AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(12): 30194–30208.
DOI:10.3934/math.20231543
Received: 08 September 2023
Revised: 09 October 2023
Accepted: 15 October 2023
Published: 07 November 2023

Research article

Matrix inverses along the core parts of three matrix decompositions

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Abstract: New characterizations for generalized inverses along the core parts of three matrix decompositions were investigated in this paper. Let A_1, $\hat{A}_1$ and $\tilde{A}_1$ be the core parts of the core-nilpotent decomposition, the core-EP decomposition and EP-nilpotent decomposition of $A \in \mathbb{C}^{n\times n}$, respectively, where EP denotes the EP matrix. A number of characterizations and different representations of the Drazin inverse, the weak group inverse and the core-EP inverse were given by using the core parts A_1, $\hat{A}_1$ and $\tilde{A}_1$. One can prove that, the Drazin inverse is the inverse along A_1, the weak group inverse is the inverse along $\hat{A}_1$ and the core-EP inverse is the inverse along $\tilde{A}_1$. A unified theory presented in this paper covers the Drazin inverse, the weak group inverse and the core-EP inverse based on the core parts of the core-nilpotent decomposition, the core-EP decomposition and EP-nilpotent decomposition of $A \in \mathbb{C}^{n\times n}$, respectively. In addition, we proved that the Drazin inverse of A is the inverse of A along U and A_1 for any $U \in \{A_1, \hat{A}_1, \tilde{A}_1\}$; the weak group inverse of A is the inverse of A along U and $\hat{A}_1$ for any $U \in \{A_1, \hat{A}_1, \tilde{A}_1\}$; the core-EP inverse of A is the inverse of A along U and $\tilde{A}_1$ for any $U \in \{A_1, \hat{A}_1, \tilde{A}_1\}$. Let X_1, X_4 and X_7 be the generalized inverses along A_1, $\hat{A}_1$ and $\tilde{A}_1$, respectively. In the last section, some useful examples were given, which showed that the generalized inverses X_1, X_4 and X_7 were different generalized inverses. For a certain singular complex matrix, the Drazin inverse coincides with the weak group inverse, which is different from the core-EP inverse. Moreover, we showed that the Drazin inverse, the weak group inverse and the core-EP inverse can be the same for a certain singular complex matrix.

Keywords: core-nilpotent decomposition; core-EP decomposition; EP-nilpotent decomposition; Drazin inverse; weak group inverse; core-EP inverse; index
Mathematics Subject Classification: 15A09

1. Introduction

Let $\mathbb{C}$ be the complex field. The set $\mathbb{C}^{m\times n}$ denotes the set of all $m \times n$ complex matrices over the complex field $\mathbb{C}$. Let $A \in \mathbb{C}^{m\times n}$. The symbol A^* denotes the conjugate transpose of A. Notations $\mathcal{R}(A) = \{y \in \mathbb{C}^m : y = Ax, x \in \mathbb{C}^n\}$ and $\mathcal{N}(A) = \{x \in \mathbb{C}^n : Ax = 0\}$ will be used in the sequel. The smallest positive integer is k, such that $\operatorname{rank}(A^k) = \operatorname{rank}(A^{k+1})$ is called the index of $A \in \mathbb{C}^{n\times n}$ and denoted by $\operatorname{ind}(A)$.

Let $A \in \mathbb{C}^{m\times n}$. If a matrix $X \in \mathbb{C}^{n\times m}$ satisfies $AXA = A$, $XAX = X$, $(AX)^* = AX$, $(XA)^* = XA$, then X is called the Moore-Penrose inverse of A [13, 17] and denoted by $X = A^\dagger$. Let $A, X \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. Then, the algebraic definition of the Drazin inverse is as follows if

$$AXA = A,\ XA^{k+1} = A^k \text{ and } AX = XA,$$

then X is called the Drazin inverse of A. If such X exists, then it is unique and denoted by A^D [7]. Note that for a square complex matrix, the algebraic definition of the Drazin inverse is equivalent to the functional definition of the Drazin inverse. We have the following lemma by the canonical form representation for A and A^D in Theorem 7.2.1 [5].

Lemma 1.1. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k > 0$, then the Drazin inverse exists.*

The core inverse and the dual core inverse for a complex matrix was introduced by Baksalary and Trenkler [4]. Let $A \in \mathbb{C}^{n\times n}$. A matrix $X \in \mathbb{C}^{n\times n}$ is called a core inverse of A if it satisfies $AX = P_A$ and $\mathcal{R}(X) \subseteq \mathcal{R}(A)$, where $\mathcal{R}(A)$ denotes the column space of A and P_A is the orthogonal projector onto $\mathcal{R}(A)$. If such a matrix exists, then it is unique (and denoted by $A^{\#\!\!\!\bigcirc}$). Baksalary and Trenkler gave several characterizations of the core inverse by using the decomposition of Hartwig and Spindelböck [10, 11]. In [12], Mary introduced a new type of generalized inverse, namely the inverse along an element. This inverse is depended on Green's relations [9]. The inverse along an element contains some known generalized inverses, such as group inverse, Drazin inverse and Moore-Penrose inverse. Many existence criterion for the inverse along an element can be found in [12, 16]. Manjunatha Prasad and Mohana [15] introduced the core-EP inverse of a matrix. Let $A \in \mathbb{C}^{n\times n}$. If there exists $X \in \mathbb{C}^{n\times n}$ such that $XAX = X$ and $\mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^k)$, then X is called the core-EP inverse of A. If such inverse exists, then it is unique and denoted by $A^{\dagger\!\!\!\bigcirc}$. The weak group inverse of a complex matrix was introduced by Wang and Chen [22], which is the unique matrix X such that $AX^2 = X$ and $AX = A^{\dagger\!\!\!\bigcirc}A$ and denoted by $X = A^{w\!\!\!\bigcirc}$.

