Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(12): 30023–30037.
DOI: 10.3934/math.20231535
Received: 14 July 2023
Revised: 21 October 2023
Accepted: 31 October 2023
Published: 06 November 2023

Research article

On the solvability of the indefinite Hamburger moment problem

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Abstract: In this paper, we present a new approach for the solvability of the indefinite Hamburger moment problem in the class of generalized Nevanlinna functions with a given negative index, which is more algebraic and completely different from the existing method [8] based on the step-by-step Schur algorithm. As a by-product of this approach, we simultaneously obtain a concrete rational solution of such an indefinite Hamburger moment problem when the solvability conditions are met. The basic strategy focuses on the structural characteristics of the Hankel matrix and the relation among the Hankel, Loewner, Bezout and some other structured matrices.

Keywords: generalized Nevanlinna function; infinite Hamburger moment problem; Hankel matrix; characteristic degree; characteristic polynomial quadruple; quasidirect decomposition; McMillan degree
Mathematics Subject Classification: 30E05, 15B57, 46C20

1. Introduction

Throughout this paper, we write $\mathbb{C}^{p\times q}$ ($\mathbb{R}^{p\times q}$, resp.) for the set of all $p\times q$ complex (real, resp.) matrices. We use the symbol $\mathbb{C}^+$ to stand for the open upper half complex plane. For a Hermitian matrix $A = A^* \in \mathbb{C}^{p\times p}$, the number of negative eigenvalues (counting algebraic multiplicities) of A is denoted by $\nu(A)$. For convenience, the symbol $\mathbb{R}_n[z]$ ($\mathbb{R}_n^0[z]$, resp.) represents the set of all real coefficient polynomials of degree at most n (real coefficient polynomials of degree n, resp.) of the variable z. For a rational function $r(z)$ of the form $r(z) = p(z)/q(z)$, in which $p(z), q(z)$ are nonzero complex polynomials such that $\gcd(p(z), q(z)) = 1$, the McMillan degree of $r(z)$ is defined by $\deg r(z) = \max\{\deg p(z), \deg q(z)\}$.

Let $f(z)$ be a function meromorphic on $\mathbb{C}\setminus\mathbb{R}$. The domain of $f(z)$ is denoted by $D(f)$. For a nonnegative integer κ, $f(z)$ is called a generalized Nevanlinna function with negative index κ if the following statements hold: (i) $f(z)$ satisfies the symmetry condition $f(\bar{z}) = \overline{f(z)}$; (ii) for each choice of a positive integer m and m distinct points $z_1, \ldots, z_m \in \mathbb{C}^+ \cap D(f)$, we have

$$\nu(P_f(z_1, \ldots, z_m)) \leq \kappa, \tag{1.1}$$

and, for some particular choice, the equality in (1.1) holds, where

$$P_f(z_1, \ldots, z_m) = \left(\frac{f(z_i) - \overline{f(z_j)}}{z_i - \overline{z}_j} \right)_{i,j=1}^{m}$$

is a Hermitian Loewner matrix. We denote by $\mathcal{N}_\kappa$ the class of all generalized Nevanlinna functions with negative index κ.

The indefinite Hamburger moment problem in the class $\mathcal{N}_\kappa$ (short for the HM($\mathcal{N}_\kappa$) problem) can be formulated in the following manner: Given a sequence of real numbers $s_0, \cdots, s_{2n-2}$, it is required to find all functions $f(z) \in \mathcal{N}_\kappa$ such that the following asymptotic expansion at infinity

$$f(z) = -\frac{s_0}{z} - \frac{s_1}{z^2} - \cdots - \frac{s_{2n-2}}{z^{2n-1}} + \mathrm{o}(z^{-2n+1}) \tag{1.2}$$

holds when z tends to ∞ in the sector $\pi_\epsilon(0) = \{z \in \mathbb{C} \mid \epsilon \leq \arg z \leq \pi - \epsilon\}$ $(0 < \epsilon < \pi/2)$.

The classical Hamburger moment problem (e.g., [1, 15]) is identical to the HM($\mathcal{N}_\kappa$) problem with $\kappa = 0$. In comparison with the classical case, the HM($\mathcal{N}_\kappa$) problem with $\kappa > 0$ seems to be much more complicated. In 2003, Derevyagin [7] applied the step-by-step Schur algorithm to give a description of the solutions of the HM($\mathcal{N}_\kappa$) problem when the Hankel matrix

$$\mathbf{H} = (s_{i+j})_{i,j=0}^{n-1} \tag{1.3}$$

determined by the asymptotic expansion (1.2) is nonsingular. In 2012, Derkach et al. [8] gave the solvability criterion of the HM($\mathcal{N}_\kappa$) problem and a complete parametrization description of the solutions by using the same algorithm. In this paper, we derive the solvability criterion for the HM($\mathcal{N}_\kappa$) problem by a new approach, which is more algebraic and different from the existing methods. As a by-product of this approach, we obtain a concrete rational solution of the HM($\mathcal{N}_\kappa$) problem with the least McMillan degree when the solvability conditions are met.

We remark that, starting from the Hankel matrix $\mathbf{H}$ given by (1.3), we can derive the solvability criterion of the HM($\mathcal{N}_\kappa$) problem and the concrete formula of the solutions when the solvability conditions are met. For this reason, we say $\mathbf{H}$ is the Hankel matrix of the HM($\mathcal{N}_\kappa$) problem. Moreover, the HM($\mathcal{N}_\kappa$) problem is said to be non-degenerate (degenerate, resp.) if its Hankel matrix is nonsingular (singular, resp.). In this paper, we divide the HM($\mathcal{N}_\kappa$) problem into the non-degenerate case and the degenerate case to derive the solvability criterion by using the structural characteristics of the Hankel matrix $\mathbf{H}$, such as the characteristic degrees and characteristic polynomial quadruple (see, e.g., [4, 5]), the quasidirect decomposition (see, e.g., [9]), and the relation among the Hankel, Loewner, Bezout and some other structured matrices (see, e.g., [4, 6, 11]).

A brief synopsis of this paper is as follows. In Section 2, we introduce the characteristic degrees, characteristic polynomial quadruple and quasidirect decomposition of the Hankel matrix of the HM($\mathcal{N}_\kappa$) problem, and we list some basic results about these structural characteristics without proofs. In Section 3, we first recall some known properties of the generalized Nevanlinna functions in [17] given by the first two authors of this paper and their collaborators, and then, we prove several new properties by using the structural characteristics of the Hankel matrix and the relation among the Hankel, Loewner, Bezout and some other structured matrices. The last section is devoted to the solvability criterion of the HM($\mathcal{N}_\kappa$) problem and a concrete rational solution with the least McMillan degree when the solvability conditions are met.

2. Structural characteristics of the Hankel matrix of the HM($\mathcal{N}_\kappa$) problem

Let $\mathbf{H} = (s_{i+j})_{i,j=0}^{n-1} \in \mathbb{R}^{n\times n}$ be the Hankel matrix of the HM($\mathcal{N}_\kappa$) problem. The first and the second characteristic degrees of $\mathbf{H}$ are defined by $n_1 = \operatorname{rank}\mathbf{H}$, $n_2 = 2n - n_1$, respectively. Clearly, $n_1 \le n \le n_2$. For a pair of positive integers k, l such that $k + l = 2n$, we write $\mathbf{H}_{kl} = (s_{i+j})_{i,j=0}^{k-1,l-1}$. In particular, $\mathbf{H} = \mathbf{H}_{nn}$. It follows from [13] that $\operatorname{rank}\mathbf{H}_{kl} = \min\{k, l, n_1\}$. We use the symbol $\mathcal{A}_l$ to stand for the subspace of $\mathbb{R}_{l-1}[z]$:

$$\mathcal{A}_l = (1, z, \cdots, z^{l-1})\operatorname{Ker}\mathbf{H}_{kl}, \quad k + l = 2n.$$

In the case $n_1 = n_2 = n$, $\mathbf{H}$ is nonsingular, and there exists a unique monic polynomial $p(z) \in \mathbb{R}_{n-1}[z]$ and a polynomial $q(z) \in \mathbb{R}_n^0[z]$, forming a basis of the space $\mathcal{A}_{n+1}$. In the case $n_1 < n < n_2$, $\mathbf{H}$ is singular and there exists uniquely a monic polynomial $p(z) \in \mathbb{R}_{n_1}[z]$, forming a basis of $\mathcal{A}_{n_1+1}$ and, moreover, a polynomial $q(z) \in \mathbb{R}_{n_2}[z]$ such that $p(z), zp(z), \ldots, z^{n_2-n_1}p(z), q(z)$ forms a basis of the space $\mathcal{A}_{n_2+1}$. For convenience, in this paper we always assume that either $\deg p(z) = n_1$ and $\deg q(z) < n_2$ or $\deg p(z) < n_1$ and $\deg q(z) = n_2$. Such a pair of polynomials $p(z)$ and $q(z)$ are referred to as the first and the second characteristic polynomials of $\mathbf{H}$, respectively.

