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***Research article***

# Generalizations of AM-GM-HM means inequalities

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**Abstract:** In this paper, we showed some generalized refinements and reverses of arithmetic-geometric-harmonic means (AM-GM-HM) inequalities due to Sababheh [J. Math. Inequal. 12 (2018), 901–920]. Among other results, it was shown that if $a, b > 0$, $0 < p \le t < 1$ and $m \in \mathbb{N}^+$, then

$$\frac{(a\nabla_p b)^m - (a!_p b)^m}{(a\nabla_t b)^m - (a!_t b)^m} \le \frac{p(1-p)}{t(1-t)}$$

and

$$\frac{(a\sharp_p b)^m - (a!_p b)^m}{(a\sharp_t b)^m - (a!_t b)^m} \le \frac{p(1-p)}{t(1-t)}$$

for $b \ge a$, and the inequalities are reversed for $b \le a$. As applications, we obtained some inequalities for operators and determinants.

**Keywords:** AM-GM-HM; determinants; operators
**Mathematics Subject Classification:** 15A45, 15A60, 47A30, 47A63

## 1. Introduction

Let $M_n(\mathbb{C})$ denote the algebra of all $n\times n$ complex matrices and $M_n^{++}(\mathbb{C})$ be the set of positive definite matrices in $M_n(\mathbb{C})$. For $A$ and $B$ are two Hermitian matrices, $A > B$ means that $A - B \in M_n^{++}(\mathbb{C})$. Let $B(\mathcal{H})$ denote the $C^*$-Algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. An operator $A \in B(\mathcal{H})$ is called positive if $\langle Ax, x\rangle \ge 0$ for all $x \in \mathcal{H}$, denoted by $A \ge 0$. The set of all positive invertible operators is denoted by $B^{++}(\mathcal{H})$. For two self-adjoint operators $A, B \in B(\mathcal{H})$, $A > B$ means $(A - B) \in B^{++}(\mathcal{H})$.

As usual, we denote the arithmetic-geometric-harmonic means (AM-GM-HM) as $A\nabla_p B = (1 - p)A + pB$, $A\sharp_p B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^p A^{\frac{1}{2}}$ and $A!_p B = ((1-p)A^{-1} + pB^{-1})^{-1}$ for $A, B \in B^{++}(\mathcal{H})$ and $0 \le p \le 1$. Similarly, we define the weighted AM-GM-HM as $a\nabla_p b = (1 - p)a + pb$, $a\sharp_p b = a^{1-p}b^p$

and $a!_p b = ((1-p)a^{-1} + pb^{-1})^{-1}$ for $a, b > 0$ and $0 \le p \le 1$.

The scalars AM-GM-HM means reads

$$a!_p b \le a\sharp_p b \le a\nabla_p b,$$

where $a, b > 0$ and $p \in [0, 1]$. In 2015, Alzer, Fonseca and Kovačec [1] presented the following AM-GM means inequalities

$$\left(\frac{p}{t}\right)^m \le \frac{(a\nabla_p b)^m - (a\sharp_p b)^m}{(a\nabla_t b)^m - (a\sharp_t b)^m} \le \left(\frac{1-p}{1-t}\right)^m \tag{1.1}$$

for $0 < p \le t < 1$ and $m \ge 1$. In fact, Alzer-Fonseca-Kovačec's inequalities have become one of the most important extensions to Young's inequalities for the past few years. Liao and Wu [5] replicated (1.1) as follows

$$\left(\frac{p}{t}\right)^m \le \frac{(a\nabla_p b)^m - (a!_p b)^m}{(a\nabla_t b)^m - (a!_t b)^m} \le \left(\frac{1-p}{1-t}\right)^m \tag{1.2}$$

for $a, b > 0$, $0 < p \le t < 1$ and $m \ge 1$. Sababheh [7] improved (1.2) under some conditions: For $a, b > 0$ and $k = 1, 2$,
(i) if $(b-a)(t-p) \ge 0$, then

$$\frac{(a\nabla_p b)^k - (a!_p b)^k}{(a\nabla_t b)^k - (a!_t b)^k} \le \frac{p(1-p)}{t(1-t)}; \tag{1.3}$$

(ii) if $(b-a)(t-p) \le 0$, then

$$\frac{(a\nabla_p b)^k - (a!_p b)^k}{(a\nabla_t b)^k - (a!_t b)^k} \ge \frac{p(1-p)}{t(1-t)}. \tag{1.4}$$

In the same paper [7], the author also showed that: If $(b-a)(t-p) \ge 0$, then

$$\frac{a\sharp_p b - a!_p b}{a\sharp_t b - a!_t b} \le \frac{p(1-p)}{t(1-t)}. \tag{1.5}$$

We refer the readers to [3, 4, 8, 9] and references therein for some other results about the AM-GM-HM means inequality.