Let $A \in \mathbb{C}^{n\times n}$. The core-nilpotent decomposition [14, see Theorem 2.2.21] of A is the sum of two matrices A_1 and A_2, i.e., $A = A_1 + A_2$, such that $\operatorname{rank}(A_1) = \operatorname{rank}(A_1^2)$, A_2 is nilpotent and $A_1A_2 = A_2A_1 = 0$. It is well known that this decomposition is unique. Moreover, $A_1 = AA^DA = A^DA^2 = A^2A^D$ by [5, Definition 7.3.1], if $\operatorname{ind}(A) \leqslant 1$, and thus A coincides with A_1. A_1 is called the core part of A. Also, $A_2 = A - AA^DA$ is the nilpotent part of A. In [21, Theorem 2.1], Wang introduced a new matrix decomposition, namely the core-EP decomposition of $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. Given a matrix $A \in \mathbb{C}^{n\times n}$, then A can be written as the sum of matrices $\hat{A}_1 \in \mathbb{C}^{n\times n}$ and $\hat{A}_2 \in \mathbb{C}^{n\times n}$. That is $A = \hat{A}_1 + \hat{A}_2$, where $\hat{A}_1$ is an index one matrix, $\hat{A}_2^k = 0$ and $\hat{A}_1^*\hat{A}_2 = \hat{A}_2\hat{A}_1 = 0$. In [21, Theorems 2.3 and 2.4], Wang proved this matrix decomposition is unique and that there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that

$$\hat{A}_1 = U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}U^* \text{ and } \hat{A}_2 = U\begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix}U^*, \tag{1.1}$$

where $T \in \mathbb{C}^{r\times r}$ is nonsingular, $N \in \mathbb{C}^{(n-r)\times(n-r)}$ is nilpotent and r is number of nonzero eigenvalues of A. In [21, Theorem 2.3], Wang proved that $\hat{A}_1$ can be described by using the Moore-Penrose inverse of A^k. The explicit expressions of $\hat{A}_1$ can be found in the following lemmas.

Lemma 1.2. *Let $A \in \mathbb{C}^{n\times n}$ with* $\text{ind}(A) = k$*. If $A = \hat{A}_1 + \hat{A}_2$ is the core-EP decomposition of A, then $\hat{A}_1 = A^k(A^k)^\dagger A$ and $\hat{A}_2 = A - A^k(A^k)^\dagger A$.*

Let $A \in \mathbb{C}^{n\times n}$ with $\text{ind}\,(A) = k$. The EP-nilpotent decomposition of A was introduced by Wang and Liu [23]. A can be written as $A = \tilde{A}_1 + \tilde{A}_2$, where $\tilde{A}_1$ is an EP matrix, $\tilde{A}_2^{k+1} = 0$ and $\tilde{A}_2\tilde{A}_1 = 0$. By the proof of [23, Theorem 2.2], one can get the following lemma.

Lemma 1.3. [23, Theorem 2.1] *Let $A \in \mathbb{C}^{n\times n}$ with* $\text{ind}\,(A) = k$ *and $A = \tilde{A}_1 + \tilde{A}_2$ be the EP-nilpotent decomposition of A. Then, there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that*

$$\tilde{A}_1 = U\begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix}U^* \text{ and } \tilde{A}_2 = U\begin{bmatrix} 0 & S \\ 0 & N \end{bmatrix}U^*, \tag{1.2}$$

where $T \in \mathbb{C}^{r\times r}$ is nonsingular, $N \in \mathbb{C}^{(n-r)\times(n-r)}$ is nilpotent and r is the number of nonzero eigenvalues of A.

The core part of the EP-nilpotent decomposition can be expressed by the Moore-Penrose inverse of A^k, where $\text{ind}(A) = k$.

Lemma 1.4. [23, Theorem 2.2] *Let $A \in \mathbb{C}^{n\times n}$ with* $\text{ind}(A) = k$ *and $A = \tilde{A}_1 + \tilde{A}_2$ be the EP-nilpotent decomposition of A as* (1.2)*, then $\tilde{A}_1 = AA^k(A^k)^\dagger$.*

Let $A, B, C \in \mathbb{C}^{n\times n}$. We say that $Y \in \mathbb{C}^{n\times n}$ is a (B, C)-inverse of A if we have

$$YAB = B,\ CAY = C,\ \mathcal{N}(C) \subseteq \mathcal{N}(Y) \text{ and } \mathcal{R}(Y) \subseteq \mathcal{R}(B).$$

If such Y exists, then it is unique (see [1, Definition 4.1] and [19, Definition 1.2]). We also call the (B, C)-inverse of A is the inverse of A along B and C. Note that the (B, C)-inverse was introduced in the setting of semigroups [8]. The (B, C)-inverse of A will be denoted by $A^{\parallel(B,C)}$. Note that Bapat et al. [2] investigated an outer inverse in Theorem 5 that is exactly the same as the (y, x)-inverse, where x and y are elements in a semigroup. In [20], Rao and Mitra showed that $A^{\parallel(B,C)} = B(CAB)^-C$, where $(CAB)^-$ stands for the arbitrary inner inverse of CAB, where CAB is the product of $A, B, C \in \mathbb{C}^{n\times n}$.

Lemma 1.5. [18, Lemma 2.2.6(g)] *Let $A, B, C \in \mathbb{C}^{n\times n}$. If* $\text{rank}\,(CAB) = \text{rank}\,(B) = \text{rank}\,(C)$*, then $B(CAB)^-C$ is invariant for any choice of $(CAB)^-$.*

The following lemma shows that the (B, C)-inverse of A is an outer inverse of A, and can be characterized by using the column space of B and the null space of C.

Theorem 1.6. [8, Theorem 2.1 (ii) and Proposition 6.1] *Let $A, B, C \in \mathbb{C}^{n\times n}$. Then, $Y \in \mathbb{C}^{n\times n}$ is the (B, C)-inverse of A if, and only if, $YAY = Y$, $\mathcal{R}(Y) = \mathcal{R}(B)$ and $\mathcal{N}(Y) = \mathcal{N}(C)$.*

The following lemma can be found in [24, Lemma 3.11] for elements in rings, which also shows that the Drazin inverse is the inverse along A^k and A^k, where k is the index of A.

Lemma 1.7. [8, p1910] *Let $A \in \mathbb{C}^{n\times n}$ with* $\text{ind}(A) = k$*, then the Drazin inverse of A coincides with the (A^k, A^k)-inverse of A. In particular, the group inverse of A coincides with the (A, A)-inverse of A.*

Lemmas 1.8 and 1.9 show that the core-EP inverse of A is a generalization of the core inverse of A. Moreover, the core inverse of A^k is the core-EP inverse of A, where k is the index of A.

Lemma 1.8. [8, p1910] *Let* $A \in \mathbb{C}^{n\times n}$ *with* $\mathrm{ind}(A) = 1$*, then the core inverse of* A *coincides with the* (A, A^*)*-inverse of* A.

