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Research article

On an axiomatization of the grey Banzhaf value

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Abstract: The Banzhaf value with grey data is a solution concept in cooperative grey games that has been extensively studied in the context of operations research. The author aims to define the traits of the Banzhaf value in cooperative grey games, where the values of coalitions are depicted as grey numbers within intervals. The grey Banzhaf value is defined by several axioms, including the grey dummy player, grey van den Brink fairness, and grey superadditivity. By presenting these axioms, this investigation contributes novel insights to the axiomatic characterization of the grey Banzhaf value, offering a distinct perspective. Finally, the study concludes by presenting applications in cooperative grey game models, thereby enriching the understanding of this concept.

Keywords: Banzhaf value; grey calculus; cooperative grey games; Shapley value; characterization
Mathematics Subject Classification: 90B70, 91A12

1. Introduction

Mathematical models have found utility across diverse domains, spanning from social sciences to economics, facilitating the resolution of intricate strategic problems and even contributing to insights in psychology. In this context, game theory emerges as a broad mathematical field, encompassing models and techniques essential for the analysis of human behavior. Notably, recent years have witnessed a surge in the incorporation of *uncertainty* into various facets of life, transforming it into a focal point in model development, control and optimization across economics, finance, data science engineering, and operations research. This shift in perspective towards uncertainty is underpinned by methodologies such as grey numbers, uncertainty quantization, robust optimization counterparts, ellipsoidal calculus, and stochastic optimal control, including applications to stochastic hybrid systems [1].

The overarching goal of the grey system and its applications is to dismantle barriers between natural sciences and sociology. In other words, grey systems theory emerges as an interdisciplinary field, transcending specific domains. The evolution of the grey system aligns with its theoretical advancements, coupled with tangible applications across diverse fields. Through its well-established theory and successful practical implementations, the concept of the grey system has gained global recognition. Some of the application areas of the grey system: Environmental science [2–4], earthquake [5], geology [6], meteorology [7], irrigation strategy [8], economic model [9], biological conservation [10], agriculture [11], traffic [12], medicine [13], industry [14], military affairs, sports [15], materials science [16], the judicial system [17], etc. [18–20]. The grey system theory, put forward by [21], emerged as a new methodology focused on examining problems with small examples and weak information. By generating and extracting valuable insights from available data, uncertain systems characterized by partially known information are scrutinized. This approach enables the precise definition and effective monitoring of the operational behavior and evolutionary laws of systems. In practical scenarios, uncertain systems often present with limited and weak information, often derived from small sample sizes. This underscores the extensive applicability of grey systems theory, making it a potential tool capable of delivering beneficial impacts on the advancement of human society. The versatility of this theory, particularly in handling situations with sparse information, positions it as a robust and practical framework with wide-ranging applications.

In recent years, the field of grey system models has witnessed substantial advancements, with scholars, as exemplified by Xie et al. [22], contributing significantly to this ongoing research. Among these models, grey prediction models stand out as a pivotal component. These models, rooted in grey correlation theory, provide a quantitative framework for exploring the relationships between two correlated factors. By emphasizing geometric analysis and the comparison of data sequence curves, the theory discerns the strength of association. A closer alignment in developmental trends, as reflected in similar curve shapes, signifies a stronger degree of correlation and, conversely, a divergence in trends indicates a weaker association [23].

Furthermore, grey decision-making, as elucidated by [24], emerges as a critical facet within this realm. It entails the utilization of decision models that either incorporate grey components or integrate traditional decision models with grey systems models. This synthesis facilitates a nuanced approach to decision-making, drawing on the strengths of both traditional and grey system methodologies. As such, the synergy between grey prediction models and decision-making frameworks underscores the multifaceted nature of grey system models and their applicability in diverse academic contexts [25].