Following the ideas of Yang and Wang [10], we will give some generalizations of inequalities (1.3)–(1.5) and a generalized reverse of inequality (1.5). As applications, we obtain some inequalities for operators and determinants.

## 2. Main results

Firstly, we give the generalization of inequalities (1.3) and (1.5). Without loss of generality, we may assume $0 < p \le t < 1$ in the following theorem.

**Theorem 1.** *Let $a, b > 0$, $0 < p \le t < 1$ and $m \in \mathbb{N}^+$. If $b \ge a$, then we have*

$$\frac{(a\nabla_p b)^m - (a!_p b)^m}{(a\nabla_t b)^m - (a!_t b)^m} \le \frac{p(1-p)}{t(1-t)} \tag{2.1}$$

*and*

$$\frac{(a\sharp_p b)^m - (a!_p b)^m}{(a\sharp_t b)^m - (a!_t b)^m} \le \frac{p(1-p)}{t(1-t)}. \tag{2.2}$$

*Proof.* Letting $f(p) = (1-p+px)^m - ((1-p+px^{-1})^{-1})^m$, then $f(p) = ((1-p+px)-(1-p+px^{-1})^{-1})h(p)$, where $h(p) = (1-p+px)^{m-1} + (1-p+px)^{m-2}(1-p+px^{-1})^{-1} + \cdots + (1-p+px)(1-p+px^{-1})^{2-m} + (1-p+px^{-1})^{1-m}$. Thus, we have $h'(p) = (x-1)[(m-1)(1-p+px)^{m-2} + (m-2)(1-p+px)^{m-3}(1-p+px^{-1})^{-1} + \cdots + (1-p+px^{-1})^{2-m}] + (1-x^{-1})[(1-p+px)^{m-2}(1-p+px^{-1})^{-2} + \cdots + (1-p+px)(m-2)(1-p+px^{-1})^{1-m} + (m-1)(1-p+px^{-1})^{-m}]$. It is clear that $h'(p) \ge 0$ when $x \ge 1$, which means that $h(p) \le h(t)$. Therefore,

$$\begin{aligned}\frac{f(p)}{f(t)} &= \frac{(1-p+px)^m - ((1-p+px^{-1})^{-1})^m}{(1-t+tx)^m - ((1-t+tx^{-1})^{-1})^m}\\ &= \frac{((1-p+px)-(1-p+px^{-1})^{-1})h(p)}{((1-t+tx)-(1-t+tx^{-1})^{-1})h(t)}\\ &\le \frac{(1-p+px)-(1-p+px^{-1})^{-1}}{(1-t+tx)-(1-t+tx^{-1})^{-1}}\\ &\le \frac{p(1-p)}{t(1-t)}. \quad \text{(by (1.3))}\end{aligned}$$

Taking $x = \frac{b}{a}$, we complete the proof of (2.1). Similarly, letting $f(p) = (x^p)^m - ((1-p+px^{-1})^{-1})^m$, then $f(p) = (x^p - (1-p+px^{-1})^{-1})h(p)$, where $h(p) = (x^p)^{m-1} + (x^p)^{m-2}(1-p+px^{-1})^{-1} + \cdots + x^p(1-p+px^{-1})^{2-m} + (1-p+px^{-1})^{1-m}$. Thus, we have $h'(p) = \ln x[(m-1)x^{(m-1)p} + (m-2)x^{(m-2)p}(1-p+px^{-1})^{-1} + \cdots + x^p(1-p+px^{-1})^{2-m}] + (1-x^{-1})[x^{(m-2)p}(1-p+px^{-1})^{-2} + \cdots + (m-2)x^p(1-p+px^{-1})^{1-m} + (m-1)(1-p+px^{-1})^{-m}]$. It is clear that $h'(p) \ge 0$ when $x \ge 1$, which means that $h(p) \le h(t)$. Therefore,

$$\begin{aligned}\frac{f(p)}{f(t)} &= \frac{x^{mp} - ((1-p+px^{-1})^{-1})^m}{x^{mt} - ((1-t+tx^{-1})^{-1})^m}\\ &= \frac{(x^p - (1-p+px^{-1})^{-1})h(p)}{(x^t - (1-t+tx^{-1})^{-1})h(t)}\\ &\le \frac{x^p - (1-p+px^{-1})^{-1}}{x^t - (1-t+tx^{-1})^{-1}}\\ &\le \frac{p(1-p)}{t(1-t)}, \quad \text{(by (1.5))}\end{aligned}$$

taking $x = \frac{b}{a}$, as desired. □

Letting $a = b$, $b = a$, $p = 1-t$ and $t = 1-p$ in Theorem 1, we have the following results:

