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Research article

Lie symmetry group, exact solutions and conservation laws for multi-term time fractional differential equations

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Abstract: In this paper, the time fractional Benjamin-Bona-Mahony-Peregrine (BBMP) equation and time-fractional Novikov equation with the Riemann-Liouville derivative are investigated through the use of Lie symmetry analysis and the new Noether's theorem. Then, we construct their group-invariant solutions by means of Lie symmetry reduction. In addition, the power-series solutions are also obtained with the help of the Erdélyi-Kober (E-K) fractional differential operator. Furthermore, the conservation laws for the time-fractional BBMP equation are established by utilizing the new Noether's theorem.

Keywords: Lie symmetry analysis; time-fractional BBMP equation; time-fractional Novikov equation; group-invariant solutions; conservation laws
Mathematics Subject Classification: 26A33, 76M60

1. Introduction

In the past few decades, fractional partial differential equations (FPDEs) have received extensive attention in studies of nonlinear fields such as physics, science, biology, chemistry, mathematics, electronics, viscoelasticity, signal processing and soft materials. Obtaining the exact solutions of nonlinear FPDEs is one of the most significant goals of the research process for FPDEs. A large number of papers have presented many effective methods for constructing explicit solutions of FPDEs, such as Lie symmetry analysis [1–5], the Laplace transform [6], the Adomian decomposition method [7], the variational iteration method [8], the homotopy analysis method [9], the separation of variables method [10], the invariant subspace method [11], the extended direct algebraic method [12], Chebyshev series method [13], etc.

Lie symmetry analysis is a very useful tool in constructing exact solutions for nonlinear partial differential equations (PDEs). As a result of applying the Lie symmetry method, many kinds of exact solutions for integer-order PDEs have been obtained, such as similarity solutions, fundamental solutions and traveling wave solutions [14, 15]. Inspired by the Lie symmetry of integer-order PDEs, researchers have extended and applied the Lie symmetry method to investigate FPDEs and constructed the group invariant solutions for numerous FPDEs. For example, Hu et al. [16] studied the exact solutions for the time-fractional KdV-type equation. Leo et al. [17] provided a general theoretical framework to extend the classical Lie theory for PDEs to the case of equations of fractional order. Wang et al. [18] performed Lie symmetry analysis on the time-fractional Harry Dym equation and constructed group invariant solutions. Lashkarian et al. [19] considered the Lie group analysis and obtained the explicit solutions of the time-fractional cylindrical Burgers equation. In addition, Erdélyi-Kober (E-K) fractional differential operators [17] can be used to establish the exact solutions for FPDEs; for instance, Cheng and Wang [20] constructed the power-series solutions to (2+1)-dimensional time-fractional Navier-Stokes equation by using an E-K fractional differential operator. In [21], Zhang and Li achieved explicit solutions to the time-fractional b-family peakon equations, which contain the mixed derivative that encompasses the time-fractional derivative and integer-order x-derivative.

Conservation laws play an important role in mathematical physics as a means to study some properties of solutions to nonlinear PDEs, and they can describe the invariable properties of the values of some physical models in nature, such as momentum, energy, mass, potential energy, electric charge and so on. It is well-known that Noether's theorem establishes a close connection between Lie symmetries and the conservation laws of PDEs [22]. Ibragimov presented a new conservation theorem in [23], and he obtained the conservation laws for PDEs without classical Lagrange equations. In recent years, Lie symmetry analysis and the conservation laws for many PDEs have been discussed, such as those for various wave equations [24], reaction-diffusion equations [25], the Rosenau-KdV-RLW equation [26], time-fractional Cahn-Allen and Klein-Gordon equations [27] and the time-fractional Caudrey-Dodd-Gibbon-Sawada-Kotera equation [28].

Nonlinear equations take the form

$$\sum_{(i)} a_i(u, u_x, \cdots) D_t^i L[u] = \sum_{(j)} b_j(u, u_x, \cdots) D_x^j L[u],$$

with arbitrary coefficients $a_i(\cdot)$, $b_j(\cdot)$ and annihilating operator $L = I + \frac{d^2}{dx^2}$, and they are called tautological PDEs [29], which are also referred to as multi-term time-fractional differential equations. Motivated by [21], in this paper, we focus on the study of multi-term time-fractional equations, particularly the time-fractional Benjamin-Bona-Mahony-Peregrine (BBMP) equation and time-fractional Novikov equation, respectively. The BBMP equation first proposed by Peregrine [30] was derived from the Korteweg-de Vries equation. The BBMP equation is commonly used to study long waves in the shallow water of ocean beaches, drift waves in plasma and the Rossby waves in rotating fluids [31]. The BBMP equation is also known as the regularized long-wave equation. This paper focuses on the time-fractional BBMP equation, which has the following form:

$$\partial_t^\alpha u - \partial_t^\alpha (u_{xx}) - u u_x = 0, \tag{1.1}$$

where $0 < \alpha < 1$, $u = u(x, t)$ is the unknown function, $x \in \mathbb{R}$ is a space variable and $t > 0$ is the time variable. In particular, when $\alpha = 1$, Eq (1.1) reduces to the classical BBMP equation, which has a wide range of applications in the development of optical equipment, semiconductors [32], etc. Khalique [33] utilized Lie group analysis to obtain the periodic solutions and soliton solutions and derived the conservation laws for the classical BBMP equation.

The Novikov equation is an extremely classical integrable evolution equation, which is thought to be a generalization of the Camassa-Holm equation and originally discovered by Vladimir Novikov [34], who has used it to describe fluid motion in shallow water environments. In [34], Novikov discovered the first few symmetries and studied a scalar Lax pair of the equation. In this paper, we pay attention to the time-fractional Novikov equation, which is given by

$$\partial_t^\alpha u - \partial_t^\alpha (u_{xx}) + 4u^2 u_x - 3uu_x u_{xx} - u^2 u_{xxx} = 0, \tag{1.2}$$

where $0 < \alpha < 1$ and $u = u(x,t)$ is the unknown function of space variable x and time variable t.

In view of the non-locality property of fractional derivatives, time-fractional PDEs can be used to describe problems with memory effects and genetic characteristics in real life. Thus, there is practical significance in a study of the time-fractional equations given by Eqs (1.1) and (1.2). To our knowledge, equations containing a time fractional-order mixed derivative and spatial integer-order derivative have rarely been studied, while Eqs (1.1) and (1.2) are such equations. Therefore, in this paper, we will construct the exact solutions of Eqs (1.1) and (1.2) through the use of Lie symmetry analysis and symmetry reduction. Moreover, in view of the difficulty in solving ordinary differential equations (ODEs), we introduce the E-K fractional differential operator to transform Eqs (1.1) and (1.2) into ODEs with only a single variable, and we obtain their power series solutions. In addition, the q-homotopy analysis method and invariant subspace method may be used to discuss the exact solutions of the time-fractional equations given by Eqs (1.1) and (1.2), while, in this paper, we take no account of these methods.

This paper is organized as follows. In Section 2, we will give some basic knowledge and properties of fractional calculus. Section 3 is fully dedicated to obtaining the Lie symmetry groups and exact solutions of the time-fractional BBMP equation and time-fractional Novikov equation. First, we will introduce the Lie symmetry method for general time-fractional PDEs with the Riemann-Liouville derivative. Next, infinitesimal generators for Eqs (1.1) and (1.2) will be constructed, respectively. Then, based on the obtained Lie algebras, we perform symmetry reduction and obtain the group-invariant solutions of the equations under consideration. In addition, by using the E-K fractional differential operator, the power-series solutions for Eqs (1.1) and (1.2) is also constructed. In Section 4, conservation laws for the time-fractional BBMP equation are obtained. At last, we will show the conclusion of this paper.

2. Preliminaries

The purpose of this section is to review some basic properties of the Riemann-Liouville fractional integral and derivative [35], which will be used throughout this paper. Meanwhile, we will give the definition of the E-K fractional differential operator [17].

