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Research article

Analysis of traveling fronts for chemotaxis model with the nonlinear degenerate viscosity

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Abstract: In this paper, we are interested in chemotaxis model with nonlinear degenerate viscosity under the assumptions of $\beta = 0$ (without the effect of growth rate) and $u_+ = 0$. We need the weighted function defined in Remark 1 to handle the singularity problem. The higher-order terms of this paper are significant due to the nonlinear degenerate viscosity. Therefore, the following higher-order estimate is introduced to handle the energy estimate:

$$U^{m-2} = \left(\frac{1}{U}\right)^{2-m} \leq Kw(z) \leq \frac{Cw(z)}{U}, \text{ if } 0 < m < 2,$$
$$U^{m-2} \leq Lu_- \leq \frac{Cu_-}{U}, \text{ if } m \geq 2,$$

where $C = max\{K, L\} = max\left\{\frac{a}{m-a}, (m+a)^m\right\}$ for $a > 0$ and $m > a$, and $w(z)$ is the weighted function. Then we show that the traveling waves are stable under the appropriate perturbations. The proof is based on a Cole-Hopf transformation and weighted energy estimates.

Keywords: chemotaxis model; nonlinear diffusion; stability; existence; weighted energy estimates
Mathematics Subject Classification: 35A01, 35B40, 35Q92, 92C17

1. Introduction

This paper is based on the following chemotaxis system with nonlinear diffusion [6]

$$\begin{cases} u_t = D(u^m)_{xx} - \chi\left(u(\ln c)_x\right)_x, \\ c_t = -uc + \beta c, \end{cases} \tag{1.1}$$

for $m > 0$ and the initial data

$$(u, c)(x, 0) = (u_0, c_0)(x) \to (u_\pm, c_\pm) \text{ as } x \to \pm\infty.$$

When $m \neq 1$, the chemotaxis system (1.1) represents the reinforced movement of cells (or bacteria) in porous media, where u, c and $\beta > 0$ are the population density of cells, concentration of chemical signals (e.g., nutrients), and growth rate, respectively. Moreover, the diffusion rate of cells and the chemotactic coefficient are denoted by $D > 0$ and χ, respectively. The chemotaxis is said to be attractive if $\chi > 0$ and repulsive if $\chi < 0$. The logarithmic sensitivity $\ln c$ was derived from Weber-Fechner law [13] and has been verified by the experimental data [11]. The above PDE-ODE system is the special case of the following Keller-Segel model with porous media type diffusion:

$$\begin{cases} u_t = \nabla \cdot (D\nabla u^m - \chi u \nabla \pi(c)), \quad \text{for} \;\; m > 0, \\ c_t = \varepsilon \Delta c + f(u, c), \end{cases}$$

where $f(u, c)$ is a function characterizing the chemical growth and degradation defined as $f(u, c) = \beta c - ug(c)$. This system describes the chemotactic dynamics, where cells move up the chemical concentration gradient and consume (or degrade) the chemical along the path. As stated in [26], the function $g(c)$ is called the consumption rate function in the form

$$g(c) = c^p = \begin{cases} \text{constant rate}, \quad p = 0, \\ \text{sublinear rate}, \quad 0 < p < 1, \\ \text{linear rate}, \quad p = 1, \\ \text{superlinear rate}, \quad p > 1. \end{cases}$$

Moreover, the typical examples of chemosensitivity function $\pi(c)$ include $\pi(c) = kc$ (linear law), $\pi(c) = k \log c$ (logarithmic law), and $\pi(c) = kc^p/(1 + c^m)$ (receptor law) where $k > 0$ and $p \in \mathbb{N}$.

The problems of chemotaxis model in porous media are extensively studied for both the experiments and mathematical modeling. The experiments of bacterial chemotaxis in porous media were investigated in [21, 25], and the nonlinear diffusion to a chemotaxis model in order to avoid overcrowding was instroduced in [1, 8]. Tao and Winkler [24] established the global existence and boundedness of solutions to a chemotaxis model of self-aggregation with arbitrary porous medium diffusion. However, few results are available to the chemotaxis model (1.1) except for the existence of compactly supported traveling waves in [2].

When $m = 1$, the system (1.1) is exactly the chemotaxis model proposed in [22] to describe the reinforced random walks. There are many other interesting analytical works with reinforced random walks. Othmer and Stevens [22] studied the model from random walk and presented the numerical simulations of the formation of spikes and blowup. The analytic results that support some numerical results in [22] were established in [23]. The global existence and blowup of classical solutions on a bounded domain with no-flux boundary conditions were studied in [29, 30]. Moreover, the further study of global existence of smooth solutions to system (1.1) was investigated by Li et al. [16]. Zhang and Zhu [31] presented the weak solutions to system (1.1) with the Robin boundary condition. Other references for global dynamics including well-posedness and large time behaviors of solutions in the whole space were presented in [3, 14, 18, 28]. The spike solution and blowup solution, traveling wave is another biological pattern observed in chemotaxis [13]. The existence of traveling fronts to (1.1) was firstly established in [27]. The stability problem of such a traveling front in the case of $u_+ > 0$ was obtained in [17]. Moreover, when $u_+ = 0$, the energy estimate has the singular term, which is extremely difficult to overcome. This singular term was presented in [10] by employing it as the weighted function in the energy estimate. Recently, the half-space problem of (1.1) under the non-zero flux boundary condition was considered in [15]. The authors showed that the system still admits traveling wave profiles on the half-space by introducing a wave selection mechanism. For other related work on traveling waves of chemotaxis models and Burger's equations, we refer the readers to these references [4, 5, 7, 9, 26].

By ignoring the effect of growth rate ($\beta = 0$), then one can derive the chemotaxis model

$$\begin{cases} u_t = D(u^m)_{xx} - \chi\left(u(\ln c)_x\right)_x, \\ c_t = -uc, \end{cases} \tag{1.2}$$

for $m > 0$ and the initial data

$$(u, c)(x, 0) = (u_0, c_0)(x) \to (u_\pm, c_\pm) \text{ as } x \to \pm\infty. \tag{1.3}$$

The major problem of this paper is concerned with the nonlinear diffusion and singularity problem. Under small perturbations and large wave amplitude, we prove the existence and stability of traveling waves to system (1.2) with $m > 0$ and $u_+ = 0$. The logarithmic singularity for the first equation of (1.2) is very difficult to study. Therefore, to handle these barriers, we employ the following Cole-Hopf transformation as in [10, 17]:

$$v = -(\ln c)_x, \tag{1.4}$$

which presents the chemotaxis system as follows:

$$\begin{cases} u_t - \chi(uv)_x = D(u^m)_{xx}, \\ v_t - u_x = 0, \end{cases} \tag{1.5}$$

and the initial data

$$(u, v)(x, 0) = (u_0, v_0)(x) \to (u_\pm, v_\pm) \text{ as } x \to \pm\infty. \tag{1.6}$$

We organize this paper as follows: In Section 2, we present the theorems of existence and stability of the transformed system (1.5) and the original system (1.2). The proofs of weighted energy estimates and the stability of transformed system (1.5) are provided in Section 3. Then, we transfer the obtained results to prove the existence and stability of the main results for original system (1.2).