In the realm of cooperative game theory, the prominent solutions often center around the Banzhaf value and the Shapley value. Many authors have undertaken efforts to delineate the Banzhaf value by introducing alternative axioms in lieu of the efficiency axiom, which is traditionally employed in the derivation of the Shapley value [26–28]. The concept of the Banzhaf value was first introduced by Banzhaf within the context of voting games. Its application later expanded to arbitrary games as shown by [29, 30]. The Banzhaf value is defined by the principle that each player has an equal probability to join any coalition. Many researchers have formulated several axiomizations of the Banzhaf value in the realm of transferable utility games (TU-games), such as the characterizations of [31–33], contain the additivity axiom. [34], on the other hand, did not use the additivity axiom, but instead uses the marginality used by [35] in characterizing the Shapley value. Further, [35] used the dummy player axiom, symmetry and 2-efficiency. Later, [36] used van den Brink's fairness/differential marginality axioms instead of marginality and symmetry axioms. Ultimately, [36] obtained a new characterization of the Banzhaf value. The aim of all these efforts is not to use the axiom of additivity axiom. [35] has given the marginality axiom of a player's payoff, which requires that a player depend on their own productivity measured only by marginal contributions. Next, [35] characterized the Shapley value with the help of the equal treatment and the efficiency axiom. Lately, [36] has put forward a differential version of marginality, differential marginality is due to the difference in returns of two players being affected only by the difference in their own productivity, and is again measured by the difference in their marginal contributions. It turns out that the differential marginality proposed by [37] is equivalent to the fairness property. The Shapley value is characterized by differential marginality, fairness properties along with the null player and efficiency axiom such as several researchers having proposed alternative characterizations of the Shapley value without the additivity axiom [38–41].

In this study, the author combines characterizations of [34, 36] and presents a novel description of the grey Banzhaf value in cooperative games, utilizing the grey van den Brink fairness axiom, the grey dummy player axiom, and the grey superadditivity axiom. For example, this characterization may be restricted to certain sub-domains, the domain of superadditive games.

This study can be briefly summarised as follows: Section 2 presents the fundamental definitions and symbols, laying the groundwork. In Section 3, the axioms and lemmas required for the innovative characterisation of the Banzhaf value are rigorously established. Finally, the study concludes with a comprehensive insight into the findings, extending the discourse with practical applications and notable observations.

2. Preliminaries

The author gives some information from cooperative game theory and grey calculus. The ordered pair $< N, v >$ is a cooperative game, where $N = \{1, 2, \cdots, n\}$ is the set of players, and it is a function that for every $S \in 2^N$ assigns a real number to the v: $2^N \rightarrow \mathbb{R}$ coalition so that $v(\emptyset) = 0$. Here, the function $v(S)$ is called the value of the coalition S in the game v and I define cooperative game $< N, v >$ with its characteristic function of v. The set of coalitional games with player set N is shown as G^N [42].

A number, the precise value of which is unknown but exists within a known range, is referred to as a grey number, denoted by $\otimes$. In practical applications, a grey number is typically represented as either a specified range or a general set of numbers. Various types of grey numbers exist to accommodate different scenarios and applications.

(1) Grey numbers with lower bounds only: This kind of grey number $\otimes$ is written as $\otimes \in \left[\underline{x}, \infty\right]$ or $\otimes\left(\underline{x}\right)$, where $\underline{x}$ is known lower bound of the grey number $\otimes$.

(2) Grey numbers with upper bound only: This kind of grey number $\otimes$ is written as $\otimes \in [-\infty, \overline{x}]$ or $\otimes(\overline{x})$, where $\overline{x}$ is known upper bound of the grey number $\otimes$.

(3) Interval grey numbers: A grey number with both a lower limit $\underline{x}$ and an upper limit $\overline{x}$ is called an interval grey number, denoted as $w \in \left[\underline{x}, \overline{x}\right]$. In this study, The author consider interval grey numbers.

(4) Continuous and discrete grey numbers: A grey number which only takes a finite number or a countable number of potenial values is known as discrete. If a grey number can potentially take any value within a interval, then it is said to be continuous.

(5) Black and white numbers: If given as $\otimes \in (-\infty, +\infty)$, then $\otimes$ is called a black number. If $\otimes \in \left[\underline{x}, \overline{x}\right]$ and $\underline{x} = \overline{x}$, $\otimes$ is known as a white number.