Lemma 1.9. [19, Theorem 1.10] *Let* $A \in \mathbb{C}^{n\times n}$ *with* $\mathrm{ind}(A) = k$*, then the core-EP inverse of* A *coincides with the* $(A^k, (A^k)^*)$*-inverse of* A.

Lemma 1.10. [3, Remark 2.2 (i)] *Let* $A, B, C, U, V \in \mathbb{C}^{n\times n}$*. If* $\mathcal{R}(B) = \mathcal{R}(U)$ *and* $\mathcal{N}(C) = \mathcal{N}(V)$*, then* A *is* (B, C)*-invertible if and only if* A *is* (U, V)*-invertible. In this case, we have* $A^{\|(B,C)} = A^{\|(U,V)}$.

Based on the core parts of the core-nilpotent decomposition, core-EP decomposition and EP-nilpotent decomposition of $A \in \mathbb{C}^{n\times n}$, respectively, three generalized inverses along two matrices are investigated, namely, the Drazin inverse, the weak group inverse and the core-EP inverse. Let X_1, X_4 and X_7 be the generalized inverses along A_1, $\hat{A}_1$ and $\tilde{A}_1$, respectively. The major contributions of the article can be highlighted as follows:

1) Three generalized inverses related the core part A_1 of the core-nilpotent decomposition are investigated.

2) Three generalized inverses related the core part $\hat{A}_1$ of the core-EP decomposition are investigated.

3) Three generalized inverses related the core part $\tilde{A}_1$ of the EP-nilpotent decomposition are investigated.

4) We show that the Drazin inverse, the weak group inverse and the core-EP inverse are different generalized inverses.

5) For a singular complex matrix, we can prove that the Drazin inverse coincides with the weak group inverse, which is different from the core-EP inverse. Moreover, we can show that the Drazin inverse, the weak group inverse and the core-EP inverse can be same for a certain singular complex matrix.

The paper is organized as follows. In section two, we prove that X_i is the same as X_j. Moreover, X_j coincides with the Drazin inverse of A, where $i, j \in \{1, 2, 3\}$. In section three, we can prove that X_i is the same as X_j and that X_j coincides with the weak group inverse of A, where $i, j \in \{4, 5, 6\}$. In section four, we can prove that X_i is the same as X_j and X_j coincides with the core-EP inverse of A, where $i, j \in \{7, 8, 9\}$. In section five, relationships between X_i and X_j for $i, j \in \{1, 2, \cdots, 9\}$ are investigated.

2. Three generalized inverses related the core part A_1 of the core-nilpotent decomposition

In this section, three generalized inverses along the core parts of matrix decompositions are introduced. In Table 1, one can see that we denoted the generalized inverse along the core parts of the core-nilpotent decomposition as X_1 by using the symbol of the generalized inverse along two matrices. In a similar way, X_2 denotes the generalized inverse along the core part of the core-EP decomposition and the core part of the core-nilpotent decomposition. X_3 denotes the generalized inverse along the core part of the EP-nilpotent decomposition and the core part of the core-nilpotent decomposition. In addition, we prove that X_i is the same as X_j and that X_j coincides with the Drazin inverse of A, where $i, j \in \{1, 2, 3\}$.

Table 1. Three generalized inverses related A_1 of the core-nilpotent decomposition.

Three generalized inverses	Core part	The generalized inverses along the core part
type I	A_1	$X_1 = A^{\parallel(A_1,A_1)}$
type II	$\hat{A}_1$ and A_1	$X_2 = A^{\parallel(\hat{A}_1,A_1)}$
type III	$\tilde{A}_1$ and A_1	$X_3 = A^{\parallel(\tilde{A}_1,A_1)}$

Theorem 2.1. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. The X_1 coincides with the Drazin inverse of A. That is the Drazin inverse of A is the inverse along A_1, where A_1 is the core part of the core-nilpotent decomposition.*

Proof. Let A_1 be the core part of the core-nilpotent decomposition, then $A_1 = A^DA^2$ and we have

$$\begin{aligned} &A_1 = A^DA^2 = (A^DA)A = (A^DA)^kA = A^k(A^D)^kA. \\ &A^k = A^DA^{k+1} = (A^DA)A^k = (A^DA)^2A^k = A^DA^2A^DA^k = A_1A^DA^k. \end{aligned} \tag{2.1}$$

Thus, we have

$$\mathcal{R}(A_1) = \mathcal{R}(A^DA^2) = \mathcal{R}(A^k). \tag{2.2}$$

For any $x \in \mathcal{N}(A_1)$, then

$$A^Dx = A^DAA^Dx = (A^DA)^2A^Dx = (A^D)^2A^DA^2x = (A^D)^2A_1x = 0. \tag{2.3}$$

For any $y \in \mathcal{N}(A^D)$, then

$$A_1y = A^DA^2y = A^2A^Dy = 0. \tag{2.4}$$

So,

$$\mathcal{N}(A^D) = \mathcal{N}(A_1) \tag{2.5}$$

by the Eqs (2.3) and (2.4). For any $u \in \mathcal{N}(A^D)$, then

$$A^ku = A^DA^{k+1}u = A^{k+1}A^Du = 0. \tag{2.6}$$

For any $v \in \mathcal{N}(A^k)$, then

$$A^Dv = A^DAA^Dv = (A^DA)^kA^Dv = (A^D)^{k+1}A^kv = 0. \tag{2.7}$$

So,

$$\mathcal{N}(A^D) = \mathcal{N}(A^k) \tag{2.8}$$

by the Eqs (2.6) and (2.7). Thus, we have

$$\mathcal{N}(A_1) = \mathcal{N}(A^k) \tag{2.9}$$

by the Eqs (2.5) and (2.8). Therefore, X_1 coincides with the Drazin inverse by Eqs (2.2) and (2.9) and Lemmas 1.7 and 1.10. □

Theorem 2.2. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. The X_2 coincides with the Drazin inverse of A. That is the Drazin inverse of A is the inverse along $\hat{A}_1$ and A_1, where $\hat{A}_1$ is the core part of the core-EP decomposition and A_1 is the core part of the core-nilpotent decomposition.*

