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Research article

On the generalized Cochrane sum with Dirichlet characters

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Abstract: In this paper, we defined a new generalized Cochrane sum with Dirichlet characters, and gave the upper bound of the generalized Cochrane sum with Dirichlet characters. Moreover, we studied the asymptotic estimation problem of the mean value of the generalized Cochrane sum with Dirichlet characters and obtained a sharp asymptotic formula for it. By using this asymptotic formula, we also gave the mean value of the generalized Dedekind sum.

Keywords: generalized Cochrane sum; Dirichlet character; upper bound; mean value
Mathematics Subject Classification: 11F20, 11L05

1. Introduction

Let h, q be integers with $q > 0$. The classical Dedekind sum $s(h, q)$ is defined by

$$s(h,q) = \sum_{a=1}^{q-1} \left(\left(\frac{a}{q}\right)\right)\left(\left(\frac{ha}{q}\right)\right),$$

where

$$((x)) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \text{ is not an integer;} \\ 0, & \text{if } x \text{ is an integer} \end{cases}$$

and $[x]$ is the largest integer not exceeding x. The Dedekind sum plays an important role in the Dedekind η function and has applications to many parts of mathematics (see [7, 11, 12]).

For any nonnegative integer n, let B_n and $B_n(X)$ be the n-th Bernoulli number and polynomial, respectively, which is defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad \frac{te^{tX}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(X) \frac{t^n}{n!},$$

where $\overline{B}_n(X) = B_n(X - [X])$ is the n-th periodic Bernoulli function in the interval $(0, 1]$ for $n > 1$, $\overline{B}_1(X) = B_1(X - [X])$ and if X is an integer, then $\overline{B}_1(X) = 0$. For positive integers m, n, we have the generalized Dedekind sum

$$S(h, m, n, q) = \sum_{a=1}^{q-1} \overline{B}_m\left(\frac{a}{q}\right)\overline{B}_n\left(\frac{ah}{q}\right).$$

Let p be an odd prime and let χ be any even Dirichlet character mod p. For any integers n, k, Xie and Zhang [8] showed that if n is an odd integer, then

$$\begin{aligned}\sum_{h=1}^{p-1}\chi(h)|S(h,n,n,p)|^{2k} =& 2\frac{(n!)^{4k}}{4^{2(n-1)k}}\left(\frac{p\zeta(2n)}{2\pi^{2n}}\right)^{2k}\frac{|L(2nk,\chi)|^2}{\zeta(4nk)}\\ &+ O\left(p^{2k-n}\exp\left(\frac{6\ln p}{\ln\ln p}\right)\right),\end{aligned} \tag{1.1}$$

they also have the following result for any even integer n

$$\sum_{h=1}^{p-1}\chi(h)|S(h,n,n,p)|^{2k} = 2\frac{(n!)^{4k}}{4^{2(n-1)k}}\left(\frac{p\zeta(2n)}{2\pi^{2n}}\right)^{2k}\frac{|L(2nk,\chi)|^2}{\zeta(4nk)} + O\left(p^{2k-1}\right). \tag{1.2}$$

In October 2000, Professor Todd Cochrane first introduced a sum analogous to the Dedekind sum as follows:

$$C(h, q) = \sum_{a=1}^{q}{}' \left(\left(\frac{\bar{a}}{q}\right)\right)\left(\left(\frac{ha}{q}\right)\right),$$

where $\bar{a}$ satisfies $a\bar{a} \equiv 1 \pmod q$, $\sum_{a=1}^{\prime q}$ denotes the summation over all $1 \le a \le q$ such that $(a, q) = 1$. Many scholars studied the properties of $C(h, q)$. Zhang and Yi [13] gave the following upper bound estimate:

$$|C(h, q)| \ll q^{\frac{1}{2}} d(q) \ln^2 q,$$

where $d(q)$ is the divisor function. Ma et al. [4] gave the upper bound estimate of the incomplete Cochrane sum.

Xu and Zhang [9] defined the high-dimensional Cochrane sum by the following equation:

$$C(h, k, q) = \sum_{a_1=1}^{q}{}' \cdots \sum_{a_k=1}^{q}{}' \left(\left(\frac{a_1}{q}\right)\right)\cdots\left(\left(\frac{a_k}{q}\right)\right)\left(\left(\frac{h\overline{a_1\cdots a_k}}{q}\right)\right).$$

For any fixed positive integer k with $(q, k(k + 1)) = 1$, they gave the following upper bound estimate:

$$|C(h, k, q)| \ll \frac{2^{(k+1)^2}}{\pi^{k+1}} q^{\frac{k}{2}} d(q)(2^{k+2}k)^{\omega(q)} \ln^{k+1} q,$$

where $\omega(q)$ denotes the number of all different prime divisors of q. Liu [5] improved this result.