Let $p(z) = p_{n_1}z^{n_1} + \cdots + p_0$ and $q(z) = q_{n_2}z^{n_2} + \cdots + q_0$. We define two real coefficient polynomials $\gamma(z)$ and $\delta(z)$ by

$$\gamma(z) = (1, z, \cdots, z^{n_1-1})\begin{pmatrix} p_1 & \cdots & p_{n_1} \\ \vdots & \iddots & \\ p_{n_1} & & \end{pmatrix}\begin{pmatrix} s_0 \\ \vdots \\ s_{n_1-1}\end{pmatrix} \in \mathbb{R}_{n_1-1}[z],$$

$$\delta(z) = (1, z, \cdots, z^{n_2-1})\begin{pmatrix} q_1 & \cdots & q_{n_2} \\ \vdots & \iddots & \\ q_{n_2} & & \end{pmatrix}\begin{pmatrix} s_0 \\ \vdots \\ s_{n_2-1}\end{pmatrix} \in \mathbb{R}_{n_2-1}[z].$$

Hereafter, $[p(z), q(z), \gamma(z), \delta(z)]$ is called the characteristic polynomial quadruple of $\mathbf{H}$. Such a quadruple, together with the first and second characteristic degrees, plays an important role in our discussion. By using the definitions, we check easily that the following asymptotic expansions at infinity hold:

$$p(z)\Big(\frac{s_0}{z} + \frac{s_1}{z^2} + \cdots + \frac{s_{2n-2}}{z^{2n-1}} + o(z^{-2n+1})\Big) = \gamma(z) + o(z^{-n_2+1}) \quad (z \to \infty),$$

$$q(z)\Big(\frac{s_0}{z} + \frac{s_1}{z^2} + \cdots + \frac{s_{2n-2}}{z^{2n-1}} + o(z^{-2n+1})\Big) = \delta(z) + o(z^{-n_1+1}) \quad (z \to \infty).$$

The following lemma comes as a direct consequence of the definitions and the last two equations (e.g., [4]).

Lemma 2.1. *Let n_1 and $[p(z), q(z), \gamma(z), \delta(z)]$ be the first characteristic degree and the characteristic polynomial quadruple of the Hankel matrix $\mathbf{H} = (s_{i+j})_{i,j=0}^{n-1}$ given by* (1.3)*, respectively. Then,*

$$\delta(z)p(z) - \gamma(z)q(z) = \sigma, \tag{2.1}$$

where σ is a nonzero constant. Moreover, if $\deg p(z) = n_1$*, then $f_{\gamma,p}(z) = -\gamma(z)/p(z)$ admits the following asymptotic expansion at infinity:*

$$f_{\gamma,p}(z) = -\frac{s_0}{z} - \frac{s_1}{z^2} - \cdots - \frac{s_{2n-2}}{z^{2n-1}} + \mathrm{o}(z^{-2n+1}) \quad (z \to \infty); \tag{2.2}$$

if $\deg p(z) < n_1$ *then $f_{\delta,q}(z) = -\delta(z)/q(z)$ admits the following asymptotic expansion at infinity:*

$$f_{\delta,q}(z) = -\frac{s_0}{z} - \frac{s_1}{z^2} - \cdots - \frac{s_{2n-2}}{z^{2n-1}} + \mathrm{o}(z^{-2n+1}) \quad (z \to \infty). \tag{2.3}$$

Remark 2.2. *By* (2.1) *and the fact that* $\deg p(z) + \deg q(z) \leq 2n - 1$*, we can show that the asymptotic expansions* (2.2) *and* (2.3) *cannot hold simultaneously. This means that if the asymptotic expansion* (2.2) *(*(2.3)*, resp.) holds, then* $\deg p(z) = n_1$ *(*$\deg p(z) < n_1$*, resp.).*

Now, we introduce the quasidirect decomposition of a singular Hankel matrix. Let $\mathbf{H} = (s_{i+j})_{i,j=0}^{n-1}$ given by (1.3) be singular. We say $\mathbf{H}$ is a proper Hankel matrix if $\Delta_{n_1} \neq 0$ and $\Delta_k = 0$ $(k = n_1 + 1, \cdots, n)$, in which Δ_i stands for the ith leading principle minor of $\mathbf{H}$. Moreover, we say that $\mathbf{H}$ is a degenerate Hankel matrix if $s_0 = s_1 = \cdots = s_{n-1} = 0$. By definitions, a $n \times n$ zero matrix is both a proper Hankel matrix and a degenerate Hankel matrix. In [9], Fielder showed that under certain conditions, each singular Hankel matrix can be uniquely decomposed into the sum of a proper Hankel matrix and a degenerate Hankel matrix.

Lemma 2.3. *Let $\mathbf{H} = (s_{i+j})_{i,j=0}^{n-1}$ given by* (1.3) *be singular. Then, $\mathbf{H}$ has a unique decomposition of the form:*

$$\mathbf{H} = \mathbf{H}^{\mathrm{p}} + \mathbf{H}^{\mathrm{d}}, \quad \mathrm{rank}(\mathbf{H}) = \mathrm{rank}(\mathbf{H}^{\mathrm{p}}) + \mathrm{rank}(\mathbf{H}^{\mathrm{d}}), \tag{2.4}$$

in which $\mathbf{H}^{\mathrm{p}}$ is a proper Hankel matrix and $\mathbf{H}^{\mathrm{d}}$ is a degenerate Hankel matrix.

The formula (2.4) is called the quasidirect decomposition of the Hankel matrix $\mathbf{H}$. By definition, together with Lemmas 2.1, 2.3 and Remark 2.2, we can give a characterization of the proper Hankel matrix by using the structural characteristics.

Lemma 2.4. *Let $\mathbf{H} = (s_{i+j})_{i,j=0}^{n-1}$ given by* (1.3) *be singular, and let n_1 and $[p(z), q(z), \gamma(z), \delta(z)]$ be the first characteristic degree and the characteristic polynomial quadruple of $\mathbf{H}$, respectively. Then, $\mathbf{H}$ is a proper Hankel matrix if, and only if, one of the following statements holds:*

(1) $\deg p(z) = n_1$*;*

(2) $f_{\gamma,p}(z) = -\gamma(z)/p(z)$ *admits the asymptotic expansion* (2.2) *at infinity.*

The following lemma shows that if the Hankel matrix of the HM($\mathcal{N}_\kappa$) problem is singular, then it is equivalent and congruent to a block diagonal matrix, where the upper left corner and the lower right corner blocks are Hankel matrices whose orders coincide with the ranks of its proper part and degenerate part, respectively, and the other blocks are zero matrices. Such a structural characteristic of the singular Hankel matrix plays an important role in deducing the solvability criterion of the HM($\mathcal{N}_\kappa$) problem for the degenerate case.