(6) Essential and non-essential grey numbers: The first type signifies a number that exhibits such

a degree of uncertainty that it cannot be temporarily encapsulated by a precise white number. On the other hand, the second type refers to a grey number that can be represented by a white number, either acquired through empirical experience or determined by a specific method. The specific white number, denoted by $\otimes$, is termed as the whitening of the grey number. The author also use $\otimes(x)$ to represent the grey number(s) with x as its whitenization. Let us now consider operations of numbers with interval grey numbers. Let

$$\otimes_1 \in [x, y], \quad x < y \text{ and } \otimes_2 \in [z, t], \ z < t.$$

The sum of these two numbers are given

$$\otimes_1 + \otimes_2 \in [x + z, y + t].$$

The additive inverse is given by

$$-\otimes_1 \in [-y, -x].$$

Therefore, the substraction is given

$$\otimes_1 - \otimes_2 = \otimes_1 + (-\otimes_2) \in [x - t, y - z].$$

The multiplication is defined as follows:

$$\otimes_1 \cdot \otimes_2 \in [\min\{xz, xt, yz, yt\}, \max\{xz, xt, yz, yt\}].$$

The reciprocal of $\otimes_1 \in [x, y]$, $x < y$, $x, y \neq 0$, $xy > 0$ is defined as:

$$\otimes_1^{-1} \in \left[\frac{1}{y}, \frac{1}{x}\right].$$

Let $z, t \neq 0$, $zt > 0$. The division is defined as follows:

$$\otimes_1 / \otimes_2 = \otimes_1 \cdot \otimes_2^{-1} = \left[\min\left\{\frac{x}{z}, \frac{x}{t}, \frac{y}{z}, \frac{y}{t}\right\}, \max\left\{\frac{x}{z}, \frac{x}{t}, \frac{y}{z}, \frac{y}{t}\right\}\right].$$

Assume that $w \in [x, y]$, $x < y$ and k is a positive real number. The scalar multiplication of k and $\otimes_1$ is defined as follows:

$$k\otimes_1 \in [kx, ky].$$

The *kth* power of the grey number $\otimes_1$ is given

$$\otimes_1^k \in \left[x^k, y^k\right],$$

where k is a positive constant. In general, the difference of $\otimes_1 \in [x, y]$ and $\otimes_2 \in [z, t]$ is defined as follows:

$$\otimes_1 \ominus \otimes_2 = \otimes_1 + (-\otimes_2) \in [x - t, y - z],$$

(see [43]). For example, let $\otimes_1 \in [4, 7]$ and $\otimes_2 \in [8, 9]$, then we have

$$\begin{aligned} \otimes_1 \ominus \otimes_2 &\in [4 - 9, 7 - 8] = [-5, -1], \\ \otimes_2 \ominus \otimes_1 &\in [8 - 7, 9 - 4] = [1, 5]. \end{aligned}$$

In contrast to the subtraction operator mentioned earlier, the author uses the partial subtraction operator in this context and defines $\otimes_1 - \otimes_2$, only if

$$|y - x| \geq |t - z|,$$

by

$$\otimes_1 - \otimes_2 \in [x - z, y - t].$$

Note that $x - z \leq y - t$. I remember that $[x, y]$ is weakly better than $[z, t]$, which I denote by

$$[x, y] \succcurlyeq [z, t],$$

if and only if $x \geq z$ and $y \geq t$. Also, I use the reverse notation

$$[x, y] \preccurlyeq [z, t],$$

if and only if $x \leq z$ and $y \leq t$ [44]. If the author makes a comparison with the above example, it is clearly seen that in our case $[8, 9]–[4, 7]$ is not defined but $[4, 7]–[8, 9]$ is defined. Let $\otimes_1 \in [4, 7]$ and $\otimes_2 \in [8, 9]$, $\otimes_1 - \otimes_2$ is defined since $|7 - 4| \geq |9 - 8|$, but $\otimes_2 - \otimes_1$ is not defined since

$$|9 - 8| = 1 \ngeq 3 = |7 - 4|,$$

then we have

$$w_1 - w_2 \in [4 - 8, 7 - 9] = [-4, -2].$$

The author can give the definition of the grey Shapley value, the grey Banzhaf value and some information about the grey Banzhaf value as follows.