For positive integers m, n, the main purpose of this paper is to study the generalized Cochrane sum with any Dirichlet character χ mod q as follows:

$$C(h, m, n, q, \chi) = \sum_{a=1}^{q}{}' \chi(a)\overline{B}_m\left(\frac{\overline{a}}{q}\right)\overline{B}_n\left(\frac{ah}{q}\right),$$

which is an interesting generalization of Cochrane sum. Ren and Yi [6] studied the mean square value of $C(h,1,1,p,\chi)$, for a prime $p \equiv 1 \mod 4$ and the Legendre's symbol $\chi \mod p$, they obtained

$$\sum_{h=1}^{p-1} |C(h,1,1,p,\chi)|^2 = \frac{1}{180} p^2 \prod_{p_1 \in \mathcal{A}} \left(\frac{p_1^2+1}{p_1^2-1} \right)^2 + O(p^{1+o(1)}), \tag{1.3}$$

where $\mathcal{A}$ is the set of quadratic residues of p, p_1 is prime which is not equal to p. Liu and Zhang [2, 3] studied the mean square value of $C(h,m,n,q,\chi)$ and its hybrid mean value formula when χ is the principal Dirichlet character. In this paper, by using an upper bound estimate of the Kloosterman sum with Dirichlet characters, we show that

Theorem 1.1. *Let p be an odd prime and let χ be any Dirichlet character mod p. For any integers m, n, we have*

$$C(h,m,n,p,\chi) \ll m!n!(2\pi)^{-(m+n)} p^{\frac{1}{2}} \ln^2 p,$$

if $\chi(-1) \neq (-1)^{n+m}$, then $C(h,m,n,p,\chi) = 0$.

We also obtain the mean square value of $C(h,m,n,p,\chi)$ as follows:

Theorem 1.2. *Let p be an odd prime and let χ be any Dirichlet character mod p. For any integers m, n with $\chi(-1) = (-1)^{n+m}$, we have*

$$\begin{aligned}&\sum_{h=1}^{p-1} |C(h,m,n,p,\chi)|^2 \\ &= \frac{8(m!n!)^2 p^2}{(2\pi i)^{2(m+n)}} \frac{\zeta(2m)\zeta(2n)}{\zeta(2m+2n)} |L(m+n,\chi)|^2 + O\left(p^{2-\min(n,m)} \exp\left(\frac{4\ln p}{\ln\ln p}\right)\right).\end{aligned}$$

If χ_1 mod a prime $p \equiv 1 \mod 4$ is the Legendre's symbol, then according to the value of the Dirichlet L-function $L(2,\chi_1)$, we also can get (1.3):

Corollary 1.3. *Let $p \equiv 1 \mod 4$ be a prime and the Legendre's symbol χ_1 mod p. We have*

$$\sum_{h=1}^{p-1} |C(h,1,1,p,\chi_1)|^2 = \frac{1}{180} p^2 \prod_{p_1 \in \mathcal{A}} \left(\frac{p_1^2+1}{p_1^2-1} \right)^2 + O\left(p \exp\left(\frac{4\ln p}{\ln\ln p}\right)\right),$$

where $\mathcal{A}$ is the set of quadratic residues of p, p_1 is prime, which is not equal to p.

Moreover, we give the mean square value of $S(h,m,n,p,\chi)$ as follows:

Theorem 1.4. *Let p be an odd prime and let χ be any Dirichlet character mod p. For any integers m, n with $\chi(-1) = (-1)^{n+m}$, we have*

$$\begin{aligned}&\sum_{h=1}^{p-1} \chi(h) S(\bar{h},m,m,p) S(h,n,n,p) \\ &= \frac{8(m!n!)^2 p^2}{(2\pi i)^{2(m+n)}} \frac{\zeta(2m)\zeta(2n)}{\zeta(2m+2n)} |L(m+n,\chi)|^2 + O\left(p^{2-\min(n,m)} \exp\left(\frac{4\ln p}{\ln\ln p}\right)\right).\end{aligned}$$

Obviously, Theorem 1.4 generalizes and improves (1.1) and (1.2) when $k = 1$.

2. Some lemmas

To prove theorems, we need the following several lemmas.

Lemma 2.1. *Let p be an odd prime and an integer h with $(h,p)=1$, and let χ_1 be any Dirichlet character mod p. For any positive integers m,n, if $\chi_1(-1)\neq(-1)^{m+n}$ then $C(h,m,n,p,\chi_1)=0$, and if $\chi_1(-1)=(-1)^{n+m}$, then we have*

$$\begin{aligned}&C(h,m,n,p,\chi_1)\\ &\quad=\frac{4m!n!}{(2\pi i)^{m+n}}\frac{1}{\phi(p)}\sum_{\substack{\chi \bmod p\\ \chi\chi_1(-1)=(-1)^m}}\overline{\chi}(h)\left(\sum_{r=1}^{+\infty}\frac{G(\chi\chi_1,r)}{r^m}\right)\left(\sum_{s=1}^{+\infty}\frac{G(\chi,s)}{s^n}\right),\end{aligned}$$

and

$$\begin{aligned}&C(h,m,n,p,\chi_1)\\ &\quad=\frac{4m!n!}{(2\pi i)^{m+n}}\frac{1}{\phi(p)}\sum_{\substack{\chi \bmod p\\ \chi\chi_1(-1)=(-1)^m}}\overline{\chi}(h)\tau(\chi)\tau(\chi\chi_1)L(m,\overline{\chi\chi_1})L(n,\overline{\chi}),\end{aligned}$$

where $\tau(\chi)=G(\chi,1)$, $G(\chi,r)=\sum_{a=1}^{p-1}\chi(a)e(\frac{ra}{p})$ denotes Gauss sum, $L(n,\chi)=\sum_{t=1}^{+\infty}\frac{\chi(t)}{t^n}$ is a Dirichlet L-function.