**Corollary 2.** *Let $a, b > 0$, $0 < p \leq t < 1$ and $m \in \mathbb{N}^+$. If $a \geq b$, then we have*

$$\frac{(a\nabla_t b)^m - (a!_t b)^m}{(a\nabla_p b)^m - (a!_p b)^m} \leq \frac{t(1-t)}{p(1-p)}$$

*and*

$$\frac{(a\sharp_t b)^m - (a!_t b)^m}{(a\sharp_p b)^m - (a!_p b)^m} \leq \frac{t(1-t)}{p(1-p)}.$$

*That is*

$$\frac{(a\nabla_p b)^m - (a!_p b)^m}{(a\nabla_t b)^m - (a!_t b)^m} \geq \frac{p(1-p)}{t(1-t)} \tag{2.3}$$

*and*

$$\frac{(a\sharp_p b)^m - (a!_p b)^m}{(a\sharp_t b)^m - (a!_t b)^m} \geq \frac{p(1-p)}{t(1-t)}. \tag{2.4}$$

We notice that inequality (2.3) is the generalization of (1.4), and inequality (2.4) is the reverse of (1.5) when $m = 1$.

Next, we explain that Theorem 1 and Corollary 2 improved inequalities (1.2).

**Remark 3.** *Let $a, b > 0$, $0 < p \leq t < 1$ and $m \in \mathbb{N}^+$.*
*(i) If $b \geq a$, we have*

$$\frac{(a\nabla_p b)^m - (a!_p b)^m}{(a\nabla_t b)^m - (a!_t b)^m} \leq \frac{p(1-p)}{t(1-t)} \leq \frac{p(1-p)^m}{t(1-t)^m} \leq \left(\frac{1-p}{1-t}\right)^m.$$

*(ii) If $a \geq b$, we have*

$$\left(\frac{p}{t}\right)^m \leq \frac{p^m(1-p)}{t^m(1-t)} \leq \frac{p(1-p)}{t(1-t)} \leq \frac{(a\nabla_p b)^m - (a!_p b)^m}{(a\nabla_t b)^m - (a!_t b)^m}.$$

Next, we give some inequalities for operators and determinants by Theorem 1 and Corollary 2. We will list some necessary lemmas in front of each theorem.

**Lemma 4** ([6, p.3])**.** *Let $X \in B(\mathcal{H})$ be self-adjoint and $f$ and $g$ be continuous real functions such that $f(t) \geq g(t)$ for all $t \in Sp(X)$ (the spectrum of $X$), then $f(X) \geq g(X)$.*

**Theorem 5.** *Let $A \in B^{++}(\mathcal{H})$, $0 < p \leq t < 1$ and $m \in \mathbb{N}^+$.*

*(i) If $A \geq I$, then we have*

$$(I\nabla_p A)^m - (I!_p A)^m \leq \frac{p(1-p)}{t(1-t)}\Big((I\nabla_t A)^m - (I!_t A)^m\Big) \tag{2.5}$$

*and*

$$(I\sharp_p A)^m - (I!_p A)^m \leq \frac{p(1-p)}{t(1-t)}\Big((I\sharp_t A)^m - (I!_t A)^m\Big). \tag{2.6}$$

*(ii) The inequalities (2.5) and (2.6) are reversed if $A \leq I$.*

*Proof.* We only prove the first inequality. The other inequalities are shown similarly. Let $a = 1$ in inequality (2.1), then we get

$$(1\nabla_p b)^m - (1!_p b)^m \le \frac{p(1-p)}{t(1-t)}\Big((1\nabla_t b)^m - (1!_t b)^m\Big). \tag{2.7}$$

The operator $A$ has a positive spectrum, then by Lemma 4 and inequality (2.7) we get

$$(I\nabla_p A)^m - (I!_p A)^m \le \frac{p(1-p)}{t(1-t)}\Big((I\nabla_t A)^m - (I!_t A)^m\Big). \tag{2.8}$$

Finally, multiplying inequality (2.8) by $A^{\frac{m}{2}}$ on both left and right sides, we can get (2.5) directly. □

Before we give some inequalities for determinants as promised, we should recall some basic signs. The singular values of a matrix $A$ are defined by $s_j(A)$, $j = 1, 2, \cdots, n$, and we denote the values of $\{s_j(A)\}$ as a nonincreasing order. To obtain our results, we need a following lemma.