Lemma 2.5. *Let* $\mathbf{H} = (s_{i+j})_{i,j=0}^{n-1}$ *be a singular Hankel matrix with the quasidirect decomposition* (2.4)*, let* n_1 *and* $p(z) = z^r + p_{r-1}z^{r-1} + \cdots + p_1 z + p_0$ *be the first characteristic degree and the first characteristic polynomial of* $\mathbf{H}$*, respectively and let*

$$Q = \begin{pmatrix} & & I_r & & & \\ & & & & & \\ p_0 & \cdots & p_{r-1} & 1 & & \\ & \ddots & & \ddots & \ddots & \\ & & p_0 & \cdots & p_{r-1} & 1 \end{pmatrix} \in \mathbb{R}^{n\times n}. \tag{2.5}$$

Then, Q is a nonsingular matrix satisfying

$$\begin{gathered} Q\mathbf{H}^{\mathrm{p}}Q^* = \operatorname{diag}(\widehat{\mathbf{H}}^{\mathrm{p}}, 0_{n-r}), \quad Q\mathbf{H}^{\mathrm{d}}Q^* = \operatorname{diag}(0_{n-n_1+r}, \widehat{\mathbf{H}}^{\mathrm{d}}), \\ Q\mathbf{H}Q^* = \operatorname{diag}(\widehat{\mathbf{H}}^{\mathrm{p}}, 0_{n-n_1}, \widehat{\mathbf{H}}^{\mathrm{d}}), \end{gathered} \tag{2.6}$$

in which

$$\widehat{\mathbf{H}}^{\mathrm{p}} = (s_{i+j})_{i,j=0}^{r-1}, \quad \widehat{\mathbf{H}}^{\mathrm{d}} = \begin{pmatrix} 0 & & \widehat{s}_1 \\ & \iddots & \vdots \\ \widehat{s}_1 & \cdots & \widehat{s}_{n_1-r} \end{pmatrix} \tag{2.7}$$

are nonsingular Hankel matrices of small sizes.

Hereafter, we always assume that the Hankel matrix $\widehat{\mathbf{H}}^{\mathrm{p}}$ ($\widehat{\mathbf{H}}^{\mathrm{d}}$, resp.) does not appear in the case $r = 0$ ($r = n_1$, resp.).

3. Some properties of the generalized Nevanlinna functions

In [17], Song et al. presented an equivalent definition of the generalized Nevanlinna function with negative index κ in terms of the generalized Loewner matrix. On the basis of this definition, they have shown several properties of such a kind of function. Here, we list two interesting properties of them. The first property is stated as follows, which gives a necessary condition for the HM($\mathcal{N}_\kappa$) problem to have a solution.

Lemma 3.1. *[17] Let* $\mathbf{H} = (s_{i+j})_{i,j=0}^{n-1}$ *be the Hankel matrix of the HM($\mathcal{N}_\kappa$) problem. If $f(z) \in \mathcal{N}_\kappa$ admits the asymptotic expansion* (1.2) *at infinity, then $\kappa \geq \nu(\mathbf{H})$.*

The second property is actually a generalization of the first one. Starting from this property, we can derive the solvability criterion of the HM($\mathcal{N}_\kappa$) problem in the degenerate case.

Lemma 3.2. *[17] If $f(z) \in \mathcal{N}_\kappa$ admits the asymptotic expansion* (1.2) *at infinity, then for each positive integer m and m distinct points $z_1, \ldots, z_m \in \mathbb{C}^+ \cap D(f)$,*

$$\nu(L_f(z_1, \cdots, z_m)) \leq \kappa,$$

in which

$$L_f(z_1, \cdots, z_m) = \left(\begin{array}{c|ccc} \mathbf{H} & C(z_1)^* & \cdots & C(z_m)^* \\ \hline C(z_1) & & & \\ \vdots & & P_f(z_1, \cdots, z_m) & \\ C(z_m) & & & \end{array}\right), \tag{3.1}$$

$\mathbf{H} = (s_{i+j})_{i,j=0}^{n-1}$ *is the Hankel matrix of the HM($\mathcal{N}_\kappa$) problem and*

$$C(z) = (f(z), z(f(z) + \frac{s_0}{z}), \cdots, z^{n-1}(f(z) + \frac{s_0}{z} + \frac{s_1}{z^2} + \cdots + \frac{s_{n-2}}{z^{n-1}})) \tag{3.2}$$

for arbitrary $z \in \mathbb{C}^+ \cap D(f)$.

We observe that $\mathbf{H}$ is a principle submatrix of $L_f(z_1, \cdots, z_m)$. By the interlacing relation between the eigenvalues of a Hermitian matrix and its principle submatrices (e.g., [14, Theorem 4.3.28]), we have $\nu(\mathbf{H}) \le \nu(L_f(z_1, \cdots, z_m))$, then Lemma 3.1 is a direct consequence of Lemma 3.2. If the Hankel matrix $\mathbf{H}$ of the HM($\mathcal{N}_\kappa$) problem is singular, by Lemma 2.3 it has a unique quasidirect decomposition. In that case, applying Lemmas 2.5 and 3.2, we can prove the following property of the functions in the class $\mathcal{N}_\kappa$.

Theorem 3.3. *Let* $\mathbf{H} = (s_{i+j})_{i,j=0}^{n-1}$ *given by* (1.3) *be singular,* n_1 *and* $[p(z), q(z), \gamma(z), \delta(z)]$ *be the first characteristic degree and the characteristic polynomial quadruple of* $\mathbf{H}$*, respectively and let* $f(z) \in \mathcal{N}_\kappa$ *admit the asymptotic expansion* (1.2) *at infinity. Then, for each positive integer m and m distinct points* $z_1, \ldots, z_m \in \mathbb{C}^+ \cap D(f)$*, the structured matrix* $L_f(z_1, \cdots, z_m)$ *defined by* (3.1) *and* (3.2) *is equivalent and congruent to*

$$\widetilde{L}_f(z_1, \cdots, z_m) = \left(\begin{array}{c|c|c} \widehat{\mathbf{H}}^{\mathrm{p}} & & E^* \\ \hline & \begin{matrix} 0 & \\ & \widehat{\mathbf{H}}^{\mathrm{d}} \end{matrix} & F^* \\ \hline E & F & P_f(z_1, \cdots, z_m) \end{array}\right), \tag{3.3}$$

where $\widehat{\mathbf{H}}^{\mathrm{p}}$, $\widehat{\mathbf{H}}^{\mathrm{d}}$ *are the same as in* (2.7)*, and*

$$\begin{aligned} E &= \begin{pmatrix} f(z_1) & z_1(f(z_1) + \frac{s_0}{z_1}) & \cdots & z_1^{r-1}(f(z_1) + \frac{s_0}{z_1} + \cdots + \frac{s_{r-2}}{z_1^{r-1}}) \\ \vdots & \vdots & & \vdots \\ f(z_m) & z_m(f(z_m) + \frac{s_0}{z_m}) & \cdots & z_m^{s-1}(f(z_m) + \frac{s_0}{z_m} + \cdots + \frac{s_{r-2}}{z_m^{r-1}}) \end{pmatrix} \in \mathbb{C}^{m\times r}, \\ F &= \begin{pmatrix} p(z_1)f(z_1) + \gamma(z_1) & \cdots & z_1^{n-r-1}(p(z_1)f(z_1) + \gamma(z_1)) \\ \vdots & & \vdots \\ p(z_m)f(z_m) + \gamma(z_m) & \cdots & z_m^{n-r-1}(p(z_m)f(z_m) + \gamma(z_m)) \end{pmatrix} \in \mathbb{C}^{m\times(n-r)}. \end{aligned} \tag{3.4}$$