Proof. By the equalities $\hat{A}_1 = A^k(A^k)^\dagger A = A^k[(A^k)^\dagger A^k]^*(A^k)^\dagger A = A^k(A^k)^*[(A^k)^\dagger]^*(A^k)^\dagger A$ and $A^k(A^k)^* = A^k(A^k)^\dagger A^k(A^k)^* = A^k(A^k)^\dagger AA^{k-1}(A^k)^* = \hat{A}_1A^{k-1}(A^k)^*$, we have

$$\mathcal{R}(\hat{A}_1) = \mathcal{R}(A^k(A^k)^*). \tag{2.10}$$

Thus, X_2 coincides the inverse along $A^k(A^k)^*$ and $(A^k)^*A$. That is $X_2 = A^{\|(A^k(A^k)^*,(A^k)^*A)}$, which is equivalent to X_2 as the $(A^k(A^k)^*,(A^k)^*A)$-inverse. By $\hat{A}_1 = A^k(A^k)^\dagger A$ and $A^k = A^k(A^k)^\dagger A^k = A^k(A^k)^\dagger AA^{k-1} = A_1A^{k-1}$, we have $\mathcal{R}(\hat{A}_1) = \mathcal{R}(A^k)$. Thus, the condition (3.6) can be replaced by $\mathcal{R}(\hat{A}_1) = \mathcal{R}(A^k)$ and we have the following theorems.
Thus, X_2 coincides with the Drazin inverse of A by Lemma 1.10, and the proof of Theorem 2.1. □

Theorem 2.3. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$. The X_3 coincides with the Drazin inverse of A, that is the Drazin inverse of A is the inverse along $\tilde{A}_1$ and A_1, where $\tilde{A}_1$ is the core part of the EP-nilpotent decomposition and A_1 is the core part of the core-nilpotent decomposition.*

Proof. Since

$$\begin{aligned} A^{k+1}(A^k)^\dagger &= A^{k+1}(A^k)^\dagger A^k(A^k)^\dagger \\ &= A^{k+1}[(A^k)^\dagger)A^k]^*(A^k)^\dagger = A^{k+1}(A^k)^*[(A^k)^\dagger]^*(A^k)^\dagger, \\ A^{k+1}(A^k)^* &= A^{k+1}(A^k(A^k)^\dagger A^k)^* = A^{k+1}[(A^k)^\dagger A^k]^*(A^k)^* \\ &= A^{k+1}(A^k)^\dagger A^k(A^k)^*, \end{aligned} \tag{2.11}$$

we have $\mathcal{R}(A^{k+1}) = \mathcal{R}(\tilde{A}_1)$, which implies

$$\mathcal{R}(A^k) = \mathcal{R}(\tilde{A}_1). \tag{2.12}$$

Thus, X_2 coincides the inverse along $A^k(A^k)^*$ and $(A^k)^*A$. That is $X_2 = A^{\|(A^k(A^k)^*,(A^k)^*A)}$, which is equivalent to X_2 as the $(A^k(A^k)^*,(A^k)^*A)$-inverse. By $\hat{A}_1 = A^k(A^k)^\dagger A$ and $A^k = A^k(A^k)^\dagger A^k = A^k(A^k)^\dagger AA^{k-1} = A_1A^{k-1}$, we have $\mathcal{R}(\hat{A}_1) = \mathcal{R}(A^k)$. Thus, the condition (3.6) can be replaced by $\mathcal{R}(\hat{A}_1) = \mathcal{R}(A^k)$ and we have the following theorem. Thus, X_3 coincides with the Drazin inverse of A by Lemma 1.10, and the proof of Theorem 2.1. □

Theorem 2.4. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$, then, X_i is the same as X_j. Moreover, X_j coincides with the Drazin inverse of A, where $i, j \in \{1, 2, 3\}$.*

Proof. It is trivial by Theorems 2.1–2.3. □

3. Three generalized inverses related the core part $\hat{A}_1$ of the core-EP decomposition

In this section, three generalized inverses along the core parts of matrix decompositions are introduced. In Table 2, one can see that we denoted the generalized inverse along the core parts of the core-EP decomposition as X_4 by using the symbol of the generalized inverse along two matrices. In a similar way, X_5 denotes the generalized inverse along the core part of the core-nilpotent decomposition and the core part of the core-EP decomposition. X_6 denotes the generalized inverse along the core part of the EP-nilpotent decomposition and the core part of the core-EP decomposition decomposition. In addition, we prove that X_i is the same as X_j and that X_j coincides with the weak group inverse of A, where $i, j \in \{4, 5, 6\}$.

Table 2. Three generalized inverses related $\hat{A}_1$ of the core-EP decomposition.

Three generalized inverses	Core parts	The generalized inverses along the core part
type IV	$\hat{A}_1$	$X_4 = A^{\parallel(\hat{A}_1,\hat{A}_1)}$
type V	A_1 and $\hat{A}_1$	$X_5 = A^{\parallel(A_1,\hat{A}_1)}$
type VI	$\tilde{A}_1$ and $\hat{A}_1$	$X_6 = A^{\parallel(\tilde{A}_1,\hat{A}_1)}$

Theorem 3.1. *Let* $A \in \mathbb{C}^{n\times n}$ *with* $\operatorname{ind}(A) = k$, *then the generalized inverse* X_4 *coincides with the* $(A^k(A^k)^*, (A^k)^*A)$*-inverse of* A.

Proof. Let $\hat{A}_1$ be the core part of the core-EP decomposition as (1.1), the $\hat{A}_1 = A^k(A^k)^\dagger A$ by Lemma 1.2. For any $x \in \mathcal{N}((A^k)^\dagger A)$, we have

$$\hat{A}_1 x = A^k(A^k)^\dagger Ax = 0. \tag{3.1}$$

For any $y \in \mathcal{N}(\hat{A}_1)$, we have

$$(A^k)^\dagger Ay = (A^k)^\dagger A^k(A^k)^\dagger Ay = (A^k)^\dagger A_1 y = 0. \tag{3.2}$$

Thus, we have

$$\mathcal{N}(\hat{A}_1) = \mathcal{N}((A^k)^\dagger A) \tag{3.3}$$

by Eqs (3.1) and (3.2). Also, we have

$$\mathcal{N}((A^k)^* A) = \mathcal{N}((A^k)^\dagger A) \tag{3.4}$$

by

$$(A^k)^*A = [A^k(A^k)^\dagger A^k]^*A = (A^k)^*A^k(A^k)^\dagger A$$

and

$$(A^k)^\dagger A = (A^k)^\dagger A^k(A^k)^\dagger A = (A^k)^\dagger[A^k(A^k)^\dagger]^*A = (A^k)^\dagger[(A^k)^\dagger]^*(A^k)^*A.$$

