Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(12): 29910–29924.
DOI:10.3934/math.20231529
Received: 07 September 2023
Revised: 22 October 2023
Accepted: 29 October 2023
Published: 03 November 2023

Research article

The instability of periodic solutions for a population model with cross-diffusion

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Abstract: This paper is concerned with a population model with prey refuge and a Holling type III functional response in the presence of self-diffusion and cross-diffusion, and its Turing pattern formation problem of Hopf bifurcating periodic solutions was studied. First, we discussed the stability of periodic solutions for the ordinary differential equation model, and derived the first derivative formula of periodic functions for the perturbed model. Second, applying the Floquet theory, we gave the conditions of Turing patterns occurring at Hopf bifurcating periodic solutions. Additionally, we determined the range of cross-diffusion coefficients for the diffusive population model to form Turing patterns at the stable periodic solutions. Finally, our research was summarized and the relevant conclusions were simulated numerically.

Keywords: refuge; population model; cross-diffusion; periodic solutions; Turing pattern
Mathematics Subject Classification: 35B32, 35K57, 35Q92

1. Introduction

Since 1946, biologist Crombic proved the stability effect through experiments [1, 2] and more and more scholars analyzed the refuge effect on the population model [3–9], mainly focused on the self-diffusion effect on dynamic behavior of the population system. In addition to the effect of self-diffusion, cross-diffusion also plays an important role during the population pattern formation. About the predator-prey systems with diffusion terms, many scholars have studied the Turning instability and Hopf bifurcation of its constant equilibrium [10–17]. At present, for the reaction-diffusion predator-prey system, most literatures [18–25] focus on Turing instability of the constant equilibrium, but there are few research results on the stability of the periodic solutions. Therefore, it is significant to study the Turing pattern formation of Hopf bifurcating periodic solutions for the cross-diffusion population model with prey refuge and the Holling III functional response.

In 2015, Yang et al. [9] studied a diffusive prey-predator system in Holling type III with a prey refuge:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = D_1\Delta u + au - ru^2 - \frac{\alpha(1-m)^2u^2v}{\beta^2+(1-m)^2u^2}, \ x \in \Omega, t > 0, \\ \frac{\partial v(x,t)}{\partial t} = D_2\Delta v - cv + \frac{k\alpha(1-m)^2u^2v}{\beta^2+(1-m)^2u^2}, \ x \in \Omega, t > 0, \\ \partial_\nu u = \partial_\nu v = 0, \ x \in \partial\Omega, t > 0. \end{cases} \tag{1.1}$$

Here, u, v indicates the quantity of prey and predator respectively; $\alpha, \beta, a, r, c, k$ are all positive; a is the intrinsic growth rate of the prey; a/r represents the maximum carrying capacity of the environment on the prey; c is the mortality rate of the predator; k represents the conversion rate after the predator eating the prey; $m \in [0, 1)$ indicates the refuge coefficient, i.e., the proportion of the protected prey. Only $(1 - m)u$ can be caught by the predator. In the real world, the mobility of each species is affected not only by itself but also by the density of other species. Therefore, on the basis of (1.1), we introduce the cross-diffusion terms and establish the population model as follows:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = D_{11}\Delta u + D_{12}\Delta v + au - ru^2 - \frac{\alpha(1-m)^2u^2v}{\beta^2+(1-m)^2u^2}, \ x \in \Omega, t > 0, \\ \frac{\partial v(x,t)}{\partial t} = D_{21}\Delta u + D_{22}\Delta v - cv + \frac{k\alpha(1-m)^2u^2v}{\beta^2+(1-m)^2u^2}, \ x \in \Omega, t > 0, \\ \partial_\nu u = \partial_\nu v = 0, \ x \in \partial\Omega, t > 0, \end{cases} \tag{1.2}$$

where $\Omega = (0, l\pi)$ is a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n$ and D_{11}, D_{22} and D_{12}, D_{21} are the self-diffusivity and cross-diffusivity of u and v. We assume that the diffusion coefficients satisfy $D_{11}D_{22} - D_{12}D_{21} > 0$.

The organizational structure of the rest is as follows: In section two, we study the stability of Hopf bifurcating periodic solutions for the ordinary differential population model and derive the first derivative formula of the periodic function for the corresponding perturbed model. In section three, we give the conditions of Turing patterns occurring at Hopf bifurcating periodic solutions in the reaction-diffusion population system. In section four, we give a brief conclusion. Finally, the relevant conclusions are verified by numerical simulations.

2. Dynamics of the zero-dimensional population model

In order to research conveniently, we nondimensionalize model (1.2). Let $\hat{u} = \frac{u}{\beta}, \hat{v} = \frac{v}{k\beta}, \hat{t} = at$, and we still replace $\hat{u}, \hat{v}, \hat{t}$ with u, v, t, then model (1.2) becomes

$$\begin{cases} \frac{\partial u}{\partial t} = d_{11}\Delta u + d_{12}\Delta v + u - pu^2 - \frac{s(1-m)^2u^2v}{1+(1-m)^2u^2}, \ x \in \Omega, t > 0, \\ \frac{\partial v}{\partial t} = d_{21}\Delta u + d_{22}\Delta v - \theta v + \frac{s(1-m)^2u^2v}{1+(1-m)^2u^2}, \ x \in \Omega, t > 0, \\ \partial_\nu u = \partial_\nu v = 0, \ x \in \partial\Omega, t > 0, \end{cases} \tag{2.1}$$

where, $d_{11} = \frac{D_{11}}{a}, d_{12} = \frac{D_{12}}{a}, d_{21} = \frac{D_{21}}{a}, d_{22} = \frac{D_{22}}{a}, \theta = \frac{c}{a}$ and $p = \frac{r\beta}{a}, s = \frac{k\alpha}{a}$.

2.1. Stability of periodic solutions of the ordinary differential population model

The ordinary differential equations corresponding to the reaction-diffusion population model (2.1) are

$$\begin{cases} \frac{du}{dt} = u - pu^2 - \frac{s(1-m)^2u^2v}{1+(1-m)^2u^2}, t > 0, \\ \frac{dv}{dt} = -\theta v + \frac{s(1-m)^2u^2v}{1+(1-m)^2u^2}, \ t > 0. \end{cases} \tag{2.2}$$

By calculation, four equilibria of model (2.2) are $P_0 = (0,0), P_1 = (1/p, 0), P_+ = (\kappa, v_\kappa), P_- = (u_-, v_-)$ with

$$\kappa = \frac{1}{1-m}\sqrt{\frac{\theta}{s-\theta}}, v_\kappa = \frac{\kappa}{\theta}(1-p\kappa), u_- = -\kappa, v_- = -\frac{(1+p\kappa)\left(1+(1-m)^2\kappa^2\right)}{s(1-m)^2\kappa}.$$

Clearly, the equilibrium $P_- = (u_-, v_-)$ has no biological significance, so we do not study its dynamic behavior. Let's make the following assumptions:

($\mathbf{A_1}$) $s > \theta, 0 \le \kappa < \frac{1}{p}$;

($\mathbf{A_2}$) $s < 2\theta, \frac{2\theta-s}{2\theta p} < \kappa < \frac{1}{p}$;

($\mathbf{A_3}$) $s \ge 2\theta, \sqrt{\frac{\theta}{s-\theta}} < \kappa < \frac{1}{p}$;

($\mathbf{A_4}$) $\theta < s < 2\theta, \sqrt{\frac{\theta}{s-\theta}} < \kappa < \frac{2\theta-s}{2\theta p}$.

