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Research article

Industrial optimization using three-factor Cobb-Douglas production function of non-linear programming with application

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Abstract: This paper is about the effectiveness of the Cobb-Douglas (C-D) production function in industrial optimization, estimating the number of factors used in the production process of the water industry, for instance, capital and human labor. Moreover, we have modeled a nonlinear optimization problem for a local water industry using two and three factors of production. For this purpose, we have taken into account the Cobb-Douglas production function with different production factors using the Lagrange multiplier method with the ordinary least squares method. In the course of the solution, a linear function is used to calculate the cost function, and the C-D production function is used to calculate the production function. The Lagrange multiplier method with the ordinary least squares method is then used to solve the constrained optimization problem for the product of production. Furthermore, we compared the outcomes from both examples of two- and three-factor C-D production functions in order to validate the Lagrange multiplier method for the C-D production function. Moreover, the three-factor C-D production function is solved by the Lagrange multiplier method with the ordinary least squares method, which provides optimal results as compared to previous studies in literature. The validity of the proposed methodology is explained by using the products of a local production industry in Pakistan.

Keywords: Cobb-Douglas (C-D) production function; non-linear programming; optimization; cost function; human labors and capitals
Mathematics Subject Classification: 90C30, 90C90

1. Introduction

Optimization is a useful technique for determining the optimal solution to a problem. In other words, optimization is the problem of choosing suitable inputs under given circumstances in order to get the best possible output. For instance, optimization can be used in production models to adjust different inputs and make them more effective in order to get the best output for the production of a particular industry [1]. In this scenario, once we have modeled a problem, it can be solved using the available optimization techniques to find the optimal solution [2].

An optimization problem usually consists of three ingredients: an objective function, a set of constraints and a number of decision variables. There are two types of optimization problems: constrained optimization problems and unconstrained optimization problems. Constrained optimization problems have restriction(s) on the objective function, while unconstrained optimization problems have no restriction on the objective function [3, 4].

If at least one of the objective functions or the constraint function is nonlinear, then the problem is known as a nonlinear optimization problem. There are many techniques that have been used for the maximization or minimization of nonlinear optimization problems. Sometimes, a problem cannot be modeled correctly using linear programming; therefore, one can use nonlinear programming approaches [5, 6] to model the problem. For the constrained optimization problem under consideration in this paper, the objective function, known as the cost function, is linear while the constraint function, known as the Cobb-Douglas (C-D) production function, is nonlinear. In the course of solution, the constrained optimization problem is converted into an unconstrained optimization problem and then solved by the Lagrange multiplier method using the ordinary least squares approach.

There have been many problems in literature in which the C-D production function has been used. A mixed-integer linear programming (MILP) model was established for the optimization of production scheduling [7]. A two-factor C-D production function is carried out with the aim of picking the suitable C-D production model for calculating the production process of the selected manufacturing industries in Bangladesh [8]. By combining the allometric scaling concept, which is used to estimate the parameters of the C-D function, with the application in transportation problems in China, a novel algorithm for creating geographical C-D models is developed [9].

For the improvement of different factors in the Polish metallurgical industry, the power regression C-D function was used with the aim of developing a number of production factors, for instance, net production, production sold and volume of steel production [10]. The proper management of a country's resources is an important issue for its economic development. Along similar lines, the optimization of water management for three industries that rely on water demand prediction, subject to a number of ambiguities, is handled by the use of the C-D production function [11]. A two-factor production function was used to model sustainable economic development with the goal of labor production in relation to the commodity production system's capital-labor ratio [12]. An economic model has been presented by the application of proper Inada conditions to the C-D production function, which converges to or diverges from per capita product and the steady state of capital [13]. Moreover, a two-factor C-D production function was presented in which the effects of labor force and capital on agricultural heritage systems are carried out to maximize profit as well as the sustainability of agricultural heritage through the application of a two-factor C-D production function in which they examine the impacts of major factors on agricultural productivity [14]. An algorithmic or analytical procedure was conducted to handle the issue of optimal utilization of resources towards a feasible and profitable model via the C-D production function [15]. An application of the C-D production function model was used to find the role of land in urban economic growth [16]. A general oligopolistic market equilibrium with nonlinear programming was considered, in which each firm's factor contributes to the system, and then solved by tensor variational inequality [17, 18].

It becomes necessary in a real-world optimization problem to adopt a set of nonlinear terms in a mathematical model in order to get particular operational features of the decision-making problem. On the other hand, when nonlinear terms exist in the course of a solution, they add to the computational complexity of the problem. For this purpose, the researchers have developed proper transformation as well as linearization methods for the optimization problems that consist of nonlinear terms.