Proof. Here, we give a proof only for the case $m = 1$. The proof of the case $m > 1$ is completely analogous and, thus, omitted. Let $p(z) = z^r + p_{r-1}z^{r-1} + \cdots + p_1 z + p_0$ and $\widetilde{Q} = \mathrm{diag}(Q, 1)$, in which Q is given by (2.5). Clearly, $\widetilde{Q}$ is also a nonsingular matrix. It follows from Lemma 2.5 and (2.6) that $\widetilde{L}_f(z_1, \cdots, z_m) = \widetilde{Q}L_f(z_1, \cdots, z_m)\widetilde{Q}^*$ is of the form (3.3), in which

$$(E, F) = (f(z_1), z_1(f(z_1) + \frac{s_0}{z_1}), \cdots, z_1^{n-1}(f(z_1) + \frac{s_0}{z_1} + \cdots + \frac{s_{n-2}}{z_1^{n-1}}))Q^*.$$

By a direct calculation, we have

$$E = (f(z_1), z_1(f(z_1) + \frac{s_0}{z_1}), \cdots, z_1^{r-1}(f(z_1) + \frac{s_0}{z_1} + \cdots + \frac{s_{r-2}}{z_1^{r-1}})),$$

and the $(k+1)$-th element of F is

$$\begin{aligned}
& p_0 z_1^k (f(z_1) + \frac{s_0}{z_1} + \cdots + \frac{s_{k-1}}{z_1^k}) + \cdots + p_{r-1} z_1^{r+k-1} (f(z_1) + \frac{s_0}{z_1} + \cdots + \frac{s_{r+k-2}}{z_1^{r+k-1}}) \\
& + z_1^{r+k} \Big(f(z_1) + \frac{s_0}{z_1} + \cdots + \frac{s_{r+k-1}}{z_1^{r+k}}) \\
= \;& z_1^k \left(p(z_1) f(z_1) + \gamma(z_1) \right), \quad k = 0, 1, \cdots, n-r-1,
\end{aligned}$$

then (3.4) holds. Therefore, we complete the proof of Theorem 3.3. □

To derive the solvability criterion of the HM($\mathcal{N}_\kappa$) problem, we need some properties of the rational generalized Nevanlinna function. To introduce these properties, we recall the concept of the Bezout matrix (see, e.g., [10,16]). For a pair of complex polynomials $a(z), b(z)$ with the maximal degree n, the Bezout matrix $\mathbf{B}(a,b)$ is defined by the bilinear form

$$\frac{a(z)b(w) - a(w)b(z)}{z - w} = (1, z, \cdots, z^{n-1}) \mathbf{B}(a,b) (1, w, \cdots, w^{n-1})^{\mathrm{T}}.$$

It is well known that the Bezout matrix has many applications in the theory of system and control (e.g., [2,3,12]). The following result shows that each real rational function is a generalized Nevanlinna function, whose negative index coincides with the number of negative eigenvalues of the Bezout matrix of its denominator and numerator polynomials.

Theorem 3.4. *Let $a(z) \in \mathbb{R}_n^0[z]$, $b(z) \in \mathbb{R}_n[z]$. Then, the real rational function $f_{b,a}(z) = -b(z)/a(z) \in \mathcal{N}_\kappa$, in which $\kappa = \nu(\mathbf{B}(a,b))$.*

Proof. Since $a(z), b(z)$ are real coefficient polynomials, $f_{b,a}(z)$ is meromorphic in $\mathbb{C} \setminus \mathbb{R}$ such that for each $z \in D(f_{b,a})$, $\overline{z} \in D(f_{b,a})$ and $f_{b,a}(\overline{z}) = \overline{f_{b,a}(z)}$. Moreover, for each choice of a positive integer m and m distinct points $z_1, \cdots, z_m \in \mathbb{C}^+ \cap D(f_{b,a})$, we have

$$\begin{aligned}
P_{f_{b,a}}(z_1, \ldots, z_m) &= \left(\frac{f_{b,a}(z_i) - f_{b,a}(\overline{z}_j)}{z_i - \overline{z}_j} \right)_{i,j=1}^m = \Lambda \left(\frac{a(z_i) b(\overline{z}_j) - b(z_i) a(\overline{z}_j)}{z_i - \overline{z}_j} \right)_{i,j=1}^m \Lambda^* \\
&= \Lambda V \mathbf{B}(a,b) V^* \Lambda^*,
\end{aligned}$$

where $\Lambda = \mathrm{diag}(a(z_1)^{-1}, \cdots, a(z_m)^{-1})$ is a nonsingular diagonal matrix and $V = (z_i^{j-1})_{i,j=1}^{m,n}$ is a Vandermonde matrix. This implies that $\nu(P_{f_{b,a}}(z_1, \cdots, z_m)) \le \nu(\mathbf{B}(a,b))$. Particularly, in the case of $m = n$, V is also a nonsingular matrix and, thus, $\nu(P_{f_{b,a}}(z_1, \cdots, z_n)) = \nu(\mathbf{B}(a,b))$. By the definition of generalized Nevanlinna functions in the class $\mathcal{N}_\kappa$, we have $f_{b,a}(z) = -b(z)/a(z) \in \mathcal{N}_\kappa$, in which $\kappa = \nu(\mathbf{B}(a,b))$, then the proof of Theorem 3.4 is completed. □

We remark that there are many interesting connections between Bezout and Hankel matrices. The following lemma shows that the Bezout matrix $\mathbf{B}(a,b)$ in Theorem 3.4 is equivalent and congruent to a real Hankel matrix generated by the rational function $b(z)/a(z)$ (see, e.g., [6,11] for the general case).

Lemma 3.5. *Let $a(z) = \sum_{i=0}^{n} a_i z^i$ $(a_n \neq 0)$ and $b(z) = \sum_{i=0}^{n-1} b_i z^i$ be real coefficient polynomials, and let the asymptotic expansion of $b(z)/a(z)$ at infinity be of the form*

$$\frac{b(z)}{a(z)} = \frac{h_0}{z} + \frac{h_1}{z^2} + \cdots + \frac{h_{2n-2}}{z^{2n-1}} + o(z^{-2n+1}) \quad (z \to \infty).$$

Then, $\mathbf{B}(a,b) = S(a)\mathbf{H}(a,b)S(a)$*, in which* $\mathbf{H}(a,b) = (h_{i+j})_{i,j=0}^{n-1}$ *and*

$$S(a) = \begin{pmatrix} a_1 & \cdots & a_n \\ \vdots & \iddots & \\ a_n & & \end{pmatrix}$$

is a nonsingular real symmetric matrix.

By Lemma 3.5, the negative index κ in Theorem 3.4 can be formulated in terms of the number of negative eigenvalues of the Hankel matrix $\mathbf{H}(a,b) = (h_{i+j})_{i,j=0}^{n-1}$.

Corollary 3.6. *Let* $a(z), b(z)$ *and* $\mathbf{H}(a,b)$ *be the same as in Lemma 3.5. Then, the real rational function* $f_{b,a}(z) = -b(z)/a(z) \in \mathcal{N}_\kappa$*, in which* $\kappa = \nu(\mathbf{H}(a,b))$*.*

4. Solvability criterion for the HM($\mathcal{N}_\kappa$) problem

In this section, we apply the structural characteristics of the Hankel matrix of the HM($\mathcal{N}_\kappa$) problem and the properties of the generalized Nevanlinna functions to deduce the solvability criterion of the HM($\mathcal{N}_\kappa$) problem and a concrete rational solution with the least McMillan degree for both the non-degenerate and degenerate cases. We first derive the solvability criterion for the non-degenerate HM($\mathcal{N}_\kappa$) problem.

Theorem 4.1. *Let* $\mathbf{H} = (s_{i+j})_{i,j=0}^{n-1}$ *given by* (1.3) *be nonsingular. Then, the HM(*$\mathcal{N}_\kappa$*) problem is solvable if, and only if,* $\kappa \geq \nu(\mathbf{H})$*.*

Proof. The "only if" part is a direct consequence of Lemma 3.1. For the proof of the "if" part, we consider two cases.