Theorem 2.1. *Let $\kappa_0 = \frac{2\theta-s}{2\theta p}$ and assume that ($\mathbf{A_1}$) satisfies. The following results are true for model* (2.2).

(1) If ($\mathbf{A_2}$) (or ($\mathbf{A_3}$)) holds, then the positive equilibrium $P_+ = (\kappa, v_\kappa)$ is locally asymptotically stable. If ($\mathbf{A_4}$) holds, then the positive equilibrium $P_+ = (\kappa, v_\kappa)$ is unstable.

(2) If ($\mathbf{A_3}$) holds, the positive equilibrium $P_+ = (\kappa, v_\kappa)$ is locally asymptotically stable for $\kappa \in \left(\kappa_0, \frac{1}{p}\right)$, while unstable for $\kappa \in \left(\sqrt{\frac{\theta}{s-\theta}}, \kappa_0\right)$. When $\kappa = \kappa_0$, the model undergoes a supercritical Hopf bifurcation at $P_+ = (\kappa, v_\kappa)$, a family of periodic solutions $\left(u^T(t), v^T(t)\right)$ bifurcate from $P_+ = (\kappa, v_\kappa)$ and the bifurcating periodic solutions are stable.

Proof. If ($\mathbf{A_1}$) holds, then $P_+ = (\kappa, v_\kappa)$ is a unique positive equilibrium of (2.2). Setting the Jacobi matrix of (2.2) at (κ, v_κ) is

$$J(\kappa) := \begin{pmatrix} a(\kappa), & b(\kappa) \\ c(\kappa), & 0 \end{pmatrix},$$

where, $a(\kappa) = \frac{2\theta}{s}(1-p\kappa) - 1, b(\kappa) = -\theta$ and $c(\kappa) = \frac{2(s-\theta)}{s}(1-p\kappa)$. The characteristic equation of $J(\kappa)$ is

$$\lambda^2 - T(\kappa)\lambda + D(\kappa) = 0, \tag{2.3}$$

with

$$T(\kappa) = \frac{2\theta}{s}(1-p\kappa) - 1, \ D(\kappa) = \frac{2\theta(s-\theta)}{s}(1-p\kappa).$$

then the roots of Eq (2.3) are

$$\lambda_{1,2} = \frac{1}{2}\left[T(\kappa) \pm \sqrt{T^2(\kappa) - 4D(\kappa)}\right].$$

If ($\mathbf{A_2}$) (*or* ($\mathbf{A_3}$)) satisfies, then all the eigenvalues of $J(\kappa)$ have strictly negative real parts according to the stability theory, and $P_+ = (\kappa, v_\kappa)$ is locally asymptotically stable. If ($\mathbf{A_4}$) is true, then all the eigenvalues of $J(\kappa)$ have positive real parts, hence, $P_+ = (\kappa, v_\kappa)$ is unstable. For an arbitrary $\kappa \in \left(\sqrt{\frac{\theta}{s-\theta}}, \kappa_0\right)$, model (2.2) is unstable at $P_+ = (\kappa, v_\kappa)$, and for an arbitrary $\kappa \in \left(\kappa_0, \frac{1}{p}\right)$, $P_+ = (\kappa, v_\kappa)$ (2.2) is locally asymptotically stable. When $\kappa = \kappa_0$, $J(\kappa_0)$ has a pair of pure imaginary roots $\lambda = \pm i\omega_0$ with $\omega_0 = (s-\theta)^{\frac{1}{2}}$. Let $\lambda(\kappa) = \beta(\kappa) \pm i\omega(\kappa)$ be the roots of Eq (2.3) near $\kappa = \kappa_0$, then we have

$$\beta(\kappa) = \frac{\theta}{s}(1-p\kappa) - \frac{1}{2}, \left.\frac{d\beta(\kappa)}{d\kappa}\right|_{\kappa=\kappa_0} = -\frac{p\theta}{s} < 0.$$

According to the Poincaré-Andronov-Hopf bifurcation theorem, system (2.1) undergoes Hopf bifurcation at $\kappa = \kappa_0$. Let the eigenvectors of $J(\kappa_0)$ and $J^*(\kappa_0)$ corresponding to the eigenvalues $i\omega_0$ and $-i\omega_0$ be $q = (1, b_0)^T, q^* = \left(a_0^*, b_0^*\right)^T$, satisfying $< q^*, q >= 1$ and $< q^*, \bar{q} >= 0$, where $b_0 = -\frac{\omega_0}{\theta} i, a_0^* = \frac{1}{2l\pi}, b_0^* = -\frac{\theta}{2l\pi\omega_0}$. Denote

$$
\begin{aligned}
f(\kappa, u, v) &= u + \kappa - p(u+\kappa)^2 - \frac{s(1-m)^2(u+\kappa)^2\,(v + v_\kappa)}{1 + (1-m)^2(u+\kappa)^2},\\
g(\kappa, u, v) &= -\theta\,(v + v_\kappa) + \frac{s(1-m)^2(u+\kappa)^2\,(v+v_\kappa)}{1+(1-m)^2(u+\kappa)^2}
\end{aligned}
$$

by [26], and we give the expression of the cubic coefficient $c_1(\kappa_0)$ in normal form. Calculating Q_{qq}, $Q_{q\bar{q}}$ and $C_{qq\bar{q}}$,

$$
Q_{qq} = \begin{pmatrix} c_0 \\ d_0 \end{pmatrix}, Q_{q\bar{q}} = \begin{pmatrix} e_0 \\ f_0 \end{pmatrix}, C_{qq\bar{q}} = \begin{pmatrix} g_0 \\ h_0 \end{pmatrix},
$$

with

$$
\begin{aligned}
c_0 &= -\frac{2p\left(2s^3 - 9s^2\theta + 14s\theta^2 - 8\theta^3 + 4is(s-\theta)\theta\omega_0\right)}{s^2(s-2\theta)}, & d_0 &= \frac{2p(s-\theta)\left(s^2 - 6s\theta + 8\theta^2 + 4is\theta\omega_0\right)}{s^2(s-2\theta)},\\
e_0 &= -\frac{2p\left(2s^2 - 5s\theta + 4\theta^2\right)}{s^2}, & f_0 &= \frac{2p\left(s^2 - 5s\theta + 4\theta^2\right)}{s^2},\\
g_0 &= \frac{8p^2(s-\theta)\theta^2\left(-6(s-2\theta)^2 + is(s-4\theta)\omega_0\right)}{s^3(s-2\theta)^2}, & h_0 &= \frac{8p^2(s-\theta)\theta^2\left(6(s-2\theta)^2 - is(s-4\theta)\omega_0\right)}{s^3(s-2\theta)^2},
\end{aligned}
$$

