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Research article

Existence results for $p(x)$-biharmonic problems involving a singular and a Hardy type nonlinearities

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Abstract: In this paper, we proved the existence and the multiplicity of solutions for some $p(x)$-biharmonic problems involving singular nonlinearity and a Hardy potential. More precisely, by the use of the min-max method, we proved the existence of a nontrivial solution for such a problem. Next, diversions of the mountain pass theorem were used to prove the multiplicity of solutions.

Keywords: $p(x)$-biharmonic operator; singular equations; variational methods; mountain pass theorem; existence
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1. Introduction

In this paper, we shall study the following $p(x)$-biharmonic system:

$$\begin{cases} \Delta^2_{p(x)} u = \lambda \, \frac{|u|^{p(x)-2}u}{\delta(x)^{2p(x)}} + f(x,u) + b(x)u^{-m(x)}, & \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain and $\delta(x)$ is the distance between x and the boundary of Ω, which is denoted by $\partial\Omega$. The functions m and b are continuous on $\overline{\Omega}$. $\Delta^2_{p(x)}$ is the $p(x)$-biharmonic operator, which is defined by

$$\Delta^2_{p(x)} u = \Delta\left(|\Delta u|^{p(x)-2}\Delta u\right).$$

We noted that problems involving the $p(x)$-Laplace operator appear in several fields like thermotropic fluids (Antontsev and Rodrigues [3]), electrorheological fluids (Rajagopal and Růžička [31, 32], Růžička [34]), elastic mechanics (Zhikov [38]) and other phenomena related to image processing (Aboulaich et al. [1], Chen et al. [13]).

Due to their importance, many researchers have recently concentrated on the development of problems with the $p(x)$-growth conditions (see, for example, the papers of Drissi et al. [16], Elmokhtar [19], Laghzal and Touzani [26] and Raguza et al. [30]). These problems are discussed in the spaces $L^{p(\cdot)}$ and $W^{m,p(\cdot)}$. Different methods are used to prove the existence and the multiplicity of solutions for such problems. We refer the interested readers to Ben Ali et al. [6, 7] (min-max method), Chammem et al. [10] (variational methods and monotonicity arguments combined with the theory of the generalized Lebesgue Sobolev spaces), Chammem et al. [11] (mountain pass lemma and Ekeland's variational principle), Chammem and Sahbani [12] (mountain pass lemma and its $\mathbb{Z}_2$ symmetric version), Baroni [4] (perturbation arguments), Blanco et al. [9] (monotone operators on a new Musielak-Orlicz Sobolev space) and Wang [35] (variational methods combined with the Brezis-Lieb's lemma and Mazur's lemma).

Very recently, more attention has been paid to the study of the fourth-order elliptic equations, namely, the p-biharmonic and the $p(x)$-biharmonic operator. We cite for example Alsaedi et al. [2] (fibering maps analysis in the Nehari manifold sets), Bouraunu et al. [8] (combination of the mountain pass type theorem with several variational arguments), Dhifli and Alsaedi [14] (Nehari manifold method), El Khalil et al. [17, 18] (combination of the variational method with the Ljusternik-Schnirelmann theory), Hsini et al. [22] (combination of the variational method with the Ekland's variational principle), Kefi and Saoudi [24] (monotonicity arguments in the generalized Lebesgue Sobolev spaces). In the recent paper of El Khalil et al. [18], the authors considered the following singular $p(x)$-biharmonic problem with Hardy-type nonlinearity:

$$\begin{cases} \Delta^2_{p(x)} = \lambda \frac{|u|^{p(x)-2}u}{\delta(x)^{2p(x)}} + \mu |u|^{p(x)-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

More precisely, under some suitable conditions and using the Krasnoselskii genus, the authors proved the multiplicity of solutions. Also, Laghzal et al. [25] used a variational approach combined with min-max arguments based on Ljusternik-Schnirelmann theory and proved that problem (1.2) admits a nondecreasing sequence of positive solutions.

We note that the study of differential equations with singularities has been developed very quickly and the investigation for existence and multiplicity results attracted considerable attention of researchers. We refer to the papers of Ben Ali et al. [5, 6], Chammem et al. [10], Kefi and Saoudi [24] and references therein.

Inspired by the above results, our goal in this paper is to continue this investigation by generalizing the works of Laghzal et al. [25] by adding two types of perturbation. One of them is a singular perturbation, which means that the functional energy is not of class C^1 and so we cannot use the direct variational methods. For a way to bypass the singularity and work with a functional of class C^1, we refer to the works of Papageorgiou et al. [28, 29] and the paper of Razani and Behboudi [33].

This paper is organized as follows. In section two, we recall some basic facts about the weighted variable exponent Lesbesgue and Sobolev spaces. In section three, using the min-max method, we give and prove some existing results related to problem (1.1). In section four, using the mountain pass theorem and its $\mathbb{Z}_2$-symetric version, some multiplicity results are presented and proved.

2. Notations and terminology

In this section, we shall introduce some important definitions and properties related to the variable exponent Lebesgue and Sobolev spaces. For more details about these spaces we refer to the book of Diening et al. [15] and to the papers of Fan and Zhao [20], Fan and Fan [21], Mihăilescu [27], Yao [36] and Zang and Fu [37].

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 3$. We consider the set

$$C_+(\overline{\Omega}) = \{\varphi \in C(\overline{\Omega}) : \varphi(z) > 1, \text{for all } z \in \overline{\Omega}\}.$$

For all functions φ in the set $C_+(\overline{\Omega})$, we define

$$\varphi^- = \inf_{z\in\overline{\Omega}} \varphi(z) \text{ and } \varphi^+ = \sup_{z\in\overline{\Omega}} \varphi(z).$$

The variable exponent Lebesgue space $L^{\varphi(.)}(\Omega)$ is the set of all measurable functions $\psi : \Omega \to \mathbb{R}$ such that

$$\int_\Omega |\psi(z)|^{\varphi(z)} dz < \infty\}.$$

In the space $L^{\varphi(.)}(\Omega)$ the following Luxemburg norm:

$$|\psi|_{\varphi(z)} = \inf\{\eta > 0 : \int_\Omega |\frac{\psi(z)}{\eta}|^{\varphi(z)} dz \leq 1\}.$$

It is noted that $\left(L^{\varphi(.)}(\Omega), |.|_{\varphi(z)}\right)$ becomes a separable and reflexive Banach space if and only if

$$1 < \varphi^- \leq \varphi^+ < \infty. \tag{2.1}$$

In the rest of this paper, p denotes a function in $C_+(\overline{\Omega})$ satisfying (2.1).