Case I: $\kappa = \nu(\mathbf{H})$. Let n_1, n_2 and $[p(z), q(z), \gamma(z), \delta(z)]$ be the first characteristic degree, the second characteristic degree and the characteristic polynomial quadruple of $\mathbf{H}$, respectively. Since $\mathbf{H}$ is nonsingular, we have $n_1 = n_2 = n$, $\deg p(z) < n_1 = n$ and $\deg q(z) = n$. By Lemma 2.1, the rational function $f_{\delta,q}(z) = -\delta(z)/q(z)$ admits the asymptotic expansion (1.2) at infinity, and moreover, $\mathbf{H} = \mathbf{H}(q,\delta)$. On the other hand, by Corollary 3.6, we obtain that $f_{\delta,q}(z) \in \mathcal{N}_{\kappa'}$, in which $\kappa' = \nu(\mathbf{H}(q,\delta)) = \nu(\mathbf{H}) = \kappa$. Then, $f_{\delta,q}(z)$ is a solution of the non-degenerate HM($\mathcal{N}_\kappa$) problem.

Case II: $\kappa > \nu(\mathbf{H})$. For convenience, we denote by $\mathbf{H}_n = \mathbf{H}$ and $m = \kappa - \nu(\mathbf{H}) > 0$. Define a sequence of Hankel matrices recursively by

$$\mathbf{H}_{n+k} = (s_{i+j})_{i,j=0}^{n+k-1}, \tag{4.1}$$

in which

$$s_{2n+2k-3} = 0, \quad s_{2n+2k-2} = (s_{n+k}, \cdots, s_{2n+2k-3})\mathbf{H}_{n+k-1}^{-1}(s_{n+k}, \cdots, s_{2n+2k-3})^{\mathrm{T}} - 1, \quad k = 1, 2, \cdots, m. \tag{4.2}$$

Then, $\mathbf{H}_{n+m} = (s_{i+j})_{i,j=0}^{n+m-1}$ is equivalent and congruent to $\mathrm{diag}(\mathbf{H}, -I_m)$, which implies that $\mathbf{H}_{n+m}$ is a nonsingular Hankel matrix and $\nu(\mathbf{H}_{n+m}) = \nu(\mathbf{H}) + m = \kappa$. Let $[u(z), v(z), \alpha(z), \beta(z)]$ be the characteristic polynomial quadruple of $\mathbf{H}_{n+m}$. Then $\deg u(z) < n + m$ and $\deg v(z) = n + m$. According to the analysis in Case I, we have that $f_{\beta,v}(z) \in \mathcal{N}_\kappa$ and the asymptotic expansion of $f_{\beta,v}(z)$ at infinity is of the form:

$$f_{\beta,v}(z) = -\frac{s_0}{z} - \frac{s_1}{z^2} - \cdots - \frac{s_{2n+2m-2}}{z^{2n+2m-1}} + \mathrm{o}(z^{-2n-2m+1}) \quad (z \to \infty),$$

then $f_{\beta,\nu}(z)$ is a solution of the HM($\mathcal{N}_\kappa$) problem. Summarizing the analysis above, we complete the proof of the "if" part. $\square$

From the proof of Theorem 4.1, we obtain immediately a concrete rational solution of the non-degenerate HM($\mathcal{N}_\kappa$) problem with the least McMillan degree when the solvability conditions are met.

Theorem 4.2. *Let* $\mathbf{H} = (s_{i+j})_{i,j=0}^{n-1}$ *given by* (1.3) *be nonsingular. If* $\kappa \geq \nu(\mathbf{H})$*, then* $r(z) = -\beta(z)/\nu(z)$ *is a rational solution with the least McMillan degree among all rational solutions of the HM(*$\mathcal{N}_\kappa$*) problem, in which* $[u(z), \nu(z), \alpha(z), \beta(z)]$ *is the characteristic quadruple of the Hankel matrix* $\mathbf{H}_{n+\kappa-\nu(\mathbf{H})}$ *defined recursively by* (4.1)–(4.2).

Proof. Let $m = \kappa - \nu(\mathbf{H}) \geq 0$. We can easily see from the proof of Theorem 4.1 that $r(z)$ presented in Theorem 4.2 is a rational solution of the HM($\mathcal{N}_\kappa$) problem with McMillan degree $n + m$. It remains to prove that the HM($\mathcal{N}_\kappa$) problem has not any rational solution whose McMillan degree is less than $n + m$. If there exists a rational solution $s(z)$ of the non-degenerate HM($\mathcal{N}_\kappa$) problem such that $\deg s(z) < n + m$, then $s(z) = b(z)/a(z)$, in which $a(z), b(z)$ are two co-prime real polynomials satisfying $\deg b(z) < \deg a(z) = t < n + m$. Let $a(z) = a_0 + a_1 z + \cdots + a_t z^t$ and

$$\frac{b(z)}{a(z)} = -\frac{s'_0}{z} - \frac{s'_1}{z^2} - \cdots - \frac{s'_{2n+2m-2}}{z^{2n+2m-1}} + \mathrm{o}(z^{-2n-2m+1}) \quad (z \to \infty),$$

in which $s'_i = s_i$, $i = 0, 1, \cdots, 2n-2$. If $t < n$, then $\mathbf{H}(a_0, \cdots, a_t, 0, \cdots, 0)^{\mathrm{T}} = 0$. It contradicts to the nonsingularity of $\mathbf{H}$. If $n \leq t < n + m$, we denote by $\mathbf{H}'_t = (s'_{i+j})_{i,j=0}^{t-1}$. In this case, $\mathbf{H}$ is a nonsingular principle submatrix of $\mathbf{H}'_t$, and then $\nu(\mathbf{H}'_t) \leq \nu(\mathbf{H}) + t - n < \nu(\mathbf{H}) + m = \kappa$. On the other hand, by Corollary 3.6, we have $\kappa = \nu(\mathbf{H}'_t)$. It is a contradiction, so $r(z)$ given in Theorem 4.2 is a rational solution of the HM($\mathcal{N}_\kappa$) problem with the least McMillan degree. $\square$

To derive the solvability criterion for the HM($\mathcal{N}_\kappa$) problem for the degenerate case, we need the following result, which can be verified by a direct computation.

Lemma 4.3. *Let* $A \in \mathbb{C}^{m\times m}$ *be nonsingular,* $B = B^* \in \mathbb{C}^{m\times m}$ *and*

$$C = \begin{pmatrix} 0 & A^* \\ A & B \end{pmatrix} \in \mathbb{C}^{2m\times 2m}.$$

Then, $\nu(C) = m$.

Now, we apply the results above to deduce the solvability criterion of the HM($\mathcal{N}_\kappa$) problem for the degenerate case.

Theorem 4.4. *Let* $\mathbf{H} = (s_{i+j})_{i,j=0}^{n-1}$ *given by* (1.3) *be singular and* n_1 *be the first characteristic degree of* $\mathbf{H}$*. Then, the HM(*$\mathcal{N}_\kappa$*) problem is solvable if, and only if, one of the following statements holds:*
(1) $\kappa = \nu(\mathbf{H})$ *and* $\mathbf{H}$ *is a proper Hankel matrix;*
(2) $\kappa \geq \nu(\mathbf{H}) + n - n_1$.

Proof. Let n_1, n_2 and $[p(z), q(z), \gamma(z), \delta(z)]$ be the first characteristic degree, the second characteristic degree and the characteristic polynomial quadruple of $\mathbf{H}$, respectively, and let $\deg p(z) = r \leq n_1$. We first prove the "only if" part. Suppose that $f(z) \in \mathcal{N}_\kappa$ is a solution of the degenerate HM($\mathcal{N}_\kappa$) problem.