Proof. Applying the orthogonality of multiplicative characters, it follows that

$$\begin{aligned}C(h,m,n,p,\chi_1)&=\sum_{a=1}^{p-1}\chi_1(a)\overline{B}_m\left(\frac{\overline{a}}{p}\right)\overline{B}_n\left(\frac{ah}{p}\right)\\ &=\frac{1}{\phi(p)}\sum_{\chi \bmod p}\left\{\sum_{a=1}^{p-1}\chi\chi_1(a)\overline{B}_m\left(\frac{a}{p}\right)\right\}\left\{\sum_{b=1}^{p-1}\chi(b)\overline{B}_n\left(\frac{hb}{p}\right)\right\}.\end{aligned}$$

Noting that [1]

$$\overline{B}_n(x)=-\frac{n!}{(2\pi i)^n}\sum_{\substack{r=-\infty\\ r\neq 0}}^{+\infty}\frac{e(xr)}{r^n},$$

and $G(\chi,-hn)=\overline{\chi}(-h)G(\chi,n)$. We have

$$\begin{aligned}&C(h,m,n,p,\chi_1)\\ &=\frac{1}{\phi(p)}\sum_{\chi \bmod p}\left\{\sum_{a=1}^{p-1}\left(-\frac{m!}{(2\pi i)^m}\sum_{\substack{r=-\infty\\ r\neq 0}}^{+\infty}\chi\chi_1(a)\frac{e\left(\frac{ra}{p}\right)}{r^m}\right)\right\}\\ &\quad\times\left\{\sum_{b=1}^{p-1}\left(-\frac{n!}{(2\pi i)^n}\sum_{\substack{s=-\infty\\ s\neq 0}}^{+\infty}\chi(b)\frac{e\left(\frac{sbh}{p}\right)}{s^n}\right)\right\}\\ &=\frac{m!n!}{(2\pi i)^{m+n}}\frac{1}{\phi(p)}\sum_{\chi \bmod p}\left\{\sum_{\substack{r=-\infty\\ r\neq 0}}^{+\infty}\frac{1}{r^m}\sum_{a=1}^{p-1}\chi\chi_1(a)e\left(\frac{ra}{p}\right)\right\}\end{aligned}$$

$$
\begin{aligned}
&\times\left\{\sum_{\substack{s=-\infty\\ s\neq 0}}^{+\infty}\frac{1}{s^n}\sum_{b=1}^{p-1}\chi(b)e\left(\frac{sbh}{p}\right)\right\}\\
=&\frac{m!n!}{(2\pi i)^{m+n}}\frac{1}{\phi(p)}\sum_{\chi \bmod p}\left\{\sum_{\substack{r=-\infty\\ r\neq 0}}^{+\infty}\frac{G(\chi\chi_1,r)}{r^m}\right\}\left\{\sum_{\substack{s=-\infty\\ s\neq 0}}^{+\infty}\frac{G(\chi,sh)}{s^n}\right\}\\
=&\frac{m!n!}{(2\pi i)^{m+n}}\frac{1}{\phi(p)}\sum_{\chi \bmod p}\overline{\chi}(h)\left(1+\frac{\chi\chi_1(-1)}{(-1)^m}\right)\left(\sum_{r=1}^{+\infty}\frac{G(\chi\chi_1,r)}{r^m}\right)\\
&\times\left(1+\frac{\chi(-1)}{(-1)^n}\right)\left(\sum_{s=1}^{+\infty}\frac{G(\chi,s)}{s^n}\right)\\
=&\frac{4m!n!}{(2\pi i)^{m+n}}\frac{1}{\phi(p)}\sum_{\substack{\chi \bmod p\\ \chi\chi_1(-1)=(-1)^m}}\overline{\chi}(h)\left(\sum_{r=1}^{+\infty}\frac{G(\chi\chi_1,r)}{r^m}\right)\left(\sum_{s=1}^{+\infty}\frac{G(\chi,s)}{s^n}\right),
\end{aligned}
$$

where $\chi_1(-1)=(-1)^{m+n}$. If $\chi_1(-1)\neq(-1)^{m+n}$, then $C(h,m,n,p,\chi_1)=0$.