as well as

$$
\begin{aligned}
< \overline{q^*}, Q_{qq} > &= \frac{-i}{s^2(s-2\theta)\omega_0} p\left(\theta\left(s^3 - 7s^2\theta \;\; + 14s\theta^2 - 8\theta^3\right) + 4s(s-\theta)\theta\omega_0^2\right)\\
&\quad + \frac{1}{s^2(s-2\theta)\omega_0} p\left(-2s^3 + s^2\theta(9+4\theta) + 8\theta^3 \;\; -2s\theta^2(7+2\theta)\right)\omega_0,\\
< q^*, C_{qq\bar{q}} > &= \frac{4p^2(s-\theta)\theta^2\,(\theta + i\omega_0)\left(6i(s-2\theta)^2 + s(s-4\theta)\omega_0\right)}{s^3(s-2\theta)^2\omega_0}\\
&= \frac{4p^2(s-\theta)\theta^2}{s^3(s-2\theta)^2}\left(s\theta(s-4\theta) - 6(s-2\theta)^2\right)\\
&\quad + \frac{4p^2(s-\theta)\theta^2}{s^3(s-2\theta)^2\omega_0}\left(6\theta(s-2\theta)^2 + s(s-4\theta){\omega_0}^2\right) i,\\
< q^*, Q_{qq} > &= \frac{i}{s^2(s-2\theta)\omega_0}\left(p\theta\left(s^3 - 7s^2\theta \;\; + 14s\theta^2 - 8\theta^3\right) - 4s(s-\theta)\theta\omega_0^2\right)\\
&\quad + \frac{1}{s^2(s-2\theta)\omega_0}\left(-2s^3 + s^2(9-4\theta)\theta + 8\theta^3 \;\; + 2s\theta^2(-7+2\theta)\right),\\
< q^*, Q_{q\bar{q}} > &= p\left(i\theta\left(s^2 - 5s\theta + 4\theta^2\right) + \left(-2s^2 + 5s\theta - 4\theta^2\right)\omega_0\right).
\end{aligned}
$$

Then, we can obtain

$$
\begin{aligned}
H_{20} &= \begin{pmatrix} c_0 \\ d_0 \end{pmatrix} - < q^*, Q_{qq} > \begin{pmatrix} 1 \\ b_0 \end{pmatrix} - < \overline{q^*}, Q_{qq} > \begin{pmatrix} 1 \\ \overline{b_0} \end{pmatrix} = 0,\\
H_{11} &= \begin{pmatrix} e_0 \\ f_0 \end{pmatrix} - < q^*, Q_{q\bar{q}} > \begin{pmatrix} 1 \\ b_0 \end{pmatrix} - < \overline{q^*}, Q_{q\bar{q}} > \begin{pmatrix} 1 \\ \bar{b}_0 \end{pmatrix} = 0,
\end{aligned}
$$

so

$$c_1(\kappa_0) = \frac{i}{2\omega_0} < q^*, Q_{qq} > \cdot < q^*, Q_{q\bar{q}} > + \frac{1}{2} < \bar{q}^*, C_{qq\bar{q}} > . \tag{2.4}$$

Its real and imaginary parts are

$$\begin{aligned} \mathrm{Re}\,(c_1(\kappa_0)) = & \frac{p}{2s^2(s-2\theta){\omega_0}^2}\left(-2s^3 + s^2(9-4\theta)\theta + 8\theta^3 + 2s\theta^2(-7+2\theta)\right)\theta\left(s^2 - 5s\theta + 4\theta^2\right) \\ & - \frac{p}{2s^2(s-2\theta)\omega_0}\left(p\theta\left(s^3 - 7s^2\theta + 14s\theta^2 - 8\theta^3\right) - 4s(s-\theta)\theta\omega_0^2\right)\left(-2s^2 + 5s\theta - 4\theta^2\right) \\ & + \frac{2p^2(s-\theta)\theta^2}{s^3(s-2\theta)^2}\left(s\theta(s-4\theta) - 6(s-2\theta)^2\right) \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} \mathrm{Im}\,(c_1(\kappa_0)) = & \frac{p}{2s^2(s-2\theta)\omega_0}\left(-2s^3 + s^2(9-4\theta)\theta + 8\theta^3 + 2s\theta^2(-7+2\theta)\right)\left(-2s^2 + 5s\theta - 4\theta^2\right) \\ & - \frac{p}{2s^2(s-2\theta){\omega_0}^2}\left(p\theta\left(s^3 - 7s^2\theta + 14s\theta^2 - 8\theta^3\right) - 4s(s-\theta)\theta\omega_0^2\right)\theta\left(s^2 - 5s\theta + 4\theta^2\right) \\ & + \frac{2p^2(s-\theta)\theta^2}{s^3(s-2\theta)^2\omega_0}\left(6\theta(s-2\theta)^2 + s(s-4\theta){\omega_0}^2\right). \end{aligned} \tag{2.6}$$

If $\mathrm{Re}\,(c_1(\kappa_0)) < 0\,(> 0)$, then the Hopf bifurcation is backward (forward) and the bifurcating periodic solutions $\left(u^T(t), v^T(t)\right)$ are stable (unstable). □

2.2. *Dynamics of the perturbed population model*

In this subsection, for the perturbed population model, we derive the first derivative formula of the periodic function about the perturbation coefficients. On the basis of model (2.1), we introduce the perturbation term τ and coefficients $\begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}$. The corresponding perturbed population model is

$$\left(I + \tau\begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}\right)\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} u - pu^2 - \frac{s(1-m)^2u^2v}{1+(1-m)^2u^2} \\ -\theta v + \frac{s(1-m)^2u^2v}{1+(1-m)^2u^2} \end{pmatrix}, \tag{2.7}$$

where τ is sufficiently small such that $\begin{pmatrix} 1+\tau k_{11} & \tau k_{12} \\ \tau k_{21} & 1+\tau k_{22} \end{pmatrix}$ is reversible, then (2.7) can be reduced to

$$\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \frac{1}{K(\tau)}\begin{pmatrix} 1+k_{22}\tau & -k_{12}\tau \\ -k_{21}\tau & 1+k_{11}\tau \end{pmatrix}\begin{pmatrix} u - pu^2 - \frac{s(1-m)^2u^2v}{1+(1-m)^2u^2} \\ -\theta v + \frac{s(1-m)^2u^2v}{1+(1-m)^2u^2} \end{pmatrix}, \tag{2.8}$$

where

$$K(\tau) := \left|\begin{pmatrix} 1+\tau k_{11} & \tau k_{12} \\ \tau k_{21} & 1+\tau k_{22} \end{pmatrix}\right| = (k_{11}k_{22} - k_{12}k_{21})\,\tau^2 + (k_{11}+k_{22})\,\tau + 1 > 0.$$

At $P_+ = (\kappa, v_\kappa)$, the Jacobian matrix of (2.8) is

$$J(\kappa,\tau) := \frac{1}{K(\tau)}\begin{pmatrix} J_{11}(\kappa,\tau) & J_{12}(\kappa,\tau) \\ J_{21}(\kappa,\tau) & J_{22}(\kappa,\tau) \end{pmatrix}, \tag{2.9}$$

where,

$$\begin{aligned} &J_{11}(\kappa,\tau) := (1+k_{22}\tau)\,a(\kappa) - k_{12}c(\kappa)\,\tau, J_{12}(\kappa,\tau) := (1+k_{22}\tau)\,b(\kappa),\\ &J_{21}(\kappa,\tau) := (1+k_{11}\tau)\,c(\kappa) - k_{21}a(\kappa)\,\tau, J_{22}(\kappa,\tau) := -k_{21}b(\kappa)\,\tau,\\ &a(\kappa) = \frac{2\theta}{s}(1-p\kappa) - 1, b(\kappa) = -\theta, c(\kappa) = \frac{2(s-\theta)}{s}(1-p\kappa). \end{aligned} \tag{2.10}$$