Definition 2.1. *[35] The αth-order Riemann-Liouville integral operator of $v(t)$ is defined by*

$$I_t^\alpha v(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s) \mathrm{d}s, \tag{2.1}$$

where $\Gamma(x) = \int_0^\infty e^{-z} z^{x-1} \mathrm{d}z$ is the gamma function.

Definition 2.2. *[35] The symbol $\partial_t^\alpha u$ represents the αth-order Riemann-Liouville derivative, denoted as*

$$\partial_t^\alpha u = \begin{cases} \dfrac{\partial^n u}{\partial t^n}, & \alpha = n, \\ \dfrac{1}{\Gamma(n-\alpha)} \dfrac{\partial^n}{\partial t^n} \displaystyle\int_0^t (t-s)^{n-\alpha-1} u(x,s)\mathrm{d}s, & 0 < n-1 < \alpha < n, \end{cases} \tag{2.2}$$

where $u = u(x,t)$ is the function of the space variable x and time variable t, and $\frac{\partial^n}{\partial t^n}$ is the nth-order integer derivative. Moreover, the Riemann-Liouville derivative can be expressed as

$$\partial_t^\alpha u = D^n (I_t^{n-\alpha}) u, \tag{2.3}$$

where D is the differential operator and $n = [\alpha] + 1$.

Definition 2.3. *[35] The adjoint operator $(\partial_t^\alpha)^*$ of the Riemann-Liouville fractional differential operator ∂_t^α is given by*

$$(\partial_t^\alpha)^* f(t) = (-1)^n I^{n-\alpha} \frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \frac{\mathrm{d}^n}{\mathrm{d}s^n} f(s)\mathrm{d}s, \tag{2.4}$$

where $I^{n-\alpha}$ denotes the Riemann-Liouville integral operator.

Definition 2.4. *[17] The E-K fractional differential operator $\mathcal{P}_\beta^{\tau,\alpha}$ of order α is defined by*

$$(\mathcal{P}_\beta^{\tau,\alpha} F)(\xi) = \prod_{j=0}^{m-1} \left(\tau + j - \frac{1}{\beta} \xi \frac{\mathrm{d}}{\mathrm{d}\xi} \right) (\mathcal{K}_\beta^{\tau+\alpha, m-\alpha} F)(\xi), \tag{2.5}$$

where

$$m = \begin{cases} [\alpha] + 1, & \alpha \notin \mathbb{N}, \\ \alpha, & \alpha \in \mathbb{N}, \end{cases}$$

and

$$(\mathcal{K}_\beta^{\tau,\alpha} F)(\xi) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_1^\infty (v-1)^{\alpha-1} v^{-(\tau+\alpha)} F(\xi v^{\frac{1}{\beta}})\mathrm{d}v, & \alpha > 0, \\ F(\xi), & \alpha = 0. \end{cases} \tag{2.6}$$

Lemma 2.1. *[36] Let $0 < \gamma \leq 1$ and $k_1 \in \mathbb{R}$. The solution of the fractional ODE*

$$\partial_t^\gamma g(t) = 0$$

is given by

$$g(t) = k_1 t^{\gamma-1}.$$

3. Lie symmetry and group-invariant solutions

3.1. Lie symmetry for time-fractional PDEs

Consider the symmetry group for a time-fractional PDE of the following form:

$$F(\partial_t^\alpha u, \partial_t^\alpha (u_{xx}), t, x, u, u_x, u_{xx}, \cdots) = 0, \quad 0 < \alpha < 1, \tag{3.1}$$

where $u = u(x,t)$ denotes the unknown function of the space variable x and time variable t. Suppose that Eq (3.1) remains invariant under the one-parameter Lie group of infinitesimal transformations given by

$$\begin{cases} x^* = x + \epsilon\xi(x,t,u) + o(\epsilon^2), \\ t^* = t + \epsilon\tau(x,t,u) + o(\epsilon^2), \\ u^* = u + \epsilon\eta(x,t,u) + o(\epsilon^2), \end{cases}$$

where $\epsilon \ll 1$ is the group parameter and ξ, τ and η are smooth functions called infinitesimal generators. The admitted one-parameter Lie group of infinitesimal transformation has the following form:

$$V = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}. \tag{3.2}$$

The Lie invariance criterion for time-fractional PDE (3.1) is

$$\mathrm{Pr}^{(\alpha,\mathrm{n})}\mathrm{V}(\Delta)|_{\Delta=0} = 0, \tag{3.3}$$

where $\Delta = F(\partial_t^\alpha u, \partial_t^\alpha(u_{xx}), t, x, u, u_x, u_{xx}, \cdots)$ and n is a nonnegative integer.

The operator $\mathrm{Pr}^{(\alpha,\mathrm{n})}\mathrm{V}$ denotes the prolongation of field vector V, and it is defined by

$$\mathrm{Pr}^{(\alpha,\mathrm{n})}\mathrm{V} = \mathrm{V} + \eta^{\alpha,\mathrm{t}}\frac{\partial}{\partial(\partial_\mathrm{t}^\alpha \mathrm{u})} + \eta^{\alpha,\mathrm{xx}}\frac{\partial}{\partial(\partial_\mathrm{t}^\alpha \mathrm{u_{xx}})} + \eta^{\mathrm{x}}\frac{\partial}{\partial \mathrm{u_x}} + \eta^{\mathrm{xx}}\frac{\partial}{\partial \mathrm{u_{xx}}} + \cdots, \tag{3.4}$$

where

$$\begin{aligned} \eta^{\alpha,t} &= D_t^\alpha(\eta) + \xi D_t^\alpha(u_x) - D_t^\alpha(\xi u_x) + D_t^\alpha(u(D_t\tau)) - D_t^{\alpha+1}(\tau u) + \tau D_t^{\alpha+1}(u), \\ \eta^{\alpha,xx} &= D_t^\alpha(D_x^2(\eta - \xi u_x - \tau u_t)) + \xi\partial_t^\alpha(u_{xxx}) + \tau\partial_t^\alpha(u_{txx}), \\ \eta^x &= D_x(\eta - \xi u_x - \tau u_t) + \xi u_{xx} + \tau u_{tx}, \\ \eta^{xx} &= D_{xx}(\eta - \xi u_x - \tau u_t) + \xi u_{xxx} + \tau u_{txx}, \\ &\vdots \end{aligned} \tag{3.5}$$

In Eq (3.5), D_t and D_x represent the total derivatives, respectively given by

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{xt}\frac{\partial}{\partial u_x} + \cdots, \\ D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{tx}\frac{\partial}{\partial u_t} + \cdots. \end{aligned} \tag{3.6}$$

The αth-order extended infinitesimal $\eta^{\alpha,t}$ given by Eq (3.5) can be expressed as follows [37]:

$$\begin{aligned} \eta^{\alpha,t} =& \partial_t^\alpha\eta + (\eta_u - \alpha D_t(\tau))\partial_t^\alpha u - u\partial_t^\alpha\eta_u - \sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\xi)D_t^{\alpha-n}(u_x) \\ &+ \sum_{n=1}^{\infty}\left(\binom{\alpha}{n}\partial_t^\alpha\eta_u - \binom{\alpha}{n+1}D_t^{n+1}(\tau)\right)D_t^{\alpha-n}(u) + \mu_\alpha, \end{aligned} \tag{3.7}$$

where

$$\mu_\alpha = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}(-u)^r \times \frac{\partial^m}{\partial t^m}(u^{k-r})\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^k} \tag{3.8}$$

and

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}.$$

Similarly, the expression of $\eta^{\alpha,xx}$ is defined by [21]

$$\begin{aligned}\eta^{\alpha,xx} =& \partial_t^\alpha(\eta_{xx} + (2\eta_{ux} - \xi_{xx})u_x - \tau_{xx}u_t + (\eta_{uu} - 2\xi_{ux})u_x^2 - \xi_{uu}u_x^3 - 2\tau_{ux}u_xu_t \\ & - \tau_{uu}u_x^2u_t) + \partial_t^\alpha(u_{xx})(\eta_u - \alpha D_t(\tau) - 2\xi_x - \tau_u u_t - 3\xi_u u_x) \\ & + \sum_{n=1}^{\infty}\left(\binom{\alpha}{n}D_t^n(\eta_u - 2\xi_x - \tau_u u_t - 3\xi_u u_x) - \binom{\alpha}{n+1}D_t^{n+1}(\tau)\right)\partial_t^{\alpha-n}(u_{xx}) \\ & - 2\sum_{n=0}^{\infty}\binom{\alpha}{n}D_t^n D_x(\tau)\partial_t^{\alpha-n}(u_{xt}) - \sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\xi)\partial_t^{\alpha-n}(u_{xx}).\end{aligned} \tag{3.9}$$

3.1.1. Time-fractional BBMP equation

In this subsection, through the Lie symmetry analysis given in Section 3.1, we obtain the Lie symmetry group and infinitesimal generators of the time-fractional BBMP equation given by Eq (1.1).