In this paper, a special type of production function, which was founded by Cobb and Douglas in 1928 and is known as the C-D production function, is under consideration. This function is based on empirical studies that have been applied to the economy for optimal production [19]. This function supplies a number of different inputs to the problem and, as a result, produces a unique output for the problem. These inputs may be two or more than two in number, depending on the factors of production used in the industry. Also, the function having more than two factors is known as the generalized C-D production function [20, 21]. There have been many results in literature in which two-factor C-D function was presented for production. This work presents the optimal solution technique of the C-D function for three factors of production using the Lagrange multiplier method and the ordinary least squares method, with applications. Moreover, Nervole's C-D production function with three inputs using the ordinary least squares method has been presented in [22]. Furthermore, this work also developed an environment to transform and linearize an optimization problem with nonlinear objectives or nonlinear constraints by using existing techniques [23].

The proposed work is summarized as follows:

(1) To develop a model in order to transform as well as linearize the nonlinear terms.
(2) To choose a suitable model [1] as an optimization problem and to solve it using a novel methodology known as C-D production function using the Lagrange multiplier method with the ordinary least squares method.
(3) To find whether the best possible solution to the problem is obtained or not [2].
(4) To apply the C-D production function using the Lagrange multiplier method with the ordinary least squares method for the solution of the two-factor C-D production function as well as for the three-factor C-D production function.
(5) To apply the case study to industrial optimization in order to minimize costs more efficiently.

First we solved the two-factor C-D production function [24] using the Lagrange multiplier method with the ordinary least squares method. Second we used three factors of the C-D production function with the same technique. The obtained results show that cost minimization in cases of three factors of production is more efficient as compared to cost minimization in cases of two factors of production.

2. Cost minimization

The problem under consideration depends on three basic factors:

- Cost of the company for manufacturing production.

- Quantity of production.
- Income from the sales of production according to market prices.

For instance, the role of the water industry is to filter water in different purifying tanks, then add chemicals, pack them properly, and then sell them in the market. In this task, the main objective of the industry is to minimize costs or maximize production. There have been many approaches in literature in which the total cost of an industry is presented as a linear function [24, 25].

$$K = \sum_{i=1}^{n} q_i x_i + R_s = q^T x + R_s \tag{2.1}$$

where n represents production factors.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}$$

is a vector of production factors.

$$q = \begin{bmatrix} q_1 \\ q_2 \\ . \\ . \\ . \\ q_n \end{bmatrix}$$

is the vector of prices of production factors.

The following relationships present the production volume [26]:

$$y = p(x) = p(x_1, x_2 ... x_n). \tag{2.2}$$

The income of the company from production sales is given by:

$$S(y) = b^T y \tag{2.3}$$

where,

$$y = \begin{bmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_n \end{bmatrix}$$

is a vector of the quantity of produced goods.

$$b = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_n \end{bmatrix}$$

is the vector prices of produced goods.

The profit of the company is the difference between its income and its cost of production.

$$Z(x, y) = S(y) - K(x). \tag{2.4}$$

The main focus of this paper is to minimize the cost at a specific level of production and, consequently, to maximize production with more factors of production.

The optimization problem originating from our data analysis consists of a linear function and a nonlinear function. We have used the linear function as a cost function, while the nonlinear C-D production function is used as production output. Let us consider a linear cost function as an objective function and the nonlinear C-D production function as a constrained function. Then, this becomes a constrained optimization problem with an equality constraint. For the sake of simplicity, we convert this constrained optimization problem to an unconstrained optimization problem by using the Lagrange multiplier method, and then we solve this constrained optimization problem for stationary points. Let us start with a function Z of two independent variables that is subject to an equality constraint function g. The objective and constraint function are given as follows:

$$min\ z(x, y).$$

Subject to:

$$g(x, y) = 0.$$

We have, using the Lagrange multiplier,

$$L(x, y, \lambda) = z(x, y) + \lambda g(x, y). \tag{2.5}$$

In order to solve the problem, we must first determine the values of x, y, and λ. Note that we consider the cost function to be an objective function and the C-D production function as a constraint function, and solve it for different factors of production. Furthermore, we extend this work as an application to optimize production factors in a specific industry.