Let the characteristic equation corresponding to $J(\kappa,\tau)$ be

$$\lambda^2 - H(\kappa,\tau)\lambda + D(\kappa,\tau) = 0, \tag{2.11}$$

where

$$\begin{aligned} H(\kappa,\tau) &= \frac{1}{K(\tau)}\tau\cdot(k_{22}a(\kappa) - k_{12}c(\kappa) - k_{21}b(\kappa)) + \frac{1}{K(\tau)}a(\kappa),\\ D(\kappa,\tau) &= \frac{1}{K(\tau)}\frac{2\theta(s-\theta)}{s}(1-p\kappa). \end{aligned} \tag{2.12}$$

When $\kappa \to \kappa_\tau$, let $\bar{\beta}(\kappa_\tau) \pm i\bar{\omega}(\kappa_\tau)$ be the roots of the characteristic Eq (2.11), then

$$\bar{\beta}(\kappa_\tau) = \frac{1}{2}H(\kappa,\tau), \bar{\omega}(\kappa_\tau) = \frac{1}{2}\sqrt{4D(\kappa,\tau) - H^2(\kappa,\tau)}. \tag{2.13}$$

Lemma 2.1. *(See [26]) When $\kappa \to \kappa_0$, the population model* (2.2) *has a stable periodic solution $\left(u^T(t), v^T(t)\right)$ bifurcating from $P_+ = (\kappa, v_\kappa)$ and T is the minimum positive period of $\left(u^T(t), v^T(t)\right)$. Then there is a positive number τ_1 such that for any $\tau \in (-\tau_1, \tau_1)$, the perturbed population model* (2.7) *has a periodic solution $\left(u^T(t,\tau), v^T(t,\tau)\right)$ depending on if τ, $T(\tau)$ is the minimum positive periodic function. When $\tau \to 0$, $\left(u^T(t,\tau), v^T(t,\tau)\right) \to \left(u^T(t), v^T(t)\right)$ and $T(\tau) \to T$, then*

$$\begin{aligned} T(\tau) &= \frac{2\pi}{\bar{\omega}(\kappa_\tau)}\left(1 + \left(\frac{\bar{\beta}'(\kappa_\tau)\,\mathrm{Im}(c_1(\kappa_\tau))}{\bar{\omega}(\kappa_\tau)\,\mathrm{Re}(c_1(\kappa_\tau))} - \frac{\bar{\omega}'(\kappa_\tau)}{\bar{\omega}(\kappa_\tau)}\right)(\kappa-\kappa_\tau) + O\left((\kappa-\kappa_\tau)^2\right), \kappa \to \kappa_\tau,\right.\\ c_1(\kappa_\tau) &:= \frac{i}{2\bar{\omega}(\kappa_\tau)}\left(g_{20}(\tau)g_{11}(\tau) - 2|g_{11}(\tau)|^2 - \frac{1}{3}|g_{02}(\tau)|^2\right) + \frac{g_{21}(\tau)}{2}. \end{aligned}$$

$\mathrm{Re}(c_1(\kappa_\tau))$ *and* $\mathrm{Im}(c_1(\kappa_\tau))$ *are the real and imaginary parts of $c_1(\kappa_\tau)$, and $\bar{\beta}(\kappa_\tau)$ and $\bar{\omega}(\kappa_\tau)$ are defined by* (2.13).

Theorem 2.2. *When $\kappa \to \kappa_0$ for the perturbed population model* (2.7)*, the first-order derivative formula of the periodic function, with respect to the perturbation coefficients, is*

$$T'(0) = \sqrt{D(\kappa_0)}k_{11} + \sqrt{D(\kappa_0)}k_{22} - \frac{\mathrm{Im}(c_1(\kappa_0))}{\mathrm{Re}(c_1(\kappa_0))}c(\kappa_0)k_{12} - \frac{\mathrm{Im}(c_1(\kappa_0))}{\mathrm{Re}(c_1(\kappa_0))}b(\kappa_0)k_{21},$$

where $b(\kappa_0) = -\theta, c(\kappa_0) = \frac{s-\theta}{\theta}, D(\kappa_0) = s-\theta$. $\mathrm{Re}(c_1(\kappa_0))$ *and* $\mathrm{Im}(c_1(\kappa_0))$ *are defined in* (2.5) *and* (2.6).

Proof. By Lemma 2.1, differentiating the periodic function $T(\tau)$, we have

$$T'(\tau) = -\frac{2\pi}{\bar{\omega}^2(\kappa_\tau)}\frac{d\bar{\omega}(\kappa_\tau)}{d\tau} - \frac{2\pi}{\bar{\omega}(\kappa_\tau)}\left(\frac{\bar{\beta}'(\kappa_\tau)\,\mathrm{Im}(c_1(\kappa_\tau))}{\bar{\omega}(\kappa_\tau)\,\mathrm{Re}(c_1(\kappa_\tau))} - \frac{\bar{\omega}'(\kappa_\tau)}{\bar{\omega}(\kappa_\tau)}\right)\frac{d\kappa_\tau}{d\tau} + O(\kappa-\kappa_\tau).$$

If $\kappa \to \kappa_\tau$, then $O(\kappa-\kappa_\tau) \to 0$, and setting $\tau = 0$, then $\bar{\omega}(\kappa_0) = \omega(\kappa_0) = \sqrt{D(\kappa_0)}$.

We first compute $\left.\frac{d\kappa_\tau}{d\tau}\right|_{\tau=0}$. At $\kappa = \kappa_\tau$, by the expression of $H(\kappa, \tau)$ defined in (2.12), we can gain

$$\tau\left(k_{22}a(\kappa_\tau) - k_{12}c(\kappa_\tau) - k_{21}b(\kappa_\tau)\right) + a(\kappa_\tau) = 0. \tag{2.14}$$

Differentiating (2.14) with respect to τ, we obtain

$$\left(k_{22}a(\kappa_\tau) - k_{12}c(\kappa_\tau) - k_{21}b(\kappa_\tau)\right) + a'(\kappa_\tau)\frac{d\kappa_\tau}{d\tau} = 0, \tag{2.15}$$

and setting $\tau = 0$, we have

$$\left.\frac{d\kappa_\tau}{d\tau}\right|_{\tau=0} = \frac{k_{12}c(\kappa_0) + k_{21}b(\kappa_0)}{a'(\kappa_0)}, \tag{2.16}$$

with

$$b(\kappa_0) = -\theta, c(\kappa_0) = \frac{s-\theta}{\theta}, a'(\kappa_0) = -\frac{2\theta p}{s}.$$

Second, we calculate $\bar{\omega}'(\kappa_0)$. When $\kappa \to \kappa_\tau$, we derive

$$\bar{\omega}(\kappa) = \frac{1}{2}\sqrt{4D(\kappa,\tau) - H^2(\kappa,\tau)},$$

thereby,

$$\bar{\omega}'(\kappa) = \frac{\partial_\kappa D(\kappa,\tau) - \frac{1}{2}H(\kappa,\tau)\partial_\kappa H(\kappa,\tau)}{\sqrt{4D(\kappa,\tau) - H^2(\kappa,\tau)}}.$$

Since $H(\kappa_\tau, \tau) = 0$ and $\partial_\kappa D(\kappa_\tau, \tau) = \frac{1}{K(\tau)}D'(\kappa_\tau)$, we have

$$\bar{\omega}'(\kappa_0) = \left.\frac{\partial_\kappa D(\kappa_\tau,\tau)}{2\sqrt{D(\kappa_\tau,\tau)}}\right|_{\tau=0} = \left.\frac{D'(\kappa_\tau)}{2\sqrt{K(\tau)D(\kappa_\tau)}}\right|_{\tau=0} = \frac{D'(\kappa_0)}{2\sqrt{D(\kappa_0)}}. \tag{2.17}$$