Notation 1. *The norms in the Sobolev space of $H^r(\mathbb{R})$ are stated as $\|q\|_r := \sum_{k=0}^{r} \|\partial_x^k q\|$ and $\|q\| := \|q\|_{L^2(\mathbb{R})}$. Moreover, the weighted norms in the Sobolev space of $H^r_w(\mathbb{R})$ are given by $\|q\|_{r,w} := \sum_{k=0}^{r} \|\sqrt{w(x)}\partial_x^k q\|$ and $\|q\|_w := \|q\|_{L^2_w(\mathbb{R})}$.*

2. Main results

We first establish the solutions of traveling wave $(U, V)(x - st)$ of the parabolic-hyperbolic system (1.5). Substituting the following traveling wave ansatz

$$(u, v)(x, t) = (U, V)(z), \; z = x - st, \tag{2.1}$$

into (1.5), where s and z are the traveling wave speed and moving coordinate, respectively. Then, we have

$$\begin{cases} -sU' - \chi(UV)' = D\left(U^m\right)'', \\ -sV' = U', \end{cases} \tag{2.2}$$

where $' := \frac{d}{dz}$, and the boundary conditions are given as follows:

$$(U,V)(z) \to (u_\pm, v_\pm) \text{ as } z \to \pm\infty. \tag{2.3}$$

Integrating (2.2) in z over $(-\infty, z)$ and $(z, +\infty)$, then one has

$$-\left(\int_{-\infty}^{z} sU' + \int_{z}^{+\infty} sU'\right) - \left(\int_{-\infty}^{z} \chi(UV)' + \int_{z}^{+\infty} \chi(UV)'\right) = D\left(\int_{-\infty}^{z} (U^m)'' + \int_{z}^{+\infty} (U^m)''\right)$$

and

$$-\left(\int_{-\infty}^{z} sV' + \int_{z}^{+\infty} sV'\right) = \left(\int_{-\infty}^{z} U' + \int_{z}^{+\infty} U'\right).$$

Moreover, we can rewrite the above results as follows:

$$\begin{aligned} &\left(\lim_{z\to-\infty} sU(z) - \lim_{z\to+\infty} sU(z)\right) + \left(\lim_{z\to-\infty} \chi(UV)(z) - \lim_{z\to+\infty} \chi(UV)(z)\right) \\ &= D\left(-\lim_{z\to-\infty} (mU^{m-1}U')(z) + \lim_{z\to+\infty} (mU^{m-1}U')(z)\right) \end{aligned}$$

and

$$\left(\lim_{z\to-\infty} sV(z) - \lim_{z\to+\infty} sV(z)\right) = \left(-\lim_{z\to-\infty} U(z) + \lim_{z\to+\infty} U(z)\right).$$

Employing (2.3), the fact $u_+ = 0$, and $U'(z) \to 0$ as $z \to \pm\infty$, then one has the following Rankine-Hugoniot conditions

$$\begin{aligned} -s &= \chi v_-, \\ s(v_+ - v_-) &= u_-, \end{aligned} \tag{2.4}$$

which presents

$$s^2 + s\chi v_+ - \chi u_- = 0. \tag{2.5}$$

In this paper, we only consider $s > 0$ and

$$s = -\frac{\chi v_+}{2} + \frac{\sqrt{\chi^2 v_+^2 + 4\chi u_-}}{2}. \tag{2.6}$$

Remark 1. *To provide the weighted energy estimate, the weighted function is defined as follows:*

$$w(z) = 1 + e^{\eta z} \ \text{ with } \ \eta := \frac{u_-^{1-m}}{Dm} \cdot (s + \chi v_+), \ \text{ for all } \ z \in \mathbb{R}. \tag{2.7}$$

According to the weighted function in (2.7)*, the following condition is given*

$$C_1 w(z) \le \frac{1}{U(z)} \le C_2 w(z) \ \text{ for all } \ z \in \mathbb{R}, \tag{2.8}$$

where $C_2 > C_1 > 0$.

Lemma 1. *Let* (2.4) *and* $u_+ = 0$ *hold. Then, the system (2.2) has a monotone traveling wave solution* $(U, V)(x - st)$*, which is unique up to a translation satisfying* $U' < 0$*,* $V' > 0$*. Moreover,* (U, V) *has the following monotonicity behavior:*

$$\begin{aligned} &U - u_- \sim e^{\frac{\chi u_-}{Ds}z} \text{ as } z \to -\infty,\ U \sim e^{-\frac{\chi u_-}{Ds}z} \text{ as } z \to +\infty, \\ &V - v_\pm \sim e^{\frac{\chi u_-}{Ds}\cdot(v_\mp - v_\pm)z} \text{ as } z \to \pm\infty,\ \text{for } v_+ > v_-. \end{aligned} \tag{2.9}$$

Proof. To show (2.9), we first assume that

$$H = (U^m)^{'},\quad V' = -\frac{HU^{1-m}}{sm}.$$

Then, one has

$$\begin{aligned} U' &= \frac{HU^{1-m}}{m}, \\ H' &= \frac{HU^{1-m}}{Dsm}\cdot\left(2\chi U - (\chi F + s^2)\right), \end{aligned} \tag{2.10}$$

which provides

$$\begin{cases} \frac{dH}{dU} = \frac{\chi}{Ds}\cdot(2U - u_-), \\ H(u_-) = 0. \end{cases} \tag{2.11}$$

Let H_s be the solution of Eq (2.11), then one can derive

$$H_s(U) \sim mu_-^{m-1}\cdot\frac{\chi}{Ds}(2U - u_-)(U - u_-),\ \text{as } U \to u_-. \tag{2.12}$$

By employing the first equation of Eqs (2.10), (2.12) and L'Hospital's rule, one has

$$\begin{aligned} \lim_{U\to u_-}\frac{z}{\ln(U - u_-)} &= \lim_{U\to u_-}\frac{mu_-^{m-1}(U - u_-)}{H_s} \\ &= \lim_{U\to u_-}\frac{mu_-^{m-1}(U - u_-)}{mu_-^{m-1}\cdot\frac{\chi}{Ds}(2U - u_-)(U - u_-)} \\ &= \frac{1}{\frac{\chi u_-}{Ds}}, \end{aligned}$$

which provides

$$U - u_- \sim e^{\frac{\chi u_-}{Ds}z},\ \text{as } z \to -\infty.$$

Moreover, we assume that H_s is the solution of Eq (2.11), then one has

$$H_s(U) \sim \frac{\chi}{Ds}\cdot(2U - u_-)mU^m,\ \text{as } U \to 0. \tag{2.13}$$

Similarly, by employing the first equation of Eq (2.10), L'Hospital's rule and Eq (2.13), one gets

$$\begin{aligned} \lim_{U\to 0}\frac{z}{\ln U} &= \lim_{U\to 0}\frac{mU^m}{H_s} \\ &= \lim_{U\to 0}\frac{mU^m}{\frac{\chi}{Ds}\cdot(2U - u_-)mU^m} \\ &= -\frac{1}{\frac{\chi u_-}{Ds}}, \end{aligned}$$

which gives

$$U \sim e^{-\frac{\chi u_-}{Ds} z}, \text{ as } z \to +\infty.$$

Therefore, we have

$$V - v_\pm \sim e^{\frac{\chi u_-}{Ds} \cdot (v_\mp - v_\pm) z}, \text{ as } z \to \pm\infty, \text{ for } v_+ > v_-.$$

□

Remark 2. *Based on the second equation of Eq* (2.10)*, one can derive*

$$U' = \frac{U^{1-m}}{Dms} \cdot (\chi U^2 - (s^2 + \chi F)U), \tag{2.14}$$

where $F = sv_- + u_- = sv_+ + u_+$ and the wave speed s is given by (2.6).