Equations (3.3) and (3.4) imply

$$\mathcal{N}(\hat{A}_1) = \mathcal{N}((A^k)^*A). \tag{3.5}$$

By $\hat{A}_1 = A^k(A^k)^\dagger A = A^k(A^k)^\dagger A^k(A^k)^\dagger A = A^k[(A^k)^\dagger A^k]^*(A^k)^\dagger A = A^k(A^k)^*[(A^k)^\dagger]^*(A^k)^\dagger A$ and $A^k(A^k)^* = A^k(A^k)^\dagger A^k(A^k)^* = A^k(A^k)^\dagger AA^{k-1}(A^k)^* = \hat{A}_1A^{k-1}(A^k)^*$, we have

$$\mathcal{R}(\hat{A}_1) = \mathcal{R}(A^k(A^k)^*). \tag{3.6}$$

Thus, X_4 coincides the inverse along $A^k(A^k)^*$ and $(A^k)^*A$. That is, $X_4 = A^{\parallel(A^k(A^k)^*,(A^k)^*A)}$, which is equivalent to X_4 as the $(A^k(A^k)^*, (A^k)^*A)$-inverse. □

We have $\hat{A}_1 = A^k(A^k)^\dagger A$ and $A^k = A^k(A^k)^\dagger A^k = A^k(A^k)^\dagger AA^{k-1} = A_1A^{k-1}$ by Lemma 1.2, so $\mathcal{R}(\hat{A}_1) = \mathcal{R}(A^k)$. Thus, condition (3.6) can be replaced by $\mathcal{R}(\hat{A}_1) = \mathcal{R}(A^k)$ and we have the following theorem.

Theorem 3.2. *Let* $A \in \mathbb{C}^{n\times n}$ *with* $\operatorname{ind}(A) = k$, *then* X_4 *coincides with the* $(A^k, (A^k)^*A)$*-inverse.*

For the square matrix A_1, an inner inverse of A_1 with columns belonging to the linear manifold generated by the columns of A_1 and rows belonging to the linear manifold generated by the rows of A_1 will be called a generalized constrained inverse of A and denoted by A^-_{gRC} [6, Definition 3.1]. That is, if $X \in \mathbb{C}^{n\times n}$ satisfies $A_1XA_1 = A_1$, $\mathcal{R}(X) \subseteq \mathcal{R}(A_1)$ and $\mathcal{RS}(X) \subseteq \mathcal{RS}(A_1)$, then $X = A^-_{gRC}$. In the following lemmas, one can see that the generalized constrained inverse of A coincides with the weak group inverse by Lemma 3.3. Moreover, the weak group inverse of A coincides with the group inverse of $\hat{A}_1$ by Lemma 3.5, thus the generalized constrained inverse of A coincides with the group inverse of $\hat{A}_1$. By Lemma 3.4 and Theorem 3.2, we have that X_4 coincides with the generalized constrained inverse of A.

Lemma 3.3. [6, Theorem 3.4] *Let $A \in \mathbb{C}^{n\times n}$. If $X \in \mathbb{C}^{n\times n}$ is a generalized constrained inverse of A, then this generalized constrained inverse of A is unique. Moreover, the generalized constrained inverse of A coincides with the weak group inverse; that is, $A^-_{gRC} = A^{ⓦ}$.*

Lemma 3.4. [6, Theorem 4.4] *Let $A \in \mathbb{C}^{n\times n}$ with* $\mathrm{ind}\,(A) = k$*. The generalized constrained inverse of A coincides with the $(A^k, (A^k)^*A)$-inverse of A.*

Lemma 3.5. [22, Theorem 3.7] *Let $A \in \mathbb{C}^{n\times n}$ with* $\mathrm{ind}(A) = k$ *and $A = \hat{A}_1 + \hat{A}_2$ be the core-EP decomposition of A as given in* (1.1)*. The weak group inverse of A coincides with the group inverse of $\hat{A}_1$; that is, $A^{ⓦ} = \hat{A}_1^{\#}$.*

Lemma 3.6. *Let $A \in \mathbb{C}^{n\times n}$ with* $\mathrm{ind}(A) = k$ *and $A = \hat{A}_1 + \hat{A}_2$ be the core-EP decomposition of A as given in* (1.1)*. The weak group inverse of A coincides with the $(\hat{A}_1, \hat{A}_1)$-inverse of A.*

Proof. It is trivial by Lemmas 3.5 and 1.7. □

Theorem 3.7. *Let $A \in \mathbb{C}^{n\times n}$ with* $\mathrm{ind}(A) = k$ *and $A = \hat{A}_1 + \hat{A}_2$ be the core-EP decomposition of A as* (1.1)*. Then, the inverse X_4 coincides with the weak group inverse of A.*

Proof. It is trivial by Lemma 3.6 and the definition of the inverse of X_2. □

Theorem 3.8. *Let $A \in \mathbb{C}^{n\times n}$ with* $\mathrm{ind}(A) = k$*, then the generalized inverse X_5 coincides with the $(A^k, (A^k)^*A)$-inverse of A.*

Proof. It is trivial by Theorems 2.1 and 3.1. □

Theorem 3.9. *Let $A \in \mathbb{C}^{n\times n}$ with* $\mathrm{ind}(A) = k$*, then the generalized inverse X_6 coincides with the $(A^k, (A^k)^*A)$-inverse of A.*

Proof. It is trivial by Theorems 2.3 and 3.1. □

Theorem 3.10. *Let $A \in \mathbb{C}^{n\times n}$ with* $\mathrm{ind}(A) = k$*, then X_i is the same as X_j. Moreover, X_j coincides with the weak group inverse of A, where $i, j \in \{4, 5, 6\}$.*

Proof. It is obvious by Theorems 3.1, 3.8 and 3.9. □

4. Three generalized inverses related the core part $\tilde{A}_1$ of the EP-nilpotent decomposition

In this section, three generalized inverses along the core parts of matrix decompositions are introduced. In Table 3, one can see that we denoted the generalized inverse along the core parts of the EP-nilpotent decomposition as X_7 by using the symbol of the generalized inverse along two matrices. In a similar way, X_8 denotes the generalized inverse along the core part of the core-nilpotent decomposition and the core part of the EP-nilpotent decomposition. X_9 denotes the generalized inverse along the core part of the core-EP decomposition and the core part of the EP-nilpotent decomposition decomposition. In addition, we prove that X_i is the same as X_j and that X_j coincides with the core-EP inverse of A, where $i, j \in \{7, 8, 9\}$.