The author refers to a game $< N, w >$ grey size monotonic if $< N, |w| >$ is monotonic, i.e., $|w|(S) \leq |w|(T)$ for all $S, T \in 2^N$ with $S \subset T$. Additionally, I express by $SMGG^N$ the class of all grey size monotonic games with player set N. The grey Shapley value, the grey Banzhaf value and the grey marginal operators are defined on $SMGG^N$.

The *grey Shapley value* $\Phi'\colon SMGG^N \to G(\mathbb{R})^N$ is defined by

$$\Phi'(w) := \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^{\sigma}(w) \in [\frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^{\sigma}(\underline{w}), \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^{\sigma}(\overline{w})],$$

for each $w \in SMGG^N$ [30]. The grey Banzhaf value $\beta'\colon SMGG^N \to G(\mathbb{R})^N$ is defined by

$$\begin{aligned} \beta'(w) \quad : \quad &= \frac{1}{2^{|N|-1}} \sum_{i \in S} (w(S) - w(S \setminus \{i\})) \qquad (2.1) \\ &\in \frac{1}{2^{|N|-1}} \left[\sum_{i \in S} \left(\underline{w}(S) - \underline{w}(S \setminus \{i\}) \right), \sum_{i \in S} (\overline{w}(S) - \overline{w}(S \setminus \{i\})) \right], \end{aligned}$$

for each $w \in SMGG^N$ [44]. Let $S \in 2^N \setminus \{\emptyset\}$, $w \in G(\mathbb{R})$ and u_S be the classical unanimity game based on S [42]. The cooperative grey game $< N, wu_S >$ is defined by

$$(wu_S)(T) = wu_S(T),$$

for each $T \in 2^N \setminus \{\emptyset\}$. I denote by $K\mathcal{G}G^N$ the additive cone generated by the set

$$K = \left\{ w_S u_S \mid S \in 2^N \setminus \{\emptyset\}, w_S \in \mathcal{G}(\mathbb{R}) \right\}.$$

That is, each element of the cone is a finite sum of elements of K. I notice that $K\mathcal{G}G^N \subset SM\mathcal{G}G^N$, and axiomatically characterize the restriction of the grey Banzhaf value to the cone $K\mathcal{G}G^N$ [44]. Note that here, the classic game is denoted by v and the grey game by w. A table of the terms used can be found in Table 1.

Table 1. Table of terminology.

Symbol	Meaning
N	The set of players
2^N	The collection of subsets of N
$\mathbb{R}$	Real numbers
S	The coalition: each element of 2^N
$v(S)$	The value of coalition S
v	Classical cooperative game
G^N	The set of classical cooperative games
w	Cooperative grey game
$\mathcal{G}(\mathbb{R})^N$	The set of grey payoff vectors
$\mathcal{G}G^N$	The family of all cooperative grey games
β'	The grey Shapley value
K	The base of cooperative grey game
$K\mathcal{G}G^N$	The additive cone obtained by set of K
$SM\mathcal{G}G^N$	The class of all grey size monotonic games

3. The new characterization of Banzhaf value

This section introduces a new interpretation of the grey Banzhaf value for use in the context of cooperative grey games. The author proposes that this characterization is grounded in the application of specific axioms, lemmas, and theorems. The (single-valued) grey solution is a function f: $\mathcal{G}G^N \to \mathcal{G}(\mathbb{R})^N$ which assigns an $|N|$-dimensional real vector to every grey game on N. This vector is conceptualized as a distribution representing the grey payoffs achievable through collaboration among individual players within the game. The author first states the well-known axioms for solutions f: $\mathcal{G}G^N \to \mathcal{G}(\mathbb{R})^N$.

Axiom 3.1. (Grey efficiency ($\mathcal{G}$-**EFF**)) For $\forall\omega \in \mathcal{G}G^N$, it holds that

$$\sum_{i \in N} f_i(\omega) = \omega(N).$$

Player $i \in N$ is a $\mathcal{G}$-null player in $\omega \in \mathcal{G}G^N$ if $\omega(S) = \omega(S \setminus \{i\})$ for $\forall S \subset N$.