For the parabolic-hyperbolic system (1.5), we define

$$(\pi_0, \rho_0)(z) := \int_{-\infty}^{z} (u_0 - U, v_0 - V)(y) dy,$$

where the zero perturbation is obtained (see [12,19]). Then the stability result can be stated as follows:

Theorem 1. *Assume that $D > 0$, $m > 0$ and $\chi > 0$. Let $(U, V)(x - st)$ be the traveling waves from Lemma 1. If $\|u_0 - U\|_{2,w} + \|v_0 - V\|_{2,w} + \|(\pi_0, \rho_0)\|_{3,w} \le \varepsilon_0$ for constant $\varepsilon_0 > 0$, then the Cauchy problem* (1.5) *and* (1.6) *has a unique global solution $(u, v)(x, t)$, which satisfies*

$$(u - U, v - V) \in C([0, \infty); H_w^2) \cap L^2([0, \infty); H_w^2)$$

and

$$\sup_{x \in \mathbb{R}} |(u, v)(x, t) - (U, V)(x - st)| \to 0 \text{ as } t \to +\infty.$$

By applying the change of variables $(x, t) \to (z = x - st, t)$, the system (1.5) becomes

$$\begin{cases} u_t - su_z - \chi(uv)_z = D\,(u^m)_{zz}, \\ v_t - sv_z - u_z = 0. \end{cases} \tag{2.15}$$

The solutions (u, v) of (2.15) are decomposed as follows:

$$(u, v)(z, t) = (U, V)(z) + (\pi_z, \rho_z)(z, t). \tag{2.16}$$

Then

$$\pi(z, t) = \int_{-\infty}^{z} (u(y, t) - U(y)) dy, \ \rho(z, t) = \int_{-\infty}^{z} (v(y, t) - V(y)) dy. \tag{2.17}$$

Substituting (2.16) into (2.15) and integrating the results with respect to z, one has

$$\begin{cases} \pi_t - (s + \chi V)\pi_z - \chi U \rho_z = Dm \left(U^{m-1} \pi_z \right)_z + G + \chi \pi_z \rho_z, \\ \rho_t - s\rho_z - \pi_z = 0, \end{cases} \tag{2.18}$$

where $G = D\Big((U+\pi_z)^m - U^m - mU^{m-1}\pi_z\Big)_z$, which is one of the barriers for the nonlinear diffusion. Moreover, the initial perturbation of (π,ρ) is given by

$$(\pi,\rho)(z,0) = (\pi_0,\rho_0)(z) = \int_{-\infty}^{z} (u_0 - U, v_0 - V)dy, \tag{2.19}$$

with $(\pi_0,\rho_0)(\pm\infty) = 0$. We present the solution of reformulated problem (2.18) and (2.19) in the space

$$X(0,T) := \left\{(\pi,\rho) \in C([0,T),H_w^3) : \pi_z \in L^2((0,T);H_w^3)), \rho_z \in C([0,T);H_w^2) \cap L^2((0,T);H_w^2)\right\},$$

where $0 < T \le +\infty$ and w is the weighted function defined in (2.7).

Let

$$N(t) := \sup_{0\le\tau\le t} \{\|\pi(.,\tau)\|_{3,w} + \|\rho(.,\tau)\|_3 + \|\rho(.,\tau)\|_{3,w}\}.$$

From the Sobolev inequality $\|f\|_{L^\infty} \le \sqrt{2}\|f\|_{L_w^2}^{\frac{1}{2}}\|f_x\|_{L_w^2}^{\frac{1}{2}}$, it follows that

$$\sup_{\tau\in[0,t]} \{\|\pi(\cdot,\tau)\|_{W^{2,\infty}}, \|\rho(\cdot,\tau)\|_{W^{2,\infty}}\} \le N(t).$$

Then, for system (2.18) and (2.19), we have the following global well-posedness:

Theorem 2. *There exists a constant $\delta_1 > 0$ such that if $N(0) \le \delta_1$, then the Cauchy problem* (2.18) *and* (2.19) *has a unique global solution $(\pi,\rho) \in X(0,+\infty)$ such that*

$$\begin{aligned}&\|\pi(.,t)\|_{3,w}^2 + \|\rho(.,t)\|_3^2 + \|\rho(.,t)\|_{3,w}^2 + \int_0^t \left(\|\pi_z(.,\tau)\|_{3,w}^2 + \|\rho_z(.,\tau)\|_{2,w}^2\right) d\tau \\ &\le C(\|\pi_0\|_{3,w}^2 + \|\rho_0\|_3^2 + \|\rho_0\|_{3,w}^2) \le CN^2(0),\end{aligned} \tag{2.20}$$

for any $t > 0$. Moreover, it holds that

$$\sup_{z\in R} |(\pi_z,\rho_z)(z,t)| \to 0 \text{ as } t \to +\infty. \tag{2.21}$$

According to the classical works (see [17]), the global smooth solution can be constructed by the local well-posedness, the a priori estimate and an extension procedure. By the standard ways, the local well-posedness can be inferred (e.g., see [20]).

Proposition 1. *Let $(\pi,\rho) \in X(0,T)$ be a solution of (2.18) and (2.19) for several times $T > 0$. Then a constant $\varepsilon_1 > 0$ is presented, which is independent on T, such that if $N(T) < \varepsilon_1$, then (π,ρ) satisfies (2.20) for any $0 \le t \le T$.*

We further transfer the results of the transformed system (1.5) to the original chemotaxis model (1.2). Finally, our main results on the existence and stability of traveling waves (1.2) are stated in the following theorems:

Theorem 3 (Existence)**.** *Let $\chi \in \mathbb{R}$ $(\neq 0)$. Then, the chemotaxis model with the nonlinear degenerate viscosity* (1.2) *does not have a traveling wave solution if $\chi < 0$. If $\chi > 0$ then the chemotaxis model with the nonlinear degenerate viscosity* (1.2) *has a unique monotone traveling wave solution $(U,C)(z)$ such that $U_z < 0, C_z > 0$ for any given $0 = u_+ < u_-$ and $0 = c_- < c_+$.*

Proof. The existence of U is from Lemma 1. We further show the existence of C. It follows from the second equation of (1.2), which implies that

$$sC' = CU, \tag{2.22}$$

which can be further calculated as follows:

$$C(z) = c_+ e^{\frac{1}{s}\int_0^z U(y)dy}. \tag{2.23}$$

Since U converges to $u_\pm$ exponentially as $z \to \pm\infty$, and $s > 0$, then $C(z)$ is bounded for any $z \in \mathbb{R}$. It has the consequence $C(z) > 0$ for any $z \in \mathbb{R}$, since otherwise $c_+ \equiv 0$ and hence $C(z) = 0$, which is not desired.