Table 3. Three generalized inverses related $\tilde{A}_1$ of the EP-nilpotent decomposition.

Three generalized inverses	Core parts	The generalized inverses along the core part
type VII	$\tilde{A}_1$	$X_7 = A^{\|(\tilde{A}_1,\tilde{A}_1)}$
type VIII	A_1 and $\tilde{A}_1$	$X_8 = A^{\|(A_1,\tilde{A}_1)}$
type IX	$\hat{A}_1$ and $\tilde{A}_1$	$X_9 = A^{\|(\hat{A}_1,\tilde{A}_1)}$

Theorem 4.1. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. The X_7 coincides with the inverse of A along $A^{k+1}(A^k)^*$ and $(A^k)^*$, that is X_7 is the $(A^k, (A^k)^*)$-inverse. Moreover, the generalized inverse X_7 is the core-EP inverse of A.*

Proof. Let X_3 be the $(\tilde{A}_1, \tilde{A}_1)$-inverse of A. That is the $(A^{k+1}(A^k)^\dagger, A^{k+1}(A^k)^\dagger)$-inverse of A by Lemma 1.4. Since

$$\begin{aligned} A^{k+1}(A^k)^\dagger &= A^{k+1}(A^k)^\dagger A^k(A^k)^\dagger \\ &= A^{k+1}[(A^k)^\dagger)A^k]^*(A^k)^\dagger = A^{k+1}(A^k)^*[(A^k)^\dagger]^*(A^k)^\dagger, \\ A^{k+1}(A^k)^* &= A^{k+1}(A^k(A^k)^\dagger A^k)^* = A^{k+1}[(A^k)^\dagger A^k]^*(A^k)^* \\ &= A^{k+1}(A^k)^\dagger A^k(A^k)^*, \end{aligned} \tag{4.1}$$

we have $\mathcal{R}(A^{k+1}) = \mathcal{R}(\tilde{A}_1)$, which implies

$$\mathcal{R}(A^k) = \mathcal{R}(\tilde{A}_1). \tag{4.2}$$

For any $u \in \mathcal{N}(\tilde{A}_1)$,

$$\begin{aligned} (A^k)^*u &= [A^k(A^k)^\dagger A^k]^* = (A^k)^*A^k(A^k)^\dagger u \\ &= (A^k)^*A^DA^{k+1}(A^k)^\dagger u = (A^k)^*A^D\tilde{A}_1u = 0 \end{aligned} \tag{4.3}$$

by Lemma 1.1. For any $v \in \mathcal{N}((A^k)^*)$,

$$\begin{aligned} \tilde{A}_1v &= A^{k+1}(A^k)^\dagger v = A^{k+1}(A^k)^\dagger A^k(A^k)^\dagger v \\ &= A^{k+1}(A^k)^\dagger((A^k)^\dagger)^*(A^k)^*v = 0, \end{aligned} \tag{4.4}$$

and we have

$$\mathcal{N}(\tilde{A}_1) = \mathcal{N}((A^k)^*). \tag{4.5}$$

Thus, X_7 coincides with the inverse of A along $A^{k+1}(A^k)^*$ and $(A^k)^*$ by (4.2), (4.5) and Lemma 1.10. Therefore, the generalized inverse X_7 is the core-EP inverse of A by Lemma 1.8 and the condition X_7 is the $(A^k, (A^k)^*)$-inverse. □

Theorem 4.2. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$, then X_8 coincides with the inverse of A along $A^{k+1}(A^k)^*$ and $(A^k)^*$, that is X_8 is the $(A^k, (A^k)^*)$-inverse. Moreover, the generalized inverse X_8 is the core-EP inverse of A.*

Proof. It is trivial by Theorems 2.1 and 4.1. □

Theorem 4.3. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$, then X_9 coincides with the inverse of A along $A^{k+1}(A^k)^*$ and $(A^k)^*$, that is X_9 is the $(A^k, (A^k)^*)$-inverse. Moreover, the generalized inverse X_9 is the core-EP inverse of A.*

Proof. It is trivial by Theorems 3.1 and 4.1. □

Theorem 4.4. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$, then X_i is the same as X_j. Moreover, X_j coincides with the core-EP inverse of A, where $i, j \in \{7, 8, 9\}$.*

Proof. It is obvious by Theorems 4.1–4.3. □

Let X_1, X_4 and X_7 be the generalized inverses along A_1, $\hat{A}_1$ and $\tilde{A}_1$, respectively. Note that X_1 denotes the inverse along A_1 and A_1; X_2 denotes the inverse along $\hat{A}_1$ and A_1; X_3 denotes the inverse along $\tilde{A}_1$ and A_1; X_4 denotes the inverse along A_1 and $\hat{A}_1$; X_5 denotes the inverse along $\hat{A}_1$ and $\hat{A}_1$; X_6 denotes the inverse along $\tilde{A}_1$ and $\hat{A}_1$; X_7 denotes the inverse along A_1 and $\tilde{A}_1$; X_8 denotes the inverse along $\hat{A}_1$ and $\tilde{A}_1$ and X_9 denotes the inverse along $\tilde{A}_1$ and $\tilde{A}_1$. Table 4 shows that X_1, X_2 and X_3 have the same column and nilpotent parts and $\mathcal{R}(X_i) = \mathcal{R}(A^k)$ and $\mathcal{N}(X_i) = \mathcal{N}(A^k)$ for $i = 1, 2, 3$; X_4, X_5 and X_6 have the same column and nilpotent parts and that $\mathcal{R}(X_j) = \mathcal{R}(A^k)$ and $\mathcal{N}(X_j) = \mathcal{N}((A^k)^*A)$ for $j = 4, 5, 6$ and X_7, X_8 and X_9 have the same column and nilpotent parts and $\mathcal{R}(X_k) = \mathcal{R}(A^k)$ and that $\mathcal{N}(X_k) = \mathcal{N}((A^k)^*)$ for $k = 7, 8, 9$.

Table 4. Relationships between X_i and $X_j (i, j \in \{1, 2, ..., 9\})$.