In the space $L^{p(\cdot)}(\Omega)$, we have an equivalent Hölder inequality, which is given in the following proposition.

Proposition 2.1. (*See [37]*) *For any $\varphi \in L^{p(\cdot)}(\Omega)$ and any $\psi \in L^{p'(\cdot)}(\Omega)$, we have*

$$\left|\int_\Omega \varphi(x)\psi(x)\, dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right) |\varphi|_{p(x)} |\psi|_{p'(x)},$$

where $p'(x)$ is the conjugate function of $p(x)$, which is given by

$$p'(x) = \frac{p(x)}{p(x) - 1}.$$

Another interesting property of the space $L^{p(\cdot)}(\Omega)$ is presented in the following proposition.

Proposition 2.2. (*See [20]*) *If q is a measurable function in $L^\infty(\mathbb{R}^N)$, such that for all $x \in \mathbb{R}^N$ we have $1 \leq p(x)q(x) \leq \infty$, then for any nontrivial function $\varphi \in L^{p(\cdot)}(\mathbb{R}^N)$, the following statements hold true:*

(i) $|\varphi|_{p(x)q(x)} \leq 1 \Rightarrow |\varphi|^{p^+}_{p(x)q(x)} \leq ||\varphi|^{p(x)}|_{q(x)} \leq |\varphi|^{p^-}_{p(x)q(x)}$.

(ii) $|\varphi|_{p(x)q(x)} \geq 1 \Rightarrow |\varphi|^{p^-}_{p(x)q(x)} \leq ||\varphi|^{p(x)}|_{q(x)} \leq |\varphi|^{p^+}_{p(x)q(x)}$.

The modular on the space $L^{p(\cdot)}(\Omega)$ is defined by the mapping

$$\rho_{p(x)}(\varphi) = \int_{\Omega} |\varphi(x)|^{p(x)} dx,$$

and it satisfies the following properties.

Proposition 2.3. (*See [20, Theorem 1.3]*) *For all $\varphi \in L^{p(\cdot)}(\Omega)$, we have:*

(i) $|\varphi|_{p(x)} < 1$ if and only if $\rho_{p(x)}(\varphi) < 1$. Moreover, the last equivalence holds if we replace $<$ by $=$ or by $>$.
(ii) If $|\varphi|_{p(x)} > 1$, then $|\varphi|_{p(x)}^{p^-} \leq \rho_{p(x)}(\varphi) \leq |\varphi|_{p(x)}^{p^+}$.
(iii) If $|\varphi|_{p(x)} < 1$, then $|\varphi|_{p(x)}^{p^+} \leq \rho_{p(x)}(\varphi) \leq |\varphi|_{p(x)}^{p^-}$.

Let us define the Sobolev space with a variable exponent by

$$W^{m,p(\cdot)}(\Omega) = \{\varphi \in L^{p(\cdot)}(\Omega) | \, D^{\alpha}\varphi \in L^{p(\cdot)}(\Omega), |\alpha| \leq m\},$$

equipped with the norm

$$||u||_{m,p(\cdot)} = \sum_{|\alpha| \leq m} |D^{\alpha}u|_{p(\cdot)},$$

where $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a multi-index, $|\alpha| = \sum_{i=1}^{N} \alpha_i$, and $D^{\alpha}\varphi$ is given as follows:

$$D^{\alpha}\varphi = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}} \varphi.$$

It is well-known (see [37]) that $\left(W^{m,p(\cdot)}(\Omega), ||.||_{m,p(\cdot)}\right)$ is a separable, reflexive, and uniformly convex Banach space. Moreover, if we denote $W_0^{m,p(\cdot)}(\Omega)$ the closure of $C_0^{\infty}(\Omega)$ in $W^{m,p(\cdot)}(\Omega)$, then $W_0^{m,p(\cdot)}(\Omega)$ has the same properties as $W^{m,p(\cdot)}(\Omega)$.

Put

$$X = W_0^{2,p(\cdot)}(\Omega),$$

endowed with the norm

$$||\varphi|| = \inf\{\beta > 0 : \int_{\Omega} |\frac{\Delta\varphi(x)}{\beta}|^{p(x)} dx \leq 1\}.$$

We recall from Zang [37] that X endowed with the above norm is a separable and reflexive Banach space. Next, we give a compact embedding theorem related to the space $W^{m,p(\cdot)}(\Omega)$.

Theorem 2.1. *(See [37]) If $q \in C_+(\overline{\Omega})$ such that $q(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then the embedding from $W^{2,p(\cdot)}(\Omega)$ into $L^{q(\cdot)}(\Omega)$ is compact and continuous, where*

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-2p(x)}, & if \ p(x) < \frac{N}{2}, \\ \infty, & if \ p(x) \geq \frac{N}{2}. \end{cases}$$

We note that Theorem 2.1 remains true if we replace $W^{2,p(x)}(\Omega)$ by X. Moreover, if we denoted by $M(u)$ the following expression:

$$M(u) = \int_{\Omega} |\Delta u|^{p(x)} dx,$$

then we have the following proposition.

Proposition 2.4. *(See [37]) For all $\varphi \in X$, we have:*

(i) *If* $M(\varphi) \geq 1$, *then* $\|\varphi\|^{p^-} \leq M(\varphi) \leq \|\varphi\|^{p^+}$,
(ii) *If* $M(\varphi) \leq 1$, *then* $\|\varphi\|^{p^+} \leq M(\varphi) \leq \|\varphi\|^{p^-}$,
(iii) $M(\varphi) \geq 1 (= 1, \leq 1) \Leftrightarrow \|\varphi\| \geq 1 (= 1, \leq 1)$.