By Lemma 3.1, we have $\kappa \geq \nu(\mathbf{H})$. In the case of $\kappa = \nu(\mathbf{H})$, it follows from Lemma 3.2 that for each $z \in \mathbb{C}^+ \cap D(f)$, we have $\nu(L_f(z)) \leq \kappa$, in which

$$L_f(z) = \begin{pmatrix} \mathbf{H} & C(z)^* \\ C(z) & P_f(z) \end{pmatrix},$$

and $C(z)$ is the same as in (3.2). Since $\mathbf{H}$ is a principle submatrix of $L_f(z)$, we have $\nu(L_f(z)) \geq \nu(\mathbf{H}) = \kappa$ and thus $\nu(L_f(z)) = \kappa$ for all $z \in \mathbb{C}^+ \cap D(f)$. By Theorem 3.3, $L_f(z)$ is equivalent and congruent to

$$\widetilde{L}_f(z) = \left(\begin{array}{c|cc|c} \widehat{\mathbf{H}}^{\mathrm{p}} & & & E^* \\ \hline & 0_{n-n_1} & & F^* \\ & & \widehat{\mathbf{H}}^{\mathrm{d}} & \\ \hline E & & F & P_f(z) \end{array}\right),$$

in which $\widehat{\mathbf{H}}^{\mathrm{p}}$, $\widehat{\mathbf{H}}^{\mathrm{d}}$ are the same as in (2.7), and

$$\begin{aligned} E &= (f(z), z(f(z) + \frac{s_0}{z}), \cdots, z^{r-1}(f(z) + \frac{s_0}{z} + \cdots + \frac{s_{r-2}}{z^{r-1}})), \\ F &= (p(z)f(z) + \gamma(z), \cdots, z^{n-r-1}(p(z)f(z) + \gamma(z))). \end{aligned}$$

We check easily that $\widetilde{L}_f(z)$ is furtherly equivalent and congruent to the following block diagonal matrix:

$$\widehat{L}_f(z) = \mathrm{diag}(\widehat{\mathbf{H}}^{\mathrm{p}}, \widehat{\mathbf{H}}^{\mathrm{d}}, 0_{n-n_1-1}, \begin{pmatrix} 0 & \overline{d(z)} \\ d(z) & e(z) \end{pmatrix}),$$

in which $d(z) = p(z)f(z) + \gamma(z)$ and $e(z) = \overline{e(z)}$. Note that $\nu(L_f(z)) = \nu(\widetilde{L}_f(z)) = \nu(\widehat{L}_f(z)) = \kappa = \nu(\mathbf{H}) = \nu(\widehat{\mathbf{H}}^{\mathrm{p}}) + \nu(\widehat{\mathbf{H}}^{\mathrm{d}})$, then by Lemma 4.3, we have $d(z) = p(z)f(z) + \gamma(z) = 0$, $z \in \mathbb{C}^+ \cap D(f)$. This implies that $f(z) = f_{\gamma,p}(z) = -\gamma(z)/p(z)$ for a sufficiently large $|z|$ and $z \in \mathbb{C}^+ \cap D(f)$. Since $f(z)$ admits the asymptotic expansion (1.2) at infinity, we have

$$f_{\gamma,p}(z) = -\frac{\gamma(z)}{p(z)} = -\frac{s_0}{z} - \frac{s_1}{z^2} - \cdots - \frac{s_{2n-2}}{z^{2n-1}} + \mathrm{o}(z^{-2n+1}) \quad (z \to \infty). \tag{4.3}$$

By Lemma 2.4, we have $\deg p(z) = n_1$ and, thus, $\mathbf{H} = \mathbf{H}^{\mathrm{p}}$ is a proper Hankel matrix.

When $\kappa > \nu(\mathbf{H})$, $f(z) \not\equiv f_{\gamma,p}(z) = -\gamma(z)/p(z)$. Otherwise, $f_{\gamma,p}(z) \in \mathcal{N}_\kappa$ and the asymptotic expansion (4.3) holds. In this case, by Lemma 2.4, $\mathbf{H} = \mathbf{H}^p$ is a proper Hankel matrix. Moreover, by Corollary 3.6, we have $\kappa = \nu(\mathbf{H}(p,\gamma)) = \nu(\widehat{\mathbf{H}}^{\mathrm{p}}) = \nu(\mathbf{H}^{\mathrm{p}}) = \nu(\mathbf{H})$, which contradicts the assumption $\kappa > \nu(\mathbf{H})$. We write $m = n - n_1$ and $g(z) = p(z)f(z) + \gamma(z)$ for short, then there exist m distinct points $z_1, \cdots, z_m \in \mathbb{C}^+ \cap D(f)$ such that $g(z_k) \neq 0$, $k = 1, \cdots, m$. By Lemma 3.2 and Theorem 3.3, we have $\nu(\widetilde{L}_f(z_1, \cdots, z_m)) \leq \kappa$, in which

$$\widetilde{L}_f(z_1, \cdots, z_m) = \begin{pmatrix} \widehat{\mathbf{H}}^{\mathrm{p}} & 0 & 0 & * \\ 0 & 0_m & 0 & A^* \\ 0 & 0 & \widehat{\mathbf{H}}^{\mathrm{d}} & * \\ * & A & * & P_f(z_1, \cdots, z_m) \end{pmatrix} \in \mathbb{C}^{(n+m)\times(n+m)},$$

$$A = \begin{pmatrix} g(z_1) & \cdots & z_1^{m-1} g(z_1) \\ \vdots & & \vdots \\ g(z_m) & \cdots & z_m^{m-1} g(z_m) \end{pmatrix} \in \mathbb{C}^{m\times m}.$$

We check easily that $\widetilde{L}_f(z_1, \cdots, z_m)$ is equivalent and congruent to the block diagonal matrix $\mathrm{diag}(\widehat{\mathbf{H}}^{\mathrm{p}}, \widehat{\mathbf{H}}^{\mathrm{d}}, D)$, where

$$D = \begin{pmatrix} O & A^* \\ A & B \end{pmatrix}$$

and $B = B^* \in \mathbb{C}^{m\times m}$. Since $\det A = g(z_1) \cdots g(z_m) \prod_{1\le i<j\le m}(z_j - z_i) \neq 0$, A is a nonsingular $m \times m$ matrix. Applying Lemma 4.3, we obtain that $\nu(L_f(z_1, \cdots, z_m)) = \nu(\widehat{\mathbf{H}}^{\mathrm{p}}) + \nu(\widehat{\mathbf{H}}^{\mathrm{d}}) + m = \nu(\mathbf{H}) + n - n_1$. Therefore, $\kappa \geq \nu(\mathbf{H}) + n - n_1$, as needed.

Let us turn to prove the "if" part. First we suppose that $\kappa = \nu(\mathbf{H})$ and $\mathbf{H}$ is a proper Hankel matrix. Then, $\deg p(z) = n_1 < n$, and by Lemma 2.1, $f_{\gamma,p}(z) = -\gamma(z)/p(z)$ admits the asymptotic expansion (1.2) at infinity. By Corollary 3.6, we have $f_{\gamma,p}(z) \in \mathcal{N}_{\kappa'}$, in which $\kappa' = \nu(\mathbf{H}(p,\gamma)) = \nu(\widehat{\mathbf{H}}^{\mathrm{p}}) = \nu(\mathbf{H}^{\mathrm{p}}) = \nu(\mathbf{H}) = \kappa$, then $f_{\gamma,p}(z)$ is a solution of the degenerate $\mathrm{HM}(\mathcal{N}_\kappa)$ problem.

Now, we suppose that $\kappa \geq \nu(\mathbf{H}) + n - n_1$. The proof is divided into two cases.