Moreover, we also have

$$
\begin{aligned}
&C(h,m,n,p,\chi_1)\\
&\quad=\frac{4m!n!}{(2\pi i)^{m+n}}\frac{1}{\phi(p)}\sum_{\substack{\chi \bmod p\\ \chi\chi_1(-1)=(-1)^m}}\overline{\chi}(h)\tau(\chi)\tau(\chi\chi_1)L(m,\overline{\chi\chi_1})L(n,\overline{\chi}),
\end{aligned}
$$

where $\tau(\chi)=G(\chi,1)$. □

Lemma 2.2. *Let p be a prime and let χ be any Dirichlet character mod p. For any integers r,s, we have*

$$\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ra+s\overline{a}}{p}\right)\right|\leq 2\sqrt{p}.$$

Proof. See Lemma 1 of [10]. □

Lemma 2.3. *Let p be an odd prime and an integer h with $(h,p)=1$, and let χ be any Dirichlet character mod p, then we have*

$$\sum_{\substack{\chi \bmod p\\ \chi\chi_1(-1)=(-1)^m}}\overline{\chi}(h)\left(\sum_{r=1}^{\infty}\frac{G(\chi\chi_1,r)}{r^m}\right)\left(\sum_{s=1}^{\infty}\frac{G(\chi,s)}{s^n}\right)\ll p^{\frac{3}{2}}\ln^2 p.$$

Proof. For any fixed parameter $N\geq p$, according to Abel's identity, we have

$$\sum_{r=1}^{\infty}\frac{G(\chi\chi_1,r)}{r^m}=\sum_{1\leq r\leq N}\frac{G(\chi\chi_1,r)}{r^m}+m\int_N^{+\infty}\frac{\sum_{N<r\leq y}G(\chi\chi_1,r)}{r^{m+1}}dy.$$

Since $|G(\chi,r)|\leq p^{\frac{1}{2}}$, we have

$$\sum_{1\leq r\leq N}\frac{G(\chi\chi_1,r)}{r^m}\ll p^{\frac{1}{2}}\sum_{1\leq r\leq N}\frac{1}{r^m}\ll p^{\frac{1}{2}}\ln N,$$

and from the estimates for trigonometric sums we have

$$\begin{aligned}\sum_{N<r\le y} G(\chi\chi_1, r) &= \sum_{a=1}^{p-1}\chi\chi_1(a)\sum_{N<r\le y} e\left(\frac{ar}{p}\right)\\ &= \sum_{a=1}^{p-1}\chi\chi_1(a)\frac{e\left(\frac{(N+1)a}{p}\right)-e\left(\frac{([y]+1)a}{p}\right)}{1-e\left(\frac{a}{p}\right)} \ll \sum_{a=1}^{p-1}\frac{1}{\left|\sin\frac{\pi a}{p}\right|}\\ &\ll \sum_{a=1}^{p-1}\frac{p}{a} \ll p\ln p,\end{aligned}$$

it follows that

$$m\int_N^{+\infty}\frac{\sum_{N<r\le y} G(\chi\chi_1, r)}{r^{m+1}}dy \ll \frac{p\ln p}{N^m},$$

then we have

$$\begin{aligned}&\sum_{\substack{\chi \bmod p\\ \chi\chi_1(-1)=(-1)^m}}\overline{\chi}(h)\left(\sum_{r=1}^{+\infty}\frac{G(\chi\chi_1, r)}{r^m}\right)\left(\sum_{s=1}^{+\infty}\frac{G(\chi, s)}{s^n}\right)\\ =&\sum_{\substack{\chi \bmod p\\ \chi\chi_1(-1)=(-1)^m}}\overline{\chi}(h)\left(\sum_{1\le r\le N}\frac{G(\chi\chi_1, r)}{r^m}+m\int_N^{+\infty}\frac{\sum_{N<r\le y} G(\chi\chi_1, r)}{r^{m+1}}dy\right)\\ &\times\left(\sum_{1\le r\le N}\frac{G(\chi, s)}{s^n}+n\int_N^{+\infty}\frac{\sum_{N<r\le y} G(\chi, s)}{s^{n+1}}dy\right)\\ =&\sum_{\substack{\chi \bmod p\\ \chi\chi_1(-1)=(-1)^m}}\overline{\chi}(h)\left(\sum_{r=1}^{N}\frac{G(\chi\chi_1, r)}{r^m}\right)\left(\sum_{s=1}^{N}\frac{G(\chi, s)}{s^n}\right)\\ &+n\sum_{\substack{\chi \bmod p\\ \chi\chi_1(-1)=(-1)^m}}\overline{\chi}(h)\left(\sum_{r=1}^{N}\frac{G(\chi\chi_1, r)}{r^m}\right)\left(\int_N^{+\infty}\frac{\sum_{N<r\le y} G(\chi, s)}{s^{n+1}}dy\right)\\ &+m\sum_{\substack{\chi \bmod p\\ \chi\chi_1(-1)=(-1)^m}}\overline{\chi}(h)\left(\sum_{s=1}^{N}\frac{G(\chi, s)}{s^n}\right)\left(\int_N^{+\infty}\frac{\sum_{N<r\le y} G(\chi\chi_1, r)}{r^{m+1}}dy\right)\\ &+mn\left(\int_N^{+\infty}\frac{\sum_{N\le r\le y} G(\chi\chi_1, r)}{r^{m+1}}dy\right)\left(\int_N^{+\infty}\frac{\sum_{N<r\le y} G(\chi, s)}{s^{n+1}}dy\right)\\ \ll&\sum_{\substack{\chi \bmod p\\ \chi\chi_1(-1)=(-1)^m}}\overline{\chi_1}(h)\left(\sum_{r=1}^{N}\frac{G(\chi\chi_1, r)}{r^m}\right)\left(\sum_{s=1}^{N}\frac{G(\chi, s)}{s^n}\right)\\ &+O\left(\frac{p^{\frac{3}{2}}\ln N\ln p}{\min\left(N^m, N^n\right)}\right).\end{aligned}$$