Lemma 2.1. *(See [23]) The mapping* $\Delta^2_{p(x)} : W_0^{2,p(x)}(\Omega) \to W_0^{-2,p'(x)}(\Omega)$ *is of type* S^+, *which means that if* $u_n \rightharpoonup u$, *weakly in* $W_0^{2,p(\cdot)}(\Omega)$ *and* $\limsup_{n\to\infty} \prec \Delta^2_{p(x)}(u_n), u_n - u \succ \leq 0$, *then* $u_n \to u$ *is strongly in* $W_0^{2,p(\cdot)}(\Omega)$, *where* $\prec \cdot \succ$ *is the dual product between X and its dual.*

To manipulate the Hardy term we assume that $0 < \lambda < C_H$, where

$$C_H = \frac{p^-}{p^+} \min\left(\left(\frac{N(p^- - 1)(N - 2p^-)}{(p^-)^2} \right)^{p^-}, \left(\frac{N(p^+ - 1)(N - 2p^+)}{(p^+)^2} \right)^{p^+} \right).$$

We recall the $p(\cdot)$-Hardy inequality (see El Khalil et al. [17] and Laghzal et al. [25]), which is given by

$$\int_\Omega \frac{|\Delta u(x)|^{p(x)}}{p(x)} dx \geq C_H \int_\Omega \frac{|u(x)|^{p(x)}}{p(x)\delta(x)^{2p(x)}} dx, \tag{2.2}$$

for all $u \in W_0^{2,p(x)}(\Omega)$.

3. Problem with a singular term

In this section, we will use the min-max method to prove the existence of solutions for problem (1.1). To this aim, we assume the following hypotheses:

(H_1) The function b is almost everywhere positive in Ω, such that

$$b \in L^{\frac{\tau}{\tau+m-1}}(\Omega), \text{ for some } 1 < \tau < p^*(x).$$

(H_2) There exist $l, \sigma \in C(\overline{\Omega})$ and $h \in L^{l(x)}(\Omega)$ such that for all $x \in \overline{\Omega}$ and all $\varphi \in X$, we have

$$1 < \sigma(x) < p(x) < \frac{N}{2} < l(x) < p^*(x),$$

and

$$f(x, \varphi) = h(x)|\varphi|^{\sigma(x)-2}\varphi.$$

(H_3) There exists $\Omega_1 \subset\subset \Omega$ such that $|\Omega_1| > 0$ and

$$f(x, y) \geq 0 \text{ for all } (x, y) \in \Omega_1 \times \mathbb{R}.$$

In this part, we shall use the min-max method to prove the existence of solutions. More precisely, we will prove the following theorem.

Theorem 3.1. *Assume that hypotheses* $(H_1) - (H_3)$ *hold. If* $0 < \lambda < C_H$, *then problem* (1.1) *admits a nontrivial weak solution with negative energy, where* C_H *is given in Eq* (2.2).

We note that a function $\varphi \in X$ is said to be a weak solution of problem (1.1). If for all $\psi \in X$, we have

$$\int_\Omega |\Delta\varphi|^{p(x)-2}\Delta\varphi\Delta\psi\, dx - \lambda\int_\Omega \frac{|\varphi(x)|^{p(x)-2}}{\delta(x)^{2p(x)}}\varphi(x)\psi(x)\, dx$$
$$- \int_\Omega b(x)|\varphi|^{-m(x)}\psi(x)\, dx - \int_\Omega f(x,\varphi(x))\psi(x)\, dx = 0.$$

Associated to problem (1.1), we define the functional $\Phi_\lambda : X \to \mathbb{R}$ by

$$\Phi_\lambda(u) \;=\; I_\lambda(u) - \int_\Omega \frac{b(x)}{1-m(x)}|u|^{1-m(x)}dx - \int_\Omega F(x,u(x))dx,$$

where $F(x,t) = \int_0^t f(x,s)ds$, and $I_\lambda(u)$ is given as follows:

$$I_\lambda(u) = \int_\Omega \frac{|\Delta u(x)|^{p(x)}}{p(x)}dx - \lambda\int_\Omega \frac{|u(x)|^{p(x)}}{p(x)\delta(x)^{2p(x)}}dx. \tag{3.1}$$

Remark 3.1. *Φ_λ is well-defined and differentiable, but due to the singular term, it is not in $C^1(X,\mathbb{R})$.*

Lemma 3.1. *Assume that (H_1) and (H_2) hold and $0 < \lambda < C_H$, then the functional Φ_λ is coercive in X.*

Proof. Let $u \in X$ with $\|u\| > 1$ and assume that $0 < \lambda < C_H$, then by (2.2), we have

$$\frac{\lambda}{C_H}\int_\Omega \frac{|\Delta u(x)|^{p(x)}}{p(x)}dx \geq \lambda\int_\Omega \frac{|u(x)|^{p(x)}}{p(x)\delta(x)^{2p(x)}}dx,$$

so

$$I_\lambda(u) \geq \left(1-\frac{\lambda}{C_H}\right)\int_\Omega \frac{|\Delta u(x)|^{p(x)}}{p(x)}dx \geq \frac{1}{p^+}(1-\frac{\lambda}{C_H})\int_\Omega |\Delta u(x)|^{p(x)}dx.$$