The production function for three inputs is as follows:

$$P = AX^{a_1} Y^{a_2} Z^{a_3}. \tag{2.6}$$

After linearizing, we get

$$\ln P = \ln A + a_1 \ln X + a_2 \ln Y + a_3 \ln Z. \tag{2.7}$$

After this, we will use least squares linear regression to find out the structural parameters, which are A, a_1, a_2, a_3. The corresponding cost functions is given by

$$C(X, Y, Z) = \alpha_1 + \alpha_2 X + \alpha_3 Y + \alpha_4 Z. \tag{2.8}$$

Along similar lines as Eq (2.5), we can get the corresponding Lagrange function for three-factor C-D production function.

In other words, the overall methodology can also be summarized as follows:

(1) First, model an objective and constraint function using two inputs for the given industry.
(2) Transform a constrained optimization problem into an unconstrained optimization problem.
(3) Solve the unconstrained problem using the Lagrange multiplier method with ordinary least squares.
(4) Repeat the above procedure for three factors of production for the given industry.
(5) Compare the outcomes in both cases.

3. Production function and its structural parameters

Given input prices, a cost function shows how much it costs to produce various output levels. In the course of solving such problems, one or both productions and factors of production may be stated by using their values. It is good practice to present the products of an industry in proper units that have a number of production components. In a similar way, human labor can be calculated when needed. When the availability of aggregated data is smaller, it can be measured with headcount or work time. When a higher level of aggregation is available, then the value of human labor seems more suitable. The most challenging task is the quantitative description of the capital used in the industry. In the majority of analyses, it becomes challenging due to the use of a number of factors of production. The greater a company's asset base, the less it is associated with high productivity [27]. In the following, we analyzed a local water industry with a two-factor C-D production function and a three-factor C-D production function using the Lagrange multiplier method with the ordinary least squares method and compared their results.

3.1. Cost minimization in the water industry using the two-factor C-D production function

For this problem, the data is taken from a local water industry Abysin Water Industry, which is one of the local registered branches of Chemtronics Water Services in Lahore, Pakistan. First, the problem is solved for two factors, and then it is generalized. In this case, the cost function consists of two factors (labor and capital) of production, which is given by:

$$C(X, Y) = \alpha_1 + \alpha_2 X_{+} \alpha_3 Y \tag{3.1}$$

where,

C(X,Y) represents the cost of the industry in rupees,
α_1 represents the fixed cost of the industry,
α_2 represents the unit price of labor per hour,
α_3 represents the unit price of capital per kg,
X represents the number of labor hours,

Y represents the amount of capital in kg.

All the prices in the industry are in Pakistani currency (the rupee). The raw materials used as capital are taken in kilograms, i.e., the unit for capital is kg.

Here,

$$\alpha_1 = 75000, \quad \alpha_2 = 1400, \quad \alpha_3 = 100.$$

Putting these values in Eq (3.1), we get,

$$C(X, Y) = 75000 + 1400X + 100Y. \tag{3.2}$$

For production, we use the C-D production function, which is given by:

$$P = AX^a Y^b \tag{3.3}$$

where

P represents the amount of production in liters.

After Eq (3.3) we have,

$$\ln P = \ln A + a \ln X + b \ln Y. \tag{3.4}$$

The analyzed data for the water industry with two inputs, i.e., human labor and capital, and production as an output for the year 2022 is given in Table 1.

We have used ordinary least squares regression in order to find the structural parameters of the given C-D production function. The data analysis has been done using Microsoft Excel, which is given in Regressions 1 and 2 for two and three inputs, respectively.

Structural parameters for two inputs are as follow:

$$A = 3212.468,$$
$$a = 0.3568,$$
$$b = 0.0542,$$

$$L(X, Y, \lambda) = C(X, Y) + \lambda H(X, Y), \tag{3.5}$$

where,

$$H(X, Y) = 36000 - 3212.468X^{0.3568}Y^{0.0542}.$$

Using Eq (3.2) and H(x, y) in Eq (3.5), we have,

$$L = 75000 + 1400X + 100Y + \lambda(36000 - 3212.468X^{0.3568}Y^{0.0542}). \tag{3.6}$$

Table 1. Inputs and output values of the water industry with two inputs.