Axiom 3.2. (Grey 2-efficiency ($\mathcal{G}$-**2EFF**)) For $\forall\omega \in \mathcal{G}G^N$ and $i, j \in N, i \neq j$,

$$f_i\left(\omega_{ij}\right) = f_i(\omega) + f_j(\omega).$$

Axiom 3.3. (Grey dummy player ($\mathcal{G}$-**DUMMY**)) If $i \in N$ is a $\mathcal{G}$-dummy player in game $\omega \in \mathcal{G}G^N$, then,

$$f_i(\omega) = \omega(i) \in \left[\underline{\omega}(i), \overline{\omega}(i)\right].$$

Axiom 3.4. (Grey null player ($\mathcal{G}$-**NULL**)) If $i \in N$ is a $\mathcal{G}$-null player in game $\omega \in \mathcal{G}G^N$, then $f_i(\omega) \in [0,0]$. If

$$\omega(S \cup \{i\}) = \omega(S \cup \{j\}),$$

for all $S \subseteq N \backslash \{i, j\}$, two players $i, j \in N$ are called $\mathcal{G}$-symmetric in $\omega \in \mathcal{G}G^N$.

Axiom 3.5. (Grey symmetry ($\mathcal{G}$-**SYM**)) If i and j are $\mathcal{G}$-symmetric in $\omega \in \mathcal{G}G^N$, then

$$f_i(\omega) = f_j(\omega).$$

Axiom 3.6. (Grey additivity ($\mathcal{G}$-**ADD**)) If

$$f(\omega + w) = f(\omega) + f(w),$$

for $\forall \omega, w \in \mathcal{G}G^N$, where $(\omega + w) \in \mathcal{G}G^N$ is given by

$$(\omega + w)(S) = \omega(S) + w(S),$$

for all $S \subseteq N$.

Axiom 3.7. (Grey marginality ($\mathcal{G}$-**MAR**)) For $\forall \omega, w \in \mathcal{G}G^N$ and all $i \in N$ such that

$$\omega(S \cup \{i\}) - \omega(S) = w(S \cup \{i\}) - w(S),$$

for all $S \subseteq N \backslash \{i\}$,

$$f_i(\omega) = f_i(w).$$

Axiom 3.8. (Grey superadditivity ($\mathcal{G}$-**SA**)) For $\forall \omega \in \mathcal{G}G^N$ and $i, j \in N,\ i \neq j$,

$$f_i(\omega) + f_j(\omega) \leq f_i\left(\omega_{ij}\right).$$

Axiom 3.9. (Grey van den Brink fairness ($\mathcal{G}$-**BF**)) For $\forall \omega, w \in \mathcal{G}G^N$ and $i, j \in N$, the following statement is valid: if i and j display $\mathcal{G}$-symmetric in $w \in \mathcal{G}G^N$

$$f_i(\omega + w) - f_i(\omega) = f_j(\omega + w) - f_j(\omega).$$

Axiom 3.10. (Grey null player out ($\mathcal{G}$-**NPO**)) For $\forall \omega \in \mathcal{G}G^N$ and $i \in N$ and $o \notin N$, $f_i(\omega|^o) = f_i(\omega)$.

Currently, the author characterises the Banzhaf value for grey games within a specific grey subclass. To explain this characterisation, three lemmas must be introduced and applied.

Theorem 3.1. *If f satisfies $\mathcal{G}$-**NULL** and $\mathcal{G}$-**BF**, then*

$$f_i(\omega + w) - f_j(\omega + w) = f_i(\omega) - f_j(\omega) + f_i(w) - f_j(w),$$

for all $\omega, w \in \mathcal{G}G^N$, $|N| > 2$ and $i, j \in N$.