Thus, by the last inequality and using Proposition 2.4, we get

$$I_\lambda(u) \geq \frac{1}{p^+}(1-\frac{\lambda}{C_H})\|u\|^{p^-}. \tag{3.2}$$

On the other hand, since $1 < \tau < p^*(x)$, then by Propositions 2.1 and 2.3, we obtain

$$\begin{aligned}\int_\Omega \frac{b(x)}{1-m(x)}u^{1-m(x)}dx &\leq \frac{1}{1-m^+}\int_\Omega b(x)u^{1-m(x)}dx \leq \frac{1}{1-m^+}|b|_{\frac{\tau}{\tau+m(x)-1}}\||u|^{1-m(x)}|_{\frac{\tau}{1-m(x)}}\\ &\leq \frac{C}{1-m^+}|b|_{\frac{\tau}{\tau+m(x)-1}}\max\left(\|u\|^{1-m^+},\|u\|^{1-m^-}\right).\end{aligned} \tag{3.3}$$

Now, from (H_2), Propositions 2.1 and 2.2, we have

$$\begin{aligned}\int_\Omega F(x,u(x))dx &\leq \int_\Omega h(x)|u(x)|^{\sigma(x)}dx \leq |h|_{l(x)}||u|^{\sigma(x)}|_{l'(x)}\\ &\leq |h|_{l(x)}\max\left(|u|^{\sigma^+}_{l'(x)\sigma(x)},|u|^{\sigma^-}_{l'(x)\sigma(x)}\right),\end{aligned}$$

where l' is such as

$$\frac{1}{l(x)} + \frac{1}{l'(x)} = 1.$$

Next, using hypothesis (H_2) and the fact that $l'(x)\sigma(x) < p^*(x)$, we conclude by Theorem 2.1, that

$$\int_\Omega F(x, u(x))dx \leq C'|h|_{l(x)} \max\left(||u||^{\sigma^+}, ||u||^{\sigma^-}\right). \tag{3.4}$$

Finally, combining (3.2)–(3.4), we get

$$\Phi_\lambda(u) \geq \frac{1}{p^+}(1 - \frac{\lambda}{C_H})||u||^{p^-} - \frac{C|b|_{\frac{\tau}{\tau+m(x)-1}}}{1-m^+} \max\left(||u||^{1-m^+}, ||u||^{1-m^-}\right) - C'\lambda|h|_{l(x)} \max\left(||u||^{\sigma^+}, ||u||^{\sigma^-}\right).$$

Since $1 - m^- < \sigma^+ < p^-$ and $0 < \lambda < C_H$, then $\lim_{||u||\to\infty} \Phi_\lambda(u) = \infty$, which means that Φ_λ is coercive and bounded below on X. □

Lemma 3.2. *Under assumption (H_3), there exists $\psi \in X$, such that $\psi \geq 0$, $\psi \neq 0$ and $\Phi_\lambda(t\psi) < 0$ for sufficiently small $t > 0$.*

Proof. Let $\psi \in C_0^\infty(\Omega)$, such that $supp(\psi) \subset \Omega_1 \subset\subset \Omega$, $\psi = 1$ in a subset $\Omega' \subset supp(\psi)$ and $0 \leq \psi \leq 1$ in Ω.

Let $t \in (0, 1)$, then by Propositions 2.3, 2.4 and Theorem 2.1, there exists a constant $C_3 > 0$, such that

$$I_\lambda(t\psi) \leq \int_\Omega t^{p(x)} \frac{|\Delta\psi(x)|^{p(x)}}{p(x)} dx \leq C_3 t^{p^-} \max\left(||\psi||^{p^+}, ||\psi||^{p^-}\right).$$

So, by hypothesis (H_3), we get

$$\begin{aligned}\Phi_\lambda(t\psi) &\leq C_3 t^{p^-} \max\left(||\psi||^{p^+}, ||\psi||^{p^-}\right) - t^{1-m^-} \int_\Omega \frac{b(x)}{1-m(x)} |\psi|^{1-m(x)} dx \\ &\leq t^{1-m^-}\left(t^{p^- -(1-m^-)} C_3 \max\left(||\psi||^{p^+}, ||\psi||^{p^-}\right) - \int_\Omega \frac{b(x)}{1-m(x)} |\psi|^{1-m(x)} dx\right).\end{aligned}$$

Consequently, using the fact that $p^- > 1 - m^-$, we deduce that $\Phi_\lambda(t\psi) < 0$ for $t < \min(1, B)$, where

$$B = \left(\frac{\int_\Omega \frac{b(x)}{1-m(x)} |\psi|^{1-m(x)} dx}{C_3 \max\left(||\psi||^{p^+}, ||\psi||^{p^-}\right)}\right)^{\frac{1}{p^- -(1-m^-)}}.$$

Put

$$\theta_\lambda = \inf_{u\in X} \Phi_\lambda(u).$$

Proposition 3.1. *Assume that hypotheses $(H_1) - (H_3)$ hold, then for all $0 < \lambda < C_H$, the functional Φ_λ reaches its global minimizer in X and there exists $u_* \in X$ such that $\Phi_\lambda(u_*) = \theta_\lambda$.*

Proof. Let $\{u_n\}$ be a minimizing sequence, which means that

$$\lim_{n\to\infty} \Phi_\lambda(u_n) = \theta_\lambda.$$

Since Φ_λ is coercive, we conclude that $\{u_n\}$ is bounded in a reflexive space X, so there exists a subsequence (still denoted by $\{u_n\}$) and $u_* \in X$ such that

$$\begin{cases} u_n \rightharpoonup u_* \text{ weakly in } X, \\ u_n \to u_* \text{ strongly in } L^{\beta(x)}(\Omega), \quad 1 \le \beta(x) < p^*(x), \\ u_n \to u_* \text{ a.e. in } \Omega. \end{cases}$$

Since $u_n \to u_*$ a.e. in Ω, then Fatou's lemma implies that

$$I_\lambda(u_*) \le \liminf_{n\to\infty} I_\lambda(u_n). \tag{3.5}$$

Now, we claim that

$$\lim_{n\to+\infty} \int_\Omega b(x)|u_n|^{1-m(x)}dx = \int_\Omega b(x)|u_*|^{1-m(x)}dx. \tag{3.6}$$

Indeed, let $\epsilon > 0$, then by the fact that $\int_\Omega |b(x)|^{\frac{\tau}{\tau+m(x)-1}}\, dx$ is absolutely continuous and using Proposition 2.3, there exists $\alpha, \xi > 0$, such that

$$|b|^\alpha_{\frac{\tau}{\tau+m(x)-1}} \le \int_{\Omega_2} |b(x)|^{\frac{\tau}{\tau+m(x)-1}} dx \le \epsilon^\alpha$$

for every $\Omega_2 \subset \Omega$ with $|\Omega_2| < \xi$.