Months	Production(P) in litres	Labors(X) in hours	Capital(Y) in kg	$Ln(P)$	$Ln(X)$	$Ln(Y)$
Jan.	36,000	300	900	10.49127	5.703782	6.802395
Feb.	34,000	272	833	10.43412	5.605802	6.725034
March	36,700	315	924	10.51053	5.752573	6.828712
April	37,000	330	942	10.51867	5.799093	6.848005
May	35,700	278.5	874	10.48291	5.629418	6.77308
June	36,400	321	922	10.50232	5.771441	6.826545
July	33,600	269	860	10.42228	5.594711	6.756932
Aug.	35,500	308	950	10.47729	5.7301	6.856462
Sep.	36,875	338	957	10.51529	5.823046	6.863803
Oct.	37,000	321	914	10.51867	5.771441	6.817831
Nov.	33,600	266	875	10.42228	5.583496	6.774224
Dec.	36,500	336	890	10.50507	5.817111	6.791221

SUMMARY OUTPUT

Regression Statistics	
Multiple R	0.91455
R Square	0.83641
Adjusted R Square	0.80005
Standard Error	0.01655
Observations	12

ANOVA

	df	*SS*	*MS*	*F*	*Significance F*
Regression	2	0.0126035	0.0063017	23.00737	0.00029
Residual	9	0.0024651	0.0002739		
Total	11	0.0150686			

	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*	*Lower 95.0%*	*Upper 95.0%*
A	8.07479	0.9524368	8.478037	1.39E-05	5.920233	10.22936	5.920233	10.22936
a	0.35688	0.0921322	3.8735297	0.003768	0.148459	0.565294	0.148459	0.565294
b	0.05422	0.1935459	0.2801409	0.785702	-0.38361	0.492051	-0.38361	0.492051

Regression 1. Linear regression for two inputs.

SUMMARY OUTPUT

Regression Statistics	
Multiple R	0.915252
R Square	0.837687
Adjusted R Square	0.776819
Standard Error	0.017485
Observations	12

ANOVA

	df	SS	MS	F	Significance F
Regression	3	0.012623	0.004208	13.76247	0.001593
Residual	8	0.002446	0.000306		
Total	11	0.015069			

	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%	Lower 95.0%	Upper 95.0%
A	8.278476	1.066138	7.764918	5.41E-05	5.819957	10.737	5.819957	10.737
a_1	0.294408	0.184645	1.594455	0.1495	-0.13138	0.720199	-0.13138	0.720199
a_2	0.067354	0.107888	0.624294	0.549819	-0.18144	0.316145	-0.18144	0.316145
a_3	0.045783	0.231899	0.197428	0.848417	-0.48898	0.580543	-0.48898	0.580543

Regression 2. Linear regression for three inputs.

Now taking partial derivatives of L with respect to X, Y and λ respectively and equating to zero, i.e.,

$$\frac{\partial L}{\partial X} = 0,$$

we get,

$$1400 - 1146.2085\lambda X^{-0.6432} Y^{0.0542} = 0,$$

which implies that

$$\lambda = \frac{1.2214}{X^{-0.6432} Y^{0.0542}}. \tag{3.7}$$

Now, when

$$\frac{\partial L}{\partial Y} = 0,$$

we get

$$100 - 174.1157\lambda X^{0.3568} Y^{-0.9458} = 0,$$

which implies

$$\lambda = \frac{100}{X^{0.3568} Y^{-0.9458}}. \tag{3.8}$$

Similarly, when

$$\frac{\partial L}{\partial \lambda} = 0,$$

we get,

$$36000 - 3212.468X^{0.3568}Y^{0.0542} = 0. \tag{3.9}$$

Comparing (3.7) and (3.8),

$$\frac{1.2214}{X^{-0.6432}Y^{0.0542}} = \frac{100}{X^{0.3568}Y^{-0.9458}},$$

we get

$$Y = 2.1266X. \tag{3.10}$$

Putting Eq (3.10) in Eq (3.9), we have

$$36000 - 3212.468X^{0.3568}(2.1266X)^{0.0542} = 0.$$

Solving for X, we have

$$X = 323.7554. \tag{3.11}$$

From (3.10), we have

$$Y = 688.4982, \tag{3.12}$$

and using the values of X and Y in Eq (3.2), we have,

$$C(X, Y) = 597107. \tag{3.13}$$

Table 2 compares the actual cost with the estimated cost.

Table 2. Actual and estimated cost values for water industry using two inputs.