At last, we calculate $\left.\frac{d}{d\tau}(\bar{\omega}(\kappa_\tau))\right|_{\tau=0}$. By $\bar{\omega}(\kappa_\tau) = \sqrt{D(\kappa_\tau,\tau)}$, we can get

$$\frac{d}{d\tau}(\bar{\omega}(\kappa_\tau)) = \frac{1}{2\sqrt{D(\kappa_\tau,\tau)}}\frac{d}{d\tau}(D(\kappa_\tau,\tau)). \tag{2.18}$$

According to $D(\kappa_\tau,\tau) = \frac{D(\kappa_\tau)}{K(\tau)}$, we have

$$\frac{d}{d\tau}(D(\kappa_\tau,\tau)) = -\frac{K'(\tau)}{K^2(\tau)}D(\kappa_\tau) + \frac{d}{d\tau}(D(\kappa_\tau))\frac{1}{K(\tau)}. \tag{2.19}$$

Setting $\tau = 0$, we can obtain

$$\begin{aligned} -\frac{K'(\tau)}{K^2(\tau)}D(\kappa_0) &= -(k_{11}+k_{22})D(\kappa_0), \\ \left.\frac{d}{d\tau}(D(\kappa_\tau))\frac{1}{K(\tau)}\right|_{\tau=0} &= D'(\kappa_0)\frac{d\kappa_\tau}{d\tau}(0). \end{aligned} \tag{2.20}$$

Therefore, from (2.16) and (2.20), we have

$$\left.\frac{d}{d\tau}(D(\kappa_\tau,\tau))\right|_{\tau=0} = -(k_{11}+k_{22})D(\kappa_0) + \frac{k_{12}c(\kappa_0) + k_{21}b(\kappa_0)}{a'(\kappa_0)}D'(\kappa_0). \tag{2.21}$$

By (2.18) and (2.21), we obtain

$$\frac{d}{d\tau}\left(\bar{\omega}\left(\kappa_{\tau}\right)\right)\Big|_{\tau=0} = \frac{1}{2\sqrt{D(\kappa_0)}}\left(-\left(k_{11}+k_{22}\right)D(\kappa_0) + \frac{k_{12}c(\kappa_0)+k_{21}b(\kappa_0)}{a'(\kappa_0)}D'(\kappa_0)\right). \tag{2.22}$$

Again, by (2.16), (2.17) and (2.22), we can derive

$$T'(0) = \sqrt{D(\kappa_0)}k_{11} + \sqrt{D(\kappa_0)}k_{22} - \frac{\mathrm{Im}\left(c_1(\kappa_0)\right)}{\mathrm{Re}\left(c_1(\kappa_0)\right)}c(\kappa_0)k_{12} - \frac{\mathrm{Im}\left(c_1(\kappa_0)\right)}{\mathrm{Re}\left(c_1(\kappa_0)\right)}b(\kappa_0)k_{21}.$$

□

3. Turing patterns of periodic solutions for the reaction-diffusion population model

With respect to the population model (1.2) and according to the theory expounded in [27], we study the mathematical mechanisms of Turing patterns occurring at the stable periodic solution $\left(u^T(t), v^T(t)\right)$. By the first derivative formula of the periodic function of the perturbed population model (2.7), we give the following theorem.

Theorem 3.1. *If hypothesis* $(\mathbf{A_4})$ *and* $\mathrm{Re}\left(c_1\left(\kappa_0\right)\right) < 0$ *hold, when* $\kappa \to \kappa_0$*, the stable spatially homogeneous Hopf bifurcating periodic solution bifurcates* $\left(u^T(t), v^T(t)\right)$ *from* $P_+ = (\kappa, v_\kappa)$*. If the domain* Ω *is large enough and*

$$\sqrt{D(\kappa_0)}d_{11} + \sqrt{D(\kappa_0)}d_{22} - \frac{\mathrm{Im}\left(c_1(\kappa_0)\right)}{\mathrm{Re}\left(c_1(\kappa_0)\right)}c(\kappa_0)d_{12} - \frac{\mathrm{Im}\left(c_1(\kappa_0)\right)}{\mathrm{Re}\left(c_1(\kappa_0)\right)}b(\kappa_0)d_{21} < 0,$$

then the following conclusions are true:
(1) The reaction-diffusion population model (1.2) *produces Turing patterns at the periodic solution* $\left(u^T(t), v^T(t)\right)$*;*
(2) If $\mathrm{Im}\left(c_1\left(\kappa_0\right)\right) < 0\,(>0)$*, then when* $k_{12} > M_1\,(k_{21} > M_2)$*, the reaction-diffusion population model* (1.2) *produces Turing patterns. That is, Turing patterns occuring at the periodic solution are determined by the cross-diffusion coefficients* $k_{12}\,(k_{21})$*, where*

$$b(\kappa_0) = -\theta, c(\kappa_0) = \frac{s-\theta}{\theta}, D(\kappa_0) = s-\theta.$$

$\mathrm{Re}\left(c_1\left(\kappa_0\right)\right)$ *and* $\mathrm{Im}\left(c_1\left(\kappa_0\right)\right)$ *from* (2.5) *and* (2.6)*:*

$$M_1 := \frac{\sqrt{D(\kappa_0)}d_{11} + \sqrt{D(\kappa_0)}d_{22} - \frac{\mathrm{Im}(c_1(\kappa_0))}{\mathrm{Re}(c_1(\kappa_0))}b(\kappa_0)d_{21}}{\frac{\mathrm{Im}(c_1(\kappa_0))}{\mathrm{Re}(c_1(\kappa_0))}c(\kappa_0)},$$

$$M_2 := \frac{\sqrt{D(\kappa_0)}d_{11} + \sqrt{D(\kappa_0)}d_{22} - \frac{\mathrm{Im}(c_1(\kappa_0))}{\mathrm{Re}(c_1(\kappa_0))}c(\kappa_0)d_{12}}{\frac{\mathrm{Im}(c_1(\kappa_0))}{\mathrm{Re}(c_1(\kappa_0))}b(\kappa_0)}.$$

Proof. Let the linearized vector form of the population model (2.1) at $\left(u^T(t), v^T(t)\right)$ be

$$\left(\frac{\partial \phi}{\partial t}, \frac{\partial \varphi}{\partial t}\right)^T = diag\left(D\Delta\phi, D\Delta\varphi\right) + J^T(t)(\phi,\varphi)^T), \tag{3.1}$$

where,

$$J^T(t) := \begin{pmatrix} 1-2pu^T(t) - \frac{2s(1-m)^2u^T(t)v^T(t)}{\left(1+(1-m)^2\left(u^T(t)\right)^2\right)^2} & -\frac{s(1-m)^2\left(u^T(t)\right)^2}{1+(1-m)^2\left(u^T(t)\right)^2} \\ \frac{2s(1-m)^2u^T(t)v^T(t)}{\left(1+(1-m)^2\left(u^T(t)\right)^2\right)^2} & -\theta + \frac{s(1-m)^2\left(u^T(t)\right)^2}{1+(1-m)^2\left(u^T(t)\right)^2} \end{pmatrix}$$

is the Jacobian matrix of model (2.1) at $\left(u^T(t), v^T(t)\right)$. $D := \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$, Δ is the Laplace operator. Let α_n and $\eta_n(x)$ be the eigenvalues and eigenfunctions of $-\Delta$ in region Ω, respectively, and $(\phi, \varphi)^T = (h(t), g(t))^T \sum_{n=0}^{\infty} k_n\eta_n(x)$. For the sake of convenience, we set $\varsigma := \alpha_n \geq 0, n \in \mathbb{N}_0 := \{0\} \bigcup \mathbb{N}$, then

$$\left(\frac{dh(t)}{dt}, \frac{dg(t)}{dt}\right)^T = -\varsigma D \begin{pmatrix} h(t) \\ g(t) \end{pmatrix} + J^T(t) \begin{pmatrix} h(t) \\ g(t) \end{pmatrix}. \tag{3.2}$$