On the other hand, by Propositions 2.1 and 2.2, we obtain

$$\int_\Omega b(x)|u_n|^{1-m(x)}dx \le |b|_{\frac{\tau}{\tau+m(x)-1}} \||u_n|^{1-m(x)}|_{\frac{\tau}{1-m(x)}} \le |b|_{\frac{\tau}{\tau+m(x)-1}} \max\left(|u_n|_\tau^{1-m^-}, |u_n|_\tau^{1-m^+}\right).$$

Thus, we obtain

$$\int_{\Omega_2} b(x)|u_n|^{1-m(x)}dx < \epsilon \max\left(|u_n|_\tau^{1-m^-}, |u_n|_\tau^{1-m^+}\right).$$

Since (u_n) is bounded in X and $\tau < p^*(x)$, by Theorem 2.1 we can deduce that $|u_n|_\tau$ is bounded. Using Vitali's convergence theorem, we conclude that Eq (3.6) holds.

Next, we claim that

$$\lim_{n\to+\infty} \int_\Omega F(x, u_n(x))dx = \int_\Omega F(x, u_*(x))dx. \tag{3.7}$$

Indeed, from hypothesis (H_2) there exists $c > 0$ such that

$$|F(x, u_n(x))| \le \frac{c}{\sigma^-}|h(x)||u_n|^{\sigma(x)}.$$

Since $u_n \rightharpoonup u_*$ in X and $l'(x)\sigma(x) < p^*(x)$, we have the strong convergence in $L^{l'(x)\sigma(x)}(\Omega)$. Hence, for a subsequence again denoted by $\{u_n\}$, we get $u_n \to u_*$ a.e in Ω and there exists $\kappa \in L^{\sigma(x)l'(x)}(\Omega)$, such that $|u_n(x)| \le \kappa(x)$.

So, we get

$$|F(x,u_n(x))| \leq \frac{c}{\sigma^-}|h(x)||\kappa(x)|^{\sigma(x)}.$$

Therefore, by Proposition 2.1 we obtain

$$\int_\Omega |F(x,u_n(x))|dx \leq \frac{c}{\sigma^-}|h|_{l(x)}|\kappa|^{\sigma(x)}_{l'(x)}.$$

Hence, the Lebesgue-dominated convergence theorem and Proposition 2.3 imply that Eq (3.7) holds. Now, by combining equations (3.5) and (3.6) with Eq (3.7), we concluded that Φ_λ is weakly lower semi-continuous. Finally, we deduce that

$$\theta_\lambda \leq \Phi_\lambda(u_*) \leq \liminf_{n\to\infty} \Phi_\lambda(u_n) = \theta_\lambda.$$

Proof of Theorem 3.1. By Proposition 3.1, Φ_λ has a global minimizer $u_* \in X$, so for all $t > 0$ and all $v \in X$, we have

$$\Phi_\lambda(u_* + tv) - \Phi_\lambda(u_*) \geq 0.$$

Dividing the last inequality by $t > 0$ and letting $t \to 0^+$, we obtain

$$\begin{aligned} &\int_\Omega |\Delta u_*|^{p(x)-2}\Delta u_* \Delta v dx \\ \geq\ & \lambda \int_\Omega \frac{|u_*(x)|^{p(x)-2}}{\delta(x)^{2p(x)}} u_*(x)v(x)dx + \int_\Omega b(x)|u_*|^{-m(x)}v(x)dx + \mu \int_\Omega f(x,u_*(x))v(x)dx. \end{aligned}$$

Since v is arbitrary in X, we can replace it by $-v$ which yields to

$$\begin{aligned} &\int_\Omega |\Delta u_*|^{p(x)-2}\Delta u_* \Delta v dx \\ =\ & \lambda \int_\Omega \frac{|u_*(x)|^{p(x)-2}}{\delta(x)^{2p(x)}} u_*(x)v(x)dx + \int_\Omega b(x)|u_*|^{-m(x)}v(x)dx + \mu \int_\Omega f(x,u_*(x))v(x)dx. \end{aligned}$$

Hence, u_* is a weak solution of problem (1.1). Now, by Lemma 3.2 we have $\Phi_\lambda(u_*) < 0$, so we conclude that u_* is a nontrivial weak solution of problem (1.1).

4. Problem without singular term

In this section, we shall use the mountain pass theorem and its $Z2$-symetric version to prove the existence and multiplicity of solutions for the following problem:

$$\begin{cases} \Delta^2_{p(x)}u = \lambda \frac{|u|^{p(x)-2}u}{\delta(x)^{2p(x)}} + \phi(x)\psi(u), & \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where ϕ and ψ are measurable functions satisfying the following hypotheses:

(A_1) There exists $c > 0$, α, $S \in C_+(\overline{\Omega})$, such that for all $(x,u) \in \Omega \times \mathbb{R}$,

$$\phi \in \frac{S(x)}{S(x) - \alpha(x)}(\Omega), \quad \psi(u) \leq c|u|^{\alpha(x)-1}$$

and

$$p^{+} < \alpha(x) < S(x) < p^{*}(x). \tag{4.2}$$

(A_2) There exists $M > 0, \theta > p^{+}$ such that for $x \in \Omega$,

$$0 < \theta\phi(x)\Psi(u) \leq \phi(x)\psi(u)u, \;\; |u| \geq M,$$

where $\Psi(t) = \int_0^t \psi(s)ds$.

(A_3) For all $x \in \overline{\Omega}$,

$$\psi(-u) = -\psi(u).$$

To show the existence and the multiplicity of solutions to problem (4.1), we will use the following theorems.