Months	Estimated cost in rupees	Actual cost in rupees	Error (R)	Square of the error (R^2)
Jan.	585,000	608,000	23000	529000000
Feb.	539,100	570,000	30900	954810000
March	608,400	588,000	-20400	416160000
April	631,200	608,000	-23200	538240000
May	552,300	598,000	45700	2088490000
June	616,600	621,000	4400	19360000
July	537,600	574,000	36400	1324960000
Aug.	601,200	564,000	-37200	1383840000
Sep.	643,900	622,000	-21900	479610000
Oct.	615,800	598,000	-17800	316840000
Nov.	534,900	582,000	47100	2218410000
Dec.	634,400	632,000	-2400	5760000
			$\sum R^2$	10275480000
			Mean square error	856290000

Figure 1 represents the actual and theoretical costs of the industry. For example, in January 2022, we can see that the actual cost was greater than the estimated cost, but from the data analysis, our calculated cost is less than both the actual and estimated cost. Similarly, if we compare the cost from the table with our calculations, we can see the difference. It means that there is a sufficient difference in both costs, meaning that costs are minimized to a great extent.

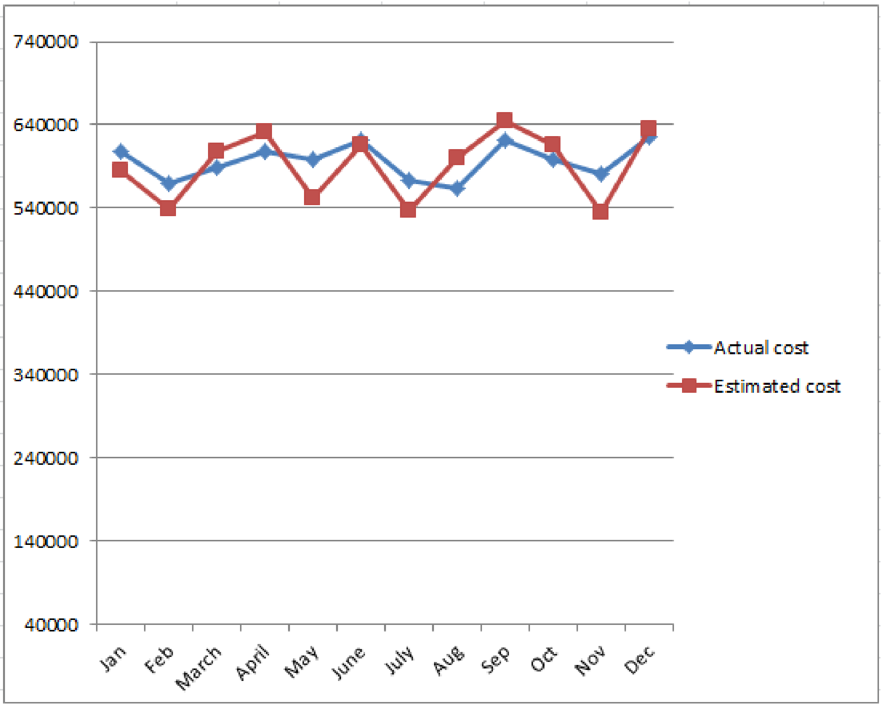

Figure 1. Estimated cost versus actual cost of the industry using two inputs.

3.2. Cost minimization in the water industry using the three-factor C-D function

In this case, the cost function consists of three factors: capital, chemicals, and labor. The cost function is given by:

$$C(X, Y, Z) = \alpha_1 + \alpha_2 X + \alpha_3 Y + \alpha_4 Z \tag{3.14}$$

where

α_1 represents the fixed cost of the industry,
α_2 represents the unit price of labor per hour,
α_3 represents the unit price of chemicals per kg,
α_4 represents the unit price of capital per kg,
X represents the number of labor hours,
Y represents the amount of chemicals in kg,
Z represents the amount of capitals in kg,

where

$$\alpha_1 = 75000, \quad \alpha_2 = 1250, \quad \alpha_3 = 2000, \quad \alpha_4 = 100.$$

Putting the above values in Eq (3.14), we get

$$C(X, Y, Z) = 75000 + 1250X + 2000Y + 100Z. \tag{3.15}$$

Now, the production function, as in [26], is given by

$$P = AX^{a_1} Y^{a_2} Z^{a_3} \tag{3.16}$$

where

P represents amount of production in liters.

After linearizing, we get

$$\ln P = \ln A + a_1 \ln X + a_3 \ln Y + a_3 \ln Z. \tag{3.17}$$

A data analysis of the water industry consisting of three inputs for the year 2022 is given in Table 3. The three inputs are human labor, capital, and chemicals, respectively.

Table 3. Inputs and output values of the water industry with three inputs.