Noting that $s > 0$ and the Eq (2.22) at $z = \pm\infty$ yields

$$c_\pm u_\pm = 0. \tag{2.24}$$

Since $C(z) > 0$ for all $z \in \mathbb{R}$, then it causes $0 \le c_- < c_+$. It follows from the fact $c_+ > 0$ and (2.24), then it leads to $u_+ = 0$, which is only possible when $U_z < 0$ and hence $0 = u_+ < u_-$. Finally, the proof of Theorem 3 is completed. □

Theorem 4 (Stability)**.** *Assume that $D > 0$, $m > 0$, and $\chi > 0$. Let $(U, C)(x - st)$ be the traveling waves obtained in Theorem 3. Then one has a constant $\varepsilon_0 > 0$ such that if $\|u_0 - U\|_{2,w} + \|(\ln c_0)_x - (\ln C)_x\|_{2,w} + \|(\pi_0, \rho_0)\|_{3,w} \le \varepsilon_0$, where*

$$\pi_0(x) = \int_{-\infty}^x (u_0 - U)(y)dy,\ \rho_0(x) = \ln C(x) - \ln c_0(x),$$

then the Cauchy problem (1.2) *and* (1.3) *has a unique global solution $(u, c)(x, t)$ satisfying*

$$(u - U, (\ln c)_x - (\ln C)_x) \in C([0, \infty); H_w^2) \cap L^2([0, \infty); H_w^2),$$

and

$$\sup_{x\in\mathbb{R}} |(u, c)(x, t) - (U, C)(x - st)| \to 0 \text{ as } t \to +\infty.$$

3. Nonlinear stability

3.1. Weighted energy estimates

We further present the a priori estimates for solutions (π, ρ) of (2.18) and (2.19), and hence prove Proposition 1. In this paper, we deal with the case $u_+ = 0$ as $z \to +\infty$ which provides the singularity of $\frac{1}{U}$. Therefore, we modify the idea of [17] by considering the singular term of $\frac{1}{U}$ as the weighted function of w in the energy estimates.

3.1.1. Estimate of (π, ρ) in L^2

Lemma 2. *Under the same assumptions of Proposition 1, if $N(t) \ll 1$, then*

$$\begin{aligned} &\|\pi(., t)\|_w^2 + \|\rho(., t)\|^2 + \int_0^t \|\pi_z(., \tau)\|_w^2 d\tau \\ &\le C(\|\pi_0\|_w^2 + \|\rho_0\|^2) + \int_0^t \int CN(t)w(z)(\rho_z^2 + \pi_{zz}^2). \end{aligned} \tag{3.1}$$

Proof. Multiplying $(2.18)_1$ by $\frac{\pi}{U}$ and $(2.18)_2$ by $\chi\rho$, adding them, and integrating the resulting equations, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\left(\frac{\pi^2}{U}+\chi\rho^2\right)+Dm\int U^{m-2}\pi_z^2\\&=-\int\frac{\pi^2}{2}\left(\left(\frac{s+\chi V}{U}\right)_z-Dm\left(U^{m-1}\left(\frac{1}{U}\right)_z\right)_z\right)+\int\left(\frac{G\pi}{U}+\chi\frac{\pi_z\rho_z\pi}{U}\right).\end{aligned}\tag{3.2}$$

From Eq (2.14) and $u_+=0$, one has

$$\begin{aligned}\left(\frac{s+\chi V}{U}\right)_z-Dm\left(U^{m-1}\left(\frac{1}{U}\right)_z\right)_z&=\left(\frac{s+\chi V}{U}-DmU^{m-1}\left(\frac{1}{U}\right)_z\right)_z\\&=\left(\frac{su_++\chi u_+v_+}{U^2}\right)_z\\&=-\frac{2u_+(s+\chi v_+)U_z}{U^3}=0.\end{aligned}\tag{3.3}$$

We further approximate $(U+\pi_z)^m$ in (2.18) through the following estimation:

$$(\pi_z+U)^m\le(\pi_z+u_-)^m=u_-^m\left(\frac{\pi_z}{u_-}+1\right)^m=\sum_{l=0}^{m}u_-^m\frac{P_l^m}{l!}\left(\frac{\pi_z}{u_-}\right)^l,\tag{3.4}$$

where $P_l^m=\frac{m!}{(m-l)!}$.

By dealing with $N(t)\ll 1$, one has $\|\pi_z(\cdot,t)\|_{L^\infty}\le 1$, and then (3.4) becomes

$$(\pi_z+U)^m\le u_-^m(m!)^2\pi_z^2\sum_{l=0}^{m}\frac{1}{l!}\left(\frac{1}{u_-}\right)^l=u_-^m(m!)^2\pi_z^2e^{1/u_-}\le C\pi_z^2.\tag{3.5}$$

Note that $0<U\le u_-$, $\|\pi_z(\cdot,t)\|_{L^\infty}\le N(t)\ll 1$, and the term $(\pi_z+U)^{m-1}$ consists of two conditions: $(\pi_z+U)^{m-1}\le(\pi_z+u_-)^{m-1}$ if $m\ge 1$ and $(\pi_z+U)^{m-1}\le(\pi_z+Cw(z))^{m-1}$ if $0<m<1$, where the weighted function $w(z)$ is given in Remark 1. Then we can derive

$$|G|\le C(|\pi_{zz}||\pi_z|+|\pi_z|^2).\tag{3.6}$$

By employing Young's inequality, we have

$$\left|\int\frac{G\pi}{U}\right|\le CN(t)\int\left(\frac{|\pi_z^2|+|\pi_{zz}|^2}{U}\right),\tag{3.7}$$

where $\|\pi(\cdot,t)\|_{L^\infty}\le N(t)$ has been employed. Similarly,

$$\left|\int\frac{\pi_z\rho_z\pi}{U}\right|\le CN(t)\int\left(\frac{\pi_z^2+\rho_z^2}{U}\right).\tag{3.8}$$

Substituting (3.3), (3.7) and (3.8) into (3.2), we get

$$\frac{1}{2}\frac{d}{dt}\int\left(\frac{\pi^2}{U}+\chi\rho^2\right)+\int\left(DmU^{m-2}-\frac{CN(t)}{U}\right)\pi_z^2\le CN(t)\int\frac{\rho_z^2+\pi_{zz}^2}{U}.\tag{3.9}$$

Moreover, the higher-order estimate U^{m-2} has two possibilities as follows:

$$U^{m-2} = \left(\frac{1}{U}\right)^{2-m} \leq Kw(z) \leq \frac{Cw(z)}{U}, \text{ if } 0 < m < 2,$$
$$U^{m-2} \leq Lu_- \leq \frac{Cu_-}{U}, \text{ if } m \geq 2,$$

where $C = max\{K, L\} = max\left\{\frac{a}{m-a}, (m+a)^m\right\}$ for $a > 0$ and $m > a$. Then, (3.9) becomes

$$\frac{1}{2}\frac{d}{dt}\int\left(\frac{\pi^2}{U} + \chi\rho^2\right) + \int Cw\left(Dm(w(z) + u_-) - N(t)\right)\pi_z^2 \leq \int CN(t)\frac{(\rho_z^2 + \pi_{zz}^2)}{U}. \tag{3.10}$$

Conducting $N(t) \leq Dm(w(z) + u_-)$ and $1/U(z) \leq Cw(z)$ for all $z \in \mathbb{R}$, and further calculation of the integration of (3.10) with respect to t, the proof of estimate (π, ρ) in L^2 is completed. □

3.1.2. Estimate of (π, ρ) in H^1

Lemma 3. *Under the same assumptions of Proposition 1, if $N(t) \ll 1$, then*

$$\begin{aligned} &\|\pi(.,t)\|_{1,w}^2 + \|\rho(.,t)\|_1^2 + \|\rho(.,t)\|_{1,w}^2 + \int_0^t (\|\pi_z(.,\tau)\|_{1,w}^2 + \|\rho_z(.,\tau)\|_w^2)d\tau \\ &\leq C(\|\pi_0\|_{1,w}^2 + \|\rho_0\|_1^2 + \|\rho_0\|_{1,w}^2). \end{aligned} \tag{3.11}$$