Theorem 4.1. *(Mountain pass theorem) Let E be a real Banach space and $J \in C^1(E, \mathbb{R})$ satisfy the Palais-Smale condition. Assume that*

(i) $J(0) = 0$.
(ii) There is $\rho > 0$ and $\sigma > 0$, such that $J(z) \geq \sigma$ for all $z \in E$ with $\|z\| = \rho$.
(iii) There exists $z_1 \in E$ with $\|z_1\| \geq \rho$, such that $J(z_1) < 0$.

Then ϕ_λ possesses a critical value $c \geq \sigma$. Moreover, c can be characterized as

$$c = \inf_{\gamma\in\Gamma} \max_{z\in[0,1]} \phi_\lambda(\gamma(z)),$$

where $\Gamma = \{\gamma \in C([0, 1], E) : \gamma(0) = 0, \gamma(1) = z_1\}$.

Theorem 4.2. *($\mathbb{Z}_2$-symmetric version of the mountain pass theorem) Let E be an infinite dimensional real Banach space. Let $J \in C^1(E, \mathbb{R})$, satisfying the following conditions:*

- *J is an even functional such that $J(0) = 0$.*
- *J satisfies the Palais-Smale condition.*
- *There exists positive constants ρ_0 and α_0, such that if $\|u\| = \rho_0$, then $J(u) \geq \alpha_0$.*
- *For each finite-dimensional subspace $X \subset E$, the set $\{u \in X, J(u) \geq 0\}$ is bounded in E.*

Then, J has an unbounded sequence of critical values.

We note that the functional J satisfies the Palais-Smale condition if any Palais-Smale sequence has a strongly convergent subsequence. That is, if $\{u_m\} \subset E$ such that $J(u_m)$ is bounded and $J'_\lambda(u_m)$ converges to zero in the dual space E', then $\{u_m\}$ has a convergent subsequence.

The main results of this section are summarized in the following theorems.

Theorem 4.3. *Under hypothesis (A_1) and (A_2), there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$, problem (4.1) has a nontrivial weak solution.*

Theorem 4.4. *Under the same hypotheses of Theorem 4.3, if in addition hypothesis (A_3) is satisfied, then there exists $\lambda^* > 0$, such that for any $\lambda \in (0, \lambda^*)$, problem (4.1) has infinitely many solutions.*

It is noted that a function $u \in X$, is said to be a weak solution for problem (4.1) if for any $v \in X$ we have

$$\int_\Omega |\Delta u|^{p(x)-2}\Delta u\, \Delta v dx - \lambda \int_\Omega \frac{|u|^{p(x)-2}uv}{\delta(x)^{2p(x)}}dx - \int_\Omega \phi(x)\psi(u)v dx = 0.$$

Associated to the problem (4.1), we define the functional $\chi_\lambda : X \to \mathbb{R}$ as follows:

$$\chi_\lambda(u) \;=\; \int_\Omega \frac{|\Delta u(x)|^{p(x)}}{p(x)}dx - \lambda \int_\Omega \frac{|u(x)|^{p(x)}}{p(x)\delta(x)^{2p(x)}}dx - \int_\Omega \phi(x)\Psi(u)dx.$$

Remark 4.1. *The functional χ_λ is well defined, it is in $C^1(X, \mathbb{R})$. Moreover, for all $(u, v) \in X \times X$, we have*

$$< \chi_\lambda^{'}(u), v >= \int_\Omega |\Delta u|^{p(x)-2}\Delta u\, \Delta v dx - \lambda \int_\Omega \frac{|u|^{p(x)-2}uv}{\delta(x)^{2p(x)}}dx - \int_\Omega \phi(x)\psi(u)v dx.$$

Also, weak solutions of problem (4.1) *correspond to critical points of the functional χ_λ.*

To prove our main results, we need to prove several lemmas.

Lemma 4.1. *Under hypothesis (A_1), there exists $\eta, \varrho > 0$ such that for $u \in X$:*

$$\textit{If} \quad \|u\| = \eta, \;\; \textit{then}, \;\; \chi_\lambda(u) \geq \varrho.$$

Proof. Let $x \in \Omega$ and $u \in X$ with $\|u\| < 1$, then from (A_1) we get

$$F(x, u) \leq c \int_0^u |\phi(x)||s|^{\alpha(x)-1}ds \leq \frac{c}{\alpha(x)}|\phi(x)||u|^{\alpha(x)}. \tag{4.3}$$

So from (2.2), we get

$$\begin{aligned}
\chi_\lambda(u) &= \int_\Omega \frac{|\Delta u|^{p(x)}}{p(x)}dx - \lambda \int_\Omega \frac{|u|^{p(x)}}{p(x)\delta(x)^{2p(x)}}dx - \int_\Omega \phi(x)\Psi(u)dx \\
&\geq (1 - \frac{\lambda}{C_H}) \int_\Omega \frac{|\Delta u(x)|^{p(x)}}{p(x)}dx - \int_\Omega \phi(x)\Psi(u)dx \\
&\geq \frac{(1 - \frac{\lambda}{C_H})}{p^+}M(u) - \int_\Omega \phi(x)\Psi(u)dx. \qquad (4.4)
\end{aligned}$$

By combining Eqs (4.3) and (4.4) with the Hölder inequality and Proposition 2.4, there exists $c_1 > 0$ such that

$$\begin{aligned}
\chi_\lambda(u) &\geq \frac{(1 - \frac{\lambda}{C_H})}{p^+}M(u) - \frac{c}{\alpha^-}\int_\Omega |\phi(x)||u|^{\alpha(x)}dx \\
&\geq \frac{(1 - \frac{\lambda}{C_H})}{p^+}M(u) - \frac{c_1}{\alpha^-}|\phi|_{\frac{S(x)}{S(x)-\alpha(x)}}||u|^{\alpha(x)}|_{\frac{S(x)}{\alpha(x)}} \\
&\geq \frac{(1 - \frac{\lambda}{C_H})}{p^+}M(u) - \frac{c_1}{\alpha^-}|\phi|_{\frac{S(x)}{S(x)-\alpha(x)}} \max(|u|^{\alpha^-}_{S(x)}, |u|^{\alpha^+}_{S(x)}).
\end{aligned}$$