**Lemma 6** ([2], p.26)**.** *(Minkowski inequality) Let $a = [a_i]$, $b = [b_i]$, $i = 1, 2, \cdots, n$ be such that $a_i$, $b_i$ are two sets of positive numbers, then*

$$\Big(\prod_{i=1}^{n} a_i\Big)^{\frac{1}{n}} + \Big(\prod_{i=1}^{n} b_i\Big)^{\frac{1}{n}} \le \Big(\prod_{i=1}^{n}(a_i + b_i)\Big)^{\frac{1}{n}}.$$

*With equation if and only if $a = b$.*

**Theorem 7.** *Let $A, B \in M_n^{++}(\mathbb{C})$, $0 < p \le t < 1$ and $m \in \mathbb{N}^+$. If $B \ge A$, then we have*

$$\frac{t(1-t)}{p(1-p)} \det\left(A\nabla_p B - A!_p B\right)^{\frac{m}{n}} + \det(A!_t B)^{\frac{m}{n}} \le \det(A\nabla_t B)^{\frac{m}{n}} \tag{2.9}$$

*and*

$$\frac{t(1-t)}{p(1-p)} \det\left(A\sharp_p B - A!_p B\right)^{\frac{m}{n}} + \det(A!_t B)^{\frac{m}{n}} \le \det(A\sharp_t B)^{\frac{m}{n}}. \tag{2.10}$$

*Proof.* Under the conditions, we have $1 \le s_j(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})$ for $B \ge A$. Putting $a = 1$ and $b = s_j(T)$, where $T = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ in the inequality (2.1), then we have

$$\frac{(1\nabla_p s_j(T))^m - (1!_p s_j(T))^m}{(1\nabla_t s_j(T))^m - (1!_t s_j(T))^m} \le \frac{p(1-p)}{t(1-t)} \tag{2.11}$$

for $j = 1, 2, \cdots, n$. It is a fact that the determinant of a positive definite matrix is a product of its singular values, and we have

$$\begin{aligned}\det(I\nabla_t T)^{\frac{m}{n}} &= \Big(\prod_{j=1}^{n}(1\nabla_t s_j(T))^m\Big)^{\frac{1}{n}}\\ &\ge \prod_{j=1}^{n}\Big[\frac{t(1-t)}{p(1-p)}\Big((1\nabla_p s_j(T))^m - (1!_p s_j(T))^m\Big) + (1!_t s_j(T))^m\Big]^{\frac{1}{n}} \quad \text{(by (2.11))}\end{aligned}$$

$$\geq \prod_{j=1}^{n}\Big[\frac{t(1-t)}{p(1-p)}\Big((1\nabla_p s_j(T))^m-(1!_p s_j(T))^m\Big)\Big]^{\frac{1}{n}}+\prod_{j=1}^{n}\Big((1!_t s_j(T))^m\Big)^{\frac{1}{n}} \quad \text{(by Lemma 6)}$$
$$\geq \frac{t(1-t)}{p(1-p)}\prod_{j=1}^{n}(1\nabla_p s_j(T)-1!_p s_j(T))^{\frac{m}{n}}+\prod_{j=1}^{n}(1!_t s_j(T))^{\frac{m}{n}}$$
$$=\frac{t(1-t)}{p(1-p)}\det(I\nabla_p T-I!_p T)^{\frac{m}{n}}+\det(I!_t T)^{\frac{m}{n}},$$

where the last inequality is by $(a^m-b^m)\geq(a-b)^m$ for $a\geq b>0$ and $m\in\mathbb{N}^+$. Multiplying $(\det A^{\frac{1}{2}})^{\frac{m}{n}}$ on both sides of the inequalities above, we can get the desired inequality (2.9).

Using the same technique above in (2.2), we can obtain inequality (2.10) easily, so we omit the details. □

**Corollary 8.** *Let $A,B\in M_n^{++}(\mathbb{C})$, $0<p\leq t<1$ and $m\in\mathbb{N}^+$. If $B\leq A$, then we have*

$$\frac{p(1-p)}{t(1-t)}\det(A\nabla_t B-A!_t B)^{\frac{m}{n}}+\det(A!_p B)^{\frac{m}{n}}\leq\det(A\nabla_p B)^{\frac{m}{n}}$$

*and*

$$\frac{p(1-p)}{t(1-t)}\det(A\sharp_t B-A!_t B)^{\frac{m}{n}}+\det(A!_p B)^{\frac{m}{n}}\leq\det(A\sharp_p B)^{\frac{m}{n}}.$$

*Proof.* The proof comes from Corollary 2 and Theorem 7 directly. □

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

## Conflict of interest

The authors declare that they have no competing interests.

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