Proof. Differentiating (2.18) in z gives

$$\begin{cases} \pi_{zt} - \chi U\rho_{zz} - Dm\left(U^{m-1}\pi_{zz}\right)_z = Dm((U^{m-1})_z\pi_z)_z + \chi U_z\rho_z + ((s+\chi V)\pi_z)_z + (G + \chi\pi_z\rho_z)_z, \\ \rho_{zt} - s\rho_{zz} - \pi_{zz} = 0. \end{cases} \tag{3.12}$$

Multiplying $(3.12)_1$ by $\frac{\pi_z}{U}$ and $(3.12)_2$ by $\chi\rho_z$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int\left(\frac{\pi_z^2}{U} + \chi\rho_z^2\right) + Dm\int U^{m-2}\pi_{zz}^2 \\ &= \int\frac{Dm\pi_z^2}{2}\left((U^{m-1})_{zz}\frac{1}{U} - \left(U^{m-1}\right)_z\left(\frac{1}{U}\right)_z\right) \\ &\quad + \chi\int\frac{V_z\pi_z^2}{U} + \chi\int\frac{U_z\rho_z\pi_z}{U} - \int(G + \chi\pi_z\rho_z)\left(\frac{\pi_z}{U}\right)_z. \end{aligned} \tag{3.13}$$

By Young's inequality,

$$\chi\int\frac{U_z\rho_z\pi_z}{U} \leq \frac{\gamma\chi}{2}\int U\rho_z^2 + \frac{\chi}{2\gamma}\int\frac{U_z^2\pi_z^2}{U^3},$$

where γ is a small enough constant. Substituting this inequality into (3.13) leads to

$$\begin{aligned} &\int\left(\frac{\pi_z^2}{U} + \chi\rho_z^2\right) + 2Dm\int_0^t\int U^{m-2}\pi_{zz}^2 \\ &\leq C\|\pi_{0z}\|_w^2 + C\|\rho_{0z}\|^2 + C(1 + \frac{1}{\gamma})\int_0^t\int\frac{\pi_z^2}{U} + \gamma\chi\int_0^t\int U\rho_z^2 \\ &\quad + C\int_0^t\int|(G + \chi\pi_z\rho_z)|\left(|\pi_z| + \frac{|\pi_{zz}|}{U}\right). \end{aligned} \tag{3.14}$$

We further multiply the first equation of (2.18) by ρ_z to present the estimate $\int_0^t\int U\rho_z^2$, and one yields

$$\chi U\rho_z^2 = \pi_t\rho_z - (s+\chi V)\pi_z\rho_z - Dm\left(U^{m-1}\pi_z\right)_z\rho_z - (G+\chi\pi_z\rho_z)\rho_z. \tag{3.15}$$

By the second equation of (3.12), we have

$$\pi_t\rho_z = (\pi\rho_z)_t - \pi\rho_{zt} = (\pi\rho_z)_t - \pi(s\rho_{zz}+\pi_{zz}) = (\pi\rho_z)_t - s(\pi\rho_z)_z + s\pi_z\rho_z - (\pi\pi_z)_z + \pi_z^2. \tag{3.16}$$

Combining (3.15) with (3.16) and integrating the results, we get

$$\begin{aligned}\chi\int_0^t\int U\rho_z^2 = &\int \pi\rho_z - \int \pi_0\rho_{0z} + \int_0^t\int \pi_z^2 - Dm\int_0^t\int\left(U^{m-1}\pi_z\right)_z\rho_z\\ &-\chi\int_0^t\int V\pi_z\rho_z - \int_0^t\int(G+\chi\pi_z\rho_z)\rho_z.\end{aligned}$$

By Young's inequality, noting $0 < U \le u_-$, we have

$$\begin{aligned}-Dm\int\left(U^{m-1}\pi_z\right)_z\rho_z &= -Dm\int U^{m-1}\pi_{zz}\rho_z - Dm\int (U^{m-1})_z\pi_z\rho_z\\ &\le \frac{\chi}{4}\int U\rho_z^2 + \frac{D^2m^2A_m}{\chi}\int U^{m-2}\pi_{zz}^2 + C\int\pi_z^2,\end{aligned}$$

where $A_m = u_-^{m-1}$ if $m \ge 1$ and $A_m = C(1+e^{\eta z})^{m-1}$ if $0 < m < 1$, for $\eta = \frac{u_-^{1-m}}{Dm}\cdot(s+\chi v_+)$ and some constant $C > 0$. By Young's inequality again,

$$\chi\int|V\pi_z\rho_z| \le \frac{\chi}{4}\int U\rho_z^2 + \chi\int V^2\pi_z^2.$$

Thus, by employing $\pi_z^2 \le \frac{C\pi_z^2}{U}$, we have

$$\begin{aligned}\chi\int_0^t\int U\rho_z^2 \le &\int\rho_z^2 + \int\pi^2 + 2\int|\pi_0\rho_{0z}| + \frac{2D^2m^2A_m}{\chi}\int_0^t\int U^{m-2}\pi_{zz}^2\\ &+C\int_0^t\int\frac{\pi_z^2}{U} + C\int_0^t\int|(G+\chi\pi_z\rho_z)\rho_z|.\end{aligned} \tag{3.17}$$

By choosing $\gamma = \min\{\frac{\chi}{2Dmu_-^{m-1}}, \frac{\chi}{2}\}$ for $m \ge 1$, $\gamma = \min\{\frac{\chi}{2DmC(1+e^{\eta z})^{m-1}}, \frac{\chi}{2}\}$ for $0 < m < 1$, substituting (3.17) into (3.14), and through Lemma 2, when $N(t) \ll 1$, one gets

$$\begin{aligned}&\int\left(\frac{\pi_z^2}{U} + \chi\rho_z^2\right) + \int_0^t\int U^{m-2}\pi_{zz}^2\\ &\le C(\|\pi_0\|_{1,w}^2 + \|\rho_0\|_1^2) + C\int|(G+\chi\pi_z\rho_z)|\left(|\pi_z| + \frac{|\pi_{zz}|}{U} + |\rho_z|\right).\end{aligned} \tag{3.18}$$

Substituting (3.18) into (3.17) gives

$$\int_0^t\int\rho_z^2 \le C(\|\pi_0\|_1^2 + \|\rho_0\|_1^2) + C\int|(G+\chi\pi_z\rho_z)|\left(|\pi_z| + \frac{|\pi_{zz}|}{U} + |\rho_z|\right), \tag{3.19}$$

which is combined with (3.18), $\pi_z \leq C\frac{\pi_z}{U}$, $\pi_{zz} \leq C\frac{\pi_{zz}}{U}$, $\rho_z \leq C\frac{\rho_z}{U}$, one has

$$\int\left(\frac{\pi_z^2}{U}+\rho_z^2\right)+\int_0^t\int\frac{\pi_{zz}^2}{U}\leq C(\|\pi_0\|_{1,w}^2+\|\rho_0\|_1^2)+C\int_0^t\int|(G+\chi\pi_z\rho_z)|\left(|\pi_z|+\frac{|\pi_{zz}|}{U}+|\rho_z|\right). \tag{3.20}$$