Months	Labors(X) in hours	Chemicals(Y) in kg	Capitals(Z) in kg	Production(P) in litres	$Ln(X)$	$Ln(Y)$	$Ln(Z)$	$Ln(P)$
Jan.	240	60	900	36,000	5.4806	4.0943	6.8023	10.4912
Feb.	220	52	833	34,000	5.3936	3.9512	6.7250	10.4341
March	252	63	924	36,700	5.5294	4.1431	6.8287	10.5105
April	264	66	942	37,000	5.5759	4.1896	6.8480	10.5186
May	222	56.5	874	35,700	5.4026	4.0342	6.773	10.4829
June	260	61	922	36,400	5.5606	4.1108	6.8265	10.5023
July	215	54	860	33,600	5.3706	3.9889	6.7569	10.4222
Aug.	250	58	950	35,500	5.5214	4.0604	6.8564	10.4772
Sep.	268	70	957	36,875	5.5909	4.2484	6.8638	10.5152
Oct.	256	65	914	37,000	5.5451	4.1743	6.8178	10.5186
Nov.	212	54	875	33,600	5.3565	3.9889	6.7742	10.4222
Dec.	262	74	890	36,500	5.5683	4.3040	6.7912	10.5050

This data is collected from the water industry for the year 2022, and all results are calculated on a monthly basis.

From these analyses, we have,

$$A = 3938.188, \quad a_1 = 0.2944, \quad a_2 = 0.0673, \quad a_3 = 0.0457.$$

Putting these values in (3.16), we get

$$P = 3938.188X^{0.2944}Y^{0.0673}Z^{0.0457}. \tag{3.18}$$

Using the Lagrange multipliers method, we have,

$$L(X, Y, Z, \lambda) = C(X, Y, Z) + \lambda H(X, Y, Z), \tag{3.19}$$

where

$$H(X, Y, Z) = 36000 - 3938.188X^{0.2944}Y^{0.0673}Z^{0.0457}. \tag{3.20}$$

Putting Eqs (3.15) and (3.20) in Eq (3.19), we obtain

$$L = 75000 + 1250X + 2000Y + 100Z + \lambda(36000 - 3938.188X^{0.2944}Y^{0.0673}Z^{0.0457}). \tag{3.21}$$

Now, taking partial derivatives of L with respect to X, Y, Z and λ and setting all partial derivatives equal to zero, we have,

$$\frac{\partial L}{\partial X} = 0.$$

we then get

$$1250 - 1159.4925\lambda X^{-0.7056} Y^{0.0673} Z^{0.0457} = 0.$$

Solving the above equation for λ, we get

$$\lambda = \frac{1250}{1159.4925 X^{-0.7056} Y^{0.0673} Z^{0.0457}}. \tag{3.22}$$

also setting

$$\frac{\partial L}{\partial Y} = 0,$$

we get

$$2000 - 265.04\lambda X^{0.2944} Y^{-0.9327} Z^{0.0457} = 0.$$

Solving above equation for λ, we get

$$\lambda = \frac{7.5460}{X^{0.2944} Y^{-0.9327} Z^{0.0457}}. \tag{3.23}$$

Finally setting

$$\frac{\partial L}{\partial Z} = 0,$$

we get

$$100 - 179.9751\lambda X^{0.2944} Y^{0.0673} Z^{-0.9543} = 0.$$

Solving above equation for λ, we get

$$\lambda = \frac{100}{179.9751 X^{0.2944} Y^{0.0673} Z^{-0.9543}}. \tag{3.24}$$

Similarly, when

$$\frac{\partial L}{\partial \lambda} = 0,$$

we get

$$36000 - 3938.188 X^{0.2944} Y^{0.0673} Z^{0.0457} = 0. \tag{3.25}$$

Comparing (3.22) and (3.23), we get

$$\frac{1250}{1159.4925 X^{-0.7056} Y^{0.0673} Z^{0.0457}} = \frac{7.5460}{X^{0.2944} Y^{-0.9327} Z^{0.0457}}$$

which implies

$$X = 6.9996Y. \tag{3.26}$$

Now, comparing (3.23) and (3.24), we get

$$\frac{7.5460}{X^{0.2944} Y^{-0.9327} Z^{0.0457}} = \frac{100}{179.9751 X^{0.2944} Y^{0.0673} Z^{-0.9543}},$$

which implies

$$Z = 13.5809Y. \tag{3.27}$$

Using (3.25) and (3.26) in (3.27)

$$(6.9996Y)^{0.2944}Y^{0.0673}(13.5809Y)^{0.0457} = 9.1412,$$

yields,

$$Y^{0.4074} = 4.5755.$$

Thus,

$$Y = 41.7932. \tag{3.28}$$

Putting (3.28) in (3.26) and (3.27) respectively, we get

$$\begin{aligned} X &= 292.5356, \\ Z &= 567.5892, \end{aligned} \tag{3.29}$$

and using the values of X, Y and Z in Eq (3.15), we have

$$C(X, Y, Z) = 581014. \tag{3.30}$$

Table 4 presents actual and estimated costs on a monthly basis for the water industry for the year 2022.