If $D = \mathbf{0}$, then Eq (3.2) can be reduced to

$$\left(\frac{dh(t)}{dt}, \frac{dg(t)}{dt}\right)^T = J^T(t)(h(t), g(t))^T. \tag{3.3}$$

Setting $\rho(t)$ as the fundamental solution matrix of Eq (3.3), it satisfies $\rho(0) = I_2$. Let $\lambda_i, i = 1, 2$ be the eigenvalues of $\rho(T)$, the corresponding eigenfunctions are $(\xi_i, \eta_i)^T, i = 1, 2$, i.e.,

$$\rho(T)(\xi_i, \eta_i)^T = \lambda_i(\xi_i, \eta_i)^T,$$

where λ_1 and λ_2 are Floquet multipliers. Define

$$(\phi_i(t), \psi_i(t))^T := \rho(t)(\xi_i, \eta_i)^T,$$

clearly,

$$(\phi_i(0), \psi_i(0))^T = (\xi_i, \eta_i)^T, \ \rho(T)(\phi_i(0), \psi_i(0))^T = \lambda_i(\phi_i(0), \psi_i(0))^T.$$

Differentiating with respect to t in (2.2), we can obtain

$$\begin{pmatrix} u'' \\ v'' \end{pmatrix} = \begin{pmatrix} 1-2pu - \frac{2s(1-m)^2uv}{\left(1+(1-m)^2u^2\right)^2} & -\frac{s(1-m)^2u^2}{1+(1-m)^2u^2} \\ \frac{2s(1-m)^2uv}{\left(1+(1-m)^2u^2\right)^2} & -\theta + \frac{s(1-m)^2u^2}{1+(1-m)^2u^2} \end{pmatrix} \left(\frac{du}{dt}, \frac{dv}{dt}\right)^T,$$

then $\lambda_1 = 1$ is the eigenvalue of $\rho(T)$ and the eigenvector is $(\phi_1(t), \psi_1(t))^T = \left(\frac{du^T(t)}{dt}\Big|_{t=0}, \frac{dv^T(t)}{dt}\Big|_{t=0}\right)^T$. Since $\left(u^T(t), v^T(t)\right)$ is stable, $|\lambda_i| < 1$. Let $\rho(t, \varsigma)$ be the fundamental solution matrix of Eq (3.2), then we have

$$\frac{\partial\rho(t, \varsigma)}{\partial t} = -\varsigma D\rho(t, \varsigma) + J^T(t)\rho(t, \varsigma)$$

and $\rho(t, 0) = \rho(t)$. By the implicit function theorem, there is $\varsigma_1 > 0$, $\varsigma \in (-\varsigma_1, \varsigma_1)$ and continuous differentiable functions $\delta_i(\varsigma), \xi_i(\varsigma), \eta_i(\varsigma)$, such that

$$\rho(T, \varsigma)(\xi_i(\varsigma), \eta_i(\varsigma))^T = \delta_i(\varsigma)(\xi_i(\varsigma), \eta_i(\varsigma))^T, \tag{3.4}$$

where $\delta_1(\varsigma)$ and $\delta_2(\varsigma)$ are Floquet multipliers. Make the following definition

$$(\phi_i(t,\varsigma),\psi_i(t,\varsigma))^T := \rho(t,\varsigma)(\xi_i(\varsigma),\eta_i(\varsigma))^T; \tag{3.5}$$

by $\rho(0,\varsigma) = I$, we have

$$(\phi_i(0,\varsigma),\psi_i(0,\varsigma))^T = (\xi_i(\varsigma),\eta_i(\varsigma))^T. \tag{3.6}$$

From (3.4) and (3.6), we can gain

$$\rho(T,\varsigma)(\phi_i(0,\varsigma),\psi_i(0,\varsigma))^T = \delta_i(\varsigma)(\phi_i(0,\varsigma),\psi_i(0,\varsigma))^T,$$

and especially

$$\begin{pmatrix}\phi_i(t,0)\\ \psi_i(t,0)\end{pmatrix} = \rho(t,0)\begin{pmatrix}\xi_i(0)\\ \eta_i(0)\end{pmatrix} = \rho(t)\begin{pmatrix}\xi_i\\ \eta_i\end{pmatrix} = \rho(t)\begin{pmatrix}\phi_i(0)\\ \psi_i(0)\end{pmatrix} = \begin{pmatrix}\phi_i(t)\\ \psi_i(t)\end{pmatrix}.$$

Taking $i = 1$, by (3.5), we know

$$(\phi_1(t,\varsigma),\psi_1(t,\varsigma))^T := \rho(t,\varsigma)(\xi_1(\varsigma),\eta_1(\varsigma))^T.$$

Hence, we can derive

$$\left(\frac{\partial\phi_1(t,\varsigma)}{\partial t},\frac{\partial\psi_1(t,\varsigma)}{\partial t}\right)^T = -\varsigma D(\phi_1(t,\varsigma),\psi_1(t,\varsigma))^T + J^T(t)(\phi_1(t,\varsigma),\psi_1(t,\varsigma))^T.$$

Differentiating the above equation with respect to ς and setting $\varsigma = 0$, we obtain

$$\left(\frac{\partial\phi_{1\varsigma}(t,0)}{\partial t},\frac{\partial\psi_{1\varsigma}(t,0)}{\partial t}\right)^T = -D(\phi_1(t),\psi_1(t))^T + J^T(t)(\phi_{1\varsigma}(t,0),\psi_{1\varsigma}(t,0))^T, \tag{3.7}$$

where, $\phi_{1\varsigma} := \partial_\varsigma\phi_1, \psi_{1\varsigma} := \partial_\varsigma\psi_1$. On the other hand, from (3.4) and (3.5), we can get

$$(\phi_1(T,\varsigma),\psi_1(T,\varsigma))^T = \delta_1(\varsigma)(\phi_1(0,\varsigma),\psi_1(0,\varsigma))^T.$$

Differentiating with respect to ς, we have

$$(\phi_{1\varsigma}(T,\varsigma),\psi_{1\varsigma}(T,\varsigma))^T = \delta'_1(\varsigma)(\phi_1(0,\varsigma),\psi_1(0,\varsigma))^T + \delta_1(\varsigma)(\phi_{1\varsigma}(0,\varsigma),\psi_{1\varsigma}(0,\varsigma))^T.$$

Let $\varsigma = 0$ by (3.6) and $\delta_1(0) = \lambda_1 = 1$, and we can derive

$$(\phi_{1\varsigma}(T,0),\psi_{1\varsigma}(T,0))^T = \delta'_1(0)(\phi_1(0),\psi_1(0))^T + (\phi_{1\varsigma}(0,0),\psi_{1\varsigma}(0,0))^T. \tag{3.8}$$