Nine generalized inverses	The column part	The nilpotent part
X_1	A^k	A^k
X_2	A^k	A^k
X_3	A^k	A^k
X_4	A^k	$(A^k)^*A$
X_5	A^k	$(A^k)^*A$
X_6	A^k	$(A^k)^*A$
X_7	A^k	$(A^k)^*$
X_8	A^k	$(A^k)^*$
X_9	A^k	$(A^k)^*$

5. Relationships between X_i and X_j for $i, j \in \{1, 2, \cdots, 9\}$

Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind} A = k$. In this section, we will show that the generalized inverses X_1, X_4 and X_7 are different generalized inverses. For a singular complex matrix, we can prove that the Drazin inverse coincides with the weak group inverse, which is different from the core-EP inverse. Moreover, we show that the Drazin inverse, the weak group inverse and the core-EP inverse can be the same for a certain singular complex matrix.

Example 5.1. *Let* $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & \frac{1}{2} \end{bmatrix} \in \mathbb{C}^{3\times3}$. *Then, it is easy to check that* $\mathrm{ind}(A) = 2$ *and*

$$A_1 = \begin{bmatrix} \frac{4}{5} & \frac{4}{5} & \frac{2}{5} \\ \frac{6}{5} & \frac{6}{5} & \frac{3}{5} \\ 1 & 1 & \frac{1}{2} \end{bmatrix}, \hat{A}_1 = \begin{bmatrix} \frac{60}{77} & \frac{60}{77} & \frac{34}{77} \\ \frac{90}{77} & \frac{90}{77} & \frac{51}{77} \\ \frac{75}{77} & \frac{75}{77} & \frac{85}{154} \end{bmatrix}, \tilde{A}_1 = \begin{bmatrix} \frac{40}{77} & \frac{60}{77} & \frac{50}{77} \\ \frac{60}{77} & \frac{90}{77} & \frac{75}{77} \\ \frac{50}{77} & \frac{75}{77} & \frac{125}{154} \end{bmatrix},$$

$$X_1 = X_2 = X_3 = \begin{bmatrix} \frac{16}{125} & \frac{16}{125} & \frac{8}{125} \\ \frac{24}{125} & \frac{24}{125} & \frac{12}{125} \\ \frac{4}{25} & \frac{4}{25} & \frac{2}{25} \end{bmatrix}, X_4 = X_5 = X_6 = \begin{bmatrix} \frac{48}{385} & \frac{48}{385} & \frac{136}{1925} \\ \frac{72}{385} & \frac{72}{385} & \frac{204}{1925} \\ \frac{12}{77} & \frac{12}{77} & \frac{34}{385} \end{bmatrix},$$

$$X_7 = X_8 = X_9 = \begin{bmatrix} \frac{32}{385} & \frac{48}{385} & \frac{8}{77} \\ \frac{48}{385} & \frac{72}{385} & \frac{12}{77} \\ \frac{8}{77} & \frac{12}{77} & \frac{10}{77} \end{bmatrix}.$$

However, $X_1 \neq X_4$, $X_1 \neq X_7$, $X_4 \neq X_7$.

It is trivial that the generalized inverses X_1, X_4 and X_7 are different generalized inverses by Example 5.1. Thus, we have the following theorem.

Theorem 5.2. *Let* $A \in \mathbb{C}^{n\times n}$ *with* $\mathrm{ind}\,A = k$, *then generalized inverses* X_1, X_4 *and* X_7 *are different generalized inverses.*

Example 5.3. *Let* $A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & -1 & -2 \end{bmatrix} \in \mathbb{C}^{4\times4}$. *Then, it is easy to check that* $\mathrm{ind}(A) = 2$, *and*

$$A_1 = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \hat{A}_1 = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tilde{A}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$X_1 = X_2 = X_3 = X_4 = X_5 = X_6 \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$X_7 = X_8 = X_9 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

However, $X_1 \neq X_7$, $X_4 \neq X_7$.

For a singular complex matrix, Example 5.3 shows that the Drazin inverse coincides with the weak group inverse, which is different from the core-EP inverse.

Example 5.4. *Let* $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{5} & \frac{2}{5} \\ 0 & 0 & \frac{2}{5} & \frac{4}{5} \end{bmatrix} \in \mathbb{C}^{4\times 4}$*. Then, it is easy to check that* $\mathrm{ind}(A) = 1$*, and*

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{5} & \frac{2}{5} \\ 0 & 0 & \frac{2}{5} & \frac{4}{5} \end{bmatrix}, \hat{A}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{5} & \frac{2}{5} \\ 0 & 0 & \frac{2}{5} & \frac{4}{5} \end{bmatrix}, \tilde{A}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{5} & \frac{2}{5} \\ 0 & 0 & \frac{2}{5} & \frac{4}{5} \end{bmatrix}.$$

However, $X_1 = X_2 = X_3 = X_4 = X_5 = X_6 = X_7 = X_8 = X_9 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{5} & \frac{2}{5} \\ 0 & 0 & \frac{2}{5} & \frac{4}{5} \end{bmatrix}$.

Example 5.4 shows that the Drazin inverse, the weak group inverse and the core-EP inverse can be the same for a certain singular complex matrix.

Example 5.5. *Let* $A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{C}^{4\times 4}$*,* $B = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \in \mathbb{C}^{4\times 4}$*. Then, it is easy to check that* $\mathrm{ind}(A) = 2$ *and* $\mathrm{ind}(B) = 3$*, but*

$$A_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \hat{A}_1 = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tilde{A}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$X_1 = X_2 = X_3 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, X_4 = X_5 = X_6 = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$X_7 = X_8 = X_9 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

with $X_1 \neq X_4$, $X_1 \neq X_7$, $X_4 \neq X_7$, and

$$B_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \hat{B}_1 = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tilde{B}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$Y_1 = Y_2 = Y_3 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, Y_4 = Y_5 = Y_6 = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$Y_7 = Y_8 = Y_9 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

with $Y_1 \neq Y_4$, $Y_1 \neq Y_7$, $Y_4 \neq Y_7$.

Example 5.5 shows that the difference index of the complex matrices does not affect the relationships between the Drazin inverse, the weak group inverse and the core-EP inverse.

Theorem 5.2 and Example 5.1 show that the generalized inverses X_1, X_4 and X_7 are different generalized inverses. Thus, we have the following Tables 5 and 6.

Table 5. Counterexamples related the inverse X_1 to X_9.