Table 4. Actual and estimated cost values for the water industry using three inputs.

Months	Estimated cost in rupees	Actual cost in rupees	Error (R)	Square of the error (R^2)
Jan.	585,000	608,000	23000	529000000
Feb.	537,300	570,000	32700	1069290000
March	608,400	588,000	-20400	416160000
April	631,200	608,000	-23200	538240000
May	552,900	598,000	45100	2034010000
June	614,200	621,000	6800	46240000
July	537,750	574,000	36250	1314062500
Aug.	598,500	564,000	-34500	1190250000
Sep.	645,700	622,000	-23700	561690000
Oct.	616,400	598,000	-18400	338560000
Nov.	535,500	582,000	46500	2162250000
Dec.	639,500	632,000	-7500	56250000
			$\sum R^2$	10256002500
			Mean square error	854666875

Figure 2 represents the actual and estimated cost of the industry for the three-factors C-D production function.

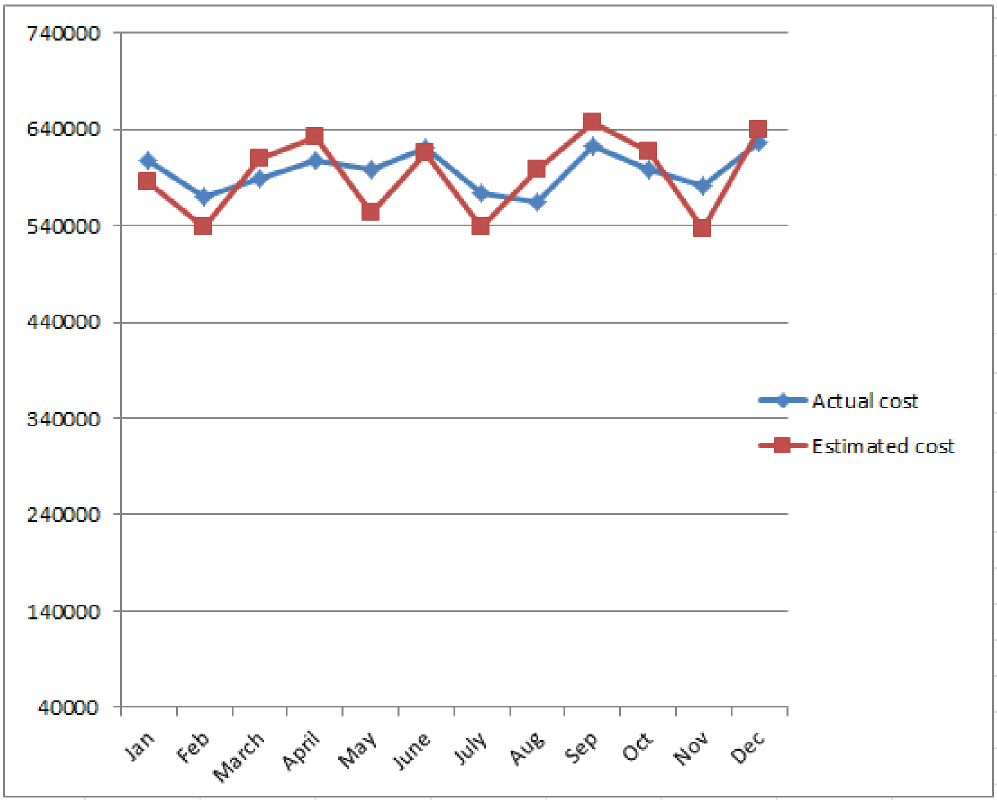

Figure 2. Estimated cost versus actual cost for the industry for three inputs.

4. Results and comparative analysis

We have solved the C-D production function with two and three factors of production using the Lagrange multiplier method with the ordinary least squares method. Moreover, this is an optimal solution approach for the C-D production function with three factors of production using the Lagrange multiplier method with the ordinary least squares method. Despite the fact that the proposed approach is a different solution technique as compared to the existing solution techniques in the literature, we still compared the general features of the proposed methodology for the water industry with different factors to Nervole's approach. In the following, we compared the presented solution approach with Nervole's approach [22]. In Table 5, we have differentiated Nerlove's C-D function and the C-D production function with the Lagrange multiplier method.