According to Lemma 2.1, $\left(u^T(t,\tau),v^T(t,\tau)\right)$ is the periodic solution of the perturbed population model (2.7), i.e.,

$$\begin{pmatrix}1+\tau d_{11} & \tau d_{12}\\ \tau d_{21} & 1+\tau d_{22}\end{pmatrix}\begin{pmatrix}\dfrac{\partial u^T(t,\tau)}{\partial t}\\ \dfrac{\partial u^T(t,\tau)}{\partial t}\end{pmatrix} = \begin{pmatrix}u^T(t,\tau) - p\left(u^T(t,\tau)\right)^2 - \frac{s(1-m)^2\left(u^T(t,\tau)\right)^2 v^T(t,\tau)}{1+(1-m)^2\left(u^T(t,\tau)\right)^2}\\ -\theta v^T(t,\tau) + \frac{s(1-m)^2\left(u^T(t,\tau)\right)^2 v^T(t,\tau)}{1+(1-m)^2\left(u^T(t,\tau)\right)^2}\end{pmatrix}.$$

Differentiating with respect to τ and letting $\tau = 0$, we have

$$\left(\frac{d\left(\partial_t u^T(t,0)\right)}{d\tau}, \frac{d\left(\partial_t v^T(t,0)\right)}{d\tau}\right)^T = -D(\phi_1(t), \psi_1(t))^T + J^T(t)\left(\frac{du^T(t,0)}{d\tau}, \frac{dv^T(t,0)}{d\tau}\right)^T, \tag{3.9}$$

where $\partial_t u^T(t,0) = \phi_1(t), \partial_t v^T(t,0) = \psi_1(t)$. Since $T(\tau)$ is the minimum positive periodic solution of $\left(u^T(t,\tau), v^T(t,\tau)\right)$, we have

$$\left(u^T(t,\tau), v^T(t,\tau)\right) = \left(u^T(t+T(\tau),\tau), v^T(t+T(\tau),\tau)\right).$$

Differentiating with respect to τ and letting $\tau = 0, t = 0$, we can gain

$$\left(\frac{du^T(T,0)}{d\tau}, \frac{dv^T(T,0)}{d\tau}\right)^T = -T'(0)(\phi_1(0), \psi_1(0))^T + \left(\frac{du^T(0,0)}{d\tau}, \frac{dv^T(0,0)}{d\tau}\right)^T, \tag{3.10}$$

where $u^T(t,0) = u^T(t), v^T(t,0) = v^T(t), T(0) = T$. Define

$$Z(t) := (\phi_{1\varsigma}(t,0), \psi_{1\varsigma}(t,0))^T - \left(\frac{du^T(t,0)}{d\tau}, \frac{dv^T(t,0)}{d\tau}\right)^T,$$

and by (3.7)–(3.10), we get

$$Z'(t) = J^T(t)Z(t), \tag{3.11}$$

$$Z(T) - Z(0) = (\delta'_1(0) + T'(0))\,(\phi_1(0), \psi_1(0))^T. \tag{3.12}$$

Let $Z(t) = \mathfrak{A}(t)(Z_1, Z_2)^T$ be the general solution of (3.11), where any vector $(Z_1, Z_2)^T \in \mathbb{R}^2$. Since $(\phi_1(0), \psi_1(0))^T$ and $(\phi_2(0), \psi_2(0))^T$ are linearly independent, there exits constants p_1 and p_2 such that

$$(Z_1, Z_2)^T = p_1(\phi_1(0), \psi_1(0))^T + p_2(\phi_2(0), \psi_2(0))^T. \tag{3.13}$$

Substituting (3.13) into (3.12), we get $\delta'_1(0) + T'(0) = 0$. According to Theorem 2.2, if $T'(0) < 0$, then $\delta'_1(0) > 0$. Assuming that Ω is sufficiently large, then there is at least one eigenvalue α_n of $-\Delta$ so that $\delta_1(\varsigma) = \delta_1(\alpha_n) > 1$. Therefore, the population model (1.2) appears to have Turing patterns at $\left(u^T(t), v^T(t)\right)$. When $T'(0) < 0$ by Theorem 2.2, we have

$$\sqrt{D(\kappa_0)}d_{11} + \sqrt{D(\kappa_0)}d_{22} - \frac{\mathrm{Im}\,(c_1(\kappa_0))}{\mathrm{Re}\,(c_1(\kappa_0))}c(\kappa_0)d_{12} - \frac{\mathrm{Im}\,(c_1(\kappa_0))}{\mathrm{Re}\,(c_1(\kappa_0))}b(\kappa_0)d_{21} < 0. \tag{3.14}$$

Since $(\mathbf{A_3})$ is true, we can obtain $b(\kappa_0) = -\theta < 0, c(\kappa_0) = \frac{s-\theta}{\theta} > 0$. If $\mathrm{Re}\,(c_1(\kappa_0)) < 0$, then when $\mathrm{Im}\,(c_1(\kappa_0)) < 0$,

$$\frac{\mathrm{Im}\,(c_1(\kappa_0))}{\mathrm{Re}\,(c_1(\kappa_0))}c(\kappa_0) > 0,\ \frac{\mathrm{Im}\,(c_1(\kappa_0))}{\mathrm{Re}\,(c_1(\kappa_0))}b(\kappa_0) < 0.$$

From (3.14), we gain

$$d_{12} > \frac{\sqrt{D(\kappa_0)}d_{11} + \sqrt{D(\kappa_0)}d_{22} - \frac{\mathrm{Im}(c_1(\kappa_0))}{\mathrm{Re}(c_1(\kappa_0))}b(\kappa_0)d_{21}}{\frac{\mathrm{Im}(c_1(\kappa_0))}{\mathrm{Re}(c_1(\kappa_0))}c(\kappa_0)} := M_1.$$

When the cross-diffusion coefficient $d_{12} > M_1$, the cross-diffusion population model (1.2) generates Turing patterns at the periodic solution $\left(u^T(t), v^T(t)\right)$. Similarly, if $\operatorname{Im}(c_1(\kappa_0)) > 0$, then

$$\frac{\operatorname{Im}(c_1(\kappa_0))}{\operatorname{Re}(c_1(\kappa_0))}c(\kappa_0) < 0, \; \frac{\operatorname{Im}(c_1(\kappa_0))}{\operatorname{Re}(c_1(\kappa_0))}b(\kappa_0) > 0,$$

so

$$d_{21} > \frac{\sqrt{D(\kappa_0)}d_{11} + \sqrt{D(\kappa_0)}d_{22} - \frac{\operatorname{Im}(c_1(\kappa_0))}{\operatorname{Re}(c_1(\kappa_0))}c(\kappa_0)d_{12}}{\frac{\operatorname{Im}(c_1(\kappa_0))}{\operatorname{Re}(c_1(\kappa_0))}b(\kappa_0)} := M_2.$$

When the cross-diffusion coefficient $d_{21} > M_2$, the cross-diffusion population model (1.2) produces Turing patterns at the periodic solution $\left(u^T(t), v^T(t)\right)$. □

4. Numerical simulations

We shall conduct numerical simulations in three cases to verify the relevant conclusions: The diffusive population model forms Turing patterns at the periodic solutions. Fix the parameters in model (2.1):

$$m = 0.6, s = 0.1, \theta = 0.09, p = 0.0592, x \in \Omega = (0, 30),$$

then $(\kappa, v_\kappa) = (7.5, 46.3)$ is a unique positive equilibrium. By calculation, $\kappa_0 = 7.505$. According to Theorem 2.1, when $\kappa \to \kappa_0$, $b(\kappa_0) = -0.09, c(\kappa_0) = \frac{1}{9}, D(\kappa_0) = 0.01$, $\operatorname{Re} c_1(\kappa_0) = -3.1316 \times 10^{-3} < 0$ and $\operatorname{Im} c_1(\kappa_0) = 4.3667 \times 10^{-3}$, simultaneously, hypothesis (A4) is true. Take the initial values as $u_0 = 8 + 0.1\cos(x), v_0 = 47 + 0.1\cos(x)$.