Theorem 3.1. *The Lie point symmetries admitted by Eq* (1.1) *are*

$$V_1 = \frac{\partial}{\partial x}, \quad V_2 = t\frac{\partial}{\partial t} - \alpha u\frac{\partial}{\partial u}. \tag{3.10}$$

Proof. In view of the Lie symmetry analysis given in Section 3.1, the Lie invariance criterion for Eq (1.1) is

$$(\eta^{\alpha,t} - \eta^{\alpha,xx} - u\eta^x - u_x\eta)|_{(1.1)} = 0. \tag{3.11}$$

Inserting Eqs (3.5), (3.7) and (3.9) into the invariance condition given by Eq (3.11) and equating the coefficients of the partial derivatives $u_t,\ u_x,\ u_{xx},\ \cdots$, fractional derivatives $\partial_t^{\alpha-n}u,\ \partial_t^{\alpha-n}u_x,\ \partial_t^\alpha u_{xx}\ \cdots$ and powers of u, we can obtain the following over determined equations:

$$\begin{cases}\xi_x = \xi_t = \xi_u = \tau_x = \tau_u = 0, \\ \binom{\alpha}{n}\partial_t^n\eta_u - \binom{\alpha}{n+1}D_t^{n+1}\tau = 0, \quad \forall n \in \mathbb{N}, \\ \eta + \alpha u D_t\tau = 0, \\ \partial_t^\alpha\eta - u\partial_t^\alpha\eta_u - u\eta_x = 0.\end{cases} \tag{3.12}$$

Solving Eq (3.12), the infinitesimal generators can be expressed as follows:

$$\xi = C_1, \quad \tau = C_2t + C_3, \quad \eta = -\alpha uC_2, \tag{3.13}$$

where C_i $(i = 1, 2, 3)$ are arbitrary constants. Moreover, noticing that the initial condition

$$\tau(t, x, u)|_{t=0} = 0 \tag{3.14}$$

holds for all time-fractional PDEs, the vector fields of Eq (1.1) can be expressed in the form of Eq (3.10). □

3.1.2. Time-fractional Novikov equation

This subsection is devoted to the investigation of Lie point symmetry of the time-fractional Novikov equation given by Eq (1.2).

Theorem 3.2. *Lie algebras admitted by the fractional Novikov equation expressed as Eq* (1.2) *are spanned by the following vector fields:*

$$V_1 = \frac{\partial}{\partial x}, \quad V_2 = 2t\frac{\partial}{\partial t} - \alpha u\frac{\partial}{\partial u}. \tag{3.15}$$

Proof. The invariance criterion given by Eq (3.3) of Eq (1.2) can be represented in the following form:

$$(\eta^{\alpha,t} - \eta^{\alpha,xx} + (4u^2 - 3uu_{xx})\eta^x - 3uu_x\eta^{xx} - u^2\eta^{xxx} + (8uu_x - 3u_xu_{xx} - 2uu_{xxx})\eta)|_{(1.2)} = 0, \tag{3.16}$$

where $\eta^{\alpha,t}$, $\eta^{\alpha,xx}$, η^x, η^{xx} and η^{xxx} are defined by Eqs (3.5), (3.7) and (3.9).

Equating the coefficients of fractional derivatives, partial derivatives and various powers of u to be zero, the over determined equations can be constructed as follows:

$$\begin{cases} \xi_x = \xi_t = \xi_u = \tau_x = \tau_u = 0, \\ \binom{\alpha}{n}\partial_t^n\eta_u - \binom{\alpha}{n+1}D_t^{n+1}\tau = 0, \quad \forall n \in \mathbb{N}, \\ \partial_t^\alpha\eta - u\partial_t^\alpha\eta_u + 4u^2\eta_x - u^2\eta_{xxx} = 0, \\ \alpha u\tau_t + 2\eta = 0. \end{cases} \tag{3.17}$$

Solving the above over determined equation given by Eq (3.17), we can deduce that

$$\xi = C_1, \quad \tau = 2tC_2 + C_3, \quad \eta = -\alpha uC_2, \tag{3.18}$$

where C_1, C_2 and C_3 are arbitrary parameters.

In view of the initial condition $\tau(t, x, u)|_{t=0} = 0$, the infinitesimal generators of Eq (1.2) are given by Eq (3.15). □

3.2. Symmetry reductions and group-invariant solutions

In the previous section, we obtained the infinitesimal generators for the time-fractional BBMP equation and Novikov equation. In this section, we are ready to perform symmetry reductions and construct their exact solutions. Moreover, the power-series solutions are also obtained through the use of the E-K fractional differential operator.

3.2.1. Similarity reduction and exact solutions of Eq (1.1)

In this subsection, we perform the similarity reduction of the time-fractional BBMP equation given by Eq (1.1), and then the exact solutions are calculated correspondingly.

Case 1. $V_1 = \frac{\partial}{\partial x}$. According to the characteristic equation of Eq (1.1), i.e.,

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{1} = \frac{\mathrm{d}u}{0},$$

we deduce two independent similarity variables t and u. Thus we get the group-invariant solution of Eq (1.1) with the form $u(x,t) = g(t)$, where $g(t)$ satisfies the fractional ODE

$$\partial_t^\alpha g(t) = 0.$$

By virtue of Lemma 2.1, we obtain

$$g(t) = k_1 t^{\alpha-1}. \tag{3.19}$$

Therefore, the group-invariant solution of Eq (1.1) is

$$u(x,t) = k_1 t^{\alpha-1}. \tag{3.20}$$

Case 2. $V_2 = t\frac{\partial}{\partial t} - \alpha u \frac{\partial}{\partial u}$. Considering the characteristic equation

$$\frac{\mathrm{d}t}{t} = \frac{\mathrm{d}x}{0} = \frac{\mathrm{d}u}{-\alpha u},$$

the similarity variables x and $t^\alpha u$ are obtained. Thus the group-invariant solution of Eq (1.1) is

$$u(x,t) = t^{-\alpha} g(x), \tag{3.21}$$

where $g(x)$ satisfies the equation

$$\frac{gg'}{g - g''} = \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}.$$

Solving the above equation, we find that

$$g(x) = C + \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)} x \tag{3.22}$$

and C is an arbitrary constant.

Plugging Eq (3.22) into Eq (3.21) yields the following exact solution of Eq (1.1):

$$u(x,t) = Ct^{-\alpha} + \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)} t^{-\alpha} x. \tag{3.23}$$

Case 3. $V_3 = \rho V_1 + V_2 = \rho\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \alpha u \frac{\partial}{\partial u}$, where ρ is a nonzero constant. In this case, we consider the power-series solution with the help of E-K fractional operators for Eq (1.1). Solving the characteristic equation

$$\frac{\mathrm{d}t}{t} = \frac{\mathrm{d}x}{\rho} = \frac{\mathrm{d}u}{-\alpha u}$$

leads to the similarity variables

$$\xi = e^x t^{-\rho}, \quad u = t^{-\alpha} g(\xi), \tag{3.24}$$

where g is an undetermined function with a new independent variable $\xi = e^x t^{-\rho}$. Thus, one can obtain the following theorem.