On the other hand, since $1 < S(x) < p^*(x)$, then by Theorem 2.1 there exists $c_2 > 0$ such that

$$|u|_{S(x)} \leq c_2\|u\|. \tag{4.5}$$

Using (4.5) we obtain

$$\begin{aligned}\chi_\lambda(u) &\geq \frac{(1-\frac{\lambda}{C_H})}{p^+}\|u\|^{p^+} - \frac{c_1c_2}{\alpha^-}|\phi|_{\frac{S(x)}{S(x)-\alpha(x)}}\|u\|^{\alpha^-}\\ &\geq \|u\|^{p^+}\left(\frac{(1-\frac{\lambda}{C_H})}{p^+} - \frac{c_1c_2}{\alpha^-}|\phi|_{\frac{S(x)}{S(x)-\alpha(x)}}\|u\|^{\alpha^- - p^+}\right).\end{aligned}$$

Let $0 < \eta < 1$ small enough such that

$$\frac{(1-\frac{\lambda}{C_H})}{p^+} - \frac{c_1c_2}{\alpha^-}|\phi|_{\frac{S(x)}{S(x)-\alpha(x)}}\eta^{\alpha^- - p^+} > 0,$$

then for $\|u\| = \eta$, we have

$$\chi_\lambda(u) \geq \eta^{p^+}\left(\frac{(1-\frac{\lambda}{C_H})}{p^+} - \frac{c_1c_2}{\alpha^-}|\phi|_{\frac{S(x)}{S(x)-\alpha(x)}}\eta^{\alpha^- - p^+}\right) =: \varrho > 0.$$

Lemma 4.2. *Assume that hypotheses (A_1) and (A_2) hold, then there exists $0 < \lambda^* < C_H$, such that for any $\lambda \in (0, \lambda^*)$, χ_λ satisfies the Palais Smale condition.*

Proof. Let $\{u_n\}$ be a sequence in X such that

$$\chi_\lambda(u_n) \to c,\ \chi_\lambda^{'}(u_n) \to 0,\ \ in\ X^*,\ as\ n \to \infty,$$

for some positive constant c.

It follows that there exists $d_1 > 0$, such that for an n large enough, we have

$$|\chi_\lambda(u_n)| \leq d_1. \tag{4.6}$$

On the other hand, using the fact that $\chi_\lambda^{'}(u_n) \to 0$ in X^*, which implies that $\prec \chi_\lambda^{'}(u_n), u_n \succ \to 0$, there exists $d_2 > 0$ such that

$$| \prec \chi_\lambda^{'}(u_n), u_n \succ | \leq d_2. \tag{4.7}$$

Next, we shall prove that $\{u_n\}$ is bounded. If not, without loss of generality we can assume that $\|u_n\| \to \infty$, so for an n large enough we have $\|u_n\| \geq 1$. Now, if we combine Eq (4.4) with Eq (4.6), we get

$$d_1 \geq \chi_\lambda(u_n) \geq \frac{(1-\frac{\lambda}{C_H})}{p^+}M(u_n) - \int_\Omega \phi(x)\Psi(u_n)dx, \tag{4.8}$$

and by (4.7), we obtain

$$\begin{aligned}d_2 &\geq -\prec \chi_\lambda^{'}(u_n), u_n \succ\\ &= -M(u_n) + \int_\Omega \phi(x)\psi(u_n)u_n dx.\end{aligned} \tag{4.9}$$

So using hypothesis (A_2) and Eqs (4.8) and (4.9), we obtain

$$\begin{aligned}\theta d_1 + d_2 &\geq \left((1-\frac{\lambda}{C_H})\frac{\theta}{p^+}-1\right)M(u_n)+\int_\Omega(\phi(x)\psi(u_n)u_n-\theta\phi(x)\Psi(u_n))dx\\ &\geq \left((1-\frac{\lambda}{C_H})\frac{\theta}{p^+}-1\right)M(u_n)\\ &\geq \left((1-\frac{\lambda}{C_H})\frac{\theta}{p^+}-1\right)\|u_n\|^{p^-}. \qquad (4.10)\end{aligned}$$

Put

$$\lambda^* = (1-\frac{p^+}{\theta})C_H.$$

Since $\theta > p^+$, for all $\lambda \in (0,\lambda^*)$ we have

$$(1-\frac{\lambda}{C_H})\frac{\theta}{p^+}-1>0.$$

Therefore, by letting n tend to infinity in equation (4.10), we obtain a contradiction. We conclude that $\{u_n\}$ is bounded in X, so there exists $\{u_n\}$ and u in X such that, $\{u_n\}$ converges weakly to u in X.

On the other hand, by Theorem 2.1 and the fact that $S(x) < p^*(x)$, we deduce that $\{u_n\}$ converges strongly to u in $S(x)(\Omega)$. Moreover, we know that

$$\prec \chi_\lambda^{'}(u_n), u_n-u \succ = \prec \Delta^2_{p(x)}(u_n), u_n-u\succ -\lambda \prec \varphi^{'}(u_n),u_n-u\succ - \int_\Omega \phi(x)\psi(u_n)(u_n-u)\,dx.$$

Next, by using hypothesis (A_1) and the Hölder's inequality, there exists $C>0$ and $C^{'}>0$, such that

$$\begin{aligned}\int_\Omega \phi(x)\psi(u_n)(u_n-u)dx &\leq \int_\Omega C|\phi(x)||u_n|^{\alpha(x)-1}|u_n-u|\,dx\\ &\leq C|u_n-u|_{S(x)}|\phi(x)|_{\frac{S(x)}{S(x)-\alpha(x)}}\,||u_n|^{\alpha(x)-1}|_{\frac{S(x)}{\alpha(x)-1}}\\ &\leq C^{'}|u_n-u|_{S(x)}|\phi(x)|_{\frac{S(x)}{S(x)-\alpha(x)}}\max\left(\|u_n\|^{\alpha^+-1}|,\|u_n\|^{\alpha^+-1}\right).\end{aligned}$$