Note that

$$\sum_{\substack{\chi \bmod p\\ \chi\chi_1(-1)=(-1)^m}} \chi(a) = \begin{cases} \frac{1}{2}\phi(p), & a \equiv 1 \bmod p;\\ (-1)^m\chi_1(-1)\frac{1}{2}\phi(p), & a \equiv -1 \bmod p;\\ 0, & \text{otherwise.}\end{cases} \tag{2.1}$$

Then, from the orthogonality of Dirichlet characters we find

$$\begin{aligned}
&\sum_{\substack{\chi \bmod p\\ \chi\chi_1(-1)=(-1)^m}} \overline{\chi_1}(h)\left(\sum_{r=1}^{N}\frac{G(\chi\chi_1,r)}{r^m}\right)\left(\sum_{s=1}^{N}\frac{G(\chi,s)}{s^n}\right)\\
&= \sum_{\substack{\chi \bmod p\\ \chi\chi_1(-1)=(-1)^m}} \overline{\chi}(h)\left(\sum_{1\le r\le N}\frac{\sum\limits_{a=1}^{p-1}\chi\chi_1(a)e\left(\frac{ra}{p}\right)}{r^m}\right)\left(\sum_{1\le s\le N}\frac{\sum\limits_{b=1}^{p-1}\chi(b)e\left(\frac{sb}{p}\right)}{s^n}\right)\\
&= \sum_{1\le r\le N}\sum_{1\le s\le N}\frac{1}{r^m s^n}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi_1(a)e\left(\frac{ra+sb}{p}\right)\sum_{\substack{\chi \bmod p\\ \chi\chi_1(-1)=(-1)^m}} \chi(a)\chi(b)\overline{\chi}(h)\\
&=\frac{\phi(p)}{2}\sum_{1\le r\le N}\sum_{1\le s\le N}\frac{1}{r^m s^n}\sum_{a=1}^{p-1}\sum_{\substack{b=1\\ ab\equiv h \bmod p}}^{p-1}\chi_1(a)e\left(\frac{ra+sb}{p}\right)\\
&\quad+(-1)^m\chi_1(-1)\frac{\phi(p)}{2}\sum_{1\le r\le N}\sum_{1\le s\le N}\frac{1}{r^m s^n}\sum_{a=1}^{p-1}\sum_{\substack{b=1\\ ab\equiv -h \bmod p}}^{p-1}\chi_1(a)e\left(\frac{ra+sb}{p}\right).
\end{aligned}$$

According to Lemma 2.2, we obtain

$$\sum_{\substack{\chi \bmod p\\ \chi\chi_1(-1)=(-1)^m}} \overline{\chi}(h)\left(\sum_{r=1}^{N}\frac{G(\chi\chi_1,r)}{r^m}\right)\left(\sum_{s=1}^{N}\frac{G(\chi,s)}{s^n}\right) \ll p^{\frac{3}{2}}\ln^2 N.$$

Taking $N = p^2$, we can get Lemma 2.3

$$\sum_{\substack{\chi \bmod p\\ \chi\chi_1(-1)=(-1)^m}} \overline{\chi}(h)\left(\sum_{r=1}^{+\infty}\frac{G(\chi\chi_1,r)}{r^m}\right)\left(\sum_{s=1}^{+\infty}\frac{G(\chi,s)}{s^n}\right) \ll p^{\frac{3}{2}}\cdot\ln^2 p.$$

□

Lemma 2.4. *Let p be a prime and let χ be any Dirichlet character mod p. For any integers m, n, we have*

$$\begin{aligned}
\sum_{\substack{\chi \bmod p\\ \chi\chi_1(-1)=(-1)^n}} |L(n,\chi\chi_1)|^2|L(m,\chi)|^2 =&\frac{(p-1)\zeta(2m)\zeta(2n)}{2\zeta(2m+2n)}|L(n+m,\chi_1)|^2\\
&+O\left(p^{1-\min(n,m)}\exp\left(\frac{4\ln p}{\ln\ln p}\right)\right).
\end{aligned}$$