Theorem 3.3. *The similarity transformations given by Eq* (3.24) *reduce the BBMP equation given by Eq* (1.1) *to the following form:*

$$\left(\mathcal{P}_{\frac{1}{\rho}}^{1-2\alpha,\alpha} g\right)(\xi) - \xi\left(\mathcal{P}_{\frac{1}{\rho}}^{1-2\alpha-\rho,\alpha} g'\right)(\xi) - \xi^2\left(\mathcal{P}_{\frac{1}{\rho}}^{1-2\alpha-2\rho,\alpha} g''\right)(\xi) - \xi g g' = 0. \tag{3.25}$$

Proof. Based on the definition of the Riemann-Liouville derivative given by Eq (2.2), we have

$$\partial_t^\alpha u = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_0^t (t-s)^{-\alpha}s^{-\alpha}g(e^x s^{-\rho})\mathrm{d}s. \tag{3.26}$$

Setting $v = \frac{t}{s}$, and due to Eqs (2.5) and (2.6), Eq (3.26) can be written as

$$\begin{aligned}\partial_t^\alpha u &= \frac{\partial}{\partial t}\left[t^{1-2\alpha}\frac{1}{\Gamma(1-\alpha)}\int_1^\infty (v-1)^{-\alpha}v^{2\alpha-2}g(\xi v^\rho)\mathrm{d}\xi\right] \\ &= \frac{\partial}{\partial t}\left[t^{1-2\alpha}\left(\mathcal{K}_{\frac{1}{\rho}}^{1-\alpha,\,1-\alpha}g\right)(\xi)\right] \\ &= t^{-2\alpha}\left(1-2\alpha-\rho\xi\frac{\mathrm{d}}{\mathrm{d}\xi}\right)\left(\mathcal{K}_{\frac{1}{\rho}}^{1-\alpha,\,1-\alpha}g\right)(\xi) \\ &= t^{-2\alpha}\left(\mathcal{P}_{\frac{1}{\rho}}^{1-2\alpha,\,\alpha}g\right)(\xi).\end{aligned} \tag{3.27}$$

As for $\partial_t^\alpha(u_{xx})$, we have

$$\partial_t^\alpha u_{xx} = \partial_t^\alpha(t^{-\alpha}\xi g'(\xi)) + \partial_t^\alpha(t^{-\alpha}\xi^2 g''(\xi)). \tag{3.28}$$

On one hand, similar to Eqs (3.26) and (3.27), one obtains

$$\begin{aligned}\partial_t^\alpha(t^{-\alpha}\xi g'(\xi)) &= \frac{\partial}{\partial t}\left[\frac{e^x}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}s^{-\alpha-\rho}g'(e^x s^{-\rho})\mathrm{d}s\right] \\ &= \frac{\partial}{\partial t}\left[\frac{1}{\Gamma(1-\alpha)}t^{1-2\alpha}\xi\int_1^\infty (v-1)^{-\alpha}v^{2\alpha+\rho-2}g'(\xi v^\rho)\mathrm{d}v\right] \\ &= \frac{\partial}{\partial t}\left[t^{1-2\alpha}\xi\left(\mathcal{K}_{\frac{1}{\rho}}^{1-\alpha-\rho,\,1-\alpha}g'\right)(\xi)\right] \\ &= t^{-2\alpha}\xi\left(1-2\alpha-\rho-\rho\xi\frac{\mathrm{d}}{\mathrm{d}\xi}\right)\left(\mathcal{K}_{\frac{1}{\rho}}^{1-\alpha-\rho,\,1-\alpha}g'\right)(\xi) \\ &= t^{-2\alpha}\xi\left(\mathcal{P}_{\frac{1}{\rho}}^{1-2\alpha-\rho,\,\alpha}g'\right)(\xi).\end{aligned} \tag{3.29}$$

On the other hand, we obtain

$$\begin{aligned}\partial_t^\alpha(t^{-\alpha}\xi^2 g''(\xi)) &= \frac{\partial}{\partial t}\left[\frac{e^{2x}}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}s^{-\alpha-2\rho}g''(e^x s^{-\rho})\mathrm{d}s\right] \\ &= \frac{\partial}{\partial t}\left[\frac{1}{\Gamma(1-\alpha)}t^{1-2\alpha}\xi^2\int_1^\infty (v-1)^{-\alpha}v^{2\alpha+2\rho-2}g''(\xi v^\rho)\mathrm{d}v\right] \\ &= \frac{\partial}{\partial t}\left[t^{1-2\alpha}\xi^2\left(\mathcal{K}_{\frac{1}{\rho}}^{1-\alpha-2\rho,\,1-\alpha}g''\right)(\xi)\right] \\ &= t^{-2\alpha}\xi^2\left(1-2\alpha-2\rho-\rho\xi\frac{\mathrm{d}}{\mathrm{d}\xi}\right)\left(\mathcal{K}_{\frac{1}{\rho}}^{1-\alpha-2\rho,\,1-\alpha}g''\right)(\xi) \\ &= t^{-2\alpha}\xi^2\left(\mathcal{P}_{\frac{1}{\rho}}^{1-2\alpha-2\rho,\,\alpha}g''\right)(\xi).\end{aligned} \tag{3.30}$$

Therefore, combine Eqs (3.28) and (3.29) with Eq (3.30) to get

$$\partial_t^\alpha u_{xx} = t^{-2\alpha}\xi\left(\mathcal{P}_{\frac{1}{\rho}}^{1-2\alpha-\rho,\,\alpha}g'\right)(\xi) + t^{-2\alpha}\xi^2\left(\mathcal{P}_{\frac{1}{\rho}}^{1-2\alpha-2\rho,\,\alpha}g''\right)(\xi). \tag{3.31}$$

Substituting Eqs (3.27) and (3.31) into Eq (1.1) yields that Eq (3.25) holds to be true. □

In what follows, we intend to use the power-series method to construct the exact solutions for Eq (3.25). Suppose that Eq (3.25) admits the power-series solution

$$g(\xi) = \sum_{n=0}^{\infty} a_n \xi^n, \tag{3.32}$$

where a_n $(n = 0, 1, 2, \cdots)$ are constants which will be determined later. Then, we have

$$g'(\xi) = \sum_{n=1}^{\infty} n a_n \xi^{n-1}, \quad g''(\xi) = \sum_{n=2}^{\infty} (n-1) n a_n \xi^{n-2}. \tag{3.33}$$

Next, with the help of Eq (3.33), we consider the following E-K fractional differential operator:

$$\begin{aligned}
\left(\mathcal{P}_{\frac{1}{\rho}}^{1-2\alpha,\, \alpha} g\right)(\xi) =& (1 - 2\alpha - \rho\xi \frac{\mathrm{d}}{\mathrm{d}\xi}) \left(\mathcal{K}_{\frac{1}{\rho}}^{1-\alpha,\, 1-\alpha} g\right)(\xi) \\
=& (1 - 2\alpha - \rho\xi \frac{\mathrm{d}}{\mathrm{d}\xi}) \frac{1}{\Gamma(1-\alpha)} \int_1^{\infty} (v-1)^{-\alpha} v^{2\alpha-2} \sum_{n=0}^{\infty} a_n (\xi v^{\rho})^n \mathrm{d}v \\
=& \sum_{n=0}^{\infty} \frac{\Gamma(1-\alpha-\rho n)}{\Gamma(1-2\alpha-\rho n)} a_n \xi^n.
\end{aligned} \tag{3.34}$$