(1) If $d_{11} = 1, d_{22} = 1, d_{12} = d_{21} = 0$, then $\sqrt{D(\kappa_0)}d_{11} + \sqrt{D(\kappa_0)}d_{22} = 0.2 > 0$. By Theorem 3.1, in model (1.2), Turing patterns do not appear at $\left(u^T(t), v^T(t)\right)$, namely, the same self-diffusion rates do not destroy the stability of the periodic solution(See [28]). If $d_{11} \neq d_{22}, d_{11} > 0, d_{22} > 0$ and $d_{12} = d_{21} = 0$, then $\sqrt{D(\kappa_0)}d_{11} + \sqrt{D(\kappa_0)}d_{22} > 0$ and the periodic solution of diffusion model (1.2) is stable (Figure 1).

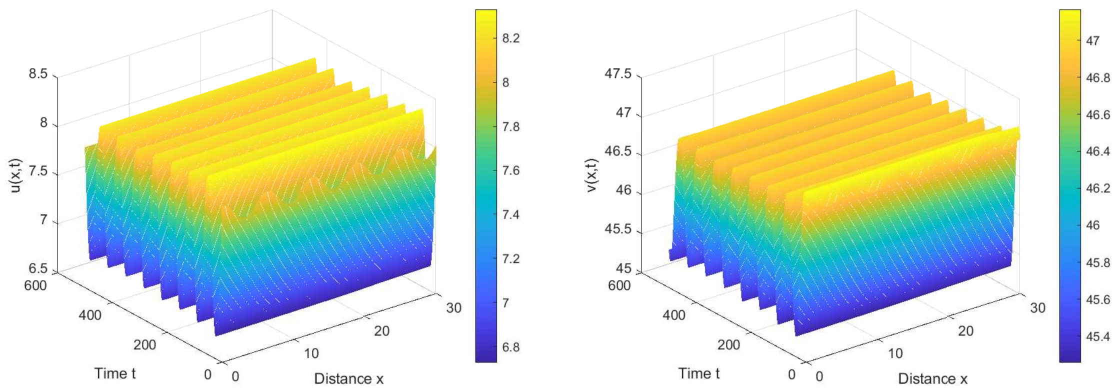

Figure 1. The periodic solution $\left(u^T(t), v^T(t)\right)$ of the reaction-diffusion equation is stable.

(2) If $d_{11} = 0.2, d_{22} = 0.5, d_{12} = 0.05$, then $M_2 = 0.6195$. Select $d_{21} = 0.7$ by calculating $T'(0) < 0$. According to Theorem 3.1, when $d_{21} > M_2 = 0.6195$, cross-diffusion induces system (1.2) to produce Turing patterns at $\left(u^T(t), v^T(t)\right)$ (Figure 2).

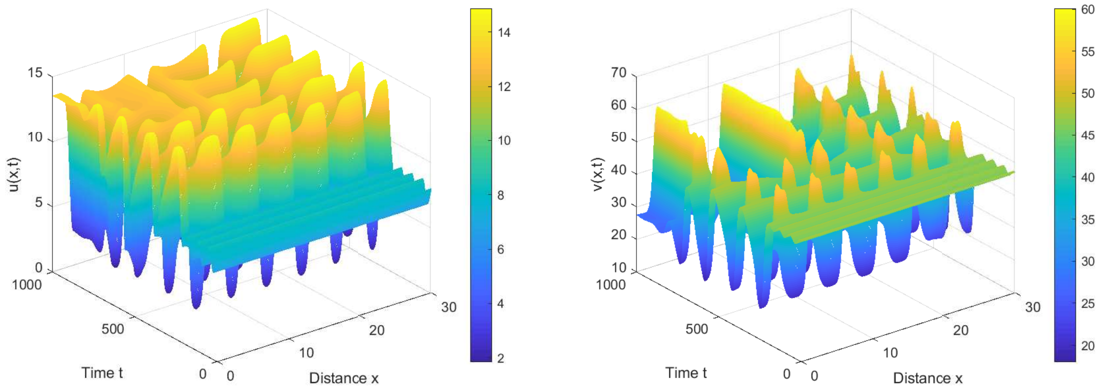

Figure 2. Cross-diffusion-induced Turing patterns.

(3) If $d_{11} = 1, d_{22} = 1, d_{12} = 0.02$, then $M_2 = 1.6184$. We choose $d_{21} = 1.7$, through computation and $T'(0) < 0$. According to Theorem 3.1, when $d_{21} > M_2$, cross-diffusion induces system (1.2) to produce Turing patterns at $\left(u^T(t), v^T(t)\right)$ (Figure 3).

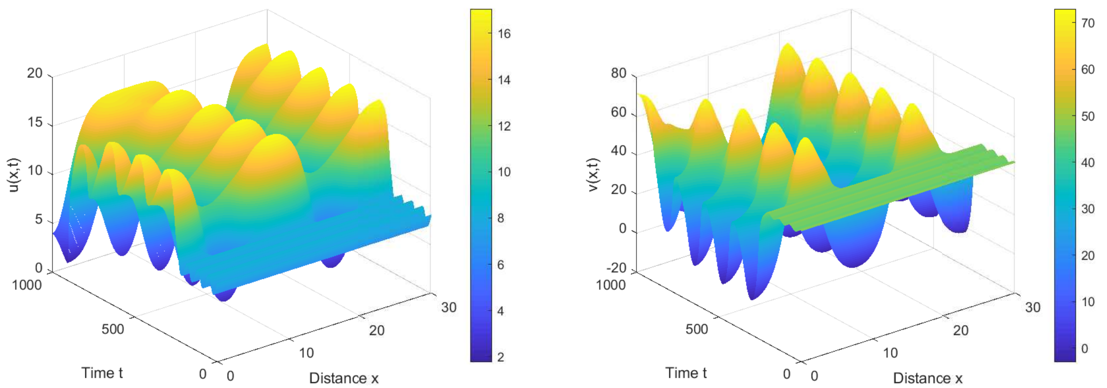

Figure 3. Turing patterns induced by diffusion coefficient d_{21}.

5. Conclusions

In this paper, we established a cross-diffusion population model with prey refuge and Holling III functional response, and studied the mathematical mechanisms of Turing patterns generated by the diffusion-driven instability of the periodic solutions. The results show that when $\operatorname{Im}(c_1(\kappa_0)) < 0\,(> 0)$, the symbol of the diffusivity expression $T'(0)$ is actually determined by the cross-diffusion coefficient $d_{21}\,(d_{12})$. That is, when $d_{21} > M_2\,(d_{12} > M_1)$ and the region Ω is sufficiently large, $T'(0) < 0$ and model (1.2) generate Turing patterns at the periodic solutions. Our research more accurately determined the range of cross-diffusion coefficients of Turing patterns occurring at the periodic solutions. This provided a new idea for model (1.2) to generate Turing instability at the periodic solutions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no competing interests.

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