Case I: $\deg p(z) = n_1$. In this case, by Lemma 2.4, $\mathbf{H} = \mathbf{H}^{\mathrm{p}}$ is a proper Hankel matrix, and the asymptotic expansion (2.2) holds. Assume that

$$f_{\gamma,p}(z) = -\frac{s_0}{z} - \frac{s_1}{z^2} - \cdots - \frac{s_{2n-2}}{z^{2n-1}} - \cdots - \frac{s_{2n_2-2}}{z^{2n_2-1}} + \mathrm{o}(z^{-2n_2+1}) \quad (z \to \infty). \tag{4.4}$$

We define an $n_2 \times n_2$ Hankel matrix by

$$\mathbf{H}_{n_2} = (\widetilde{s}_{i+j})_{i,j=0}^{n_2-1}, \quad \widetilde{s}_k = \begin{cases} s_k, & k \neq 2n-1; \\ s_k + 1, & k = 2n-1. \end{cases} \tag{4.5}$$

Analogous to the proof of Lemma 2.5, there exists a nonsingular matrix $\widetilde{Q}$ of order n_2 such that

$$\widetilde{Q}\mathbf{H}_{n_2}\widetilde{Q}^* = \begin{pmatrix} \mathbf{H}_{n_1} & 0 & 0 \\ 0 & 0 & A^* \\ 0 & A & B \end{pmatrix},$$

in which

$$\mathbf{H}_{n_1} = (s_{i+j})_{i,j=0}^{n_1-1}, \quad A = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & * \end{pmatrix} \in \mathbb{C}^{(n-n_1)\times(n-n_1)}.$$

Clearly, the Hankel matrix $\mathbf{H}_{n_2}$ defined by (4.4) and (4.5) is nonsingular. Moreover, by Lemma 4.3, we have $\nu(\mathbf{H}_{n_2}) = \nu(\mathbf{H}_{n_1}) + n - n_1 = \nu(\mathbf{H}) + n - n_1 \leq \kappa$, then by Theorem 4.1, there exists a function $f(z) \in \mathcal{N}_\kappa$ such that the following asymptotic expansion at infinity

$$f(z) = -\frac{\widetilde{s}_0}{z} - \frac{\widetilde{s}_1}{z^2} - \cdots - \frac{\widetilde{s}_{2n-2}}{z^{2n-1}} - \cdots - \frac{\widetilde{s}_{2n_2-2}}{z^{2n_2-1}} + \mathrm{o}(z^{-2n_2+1})$$

holds when z tends to ∞ in the sector $\pi_\epsilon(0)$. This means that $f(z)$ is a solution of the degenerate HM($\mathcal{N}_\kappa$) problem.

Case II: deg $p(z) < n_1$. By Lemma 2.1, $f_{\delta,q}(z) = -\delta(z)/q(z)$ admits the asymptotic expansion (2.3) at infinity. We assume that

$$f_{\delta,q}(z) = -\frac{s_0}{z} - \frac{s_1}{z^2} - \cdots - \frac{s_{2n-2}}{z^{2n-1}} - \cdots - \frac{s_{2n_2-2}}{z^{2n_2-1}} + \mathrm{o}(z^{-2n_2+1}) \quad (z \to \infty). \tag{4.6}$$

We define an $n_2 \times n_2$ Hankel matrix by

$$\mathbf{H}_{n_2} = (s_{i+j})_{i,j=0}^{n_2-1}. \tag{4.7}$$

Analogous to the proof of Lemma 2.5, there exists a nonsingular matrix $\widetilde{Q}$ of order n_2 such that

$$\widetilde{Q}\mathbf{H}_{n_2}\widetilde{Q}^* = \begin{pmatrix} \widehat{\mathbf{H}}^{\mathrm{p}} & 0 & 0 & 0 \\ 0 & 0_{n-n_1} & 0 & A \\ 0 & 0 & \widehat{\mathbf{H}}^{\mathrm{d}} & B^* \\ 0 & A & B & C \end{pmatrix},$$

in which $\widehat{\mathbf{H}}^{\mathrm{p}}$ and $\widehat{\mathbf{H}}^{\mathrm{d}}$ are the same as in (2.7), and

$$A = \begin{pmatrix} O & & \widehat{s}_1 \\ & \iddots & \vdots \\ \widehat{s}_1 & \cdots & \widehat{s}_{n-n_1} \end{pmatrix} (\widehat{s}_1 \neq 0), \quad B = (\widehat{s}_{i+j})_{i,j=1}^{n_2-n,t}, \quad C = (\widehat{s}_{i+j+t})_{i,j=1}^{n_2-n}$$

are real Hankel matrices. We check easily that the Hankel matrix $\mathbf{H}_{n_2}$ defined by (4.6) and (4.7) is nonsingular. Moreover, it is furtherly equivalent and congruent to the following block diagonal matrix:

$$\Lambda = \mathrm{diag}(\widehat{\mathbf{H}}^{\mathrm{p}}, \widehat{\mathbf{H}}^{\mathrm{d}}, \begin{pmatrix} 0_{n-n_1} & A^* \\ A & \widetilde{C} \end{pmatrix}),$$

in which $\widetilde{C}$ is a real symmetric matrix of order $n_2 - n$. Applying Lemma 4.3, we obtain that $\nu(\mathbf{H}_{n_2}) = \nu(\Lambda) = \nu(\widehat{\mathbf{H}}^p) + \nu(\widehat{\mathbf{H}}^d) + n - n_1 = \nu(\mathbf{H}) + n - n_1 \leq \kappa$, then by Theorem 4.1, there exists a function $f(z) \in \mathcal{N}_\kappa$ such that the following asymptotic expansion at infinity

$$f(z) = -\frac{s_0}{z} - \frac{s_1}{z^2} - \cdots - \frac{s_{2n-2}}{z^{2n-1}} - \cdots - \frac{s_{2n_2-2}}{z^{2n_2-1}} + \mathrm{o}(z^{-2n_2+1})$$

holds when z tends to ∞ in the sector $\pi_\epsilon(0)$. This implies that $f(z)$ is a solution of the degenerate HM($\mathcal{N}_\kappa$) problem, and the proof of Theorem 4.4 is completed. □

From the proofs of Theorems 4.1 and 4.4, we can obtain a concrete rational solution with the least McMillan degree among all rational solutions of the degenerate HM($\mathcal{N}_\kappa$) problem, when the solvability conditions are met.

Theorem 4.5. *Let* $\mathbf{H} = (s_{i+j})_{i,j=0}^{n-1}$ *given by* (1.3) *be singular, and let* n_1, n_2 *and* $[p(z), q(z), \gamma(z), \delta(z)]$ *be the first characteristic degree, the second characteristic degree and the characteristic quadruple of* $\mathbf{H}$*, respectively.*

(1) *If* $\kappa = \nu(\mathbf{H})$ *and* $\mathbf{H}$ *is a proper Hankel matrix, then* $r(z) = -\gamma(z)/p(z)$ *is a rational solution with the least McMillan degree among all rational solutions of the HM(*$\mathcal{N}_\kappa$*) problem;*

(2) *If* $\kappa \geq \nu(\mathbf{H}) + n - n_1$*, then* $r(z) = -\beta(z)/v(z)$ *is a rational solution with the least McMillan degree among all rational solutions of the HM(*$\mathcal{N}_\kappa$*) problem, where* $[u(z), v(z), \alpha(z), \beta(z)]$ *is the characteristic quadruple of the Hankel matrix* $\mathbf{H}_{\kappa+n-\nu(\mathbf{H})} = (s_{i+j})_{i,j=0}^{\kappa+n-\nu(\mathbf{H})-1}$ *defined by* (4.4) *if* $\kappa + n - \nu(\mathbf{H}) = n_2$ *and* $\deg p(z) = n_1$*, by* (4.6) *if* $\kappa + n - \nu(\mathbf{H}) = n_2$ *and* $\deg p(z) < n_1$*, and defined recursively by*

$$
\begin{aligned}
&s_{2n_2+2k-3} = 0,\\
&s_{2n_2+2k-2} = (s_{n_2+k}, \cdots, s_{2n_2+2k-3})\mathbf{H}_{n_2+k-1}^{-1}(s_{n_2+k}, \cdots, s_{2n_2+2k-3})^{\mathrm{T}} - 1, \quad k = 1, 2, \cdots
\end{aligned}
$$

if $\kappa + n - \nu(\mathbf{H}) > n_2$.

Proof. We see from the proofs of Theorems 4.1 and 4.4 that the rational function $r(z)$ presented in Theorem 4.5 for each case is a solution of the HM($\mathcal{N}_\kappa$) problem. Now it remains to prove that such a rational solution has the least McMillan degree among all rational solutions of the HM($\mathcal{N}_\kappa$) problem, when the solvability conditions are met. We divide the proof into two cases.