Similar to Eq (3.34), we can also obtain

$$\begin{aligned}
\left(\mathcal{P}_{\frac{1}{\rho}}^{1-2\alpha-\rho,\, \alpha} g'\right)(\xi) =& (1 - 2\alpha - \rho - \rho\xi \frac{\mathrm{d}}{\mathrm{d}\xi}) \left(\mathcal{K}_{\frac{1}{\rho}}^{1-\alpha-\rho,\, 1-\alpha} g'\right)(\xi) \\
=& \sum_{n=0}^{\infty} \frac{\Gamma(1-\alpha-\rho n)}{\Gamma(1-2\alpha-\rho n)} n a_n \xi^{n-1}
\end{aligned} \tag{3.35}$$

and

$$\begin{aligned}
\left(\mathcal{P}_{\frac{1}{\rho}}^{1-2\alpha-2\rho,\, \alpha} g''\right)(\xi) =& (1 - 2\alpha - 2\rho - \rho\xi \frac{\mathrm{d}}{\mathrm{d}\xi}) \left(\mathcal{K}_{\frac{1}{\rho}}^{1-\alpha-2\rho,\, 1-\alpha} g''\right)(\xi) \\
=& \sum_{n=0}^{\infty} \frac{\Gamma(1-\alpha-\rho n)}{\Gamma(1-2\alpha-\rho n)} n(n-1) a_n \xi^{n-2}.
\end{aligned} \tag{3.36}$$

Therefore, inserting Eqs (3.32)–(3.36) into Eq (3.25) yields that

$$\sum_{n=0}^{\infty} \frac{\Gamma(1-\alpha-\rho n)}{\Gamma(1-2\alpha-\rho n)} (1-n^2) a_n \xi^n - \sum_{n=1}^{\infty} \sum_{m=1}^{n} m a_m a_{n-m} \xi^n = 0. \tag{3.37}$$

When $n = 1$, by comparing coefficients of ξ, we obtain

$$a_0 = 0, \quad a_1 \text{ is an arbitrary constant.} \tag{3.38}$$

When $n \geq 2$, we have

$$a_n = \frac{1}{1-n^2} \frac{\Gamma(1-2\alpha-\rho n)}{\Gamma(1-\alpha-\rho n)} \sum_{m=1}^{n} m a_m a_{n-m}. \tag{3.39}$$

Consequently, the power series solution for Eq (3.25) can be derived as follows:

$$g(\xi) = a_1\xi + \sum_{n=2}^{\infty} \frac{1}{1-n^2} \frac{\Gamma(1-2\alpha-\rho n)}{\Gamma(1-\alpha-\rho n)} \sum_{m=1}^{n} m a_m a_{n-m} \xi^n, \quad n = 2, 3, 4, \cdots \tag{3.40}$$

Substituting Eq (3.40) into Eq (3.24) leads to the exact solution of the BBMP equation given by Eq (1.1):

$$u(x,t) = a_1 e^x t^{-\alpha-\rho} + \sum_{n=2}^{\infty} \frac{1}{1-n^2} \frac{\Gamma(1-2\alpha-\rho n)}{\Gamma(1-\alpha-\rho n)} \sum_{m=1}^{n} m a_m a_{n-m} e^{nx} t^{-\alpha-\rho n}. \tag{3.41}$$

3.2.2. Similarity reduction and exact solution of Eq (1.2)

This subsection derives the similarity variables and their reduction equations and presents the exact solutions of the time-fractional equation given by Eq (1.2).

Case 1. $V_1 = \frac{\partial}{\partial x}$. We find that the group-invariant solution of Eq (1.2) is $u(x,t) = g(t)$, where $g(t)$ is given by $\partial_t^\alpha g(t) = 0$. Then we observe that the exact solution to Eq (1.2) is the same as that presented as Eq (3.20).

Case 2. $V_2 = 2t\frac{\partial}{\partial t} - \alpha u\frac{\partial}{\partial u}$. The similarity variables of this infinitesimal generator are x and $t^{\frac{\alpha}{2}}u$. Substituting $u(x,t) = t^{-\frac{\alpha}{2}}g(x)$ into Eq (1.2) yields the following reduction equation:

$$\frac{\Gamma(1-\frac{\alpha}{2})}{\Gamma(1-\frac{3}{2}\alpha)}g - \frac{\Gamma(1-\frac{\alpha}{2})}{\Gamma(1-\frac{3}{2}\alpha)}g'' + 4g^2g' - 3gg'g'' - g^2g''' = 0.$$

Case 3. $V_3 = \lambda V_1 + V_2 = \lambda\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - \alpha u\frac{\partial}{\partial u}$, where λ is a nonzero constant. We can also obtain the following reduced equation for the field vector V_3.

Theorem 3.4. *Induced by the general infinitesimal generator $V_3 = \lambda V_1 + V_2$, Eq* (1.2) *can be reduced to the following form:*

$$\begin{aligned} &\left(\mathcal{P}_{\frac{2}{\lambda}}^{1-\frac{3}{2}\alpha,\alpha} g\right)(z) - z\left(\mathcal{P}_{\frac{2}{\lambda}}^{1-\frac{3}{2}\alpha-\frac{\lambda}{2},\alpha} g'\right)(z) - z^2\left(\mathcal{P}_{\frac{2}{\lambda}}^{1-\frac{3}{2}\alpha-\lambda,\alpha} g''\right)(z) \\ &+ z(3g^2g' - 3zg(g')^2 - 3z^2gg'g'' - 3zg^2g'' - z^2g^2g''') = 0, \end{aligned} \tag{3.42}$$

where $z = e^x t^{-\frac{\lambda}{2}}$, $g = ut^{\frac{\alpha}{2}}$ and $0 < \alpha < 1$.

Proof. For the infinitesimal generator $V_3 = \lambda V_1 + V_2 = \lambda\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - \alpha u\frac{\partial}{\partial u}$, the characteristic equation is given by

$$\frac{\mathrm{d}x}{\lambda} = \frac{\mathrm{d}t}{2t} = \frac{\mathrm{d}u}{-\alpha u}.$$

Solving the above equation yields the corresponding similarity variables $e^x t^{-\frac{\lambda}{2}}$ and $t^{\frac{\alpha}{2}}u$; then, we obtain the similarity transformation

$$u = t^{-\frac{\alpha}{2}}g(z), \tag{3.43}$$

where $z = e^x t^{-\frac{\lambda}{2}}$ and g is the function of z to be determined later. Thus, we get

$$\begin{aligned} u_x &= t^{-\frac{\alpha}{2}}zg'(z), \\ u_{xx} &= t^{-\frac{\alpha}{2}}(zg'(z) + z^2g''(z)), \\ u_{xxx} &= t^{-\frac{\alpha}{2}}(zg'(z) + 3z^2g''(z) + z^3g'''(z)). \end{aligned} \tag{3.44}$$

In the sequel, we consider the time-fractional derivatives $\partial_t^\alpha u$ and $\partial_t^\alpha(u_{xx})$ in Eq (1.2) with $0 < \alpha < 1$, respectively. According to the definition of the Riemann-Liouville fractional derivative, one can obtain

$$\partial_t^\alpha u = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_0^t (t-s)^{-\alpha} s^{-\frac{\alpha}{2}} g(e^x s^{-\frac{\lambda}{2}})\mathrm{d}s. \tag{3.45}$$

Letting $v = \frac{t}{s}$, and in view of Eq (2.6), Eq (3.45) can be converted to the following form:

$$\begin{aligned}\partial_t^\alpha u &= \frac{\partial}{\partial t}\left[t^{1-\frac{3}{2}\alpha}\frac{1}{\Gamma(1-\alpha)}\int_1^\infty (v-1)^{-\alpha} v^{\frac{3}{2}\alpha-2} g(zv^{\frac{\lambda}{2}})\mathrm{d}v\right]\\ &= \frac{\partial}{\partial t}\left[t^{1-\frac{3}{2}\alpha}\left(\mathcal{K}_{\frac{2}{\lambda}}^{1-\frac{\alpha}{2},\,1-\alpha} g\right)(z)\right]\\ &= t^{-\frac{3}{2}\alpha}\left(1-\frac{3}{2}\alpha-\frac{\lambda}{2}z\frac{\mathrm{d}}{\mathrm{d}z}\right)\left(\mathcal{K}_{\frac{2}{\lambda}}^{1-\frac{\alpha}{2},\,1-\alpha} g\right)(z).\end{aligned} \tag{3.46}$$