In view of (3.6), by Young's inequality, $\|\pi_z(\cdot,t)\|_{L^\infty}\leq N(t)$, and Lemma 2, we get

$$\int_0^t\int|(G+\chi\pi_z\rho_z)|\left(|\pi_z|+\frac{|\pi_{zz}|}{U}+|\rho_z|\right)\leq\int_0^t\int CN(t)\left(\frac{\pi_{zz}^2+\rho_z^2}{U}\right).$$

Substituting this inequality into (3.20) and employing the higher-order estimate U^{m-2} as in L^2, then one can obtain

$$\begin{aligned}&\int\left(\frac{\pi_z^2}{U}+\rho_z^2\right)+\int_0^t\int C(Dm(w(z)+u_-)-N(t))\frac{\pi_{zz}^2}{U}\\&\leq C(\|\pi_0\|_{1,w}^2+\|\rho_0\|_1^2)+\int_0^t\int CN(t)\frac{\rho_z^2}{U}.\end{aligned}\tag{3.21}$$

By the assumption $1/U(z)\leq Cw(z)$ for all $z\in\mathbb{R}$, we show the term $\int_0^t\int w\rho_z^2$. Multiplying the second equation of (3.12) by $w\rho_z$, and integrating the results in z, one provides

$$\frac{1}{2}\frac{d}{dt}w\rho_z^2=ws\rho_z\rho_{zz}+w\rho_z\pi_{zz}, \tag{3.22}$$

which is combined with (3.23)

$$ws\rho_z\rho_{zz}=\frac{1}{2}w(s\rho_z^2)_z=\left(\frac{ws\rho_z^2}{2}\right)_z-\frac{s\rho_z^2w_z}{2}, \tag{3.23}$$

then (3.22) becomes

$$\frac{1}{2}\frac{d}{dt}w\rho_z^2+\frac{s\rho_z^2w_z}{2}=\left(\frac{ws\rho_z^2}{2}\right)_z+w\rho_z\pi_{zz}. \tag{3.24}$$

Since $w=1+e^{\eta z}$, one has $1<w<2$ and $0\leq w_z=\eta e^{\eta z}\leq 2\eta w$ in $(-\infty,0)$. We further integrate (3.24) with respect to z over $(-\infty,0)$ to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{-\infty}^0w\rho_z^2+\int_{-\infty}^0\frac{s\rho_z^2}{2}\eta e^{\eta z}&\leq\frac{1}{2}\frac{d}{dt}\int_{-\infty}^0w\rho_z^2+\int_{-\infty}^0s\rho_z^2\eta w\\&=s\rho_z^2(0,t)+\int_{-\infty}^0w\rho_z\pi_{zz}\\&\leq s\rho_z^2(0,t)+\int_{-\infty}^02|\rho_z\pi_{zz}|.\end{aligned}\tag{3.25}$$

Integrating (3.24) in z over $(0,+\infty)$, and using the fact $w_z=\eta e^{\eta z}\geq\frac{\eta w}{2}$ in $(0,+\infty)$ we get

$$\frac{1}{2}\frac{d}{dt}\int_0^{+\infty}w\rho_z^2+\int_0^{+\infty}\frac{s\rho_z^2}{4}\eta w\leq\frac{1}{2}\frac{d}{dt}\int_0^{+\infty}w\rho_z^2+\int_0^{+\infty}\frac{s\rho_z^2}{2}\eta e^{\eta z}=s\rho_z^2(0,t)+\int_0^{+\infty}w\rho_z\pi_{zz}. \tag{3.26}$$

Combining (3.25) and (3.26), and then integrating the results in t, one has

$$\begin{aligned}
&\frac{1}{2}\int w\rho_z^2 + C\int_0^t\int_0^{+\infty}\frac{s\rho_z^2}{4}\eta w + C\int_0^t\int_{-\infty}^0 s\rho_z^2\eta w\\
&\le \frac{1}{2}\int w\rho_{0z}^2 + C\int_0^t\int_0^{+\infty} w\rho_z\pi_{zz} + C\int_0^t\int_{-\infty}^0 2|\rho_z\pi_{zz}| \qquad (3.27)\\
&\le \frac{1}{2}\int w\rho_{0z}^2 + C\int_0^t\int_0^{+\infty}\left(2w\rho_z^2+\frac{w\pi_{zz}^2}{2}\right) + C\int_0^t\int_{-\infty}^0\left(4w\rho_z^2+\frac{w\pi_{zz}^2}{4}\right).
\end{aligned}$$

Next, we combine (3.27) with (3.21), and we have

$$\begin{aligned}
&\int\left(w\pi_z^2+\rho_z^2\right)+\int w\rho_z^2+\int_0^t\int C(Dm(w(z)+u_-)-N(t))w\pi_{zz}^2+\int_0^t\int C(1-N(t))w\rho_z^2\\
&\le \int\left(w\pi_{0z}^2+\rho_{0z}^2\right)+\int w\rho_{0z}^2. \qquad (3.28)
\end{aligned}$$

Applying $N(t) = \min\{Dm(w(z)+u_-), 1\}$, we complete the proof of Lemma 3. □

3.1.3. Estimate of (π,ρ) in H^2

Lemma 4. *Under the same assumptions of Proposition 1, if $N(t) \ll 1$, then*

$$\begin{aligned}
&\|\pi(.,t)\|_{2,w}^2+\|\rho(.,t)\|_2^2+\|\rho(.,t)\|_{2,w}^2+\int_0^t(\|\pi_z(.,\tau)\|_{2,w}^2+\|\rho_z(.,\tau)\|_{1,w}^2)d\tau\\
&\le C(\|\pi_0\|_{2,w}^2+\|\rho_0\|_2^2+\|\rho_0\|_{2,w}^2). \qquad (3.29)
\end{aligned}$$

Proof. We differentiate (3.12) in z to present

$$\begin{cases}
\pi_{zzt}-\chi U\rho_{zzz}-Dm\left(U^{m-1}\pi_{zzz}\right)_z\\
\quad = Dm(2(U^{m-1})_z\pi_{zz}+(U^{m-1})_{zz}\pi_z)_z+\chi(2U_z\rho_{zz}+U_{zz}\rho_z)\\
\qquad +((s+\chi V)\pi_z)_{zz}+(G+\chi\pi_z\rho_z)_{zz},\\
\rho_{zzt}-s\rho_{zzz}-\pi_{zzz}=0.
\end{cases} \qquad (3.30)$$

Multiplying $(3.30)_1$ by $\frac{\pi_{zz}}{U}$ and $(3.30)_2$ by $\chi\rho_{zz}$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\left(\frac{\pi_{zz}^2}{U}+\chi\rho_{zz}^2\right)+Dm\int U^{m-2}\pi_{zzz}^2-\chi\int(2U_z\rho_{zz}+U_{zz}\rho_z)\frac{\pi_{zz}}{U}\\
&=\int\left[Dm\left(\frac{3(U^{m-1})_{zz}}{U}-\left(\frac{(U^{m-1})_z}{U}\right)_z\right)-\left(\frac{s+\chi V\pi_z}{2U}\right)_z+\frac{2\chi V_z}{U}\right]\pi_{zz}^2 \qquad (3.31)\\
&\quad+\int\left(\frac{\chi V\pi_{zz}}{U}+\frac{Dm(U^{m-1})_{zzz}}{U}\right)\pi_z\pi_{zz}+\int(G+\chi\pi_z\rho_z)_z\left(\frac{\pi_{zz}}{U}\right)_z.
\end{aligned}$$