Case I: $\kappa = \nu(\mathbf{H})$ and $\mathbf{H}$ is a proper Hankel matrix. In this case, $\deg r(z) = \deg p(z) = n_1$ and $\widehat{\mathbf{H}}^{\mathrm{p}} = (s_{i+j})_{i,j=0}^{n_1-1}$ is nonsingular. Assume that the HM($\mathcal{N}_\kappa$) problem has a rational solution $f(z)$ such that $\deg f(z) < n_1$, then there exist two co-prime real polynomials $a(z), b(z)$ such that $f(z) = b(z)/a(z)$, $\deg a(z) \leq n_1 - 1$. Let $a(z) = a_0 + a_1 z + \cdots + a_{n_1-1}z^{n_1-1}$. In view of the fact that

$$
f(z) = \frac{b(z)}{a(z)} = -\frac{s_0}{z} - \frac{s_1}{z^2} - \cdots - \frac{s_{2n-2}}{z^{2n-1}} + \mathrm{o}(z^{-2n+1}),
$$

we have $\widehat{\mathbf{H}}^{\mathrm{p}}(a_0, a_1, \cdots, a_{n_1-1})^{\mathrm{T}} = 0$. It contradicts to the nonsingularity of $\widehat{\mathbf{H}}^{\mathrm{p}}$, then $r(z) = -\gamma(z)/p(z)$ is a rational solution of the HM($\mathcal{N}_\kappa$) problem with the least McMillan degree n_1.

Case II: $\kappa \geq \nu(\mathbf{H}) + n - n_1$. In this case, $\deg r(z) = \deg v(z) = \kappa + n - \nu(\mathbf{H})$. If the HM($\mathcal{N}_\kappa$) problem has a rational solution $f(z)$ such that $\deg f(z) = t < \kappa + n - \nu(\mathbf{H})$, then there exist two coprime real polynomials $a(z), b(z)$ such that $f(z) = b(z)/a(z)$ and $\deg a(z) = t$. Let

$$
f(z) = \frac{b(z)}{a(z)} = -\frac{s'_0}{z} - \frac{s'_1}{z^2} - \cdots - \frac{s'_{2k-2}}{z^{2k-1}} + \cdots \quad (z \to \infty)
$$

in which $s'_i = s_i$, $i = 1, \cdots, 2n-2$, and let $\mathbf{H}'_k = (s'_{i+j})_{i,j=0}^{k-1}$, $k = 1, 2, \cdots$. By Corollary 3.6, we have $\nu(\mathbf{H}'_t) = \kappa \geq \nu(\mathbf{H}) + n - n_1$. If $t < n_2$, then $\mathbf{H}'_{n_2}$ is singular and, thus, $\nu(\mathbf{H}'_t) \leq \nu(\mathbf{H}'_{n_2}) < \nu(\mathbf{H}) + (n_2 - n_1)/2 = \nu(\mathbf{H}) + n - n_1$. It is a contradiction. Therefore, $t \geq n_2$ and $\mathbf{H}'_{n_2}$ is a principle submatrix of $\mathbf{H}'_t$. By the interlacing relation between the eigenvalues of a Hermitian matrix and its principle submatrices (see, e.g., [14, Theorem 4.3.28]),

$$
\nu(\mathbf{H}'_t) \leq \nu(\mathbf{H}'_{n_2}) + t - n_2 \leq \nu(\mathbf{H}) + \frac{n_2 - n_1}{2} + t - n_2 = \nu(\mathbf{H}) + t - n < \kappa.
$$

It is also a contradiction, and $r(z) = -\beta(z)/v(z)$ is a rational solution of the HM($\mathcal{N}_\kappa$) problem with the least McMillan degree $\kappa + n - \nu(\mathbf{H})$. □

5. Conclusions

In this paper, we introduced some basic structural characteristics of the Hankel matrix, such as the first and second characteristic degrees, the characteristic polynomial quadruple and the quasidirect decomposition for the singular case, and then, we applied these structural characteristics and the relation among the Hankel, Loewner, Bezout and some other structured matrices to deduce several new properties of the functions in the class $\mathcal{N}_\kappa$ and the solvability criterion of the HM($\mathcal{N}_\kappa$) problem for both the non-degenerate and degenerate cases. As a by-product, we simultaneously obtained a rational solution of the HM($\mathcal{N}_\kappa$) problem with the least McMillan degree when the solvability conditions were met.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The corresponding author is supported by the Scientific Research Fund from Beijing Normal University at Zhuhai (Grant No. 111032119).

Conflict of interest

Yongjian Hu is the Guest Editor of special issue "Matrix theory and its applications" for AIMS Mathematics. Yongjian Hu was not involved in the editorial review and the decision to publish this article. All authors declare no conflicts of interest in this paper.

References

1. N. I. Akhiezer, *The Classical Moment Problem and Some Related Questions in Analysis*, London: Oliver and Boyd, 1965.
2. S. Barnett, *Polynomials and Linear Control Systems*, New York: Marcel Dekker, 1983.
3. S. Barnett, C. Storey, *Matrix Methods in Stability Theory*, London: Nelson, 1970.
4. G. N. Chen, The general rational interpolation problem and its connection with the Nevanlinna-Pick interpolation and power moment problem, *Linear Algebra Appl.*, **273** (1998), 83–117. https://doi.org/10.1016/S0024-3795(97)00346-7
5. G. N. Chen, B. Zhao, H. P. Zhang, The general rational interpolationproblem in the scalar case and its Hankel vector, *Linear Algebra Appl.*, **244** (1996), 165–201. https://doi.org/10.1016/0024-3795(94)00223-1
6. G. N. Chen, H. P. Zhang, Note on products of Bezoutians and Hankel matrices, *Linear Algebra Appl.*, **225** (1995), 23–35. https://doi.org/10.1016/0024-3795(93)00305-J
7. M. Dereyagin, On the Schur algorithm for indefinite moment problem, *Meth. Funct. Anal. Topol.*, **9** (2003), 133–145.

8. V. Derkach, S. Hassi, H. de Snoo, Truncated moment problems in the class of generalized Nevanlinna functions, *Math. Nachr.*, **285** (2012), 1741–1769. https://doi.org/10.1002/mana.201100268
9. M. Fiedler, Quasidirect decomposition of Hankel and Toeplitz matrices, *Linear Algebra Appl.*, **61** (1984), 155–174. https://doi.org/10.1016/0024-3795(84)90028-4
10. M. Fiedler, V. Pták, Bezoutians and intertwining matrices, *Linear Algebra Appl.*, **86** (1987), 43–51. https://doi.org/10.1016/0024-3795(87)90286-2
11. M. Fiedler, V. Pták, Loewner and Bezout matrices, *Linear Algebra Appl.*, **101** (1988), 187–220. https://doi.org/10.1016/0024-3795(88)90151-6
12. P. A. Fuhrmann, Polynomial models and algebraic stability criteria, In: *Feedback Control of Linear and Nonlinear Systems: Proceedings of the Joint Workshop on Feedback and Synthesis of Linear and Nonlinear Systems*, 1982, 78–90.
13. G. Heinig, U. Jungnickel, Hankel matrices generated by the Markov parameters of rational functions, *Linear Algebra Appl.*, **76** (1986), 121–135. https://doi.org/10.1016/0024-3795(86)90217-X
14. R. A. Horn, C. R. Johnson, *Matrix Analysis*, Cambridge: Cambridge University Press, 1985.
15. M. G. Krein, A. A. Nudelman, *The Markov Moment and Extremal Problems*, New York: American Mathematical Society, 1977.
16. P. Lancaster, M. Tismenetsky, *The Theory of Matrices with Applications*, 2^{nd} edition, New York: Academic Press, 1985.
17. Y. P. Song, H. F. Hao, Y. J. Hu, G. N. Chen, Some propositions on generalized Nevanlinna functions of the class $\mathcal{N}_\kappa$, *Adv. Math. Phys.*, **2014** (2014), 605492. https://doi.org/10.1155/2014/605492