Then, utilizing the E-K fractional differential operator given by Eq (2.5), we obtain

$$\partial_t^\alpha u = t^{-\frac{3}{2}\alpha}\left(\mathcal{P}_{\frac{2}{\lambda}}^{1-\frac{3}{2}\alpha,\,\alpha} g\right)(z). \tag{3.47}$$

As for $\partial_t^\alpha(u_{xx})$, similar to Eqs (3.28), (3.29) and (3.30) in the last section, one obtains

$$\begin{aligned}\partial_t^\alpha u_{xx} &= \partial_t^\alpha(t^{-\frac{\alpha}{2}} z g'(z)) + \partial_t^\alpha(t^{-\frac{\alpha}{2}} z^2 g''(z))\\ &= t^{-\frac{3}{2}\alpha} z\left(1-\frac{3}{2}\alpha-\frac{\lambda}{2}-\frac{\lambda}{2}z\frac{\mathrm{d}}{\mathrm{d}z}\right)\left(\mathcal{K}_{\frac{2}{\lambda}}^{1-\frac{\alpha}{2}-\frac{\lambda}{2},\,1-\alpha} g'\right)(z)\\ &\quad + t^{-\frac{3}{2}\alpha} z^2\left(1-\frac{3}{2}\alpha-\lambda-\frac{\lambda}{2}z\frac{\mathrm{d}}{\mathrm{d}z}\right)\left(\mathcal{K}_{\frac{2}{\lambda}}^{1-\frac{\alpha}{2}-\lambda,\,1-\alpha} g''\right)(z)\\ &= t^{-\frac{3}{2}\alpha}\left[z\left(\mathcal{P}_{\frac{2}{\lambda}}^{1-\frac{3}{2}\alpha-\frac{\lambda}{2},\,\alpha} g'\right)(z) + z^2\left(\mathcal{P}_{\frac{2}{\lambda}}^{1-\frac{3}{2}\alpha-\lambda,\,\alpha} g''\right)(z)\right].\end{aligned} \tag{3.48}$$

Therefore, inserting Eqs (3.44), (3.47) and (3.48) into Eq (1.2), we verify Eq (3.42). □

In what follows, we are ready to obtain the explicit analytic solution for Eq (3.42) by using the power-series method. We first assume that Eq (3.42) has the power-series solution

$$g(z) = \frac{1}{z}a_0 + \sum_{n=1}^{\infty} a_n z^n, \tag{3.49}$$

where a_n $(n = 1, 2, 3, \cdots)$ are constants which will be determined later.

Then, we have

$$\begin{aligned} g'(z) &= -\frac{1}{z^2}a_0 + \sum_{n=1}^{\infty} n a_n z^{n-1},\\ g''(z) &= \frac{2}{z^3}a_0 + \sum_{n=2}^{\infty}(n-1)n a_n z^{n-2},\\ g'''(z) &= -\frac{6}{z^4}a_0 + \sum_{n=3}^{\infty}(n-2)(n-1)n a_n z^{n-3}.\end{aligned} \tag{3.50}$$

Next, we calculate the E-K fractional differential operator. Analogous to Eqs (3.34), (3.35) and (3.36), one respectively obtains

$$\left(\mathcal{P}_{\frac{2}{\lambda}}^{1-\frac{3}{2}\alpha,\,\alpha} g\right)(z) = \frac{a_0}{z}\frac{\Gamma(1-\frac{\alpha}{2}+\frac{\lambda}{2})}{\Gamma(1-\frac{3}{2}\alpha+\frac{\lambda}{2})} + \sum_{n=1}^{\infty}\frac{\Gamma(1-\frac{\alpha}{2}-\frac{\lambda}{2}n)}{\Gamma(1-\frac{3}{2}\alpha-\frac{\lambda}{2}n)}a_n z^n, \tag{3.51}$$

$$\left(\mathcal{P}_{\frac{2}{\lambda}}^{1-\frac{3}{2}\alpha-\frac{\lambda}{2},\,\alpha} g'\right)(z) = -\frac{a_0}{z^2}\frac{\Gamma(1-\frac{\alpha}{2}+\frac{\lambda}{2})}{\Gamma(1-\frac{3}{2}\alpha+\frac{\lambda}{2})} + \sum_{n=1}^{\infty}\frac{\Gamma(1-\frac{\alpha}{2}-\frac{\lambda}{2}n)}{\Gamma(1-\frac{3}{2}\alpha-\frac{\lambda}{2}n)}na_n z^{n-1}, \tag{3.52}$$

and

$$\left(\mathcal{P}_{\frac{2}{\lambda}}^{1-\frac{3}{2}\alpha-\lambda,\,\alpha} g''\right)(z) = \frac{2a_0}{z^3}\frac{\Gamma(1-\frac{\alpha}{2}+\frac{\lambda}{2})}{\Gamma(1-\frac{3}{2}\alpha+\frac{\lambda}{2})} + \sum_{n=2}^{\infty}\frac{\Gamma(1-\frac{\alpha}{2}-\frac{\lambda}{2}n)}{\Gamma(1-\frac{3}{2}\alpha-\frac{\lambda}{2}n)}(n-1)na_n z^{n-2}. \tag{3.53}$$

Thus, substituting Eqs (3.50)–(3.53) into Eq (3.42), we have the following formula:

$$\begin{aligned}&\sum_{n=2}^{\infty}\frac{\Gamma(1-\frac{\alpha}{2}-\frac{\lambda}{2}n)}{\Gamma(1-\frac{3}{2}\alpha-\frac{\lambda}{2}n)}(1-n^2)a_n z^n + \left(\frac{1}{z}a_0 + \sum_{n=1}^{\infty}a_n z^n\right)^2\left(-\frac{3a_0}{z} + \sum_{n=1}^{\infty}(4-n^2)na_n z^n\right)\\ &+\left(\frac{1}{z}a_0 + \sum_{n=1}^{\infty}a_n z^n\right)\left(-\frac{1}{z}a_0 + \sum_{n=1}^{\infty}na_n z^n\right)\left(-\frac{3}{z}a_0 - \sum_{n=1}^{\infty}3n^2a_n z^n\right) = 0.\end{aligned} \tag{3.54}$$

Then, in view of the Cauchy product formula, Eq (3.54) can be converted to

$$\begin{aligned}&\sum_{n=2}^{\infty}\frac{\Gamma(1-\frac{\alpha}{2}-\frac{\lambda}{2}n)}{\Gamma(1-\frac{3}{2}\alpha-\frac{\lambda}{2}n)}(1-n^2)a_n z^n + a_0^2\sum_{n=0}^{\infty}(1-n^2)(3+n)a_{n+2}z^n\\ &+a_0\sum_{n=0}^{\infty}\sum_{m=1}^{n+1}m(2m^2-3mn-3m+4)a_m a_{n+1-m}z^n\\ &+a_0\sum_{n=1}^{\infty}\sum_{m=1}^{n}m(4-m^2)a_{m+1}a_{n-m}z^n\\ &+\sum_{n=1}^{\infty}\sum_{k=1}^{n}\sum_{m=1}^{k}m(2m^2-3mk+4)a_m a_{k-m}a_{n-k}z^n = 0.\end{aligned} \tag{3.55}$$

Hence, by vanishing the coefficients of different powers of z in Eq (3.55), we deduce that a_0 and a_3 are arbitrary constants, and that $a_2 = -a_1$. When $n \geq 2$, one obtains

$$\begin{aligned}a_{n+2} = \frac{1}{(n^2-1)(n+3)}\Bigg[&\frac{\Gamma(1-\frac{\alpha}{2}-\frac{\lambda}{2}n)}{\Gamma(1-\frac{3}{2}\alpha-\frac{\lambda}{2}n)a_0^2}(1-n^2)a_n\\ &+\frac{1}{a_0}\sum_{m=0}^{n}(m^3+3m^2+8m-3n(m+1)^2+3)a_{m+1}a_{n-m}\\ &+\frac{1}{a_0^2}\sum_{k=1}^{n}\sum_{m=1}^{k}m(2m^2-3mk+4)a_m a_{k-m}a_{n-k}\Bigg].\end{aligned} \tag{3.56}$$