By Young's inequality,

$$\chi\left|(2U_z\rho_{zz}+U_{zz}\rho_z)\frac{\pi_{zz}}{U}\right|\le\frac{\gamma\chi}{2}U\rho_{zz}^2+\chi\left(\frac{2}{\gamma}\cdot\frac{U_z^2}{U^3}+\frac{U_{zz}^2}{U^2}\right)\pi_{zz}^2+\chi\rho_z^2, \qquad (3.32)$$

where γ is a small enough constant. Noting

$$
\begin{aligned}
G_z =& Dm((U+\pi_z)^{m-1}-U^{m-1})\pi_{zzz}+Dm(m-1)(U+\pi_z)^{m-2}\pi_{zz}^2\\
&+Dm(m-1)U_z^2((U+\pi_z)^{m-2}-U^{m-2}-(m-2)U^{m-3}\pi_z)\\
&+DmU_{zz}((U+\pi_z)^{m-1}-U^{m-1}-(m-1)U^{m-2}\pi_z)\\
&+2Dm(m-1)U_z((U+\pi_z)^{m-2}-U^{m-2})\pi_{zz},
\end{aligned}
\tag{3.33}
$$

we have

$$
\int (G+\chi\pi_z\rho_z)_z\left(\frac{\pi_{zz}}{U}\right)_z \le CN(t)\int\left(\frac{\pi_{zzz}^2+\rho_{zz}^2}{U}\right),
\tag{3.34}
$$

where $\|\pi_z(\cdot,t)\|_{L^\infty}, \|\rho_z(\cdot,t)\|_{L^\infty}, \|\pi_{zz}(\cdot,t)\|_{L^\infty} \le N(t)$ has been employed. Substituting (3.32) and (3.34) into (3.31), by (3.1) and (3.11), we get

$$
\begin{aligned}
&\int\left(\frac{\pi_{zz}^2}{U}+\chi\rho_{zz}^2\right)+2Dm\int_0^t\int U^{m-2}\pi_{zzz}^2\\
&\le C(\|\pi_0\|_{2,w}^2+\|\rho_0\|_2^2)+\gamma\chi\int_0^t\int U\rho_{zz}^2+\int_0^t\int CN(t)\left(\frac{\rho_{zz}^2+\pi_{zzz}^2}{U}\right).
\end{aligned}
\tag{3.35}
$$

Next we estimate $\int_0^t\int U\rho_{zz}^2$. Multiplying $(3.12)_1$ by ρ_{zz}, we get

$$
\begin{aligned}
\chi U\rho_{zz}^2 =& \pi_{zt}\rho_{zz}-Dm\left(U^{m-1}\pi_{zz}\right)_z\rho_{zz}-Dm((U^{m-1})_z\pi_z)_z\rho_{zz}\\
&-(\chi U_z\rho_z+((s+\chi V)\pi_z)_z+(G+\chi\pi_z\rho_z)_z)\rho_{zz}.
\end{aligned}
\tag{3.36}
$$

By the second equation of (3.30),

$$
\begin{aligned}
\pi_{zt}\rho_{zz} &= (\pi_z\rho_{zz})_t-\pi_z\rho_{zzt}\\
&= (\pi_z\rho_{zz})_t-s\pi_z\rho_{zzz}-\pi_z\pi_{zzz}\\
&= (\pi_z\rho_{zz})_t-s(\pi_z\rho_{zz})_z+s\pi_{zz}\rho_{zz}-(\pi_z\pi_{zz})_z+\pi_{zz}^2.
\end{aligned}
$$

By Young's inequality,

$$
\begin{aligned}
Dm\left(U^{m-1}\pi_{zz}\right)_z\rho_{zz} &= DmU^{m-1}\pi_{zzz}\rho_{zz}+Dm(U^{m-1})_z\pi_{zz}\rho_{zz}\\
&\le \frac{\chi U\rho_{zz}^2}{4}+\frac{2D^2m^2U^{2m-3}\pi_{zzz}^2}{\chi}+\frac{2D^2m^2|(U^{m-1})_z|^2\pi_{zz}^2}{\chi}.
\end{aligned}
$$

Similarly,

$$
|Dm((U^{m-1})_z\pi_z)_z\rho_{zz}+(\chi U_z\rho_z+((s+\chi V)\pi_z)_z| \le \frac{\chi U\rho_{zz}^2}{4}+C(\rho_z^2+\pi_{zz}^2+\pi_z^2).
$$

In view of (3.33), since $\|\pi_z(\cdot,t)\|_{L^\infty}, \|\rho_z(\cdot,t)\|_{L^\infty}, \|\pi_{zz}(\cdot,t)\|_{L^\infty} \le N(t)$, we get

$$
|(G+\chi\pi_z\rho_z)_z)\rho_{zz}| \le CN(t)(\rho_{zz}^2+\pi_{zz}^2+\pi_{zzz}^2+\pi_z^2).
$$

Thus, integrating (3.36) and using $\pi_z^2 \le \frac{C\pi_z^2}{U}$, $\pi_{zz}^2 \le \frac{C\pi_{zz}^2}{U}$, $\rho_z^2 \le \frac{C\rho_z^2}{U}$, $\rho_{zz}^2 \le \frac{C\rho_{zz}^2}{U}$, $\pi_{zzz}^2 \le \frac{C\pi_{zzz}^2}{U}$, we have

$$\begin{aligned}\chi\int_0^t\int U\rho_{zz}^2 \le \int(\frac{1}{\gamma}\pi_z^2+\gamma\rho_{zz}^2+\pi_{0z}^2+\rho_{0zz}^2)+\frac{2D^2m^2}{\chi}\int_0^t\int U^{2m-3}\pi_{zzz}^2 \\ +C\int_0^t\int\left(\frac{\pi_z^2+\rho_z^2+\pi_{zz}^2}{U}\right)+\int_0^t\int CN(t)\left(\frac{\rho_{zz}^2+\pi_{zzz}^2}{U}\right).\end{aligned} \tag{3.37}$$

Substituting (3.37) into (3.35), choosing $\gamma \ll 1$ and $N(t) \ll 1$, since $0 < U \le u_-$, by Lemmas 2 and 3, we have

$$\begin{aligned}&\int\left(\frac{\pi_{zz}^2}{U}+\chi\rho_{zz}^2\right)+2Dm\int_0^t\int U^{m-2}\pi_{zzz}^2 \\ &\le C(\|\pi_0\|_{2,w}^2+\|\rho_0\|_2^2)+\int_0^t\int CN(t)\left(1+\frac{D^2m^2}{\chi}\right)\frac{\pi_{zzz}^2}{U} \\ &\quad+\int_0^t\int CN(t)\left(\frac{\rho_{zz}^2}{U}\right)\end{aligned} \tag{3.38}$$

Substituting (3.38) into (3.37) gives

$$\int_0^t\int\rho_{zz}^2 \le C(\|\pi_0\|_{2,w}^2+\|\rho_0\|_2^2)+\int_0^t\int CN(t)\left(1+\frac{D^2m^2}{\chi}\right)\frac{\pi_{zzz}^2}{U}+\int_0^t\int CN(t)\left(\frac{\rho_{zz}^2}{U}\right), \tag{3.39}$$

which is combined with (3.38), further gives

$$\begin{aligned}&\int\left(\frac{\pi_{zz}^2}{U}+\chi\rho_{zz}^2\right)+\int_0^t\int C(Dm(w(z)+u_-)-N(t))\frac{\pi_{zzz}^2}{U} \\ &\le C(\|\pi_0\|_{2,w}^2+\|\rho_0\|_2^2)+\int_0^t\int CN(t)\frac{\rho_{zz}^2}{U}.\end{aligned} \tag{3.40}$$