Table 5. Comparative analysis between Nerlove's C-D function and the C-D production function with the Lagrange multiplier method.

No.	Nerlove's C-D cost function	C-D production function
1	Nerlove used the C-D cost model	We used the C-D production function
2	Nerlove used the function to estimate the cost	We used the given function as an output
3	Nerlove approach needs much algebra for evaluation	It has a simple implementation
4	Computational complexity is too much	Having less computational complexity

In addition, the paired t-test is used to determine if there is a significant difference between the means of two related data sets as well as to find the mean square error from the findings of both the two- and three-factor C-D production functions. In the case of the findings of the two-factor C-D function, the outcomes of a paired t-test on two data sets result in a P-value of 0.547 and degrees of freedom (df) of 11. The paired t-test compares the means of the two variables to determine if there is a significant difference between them. The null hypothesis is that there is no significant difference between the means of the two variables. This means that there is no significant difference between the means of the two variables at the 5% level of significance. This showed that the P-value is greater than the critical value of 0.05, indicating that we cannot reject the null hypothesis.

In the case of the findings from the three-factor C-D function, the findings of a paired t-test result in a P-value of 0.559 and degrees of freedom (df) of 11. Moreover, the findings from the three-factor C-D using a paired t-test were also conducted, which also showed that there is no significant difference between the means of the two variables with the same 5% level of significance.

Based on the findings of the paired t-test, we conclude that there is no significant difference between the means of the two related variables from the findings of two and three factors in the C-D production functions. This shows that the C-D production function plays a key role in the production problem when using the Lagrange multiplier method with the ordinary least squares method. Furthermore, we have worked on the cost comparison of two and three factors in the C-D production function. In the case of two factors of production, the cost value is 597107 per unit, while in the case of three factors of production, the cost value is 581014 per unit. In both cases, our calculated cost is less than the actual cost of the industry. Besides, the cost calculation for the three-factor C-D production function is less than that of the two-factor C-D production function. Clearly, we can see the differences in Tables 6–9.

Table 6. Data set for the two-factor C-D production function.

Paired samples statistics (Pair 1)				
	Mean	N	Std. deviation	Std. error mean
Estimated cost	591700.0000	12	40670.96351	11740.69586
Actual cost	597083.3333	12	21997.76159	6350.20679
Paired samples correlations (Pair 1)				
		N	Correlation	Sig.
Estimated cost and Actual cost		12	0.690	0.013

Table 7. Paired sample test.

	Paird differences							
	Mean	Std. derivation	Std. eror mean	95s% confidence interval of the difference		t	df	$Sig.$ (2-tailed)
				Lower	Upper			
Pair1 estimated cost-actual cost	5383.33333	30041.96560	8672.36846	-24471.08762	13704.42096	−0.621	11	0.547

Table 8. Data set for the three-factor C-D production function.

Paired samples statistics (Pair 1)				
	Mean	N	Std. deviation	Std. error mean
Estimated cost	591862.5000	12	41320.23016	11928.12300
Actual cost	597083.3333	12	21997.76159	6350.20679
Paired samples correlations (Pair 1)				
		N	Correlation	Sig.
Estimated cost and Actual cost		12	0.709	0.010

Table 9. Paired sample test.

	Paird differences					t	df	$Sig.$ (2-tailed)
	Mean	Std. derivation	Std. error mean	95s% Confidence interval of the difference				
				Lower	Upper			
Pair1 estimated cost-Actual cost	5220.83333	30043.78213	8672.89265	-24309.74179	13866.07513	−0.602	11	0.559

5. Conclusions

In this paper, the optimal solution developed by C-D is carried out with different production factors. From these analyses, we concluded that the C-D production function plays a key role in the production problem when using the Lagrange multiplier method with the ordinary least squares method. Moreover, we solved the constrained optimization problem with a two-factor and three-factor C-D production function using the Lagrange multiplier with the ordinary least squares method. In the case of two factors of production, the cost value is 597107 per unit, while in the case of three factors of production, the cost value is 581014 per unit. This showed that, with more production factors in the C-D production function, the cost value is minimized to a high extent. This validates that the C-D production function with more factors using the Lagrange multiplier is more effective than previous approaches in literature. Moreover, the presented solution methodology is compared to Nervole's C-D production function. This means that the individual expression of each factor as an input has a key role in obtaining the best optimized results. Moreover, we optimized the overall cost of the water industry using the three-factor C-D production function as an application of C-D production using the Lagrange multiplier method with the ordinary least squares method.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interest.

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