Proof. The proof of lemma is given in [45]. □

Theorem 3.2. *If f satisfies $\mathcal{G}$-**DUMMY** and $\mathcal{G}$-**BF**, then*

$$f_i\left(\omega|^o + w|^o\right) = f_i\left(\omega|^o\right) + f_i\left(w|^o\right),$$

for all $\omega, w \in \mathcal{G}G^N$ and $i \in N \cup \{o\}$, where $o \notin N$ and $\omega|^o, w|^o \in \mathcal{G}G^{N\cup\{o\}}$,

$$\omega|^o(K) = \omega(K \cap N), \quad w|^o(K) = w(K \cap N), \quad K \subseteq N \cup \{o\}.$$

Proof. I note that o is a null player for $v|^o$, $w|^o$ and $v|^o + w|^o$ and that i is a dummy player for $v|^o \in \mathcal{G}G^{\{i,o\}}$, $i \neq o$. Hence, for $i = o$ or $|N| = 1$, the claim holds by $\mathcal{G}$-**DUMMY**. Further, $\mathcal{G}$-**DUMMY** entails $\mathcal{G}$-**NULL**. Therefore, Theorem 3.1 entails

$$\begin{aligned}
&\varphi_i\left(N \cup \{o\}, v|^o + w|^o\right) - \varphi_i\left(N \cup \{o\}, v|^o\right) - \varphi_i\left(N \cup \{o\}, w|^o\right)\\
&= \varphi_o\left(N \cup \{o\}, v|^o + w|^o\right) - \varphi_o\left(N \cup \{o\}, v|^o\right) - \varphi_o\left(N \cup \{o\}, w|^o\right)\\
&\overset{\mathcal{G}-D}{\in} [0, 0],
\end{aligned}$$

for $|N| > 1$ and $i \in N$ which concludes the proof. □

Theorem 3.3. *If f satisfies $\mathcal{G}$-**DUMMY**, $\mathcal{G}$-**BF**, and $\mathcal{G}$-**SA**, then*

$$f_i(\omega|^o) = \beta'_i(\omega), \quad i \in N,$$

where $o \notin N$ and

$$\omega|^o \in \mathcal{G}G^{N\cup\{o\}}, \quad \omega|^o(K) = \omega(K \cap N), \quad K \subseteq N \cup \{o\}.$$

Proof. Since o is a null player in $(N \cup \{o\}, \omega|o)$, then $\mathcal{G}$-**DUMMY** entails the claim for $|N| < 2$. By ([45, Observation 2 and Proposition 4]), $\mathcal{G}$-**DUMMY** and $\mathcal{G}$-**BF** imply **($\mathcal{G}$-SYM)**. Consequently, Theorem 3.2 combined with the datas from the proof of [31] yield

$$f(\omega|^o) = \beta'(\omega|^o),$$

for $|N| \geq 2$. By Eq (2.1), β' obeys $\mathcal{G}$-**NPO** [46], which concludes the proof. □

Theorem 3.4. *The Banzhaf value is the unique value that satisfies $\mathcal{G}$-**DUMMY**, $\mathcal{G}$-**BF** and $\mathcal{G}$-**SA**.*

Proof. By [31], β' obeys $\mathcal{G}$-**DUMMY** and $\mathcal{G}$-**SA**. From Eq (2.1), it is immediate that β' also meets $\mathcal{G}$-**BF**. First, f satisfies $\mathcal{G}$-**DUMMY**, $\mathcal{G}$-**BF** and $\mathcal{G}$-**SA**. For $i \in N$, this gives

$$f_i(\omega) = f_i\left((\omega|^o)_{io}\right) \overset{\mathcal{G}-SA}{\geq} f_i\left(\omega|^o\right) + f_o\left(\omega|^o\right) \overset{\text{Lemma 3.3.}}{=} \beta'_i(\omega). \tag{3.1}$$

Here, the first equation is obtained from the definitions of $\omega|^o$ and of ω_{io} with o is a Null player in $\omega|^o$. By $\mathcal{G}$-**DUMMY**, the claim holds for $|N| = 1$. I continue by induction on $|N|$. Let

$$f(N, \omega) = \beta'(N, \omega),$$

for $|N| \leq k$, $\omega \in \mathcal{G}G^N$. I consider $|N| = k + 1$. For $i, j \in N, i \neq j$, I have

$$\beta'_i(\omega) + \beta'_j(\omega) \overset{\mathcal{G}-2E}{=} \beta'_i\left(\omega_A\right) = f_i\left(\omega_A\right) \overset{\mathcal{G}-SA}{\geq} f_i(\omega) + f_j(\omega),$$

where the induction hypothesis entails the second equation. Finally, Eq (3.1) implies $f(\omega) = \beta'(\omega)$ which concludes the proof. □

4. An application

In this section, the author presents the permutation situation for the grey game. The Banzhaf value is subsequently determined using grey data.