Insert Eq (3.56) into Eq (3.49) to obtain the power-series solution of Eq (3.42) as follows:

$$
\begin{aligned}
g(z) =&\frac{1}{z}a_0 + a_1 z - a_1 z^2 + a_3 z^3 \\
&+ \sum_{n=2}^{\infty} \frac{1}{(n^2-1)(n+3)} \left[\frac{\Gamma(1-\frac{\alpha}{2}-\frac{\lambda}{2}n)}{\Gamma(1-\frac{3}{2}\alpha-\frac{\lambda}{2}n)a_0^2}(1-n^2)a_n \right. \\
&+ \frac{1}{a_0}\sum_{m=0}^{n}(m^3+3m^2+8m-3n(m+1)^2+3)a_{m+1}a_{n-m} \\
&\left. + \frac{1}{a_0^2}\sum_{k=1}^{n}\sum_{m=1}^{k} m(2m^2-3mk+4)a_m a_{k-m} a_{n-k} \right] z^{n+2}.
\end{aligned}
\tag{3.57}
$$

As a consequence, substituting Eq (3.57) into Eq (3.43), we can construct the explicit solution to Eq (1.2) as follows:

$$
\begin{aligned}
u(x,t) =&a_0 e^{-x} t^{\frac{\lambda-\alpha}{2}} + a_1 e^{x} t^{-\frac{\lambda+\alpha}{2}} - a_1 e^{2x} t^{-\frac{2\lambda+\alpha}{2}} + a_3 e^{3x} t^{-\frac{3\lambda+\alpha}{2}} \\
&+ \sum_{n=2}^{\infty} \frac{1}{(n^2-1)(n+3)} \left[\frac{\Gamma(1-\frac{\alpha}{2}-\frac{\lambda}{2}n)}{\Gamma(1-\frac{3}{2}\alpha-\frac{\lambda}{2}n)a_0^2}(1-n^2)a_n \right. \\
&+ \frac{1}{a_0}\sum_{m=0}^{n}(m^3+3m^2+8m-3n(m+1)^2+3)a_{m+1}a_{n-m} \\
&\left. + \frac{1}{a_0^2}\sum_{k=1}^{n}\sum_{m=1}^{k} m(2m^2-3mk+4)a_m a_{k-m} a_{n-k} \right] e^{(n+2)x} t^{-\frac{(n+2)\lambda+\alpha}{2}}.
\end{aligned}
\tag{3.58}
$$

Remark 3.1. *In accordance with the method introduced in [38] and the explicit function theorem in [39], the power-series solutions given by Eqs* (3.41) *and* (3.58) *are convergent, and we omit the details of the proof.*

4. Conservation laws for the time-fractional BBMP equation

In this section, the conservation laws for the time-fractional BBMP equation given by Eq (1.1) will be investigated by using the new conservation theorem [23, 40]. The vector (C^t, C^x) for Eq (1.1) that satisfies the equation

$$
(D_t(C^t) + D_x(C^x))|_{(1.1)} = 0 \tag{4.1}
$$

is called the conserved vector, where $C^t = C^t(t, x, u, \cdots)$, $C^x = C^x(t, x, u, \cdots)$.

The formal Lagrangian of Eq (1.1) can be represented as

$$
\mathcal{L} = v(x,t)(\partial_t^\alpha u - \partial_t^\alpha(u_{xx}) - uu_x), \tag{4.2}
$$

where $v(x,t)$ is a new dependent variable.

The adjoint equation is given by

$$
\mathcal{F}^* = \frac{\delta\mathcal{L}}{\delta u} = 0; \tag{4.3}
$$

here, $\frac{\delta}{\delta u}$ symbolizes the Euler-Lagrange operator defined by

$$\frac{\delta}{\delta u} = (\partial_t^\alpha)^* \frac{\partial}{\partial(\partial_t^\alpha u)} + (\partial_t^\alpha)^* D_x^2 \frac{\partial}{\partial(\partial_t^\alpha u_{xx})} + \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} + D_x^2 \frac{\partial}{\partial u_{xx}} + \cdots, \tag{4.4}$$

where $(\partial_t^\alpha)^*$ is the adjoint operator of ∂_t^α that is defined by Eq (2.4).

In view of Eqs (4.2) and (4.3), we can derive the adjoint equation of Eq (1.1) with the following form:

$$(\partial_t^\alpha)^* v - (\partial_t^\alpha)^* D_x^2 v - u_x v + D_x(uv) = 0. \tag{4.5}$$

Regarding the nonlinear self-adjointness of Eq (1.1), Eq (4.5) must be satisfied for all solutions of Eq (1.1) with the substitution

$$v(x,t) = \psi(x,t,u), \quad \psi(x,t,u) \neq 0. \tag{4.6}$$

The derivatives of Eq (4.6) are given as

$$\begin{aligned} v_x &= \psi_x + \psi_u u_x, \\ v_{xx} &= \psi_{xx} + 2\psi_{ux} u_x + \psi_u u_{xx} + \psi_{uu} u_x^2. \end{aligned} \tag{4.7}$$

Substituting Eqs (4.6) and (4.7) into Eq (4.5), we obtain the following nonlinear self-adjointness expression:

$$(\partial_t^\alpha)^*(\psi - \psi_{xx} - 2\psi_{ux} u_x - \psi_u u_{xx} - \psi_{uu} u_x^2) + u\psi_x + u\psi_u u_x = 0. \tag{4.8}$$

Solving the above Eq (4.8), we conclude that

$$\psi(x,t,u) = C, \tag{4.9}$$

where C is a constant. Then, taking $C = 1$, according to the vector fields described by Eq (3.10), as admitted by Eq (1.1), and the formulas of Lie characteristic functions $W_i = \eta_i - \xi_i u_x - \tau_i u_t$ $(i = 1,\ 2)$, the characteristic functions can be expressed as follows

$$W_1 = -u_x, \quad W_2 = -\alpha u - t u_t. \tag{4.10}$$

The fractional Noether operator for the variable t is defined by [21]

$$\begin{aligned} C_i^t =& I_t^{1-\alpha}(W_i) \frac{\partial \mathcal{L}}{\partial(\partial_t^\alpha u)} - I_t^{1-\alpha}(D_x W_i) D_x \frac{\partial \mathcal{L}}{\partial(\partial_t^\alpha u_{xx})} \\ &+ \mathcal{J}\left(W_i, D_t \frac{\partial \mathcal{L}}{\partial(\partial_t^\alpha u)}\right) - \mathcal{J}\left(D_x W_i, D_x D_t \frac{\partial \mathcal{L}}{\partial(\partial_t^\alpha u_{xx})}\right), \end{aligned} \tag{4.11}$$

where the operator $\mathcal{J}(f,g)$ is given by

$$\mathcal{J}(f,g) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \int_t^T \frac{f(x,s) g(x,r)}{(r-s)^\alpha} dr ds. \tag{4.12}$$

Equivalently, the fractional Noether operator for the component x of the conserved vector is introduced as follows:

$$\begin{aligned} C_i^x =& D_t^\alpha D_x(W_i)\frac{\partial \mathcal{L}}{\partial(\partial_t^\alpha u_{xx})} - W_i(\partial_t^\alpha)^* D_x\frac{\partial \mathcal{L}}{\partial(\partial_t^\alpha u_{xx})} \\ &+ W_i(\frac{\partial \mathcal{L}}{\partial u_x} - D_x\frac{\partial \mathcal{L}}{\partial u_{xx}} + D_x^2\frac{\partial \mathcal{L}}{\partial u_{xxx}}) \\ &+ D_x(W_i)(\frac{\partial \mathcal{L}}{\partial u_{xx}} - D_x\frac{\partial \mathcal{L}}{\partial u_{xxx}}) + D_x^2(W_i)\frac{\partial \mathcal{L}}{\partial u_{xxx}}. \end{aligned} \tag{4.13}$$