Related generalized inverses	Counterexamples
$X_1 \neq X_4$	Example 5.1
$X_1 \neq X_7$	Example 5.1
$X_4 \neq X_7$	Example 5.1

Table 6. Examples related the inverse X_1 to X_9.

Related generalized inverses	Examples
$X_1 = X_4 \neq X_7$	Example 5.3
$X_i = X_j$ $(i, j \in 1, 2, \cdots, 9)$	Example 5.4

6. Conclusions

New characterizations for generalized inverses along the core parts of three matrix decompositions were investigated in this paper. A number of characterizations and different representations of the Drazin inverse, the weak group inverse and the core-EP inverse were given by using the core parts A_1, $\hat{A}_1$ and $\tilde{A}_1$. Some useful examples were given, which showed that the generalized inverses X_1, X_4 and X_7 are different generalized inverses. We believe that investigation related to the generalized inverses along the core parts of related matrix decompositions will attract attention, and we describe perspectives for further research:

1) Considering the matrix partial orders based on the generalized inverses can relate the core parts of

matrix decompositions.
2) Extending the generalized inverses can relate the core parts of matrix decompositions to an element in rings.
3) The column space and the null space of a complex matrix can be described by the core parts of matrix decompositions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The research article is supported by the National Natural Science Foundation of China (No. 12001223), the Qing Lan Project of Jiangsu Province, the Natural Science Foundation of Jiangsu Province of China (No. BK20220702), the Natural Science Foundation of Jiangsu Education Committee (No. 22KJB110010), "Five-Three-Three" Talents of Huaian City and College Students Innovation and Entrepreneurship Training Program (No. 202311049024Z).

Conflict of interest

The authors declare no conflict of interest.

References

1. J. Benítez, E. Boasso, H. W. Jin, On one-sided (B, C)-inverses of arbitrary matrices, *Electron. J. Linear Al.*, **32** (2017), 391–422. https://doi.org/10.13001/1081-3810.3487
2. R. B. Bapat, S. K. Jain, K. M. P. Karantha, M. D. Raj, Outer inverses: characterization and application, *Linear Algebra Appl.*, **528** (2017), 171–184. https://doi.org/10.1016/j.laa.2016.06.045
3. E. Boasso, G. Kantún-Montiel, The (b, c)-inverses in rings and in the Banach context, *Mediterr. J. Math.*, **14** (2017), 112. http://doi.org/ 10.1007/s00009-017-0910-1
4. O. M. Baksalary, G. Trenkler, Core inverse of matrices, *Linear Multilinear A.*, **58** (2010), 681–697. http://doi.org/10.1080/03081080902778222
5. S. L. Campbell, C. D. Meyer, *Generalized inverses of linear transformations*, Philadelphia: SIAM, 2009. https://doi.org/10.1137/1.9780898719048
6. X. F. Cao, S. Z. Xu, X. C. Wang, K. Liu, Two generalized constrained inverses based on the core part of the core-EP decomposition of a complex matrix, ScienceAsia,(accepted).
7. M. P. Drazin, Pseudo-inverses in associative rings and semigroup, *Am. Math. Mon.*, **65** (1958), 506–514. http://doi.org/10.1080/00029890.1958.11991949
8. M. P. Drazin, A class of outer generalized inverses, *Linear Algebra Appl.*, **436** (2012), 1909–1923. https://doi.org/10.1016/j.laa.2011.09.004
9. J. A. Green, On the structure of semigroups, *Ann. Math.*, **54** (1951), 163–172.

10. R. E. Hartwig, Block generalized inverses, *Arch. Rational Mech. Anal.*, **61** (1976), 197–251. https://doi.org/ 10.1007/BF00281485

11. R. E. Hartwig, K. Spindelböck, Matrices for which A^* and $A^\dagger$ commmute, *Linear Multilinear A.*, **14** (1983), 241–256. https://doi.org/10.1080/03081088308817561

12. X. Mary, On generalized inverse and Green's relations, *Linear Algebra Appl.*, **434** (2011), 1836–1844. https://doi.org/10.1016/j.laa.2010.11.045

13. E. H. Moore, On the reciprocal of the general algebraic matrix, *B. Am. Math. Soc.*, **26** (1920), 394–395.

14. S. K. Mitra, P. Bhimasankaram, S. B. Malik, *Matrix partial orders, shorted operators and applications*, Singaproe: Word Scientific, 2010. https://doi.org/10.1142/7170

15. K. M. Prasad, K. S. Mohana, Core-EP inverse, *Linear Multilinear A.*, **62** (2014), 792–804. https://doi.org/10.1080/03081087.2013.791690

16. X. Mary, P. Patrício, Generalized inverses modulo $\mathcal{H}$ in semigroups and rings, *Linear Multilinear A.*, **61** (2013), 1130–1135. https://doi.org/10.1080/03081087.2012.731054

17. R. Penrose, A generalized inverse for matrices, *Math. Proc. Cambridge*, **51** (1955), 406–413. http://doi.org/10.1017/S0305004100030401

18. C. R. Rao, S. K. Mitra, *Generalized inverse of matrices and its applications*, New York: Wiley, 1971.

19. D. S. Rakić, A note on Rao and Mitra's constrained inverse and Drazin's (b,c) inverse, *Linear Algebra Appl.*, **523** (2017), 102–108. https://doi.org/10.1016/j.laa.2017.02.025

20. C. R. Rao, S. K. Mitra, Generalized inverse of a matrix and its application, In: *Theory of statistics*, Berkeley: University of California Press, 1972, 601–620. https://doi.org/10.1525/9780520325883-032

21. H. X. Wang, Core-EP decomposition and its applications, *Linear Algebra Appl.*, **508** (2016), 289–300. https://doi.org/10.1016/j.laa.2016.08.008

22. H. X. Wang, J. L. Chen, Weak group inverse, *Open Math.*, **16** (2018), 1218–1232. http://doi.org/10.1515/math-2018-0100

23. H. X. Wang, X. J. Liu, EP-nilpotent decomposition and its applications, *Linear Multilinear A.*, **68** (2020), 1682–1694. https://doi.org/10.1080/03081087.2018.1555571

24. S. Z. Xu, J. Benítez, Existence criteria and expressions of the (b, c)-inverse in rings and their applications, *Mediterr. J. Math.*, **15** (2018), 14. https://doi.org/10.1007/s00009-017-1056-x

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