I consider the permutation situation [31,42]:

- There are n persons and each person i, possesses a machine M_i and has a job J_i $(i = 1, 2, \cdots, n)$ to be processed.
- Any machine M_j can process any job J_i, but no machine is allowed to process more than one job.
- Coalition formation and side payments are allowed.
- If a person does not cooperate, his job has to be processed on his own machine.
- The cost of processing job J_i on machine M_j equals k_{ij} $(i, j = 1, 2, \cdots, n)$.

This situation can be extended to the cost grey game $< N, c >$ where for each coalition $S \in 2^N \setminus \{\emptyset\}$

$$c(S) \in \min_{\sigma} \sum_{i \in S} k_{i\sigma(i)} \tag{4.1}$$

with the minimum taken over all permutations σ: $S \rightarrow S$. Such a σ corresponds to a plan where a job J_i of player $i \in S$ is processed on machine $M_{\sigma(i)}$ of player $\sigma(i)$. The grey game given by Eq (4.1) is called the permutation grey game, corresponding to the grey cost matrix

$$K = \left[k_{ij}\right],$$

for each $i, j = 1, \cdots, n$ [42,47].

The following example illustrates the calculation of the grey Banzhaf value.

Example 4.1. *Let $< N, c >$ be a cooperative grey permutation game with $N = \{1, 2, 3\}$ and the grey cost matrix*

$$K \in \begin{bmatrix} [1,2] & [2,3] & [3,4] \\ [3,4] & [4,5] & [5,6] \\ [4,5] & [6,7] & [7,8] \end{bmatrix}.$$

Now, I find the coalitonal values for this game.

$$\begin{aligned}
c(1) &\in [1,2], c(2) \in [4,5], c(3) \in [7,8], \\
c(12) &\in \min\{[1,2] + [4,5], [2,3] + [3,4]\} = \min\{[5,7], [5,7]\} = [5,7], \\
c(13) &\in \min\{[1,2] + [7,8], [3,4] + [4,5]\} = \min\{[8,10], [7,9]\} = [7,9], \\
c(23) &\in \min\{[4,5] + [7,8], [5,6] + [6,7]\} = \min\{[11,13], [11,13]\} = [11,13], \\
c(123) &\in \min\{[12,15], [12,15], [12,15], [11,14], [12,15], [11,14]\} = [11,14].
\end{aligned}$$

Now, let us look at how to calculate the grey Banzhaf value for this game. For player 1, I have

$$\begin{aligned}
\beta'_1(c) &\in \frac{1}{2^{3-1}} \sum_{1 \in S} [c(S) - c(S \setminus \{1\})] \\
&= \frac{1}{4} [c(1) + c(12) - c(2) + c(13) - c(3) + c(123) - c(23)] \\
&= \frac{1}{4} [2,6] = \left[\frac{1}{2}, 1\frac{1}{2}\right].
\end{aligned}$$

For player 2, I have

$$\beta_2'(c) \in [4,5].$$

For player 3, I have

$$\beta_3'(c) \in \left[6\frac{1}{2}, 7\frac{1}{2}\right].$$

Consequently, the grey Banzhaf value of this game which can be shown as:

$$\beta'(c) \in \left(\left[\frac{1}{2}, 1\frac{1}{2}\right], [4,5], \left[6\frac{1}{2}, 7\frac{1}{2}\right]\right).$$

Now, the author presents the computation of the grey Shapley value. The grey marginal vectors are given in the following table, where σ: $N \to N$ is identified with $\sigma(1)$–(3). First, for $\sigma_1 = (1,2,3)$, I calculate the grey marginal vectors. Then,

$$\begin{aligned} m_1^{\sigma_1}(c) &= c(1) \in [1,2], \\ m_2^{\sigma_1}(c) &= c(12) - c(1) \in [5,7] - [1,2] = [4,5], \\ m_3^{\sigma_1}(c) &= c(123) - c(12) \in [11,14] - [5,7] = [6,7]. \end{aligned}$$

The others can be calculated similarly, which is shown in Table 2.