Therefore, with the help of Eqs (4.11) and (4.13), we construct the conservation laws for Eq (1.1), whose components of the conserved vectors are as follows:

$$\begin{cases} C_1^x = D_t^\alpha(u_{xx}) + uu_x, \\ C_2^x = \alpha D_t^\alpha(u_x) + D_t^\alpha(tu_{xt}) + u(\alpha u + tu_t), \end{cases} \tag{4.14}$$

and

$$\begin{cases} C_1^t = -I_t^{1-\alpha}(u_x), \\ C_2^t = -\alpha I_t^{1-\alpha}(u) - I_t^{1-\alpha}(tu_t). \end{cases} \tag{4.15}$$

5. Discussion on exact solutions

In this section, we provide the 3-D graphical interpretations of the exact solutions given by Eqs (3.23), (3.41) and (3.58) for the time-fractional BBMP equation and Novikov equation, respectively. Moreover, we will discuss the influence of the parameter α in Eqs (1.1) and (1.2) on solutions.

Figure 1 shows the influence of fractional parameter α on $u(x,t)$ in Eq (3.23) (i.e., a solution) of Eq (1.1). From Figure 1, we can observe that the maximum value of solution $u(x,t)$ decreases as α increases when the constant $C = 1$ and the points (x,t) are fixed. Particularly, the solution $u(x,t)$ approaches zero with the increase of the parameter α from 0.25 to 0.9 when the points (x,t) are near the point (0, 0).

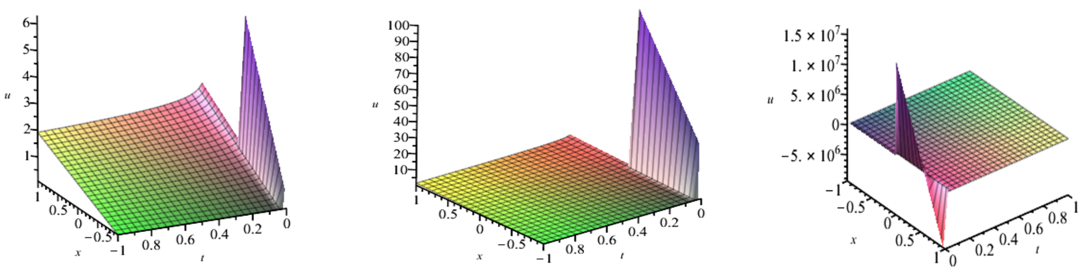

Figure 1. 3-D plots of the exact solution given by Eq (3.23) for Eq (1.1), with $C = 1$ and the parameter α as $\alpha = 0.1,\ 0.25,\ 0.9$.

Figure 2 displays the effect of fractional parameter α on $u(x,t)$ in Eq (3.23) (i.e., a solution) of Eq (1.1). The graphs show that the curvature of the graphs decreases as the fractional parameter α increases from 0.1 to 0.9.

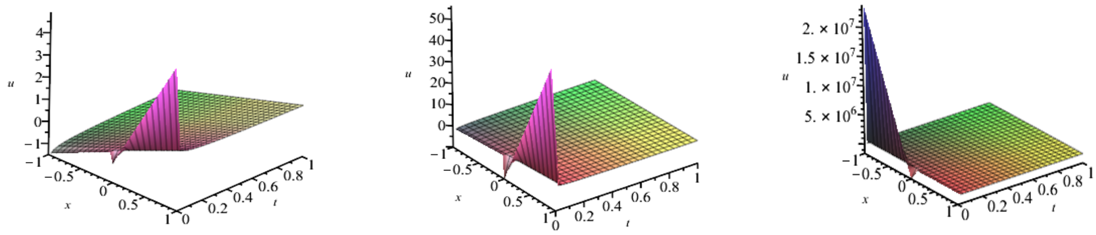

Figure 2. 3-D plots of the exact solution given by Eq (3.23) for Eq (1.1), with $C = -0.1$ and the parameter α as $\alpha = 0.1,\ 0.25,\ 0.9$.

Figure 3 shows the influence of fractional parameter α on $u(x,t)$ in Eq (3.41) (i.e., a solution) of Eq (1.1). From Figure 3, we can find that the value of the solution $u(x,t)$ increases as the fractional parameter α increases. In particular, the curvature of the plots increases as the fractional parameter α increases from 0.1 to 0.9 near the point (0, 0).

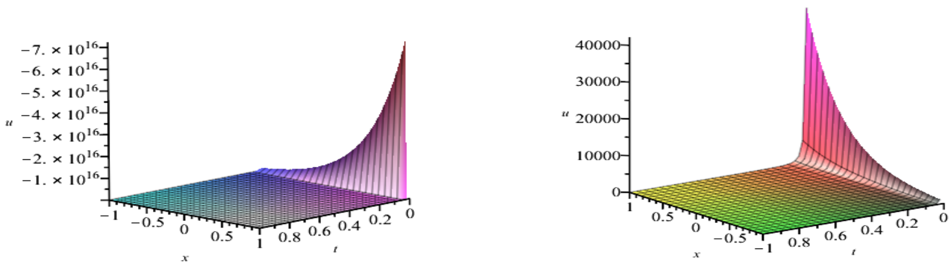

Figure 3. 3-D plots of the exact solution given by Eq (3.41) for Eq (1.1), with the constants $n = 2,\ a_0 = 0$ and $a_1 = \rho = 1$ and the parameter $\alpha = 0.1,\ 0.9$.

Figure 4 displays the effect of fractional parameter α on $u(x,t)$ in Eq (3.58) (i.e., a solution) of Eq (1.2). The plots show that the value of the solution $u(x,t)$ decreases as the fractional parameter α increases from 0.1 to 0.9.

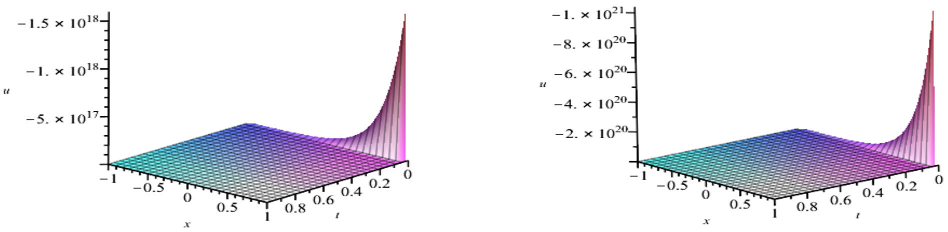

Figure 4. 3-D plots of the exact solution given by Eq (3.58) for Eq (1.2), with $n = 2,\ a_0 = a_1 = a_3 = \lambda = 1$ and $\alpha = 0.1,\ 0.9$ respectively. We notice that the plots have a mutation at $t = 0$.

6. Conclusions

In this paper, in accordance with Lie symmetry analysis and the power-series method, the explicit solutions of the time-fractional BBMP equation and time-fractional Novikov equation have been successfully constructed. More specifically, we have established the Lie algebra admitted by Eqs (1.1) and (1.2) and obtained the group-invariant solutions by means of symmetry reductions. By using the fractional E-K operator, we have transformed Eqs (1.1) and (1.2) into ODEs (3.25) and (3.42) with only one variable, respectively, and constructed their power-series solutions. Moreover, the conservation laws for the time-fractional BBMP equation have been obtained by using the new Noether's theorem. In addition, we have constructed the 3-D graphs of corresponding exact solutions by using Maple software. In the future, we will devote ourselves to constructing exact solutions of Eqs (1.1) and (1.2) with the help of other methods, such as the q-homotopy analysis method, invariant subspace method, extended direct algebraic method, Chebyshev series method, etc.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflict of interest.

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