Proof. First, we assume that $n > m$, according to Abel's identity we can write

$$L(n,\chi\chi_1)L(m,\chi) = \sum_{t=1}^{\infty}\frac{\chi(t)D(t)}{t^m} = \sum_{t=1}^{N}\frac{\chi(t)D(t)}{t^m} + m\int_N^{\infty}\frac{B(y,\chi)}{y^{m+1}}\,dy,$$

where $D(t) = \sum_{d|t}\frac{\chi_1(d)}{d^{n-m}}$, $B(y,\chi) = \sum_{N<t\le y}\chi(t)D(t)$. We know that

$$\begin{aligned}
B(y,\chi) = \sum_{N<t\le y}\chi(t)D(t) = \sum_{N<t\le y}\chi(t)\sum_{d|t}\frac{\chi_1(d)}{d^{n-m}}\\
= \sum_{t\le\sqrt{y}}\chi(t)\sum_{l\le y/t}\frac{\chi\chi_1(l)}{l^{n-m}} + \sum_{l\le\sqrt{y}}\frac{\chi\chi_1(l)}{l^{n-m}}\sum_{t\le y/l}\chi(t)\\
-\sum_{t\le\sqrt{N}}\chi(t)\sum_{l\le N/t}\frac{\chi\chi_1(l)}{l^{n-m}} - \sum_{l\le\sqrt{N}}\frac{\chi\chi_1(l)}{l^{n-m}}\sum_{t\le N/l}\chi(t)\\
-\sum_{t\le\sqrt{y}}\chi(t)\sum_{l\le\sqrt{y}}\frac{\chi\chi_1(l)}{l^{n-m}} + \sum_{l\le\sqrt{N}}\frac{\chi\chi_1(l)}{l^{n-m}}\sum_{t\le\sqrt{N}}\chi(t).
\end{aligned}$$

It follows that

$$|B(y,\chi)| \ll \sqrt{y}\ln y,$$

$$\sum_{\substack{\chi \bmod p\\ \chi\chi_1(-1)=(-1)^n}} |B(y,\chi)|^2 \ll y\phi(p)\ln^2 y.$$

From the Cauchy inequality we have

$$\begin{aligned}
&\sum_{\substack{\chi \bmod p\\ \chi\chi_1(-1)=(-1)^n}}\left|m\int_N^{\infty}\frac{B(y,\chi)}{y^{m+1}}\,dy\right|^2\\
&\ll\left\{m\int_N^{\infty}\frac{1}{y^{m+1}}\left(\sum_{\substack{\chi \bmod p\\ \chi\chi_1(-1)=(-1)^n}}|B(y,\chi)|^2\right)^{1/2}dt\right\}^2\\
&\ll\left\{m\int_N^{\infty}y^{-m-\frac{1}{2}}\phi^{\frac{1}{2}}(p)\ln y\,dt\right\}^2 \ll \frac{\phi(p)\ln^2 N}{N^{2m-1}}.
\end{aligned}$$

From (2.1), we have

$$\begin{aligned}
&\sum_{\substack{\chi \bmod p\\ \chi\chi_1(-1)=(-1)^n}}\left|\sum_{n_1=1}^{N}\frac{\chi(n_1)D(n_1)}{n_1^m}\right|^2\\
=&\frac{\phi(p)}{2}\sum_{\substack{1\le n_1,n_2\le N\\ n_1\equiv n_2(\bmod p)\\ (n_1n_2,p)=1}}\frac{D(n_1)\overline{D}(n_2)}{n_1^m n_2^m} + \frac{\chi_1(-1)(-1)^n\phi(p)}{2}\sum_{\substack{1\le n_1,n_2\le N\\ n_1\equiv -n_2(\bmod p)\\ (n_1n_2,p)=1}}\frac{D(n_1)\overline{D}(n_2)}{n_1^m n_2^m}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\phi(p)}{2} \sum_{\substack{1\le n_1\le N\\ (n_1,p)=1}} \frac{|D(n_1)|^2}{n_1^{2m}} + O\left(\phi(p)\sum_{n_2=1}^{N}\sum_{l=1}^{[N/p]} \frac{d(n_2)d(lp+n_2)}{n_2^m(lp+n_2)^m}\right)\\
&\quad + O\left(\phi(p)\sum_{n_1=1}^{p-1} \frac{d(n_1)d(q-n_1)}{n_1^m(p-n_1)^m}\right) + O\left(\phi(p)\sum_{n_1=1}^{N}\sum_{l=1+n_1/p}^{[N/p]} \frac{d(n_1)d(lp-n_1)}{n_1^m(lp-n_1)^m}\right)\\
&= \frac{\phi(p)}{2} \sum_{\substack{1\le n_1\le \infty\\ (n_1,p)=1}} \frac{|D(n_1)|^2}{n_1^{2m}} + O\left(\frac{\phi(p)}{p^m}\exp\left(\frac{2\ln N}{\ln\ln N}\right)\right),
\end{aligned}
$$

noting that $|D(n_1)| \le d(n_1) = \sum_{t|n_1} 1 \le \exp\left(\frac{(1+\epsilon)\ln 2\ln N}{\ln\ln N}\right)$, it follows that

$$
\begin{aligned}
&\sum_{\substack{\chi \bmod p\\ \chi\chi_1(-1)=(-1)^n}} \left(\sum_{t=1}^{N} \frac{\chi(t)D(t)}{t^m}\right)\left(m\int_N^\infty \frac{B(y,\chi)}{y^{m+1}}\,dy\right)\\
&\qquad \ll (\ln N)^2 \int_N^\infty \frac{1}{y^{m+1}}\left(\sum_{\substack{\chi \bmod p\\ \chi(-1)=(-1)^n}} |B(y,\chi)|\right) dy \ll \phi(p)N^{\frac{1}{2}-m}(\ln N)^3.
\end{aligned}
$$