Table 2. Grey marginal vectors.

σ	$m_1^{\sigma}(c)$	$m_2^{\sigma}(c)$	$m_3^{\sigma}(c)$
$\sigma_1 = (1,2,3)$	$m_1^{\sigma_1}(c) \in [1,2]$	$m_2^{\sigma_1}(c) \in [4,5]$	$m_3^{\sigma_1}(c) \in [6,7]$
$\sigma_2 = (1,3,2)$	$m_1^{\sigma_2}(c) \in [1,2]$	$m_2^{\sigma_2}(c) \in [4,5]$	$m_3^{\sigma_2}(c) \in [6,7]$
$\sigma_3 = (2,1,3)$	$m_1^{\sigma_3}(c) \in [1,2]$	$m_2^{\sigma_3}(c) \in [4,5]$	$m_3^{\sigma_3}(c) \in [6,7]$
$\sigma_4 = (2,3,1)$	$m_1^{\sigma_4}(c) \in [0,1]$	$m_2^{\sigma_4}(c) \in [4,5]$	$m_3^{\sigma_4}(c) \in [7,8]$
$\sigma_5 = (3,1,2)$	$m_1^{\sigma_5}(c) \in [0,1]$	$m_2^{\sigma_5}(c) \in [4,5]$	$m_3^{\sigma_5}(c) \in [7,8]$
$\sigma_6 = (3,2,1)$	$m_1^{\sigma_6}(c) \in [0,1]$	$m_2^{\sigma_6}(c) \in [4,5]$	$m_3^{\sigma_6}(c) \in [7,8]$

Table 2 displays the grey marginal vectors of the collaborative grey game in Example 4.1. The grey Shapley value of this game, which is the mean of the six grey marginal vectors, can be presented as follows:

$$\Phi'(c) \in ([\frac{1}{2}, 1\frac{1}{2}], [4,5], [6\frac{1}{2}, 7\frac{1}{2}]).$$

The Shapley value stands as one of the most coveted solutions in cooperative game theory. The closer any solution approaches the Shapley value, the more it is regarded as an ideal resolution. In permutation scenarios, it was discovered that the grey Shapley value and the grey Banzhaf value were equal upon comparing. This observation underscores that the grey Banzhaf value can be considered one of the most practical solutions for permutation cases, as evidenced by real-world applications.

5. Conclusions and discussion

The axiomatization of grey Banzhaf value in game theory provides a crucial conceptual framework for assessing power distribution among coalitions. The effort to axiomatize the grey Banzhaf value stems from the need to establish fundamental regulations governing this measurement. Within this framework, diverse axiomatic approaches aim to identify valid and consistent regulations for the grey rendition of the Banzhaf value. The grey Banzhaf value provides a model that evaluates the powers of players within an environment of uncertainty, potentially offering a more effective reflection of real-world scenarios. These attempts at axiomatization reflect endeavors to comprehend the intricacies of game theory and adapt it to more realistic applications. In this context, the axiomatization of grey Banzhaf value contributes to the evolution of the discipline by addressing gaps in game theory and tackling power distribution within the context of uncertainty.

The notion of dividends, introduced by [48], plays a pivotal role in characterizing the Banzhaf value within classical game theory. This concept is extendable to cooperative grey games, where the grey Banzhaf value can be effectively characterized through the utilization of grey dividends. In conclusion, this avenue of exploration holds promise for future research endeavors.

The purpose of the paper is to achieve the axiomatic characterization of the grey Banzhaf value through the application of the aforementioned axioms, which collectively uniquely define the grey Banzhaf value.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author has no conflict of interest.

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