

(2) Scores of edges are: $s(\eta_{1,2}) = 0.3$, $s(\eta_{2,3}) = 0.365$, $s(\eta_{1,4}) = 0.6$ and $s(\eta_{2,4}) = 0.625$. Companies v_2 and v_4 have the best collaboration. However, the worst collaboration is between company v_1 and v_2.

(3) Out of all, collaboration between companies v_2 and v_4 is the best one. In contrast, the company v_1, has the worst value.

(4) In addition, we found the deviation degree of the company v_4 and the collaboration $e_{1,4}$. We got $\delta(\eta_4) = 0.447$ and $\delta(\eta_{1,4}) = 0.274$. Hence the value of company v_4 is better than the value of collaboration between v_1 and v_4.

If we reduce a design to be HFGr i.e., we have amplitude-term only (no phase-term), then the valuations of scores (and deviation degrees, if necessary) of vertices and edges, as follows:

(1) Scores of vertices are: $s(\eta_1) = 0.25$, $s(\eta_2) = 0.35$, $s(\eta_3) = 0.4$ and $s(\eta_4) = 0.7$. Therefore, the company v_4 is the best one, in the performance of the market-worth of the company, and the company v_1 is the worst one.

(2) Scores of edges are: $s(\eta_{1,2}) = 0.2$, $s(\eta_{2,3}) = 0.33$, $s(\eta_{1,4}) = 0.7$ and $s(\eta_{2,4}) = 0.65$. Hence, the market-worth of companies' joint ventures of $\eta_{1,4}$ is the best, whereas the worst one is for $\eta_{1,2}$.

(3) Among all, the worst value is for $\eta_{1,2}$. Also, after findings deviation degree of company v_4 and collaboration between companies v_1 and v_4, we got $\delta(\eta_4) = 0.1$ and $\delta(\eta_{1,4}) = 0$. Hence the performance in the company v_4 is the best one.

(4) Finally, HFGr comparable to CHFGr has different results, and this proves that one factor gives a different decision when comparing two variables.

6. Comparative study

Since daily life decisions are changeable and affected by many factors and circumstances, finding a proper tool to represent the relation objects has become a complex task. Initially, a fuzzy graph represents the relation between networks and objects. The limitation of representing two or more sources of vagueness in fuzzy sets was the main gate to generalize FS and FGr to other tools, such as IFS, HFS, CIFS, CHFS and their graphs. Each of FGr, HFGr, IFGr, CFGr, CHFGr and CIFGr has two universes of discourse; $(\mathcal{V}, \mathcal{E})$. However, only the CFGr, CHFGr and CIFGr have a range with complex-valued lies in the unit disc in the complex plane for both universes $\mathcal{V}$ and $\mathcal{E}$. Also, all the presented graphs may convey uncertainty measurements, but only the complex graphs convey the additional periodic measurements. The Co-domain (range) of membership functions for each FGr, HFGr and IFGr has only amplitude terms. However, each of CFGr, CHFGr and CIFGr has both amplitude terms and phase terms to represent the uncertainty and the periodicity information. CHFS is considered a generalization of CFS and CIFS consequently, CHFGr carries multi values of vagueness in the form of $[0, 1]e^{[0,1]2\pi i}$ (in the Complex plane).

7. Conclusions and future works

This manuscript is an extension of CHFGr from AbuHijleh [34]. A new operation studied composition, joint and union between two CHFGrs and the complement of CHFGr. An application of CHFGr in the DM problem was given, which produced a generalization of the practical problem argued in Javid et al. [40] in Section 4. Note that if we merge two CHFGrs and present a new CHFGr such as composition, union or joint and investigate the new market-worth performance, the market worth of companies' joint ventures, and the time for both to gain more profit can as possible, then DM's can set the best equipment for companies and their collaborations from this merger. Also, one can look to the complement of CHFGr as a model and specify which model is more beneficial to each company, the original one or its complement.

As a suggestion for future work, we may improve CHFGr to represent an environmental impact assessment algorithm and a new model to identify the optimal location to build a factory, for example. Another suggestion is to improve CHFGr in the approach of intuitionistic, Pythagorean, fermatean and neutrosophic fuzzy sets.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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