Taking $N = p^2$ and $\phi(p) = p-1$, then

$$
\begin{aligned}
&\sum_{\substack{\chi \bmod p\\ \chi\chi_1(-1)=(-1)^n}} |L(n,\chi\chi_1)|^2|L(m,\chi)|^2\\
&\qquad = \frac{p-1}{2}\sum_{\substack{1\le n_1\le\infty\\ (n_1,p)=1}} \frac{|D(n_1)|^2}{n_1^{2m}} + O\left(p^{1-m}\exp\left(\frac{4\ln p}{\ln\ln p}\right)\right).
\end{aligned}
$$

From the Euler product formula, we can get

$$
\sum_{\substack{n_1=1\\ (n_1,p)=1}}^{\infty} \frac{|D(n_1)|^2}{n_1^{2m}} = \prod_{p_1\nmid p}\left(1 + \frac{|D(p_1)|^2}{p_1^{2m}} + \frac{|D(p_1^2)|^2}{p_1^{4m}} + \cdots + \frac{|D(p_1^k)|^2}{p_1^{2mk}} + \cdots\right)
$$

and

$$
D(p_1^k) = 1 + \frac{\chi_1(p_1)}{p_1^{n-m}} + \left(\frac{\chi_1(p_1)}{p_1^{n-m}}\right)^2 + \cdots + \left(\frac{\chi_1(p_1)}{p_1^{n-m}}\right)^k = \frac{1-\left(\frac{\chi_1(p_1)}{p_1^{n-m}}\right)^{k+1}}{1-\frac{\chi_1(p_1)}{p_1^{n-m}}},
$$

and it is straightforward to show that

$$
\sum_{\substack{n_1=1\\ (n_1,p)=1}}^{\infty} \frac{|D(n_1)|^2}{n_1^{2m}} = \prod_{p_1\nmid p} \frac{\frac{1}{1-\frac{1}{p_1^{2m}}} + \frac{\frac{1}{p_1^{2n-2m}}}{1-\frac{1}{p_1^{2n}}} - \frac{\frac{\chi_1(p_1)}{p_1^{n-m}}}{1-\frac{\chi_1(p_1)}{p_1^{n+m}}} - \frac{\frac{\overline{\chi_1}(p_1)}{p_1^{n-m}}}{1-\frac{\overline{\chi_1}(p_1)}{p_1^{n+m}}}}{\left|1-\frac{\chi_1(p_1)}{p_1^{n-m}}\right|^2}
$$

$$= \prod_{p_1 \nmid p} \frac{1 - \frac{1}{p_1^{2m+2n}}}{\left(1 - \frac{1}{p_1^{2m}}\right)\left(1 - \frac{1}{p_1^{2n}}\right)} \frac{1}{\left|1 - \frac{\chi_1(p_1)}{p_1^{n+m}}\right|^2}$$
$$= \frac{\zeta(2m)\zeta(2n)}{\zeta(2m+2n)} |L(n+m, \chi_1)|^2 \frac{\left(1 - \frac{1}{p^{2m}}\right)\left(1 - \frac{1}{p^{2n}}\right)}{1 - \frac{1}{p^{2m+2n}}}$$
$$= \frac{\zeta(2m)\zeta(2n)}{\zeta(2m+2n)} |L(n+m, \chi_1)|^2 + O(p^{-2m}).$$

Similarly, we also have Lemma 2.4 for $n \le m$. □

3. Proofs of theorems

Now, we prove our theorems by using the above lemmas.