Thus, we obtain

$$\lim_{n\to+\infty}\int_\Omega \phi(x)\psi(u_n)(u_n-u)\,dx=0. \qquad (4.11)$$

Now, by Lemma 2.1, we have

$$\lim_{n\to+\infty} \prec \varphi^{'}(u_n), u_n-u\succ = 0. \qquad (4.12)$$

Since $\prec \chi_\lambda^{'}(u_n), u_n-u\succ \to 0$, by combining (4.11) with (4.12), we deduce that

$$\prec \Delta^2_{p(x)}(u_n), u_n-u\succ \to 0.$$

Finally, by Lemma 2.1 and the fact that $\Delta^2_{p(x)}$ is of type (S^+), we conclude that $u_n \to u$ is strongly in X. This implies that χ_λ satisfies the Palais Smale condition.

Lemma 4.3. *If hypothesis (A_2) holds, then there exists $u_* \in X$, such that $\|u_*\| > \eta$ and $\chi_\lambda(u_*) < 0$.*

Proof. From hypothesis (A_2) there exists $m > 0$, such that for all $(x, t) \in \Omega \times \mathbb{R}$ we have

$$\phi(x)\Psi(t) \geq m\,|t|^{\theta}. \tag{4.13}$$

Let $u \in X$, such that $\int_\Omega |u|^\theta\, dx > 0$, and let $t > 1$ be large enough. Then, from (4.13) we get

$$\begin{aligned} \chi_\lambda(tu) &= \int_\Omega \frac{|\Delta tu|^{p(x)}}{p(x)}dx - \lambda \int_\Omega \frac{|tu|^{p(x)}}{p(x)\delta(x)^{2p(x)}}dx - \int_\Omega \phi(x)\Psi(tu)dx \\ &\leq \frac{t^{p^+}}{p^-}\int_\Omega |\Delta(u)|^{p(x)}dx - mt^\theta \int_\Omega |u|^\theta\, dx. \end{aligned}$$

Since $\theta > p^+$, we deduce that

$$\lim_{t\to\infty} \chi_\lambda(tu) \to -\infty,$$

so, we can choose $t_0 > 0$, such that the function $u_* = t_0 u$ satisfies

$$\|u_*\| > \eta \ \text{ and } \ \chi_\lambda(u_*) < 0.$$

Proof of Theorem 4.3. First of all, it is easy to see that $0 = \chi_\lambda(0)$, which implies that condition (*i*) of Theorem 4.1 is satisfied.

On the other hand, from Lemma 4.1 we have

$$\inf_{\|u\|=\eta} \chi_\lambda(u) \geq m > 0 = \chi_\lambda(0).$$

This implies that condition (*ii*) of Theorem 4.1 is also satisfied.

Moreover, by Lemma 4.3, there exists $u_* \in X$ such that

$$\|u_*\| > \eta \ \text{ and } \ \chi_\lambda(u_*) < 0. \tag{4.14}$$

This implies that condition (*iii*) of Theorem 4.1 is satisfied.

Finally, from Lemma 4.2, χ_λ satisfies the Palais Smale condition, and $\chi_\lambda \in C^1(X)$. Thus by the mountain pass theorem (Theorem 4.1), we concluded that the functional χ_λ has a critical point which is a weak solution for a problem (4.1). Moreover, by Eq (4.14), we see that this solution is nontrivial, so the proof of Theorem 4.3 is completed.

Next, we will use Theorem 4.2 to prove the second main result of this section, so we need to prove the following lemma.

Lemma 4.4. *Assume that hypotheses (A_1) and (A_2) hold, and let E be a finite-dimensional subspace of X, then the set*

$$H = \{u \in E, \chi_\lambda(u) \geq 0\}$$

is bounded in X.

Proof. Let $u \in H$, then we have

$$\chi_\lambda(u) \leq \frac{1}{p^-}\int_\Omega |\Delta u|^{p(x)}dx - \int_\Omega \phi(x)\Psi(u)dx.$$

On the other hand by Eq (4.13) and Proposition 2.4, we obtain

$$\chi_\lambda(u) \leq \frac{1}{p^-} M(u) - m \int_\Omega |u|^\theta dx \leq \frac{1}{p^-}(\|u\|^{p^+} + \|u\|^{p^-}) - m|u|^\theta_{L^\theta}. \tag{4.15}$$

Since E is a finite-dimensional subspace, the norms $|.|_{L^{\theta_1}}$ and $\|.\|$ are equivalent, so there exists $C > 0$ such that

$$\|u\|^\theta \leq C|u|^\theta_{L^\theta}.$$

By combining the last inequality with Eq (4.15), we obtain

$$\chi_\lambda(u) \leq \frac{1}{p^-}(\|u\|^{p^+} + \|u\|^{p^-}) - \frac{m}{C}\|u\|^\theta.$$

Since, $p^- < p^+ < \theta$, we concluded that the set H is bounded in X.

Now, we are ready to prove Theorem 4.4.

Proof of Theorem 4.4. We have $\chi_\lambda(0) = 0$. Moreover, by hypothesis (A_3) we see that χ_λ is an even functional. Therefore the proof of Theorem 4.4 is deduced by combining Lemmas 4.1, 4.2 and 4.4 with Theorem 4.2. This implies that problem (4.1) has infinitely many solutions.

5. Conclusions

This paper considered some classes of $p(x)$-biharmonic problems with singular nonlinearity and Hardy potential. More precisely, by the use of the min-max method, some existing results were proved. Moreover, some important properties of the associated functional energy were given, and after that, using diversions of the mountain pass theorem, the multiplicity of solutions was also proved. This study can be generalized to similar problems involving the $p(x, y)$-Laplacian operator.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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