Proof of Theorem 1.1. Combining Lemma 2.1 and Lemma 2.3, we have

$$C(h, m, n, p, \chi) \ll m!n!(2\pi)^{-(m+n)} p^{\frac{1}{2}} \ln^2 p.$$

□

Proof of Theorem 1.2. From Lemma 2.1 and Lemma 2.4, we have

$$\sum_{h=1}^{p-1} |C(h, m, n, p, \chi)|^2$$
$$= \frac{16(m!n!)^2}{(2\pi i)^{2(m+n)}} \frac{1}{\phi^2(p)} \sum_{h=1}^{p-1} \left| \sum_{\substack{\chi_1 \bmod p \\ \chi\chi_1(-1)=(-1)^m}} \overline{\chi_1}(h) \tau(\chi_1) \tau(\chi\chi_1) L(m, \overline{\chi\chi_1}) L(n, \overline{\chi}_1) \right|^2$$
$$= \frac{16(m!n!)^2}{(2\pi i)^{2(m+n)}} \frac{p^2}{\phi(p)} \sum_{\substack{\chi_1 \bmod p \\ \chi\chi_1(-1)=(-1)^m}} \left| L(m, \overline{\chi\chi_1}) L(n, \overline{\chi}_1) \right|^2$$
$$= \frac{8(m!n!)^2 p^2}{(2\pi i)^{2(m+n)}} \frac{\zeta(2m)\zeta(2n)}{\zeta(2m+2n)} |L(m+n, \chi)|^2 + O\left(p^{2-\min(n,m)} \exp\left(\frac{4 \ln p}{\ln \ln p}\right)\right).$$

This proves Theorem 1.2. □

Proof of Theorem 1.4. From the definition of $C(h, m, n, p, \chi)$ and the orthogonality of Dirichlet characters, we have

$$|C(h, m, n, p, \chi)|^2 = \left| \sum_{a=1}^{p-1} \chi(a) \overline{B}_m \left(\frac{\overline{a}}{p}\right) \overline{B}_n \left(\frac{ah}{p}\right) \right|^2$$
$$= \left\{ \sum_{a=1}^{p-1} \chi(a) \overline{B}_m \left(\frac{\overline{a}}{p}\right) \overline{B}_n \left(\frac{ah}{p}\right) \right\} \left\{ \sum_{b=1}^{p-1} \overline{\chi}(b) \overline{B}_m \left(\frac{\overline{b}}{p}\right) \overline{B}_n \left(\frac{bh}{p}\right) \right\}$$

$$
\begin{aligned}
&= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a)\overline{B}_m\left(\frac{\overline{ab}}{p}\right)\overline{B}_n\left(\frac{abh}{p}\right)\overline{B}_m\left(\frac{\overline{b}}{p}\right)\overline{B}_n\left(\frac{bh}{p}\right)\\
&= \frac{1}{\phi(p)}\sum_{\chi_1 \bmod p}\sum_{a=1}^{p-1}\chi(a)\left\{\sum_{b=1}^{p-1}\chi_1(b)\overline{B}_m\left(\frac{\overline{a}b}{p}\right)\overline{B}_m\left(\frac{b}{p}\right)\right\}\\
&\qquad\times\left\{\sum_{c=1}^{p-1}\chi_1(c)\overline{B}_n\left(\frac{hc}{p}\right)\overline{B}_n\left(\frac{ahc}{p}\right)\right\}\\
&= \frac{1}{\phi(p)}\sum_{\chi_1 \bmod p}\overline{\chi_1}(h)\sum_{a=1}^{p-1}\chi(a)\left\{\sum_{b=1}^{p-1}\chi_1(b)\overline{B}_m\left(\frac{\overline{a}b}{p}\right)\overline{B}_m\left(\frac{b}{p}\right)\right\}\\
&\qquad\times\left\{\sum_{c=1}^{p-1}\chi_1(c)\overline{B}_n\left(\frac{c}{p}\right)\overline{B}_n\left(\frac{ac}{p}\right)\right\},
\end{aligned}
$$

it follows that

$$
\begin{aligned}
&\sum_{h=1}^{p-1}|C(h,m,n,p,\chi)|^2\\
&=\frac{1}{\phi(p)}\sum_{h=1}^{p-1}\sum_{\chi_1 \bmod p}\overline{\chi_1}(h)\sum_{a=1}^{p-1}\chi(a)\left\{\sum_{b=1}^{p-1}\chi_1(b)\overline{B}_m\left(\frac{\overline{a}b}{p}\right)\overline{B}_m\left(\frac{b}{p}\right)\right\}\\
&\qquad\times\left\{\sum_{c=1}^{p-1}\chi_1(c)\overline{B}_n\left(\frac{c}{p}\right)\overline{B}_n\left(\frac{ac}{p}\right)\right\}\\
&=\sum_{a=1}^{p-1}\chi(a)\left\{\sum_{b=1}^{p-1}\overline{B}_m\left(\frac{\overline{a}b}{p}\right)\overline{B}_m\left(\frac{b}{p}\right)\right\}\left\{\sum_{c=1}^{p-1}\overline{B}_n\left(\frac{c}{p}\right)\overline{B}_n\left(\frac{ac}{p}\right)\right\}.
\end{aligned}
$$

Thus, from Theorem 1.2, we have

$$
\begin{aligned}
&\sum_{h=1}^{p-1}\chi(h)S(\overline{h},m,m,p)S(h,n,n,p)\\
&=\frac{8(m!n!)^2p^2}{(2\pi i)^{2(m+n)}}\frac{\zeta(2m)\zeta(2n)}{\zeta(2m+2n)}|L(m+n,\chi)|^2+O\left(p^{2-\min(n,m)}\exp\left(\frac{4\ln p}{\ln\ln p}\right)\right).
\end{aligned}
$$

□

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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