Similarly, by the assumptions $1/U(z) \le Cw(z)$ for all $z \in \mathbb{R}$ as in previous lemmas, we can consider the term $\int_0^t\int w\rho_{zz}^2$. Multiplying the second equation of (3.30) by $w\rho_{zz}$, and integrating the results in z, one has

$$\frac{1}{2}\frac{d}{dt}w\rho_{zz}^2 = ws\rho_{zz}\rho_{zzz}+w\rho_{zz}\pi_{zzz}, \tag{3.41}$$

where

$$ws\rho_{zz}\rho_{zzz} = \frac{1}{2}w(s\rho_{zz}^2)_z = \left(\frac{ws\rho_{zz}^2}{2}\right)_z-\frac{s\rho_{zz}^2w_z}{2}. \tag{3.42}$$

Then, it follows from (3.42) and (3.41), that one has

$$\frac{1}{2}\frac{d}{dt}w\rho_{zz}^2+\frac{s\rho_{zz}^2w_z}{2} = \left(\frac{ws\rho_{zz}^2}{2}\right)_z+w\rho_{zz}\pi_{zzz}. \tag{3.43}$$

Since $w = 1+e^{\eta z}$, one has $1 < w < 2$ and $0 \le w_z = \eta e^{\eta z} \le 2\eta w$ in $(-\infty, 0)$. We further integrate (3.43) in z over $(-\infty, 0)$ to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{0} w\rho_{zz}^2 + \int_{-\infty}^{0}\frac{s\rho_{zz}^2}{2}\eta e^{\eta z} &\le \frac{1}{2}\frac{d}{dt}\int_{-\infty}^{0} w\rho_{zz}^2 + \int_{-\infty}^{0} s\rho_{zz}^2\eta w \\ &= s\rho_{zz}^2(0,t) + \int_{-\infty}^{0} w\rho_{zz}\pi_{zzz} \\ &\le s\rho_{zz}^2(0,t) + \int_{-\infty}^{0} 2|\rho_{zz}\pi_{zzz}|.\end{aligned} \tag{3.44}$$

Integrating (3.43) in z over $(0, +\infty)$, and using the fact $w_z = \eta e^{\eta z} \ge \frac{\eta w}{2}$ in $(0, +\infty)$ gives one

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{0}^{+\infty} w\rho_{zz}^2 + \int_{0}^{+\infty}\frac{s\rho_{zz}^2}{4}\eta w &\le \frac{1}{2}\frac{d}{dt}\int_{0}^{+\infty} w\rho_{zz}^2 + \int_{0}^{+\infty}\frac{s\rho_{zz}^2}{2}\eta e^{\eta z} \\ &= s\rho_{zz}^2(0,t) + \int_{0}^{+\infty} w\rho_{zz}\pi_{zzz}.\end{aligned} \tag{3.45}$$

Combining (3.44) and (3.45), and then integrating the results in t, one has

$$\begin{aligned}&\frac{1}{2}\int w\rho_{zz}^2 + C\int_0^t\int_0^{+\infty}\frac{s\rho_{zz}^2}{4}\eta w + C\int_0^t\int_{-\infty}^{0} s\rho_{zz}^2\eta w \\ &\le \frac{1}{2}\int w\rho_{0zz}^2 + C\int_0^t\int_0^{+\infty} w\rho_{zz}\pi_{zzz} + C\int_0^t\int_{-\infty}^{0} 2|\rho_{zz}\pi_{zzz}| \\ &\le \frac{1}{2}\int w\rho_{0zz}^2 + C\int_0^t\int_0^{+\infty}\left(2w\rho_{zz}^2 + \frac{w\pi_{zzz}^2}{2}\right) + C\int_0^t\int_{-\infty}^{0}\left(4w\rho_{zz}^2 + \frac{w\pi_{zzz}^2}{4}\right).\end{aligned} \tag{3.46}$$

Next, we combine (3.46) with (3.40) to provide

$$\begin{aligned}&\int\left(w\pi_{zz}^2 + \rho_{zz}^2\right) + \int w\rho_{zz} + \int_0^t\int C(Dm(w(z)+u_-) - N(t))w\pi_{zzz}^2 + \int_0^t\int C(1-N(t))w\rho_{zz}^2 \\ &\le \int\left(w\pi_{0zz}^2 + \rho_{0zz}^2\right) + \int w\rho_{0zz}^2.\end{aligned} \tag{3.47}$$

Finally, employing $N(t) = \min\{Dm(w(z) + u_-), 1\}$, then the Lemma 4 is proved. □

Under the influence of nonlinear diffusion, one needs to establish the third order derivative of (π, ρ) in order to make sense with the energy estimates stated in Theorem 2. In similar ways to Lemmas 3 and 4, the estimate of (π, ρ) in H^3 can be inferred, where the details are omitted here. Proposition 1 follows from Lemma 2 to Lemma 4.

Proof of Theorem 2. The a priori estimate (2.20) states that small enough $N(0)$ gives small $N(t)$. Thus, applying the procedure in a standard way, the global well-posedness of (2.18) and (2.19) in $X(0, +\infty)$ is established. Then, the convergence (2.21) is proved. Clearly, if $(\pi, \rho) \in H_w^3$ then $(\pi, \rho) \in H^3$ since $w \ge 1$. By dealing with the global estimate (2.20), we get

$$\int_0^t\int_{-\infty}^{\infty}\pi_z^2(z,\tau)dzd\tau \le C(\|\pi_0\|_{3,w} + \|\rho_0\|_3 + \|\rho_0\|_{3,w}) \le CN^2(0),\ \forall\ t > 0. \tag{3.48}$$

Because of the first equation of (2.18) and Young's inequality, one has

$$\begin{aligned}
&\frac{d}{dt}\int_{-\infty}^{\infty}\pi_z^2(z,t)dz\\
&=-2\int_{-\infty}^{\infty}\pi_{zz}(Dm\left(U^{m-1}\pi_z\right)_z+(s+\chi V)\pi_z+\chi U\rho_z+G+\chi\pi_z\rho_z)\\
&\le\int_{-\infty}^{\infty}\pi_{zz}(Dm\left(U^{m-1}\pi_z\right)_z+(s+\chi V)\pi_z+\chi U\rho_z+\chi\pi_z\rho_z)\\
&+\int_{-\infty}^{\infty}\pi_{zz}(Dm(\pi_z+U)^{m-1}(\pi_{zz}+U_z)+mU_zB_m+Dm((m-1)U^{m-2}\pi_zU_z+U^{m-1}\pi_{zz}))\\
&\le C\int_{-\infty}^{\infty}(\pi_{zz}^2+\pi_z^2+\rho_z^2),
\end{aligned}$$

where for some constants $C>0$, we have used $B_m=u_-^{m-1}$, $(\pi_z+U)^{m-1}\le(\pi_z+u_-)^{m-1}$ if $m\ge1$ and $B_m=C(1+e^{\eta z})^{m-1}$, $(\pi_z+U)^{m-1}\le(\pi_z+C(1+e^{\eta z}))^{m-1}$ for $\eta=\frac{u_-^{1-m}}{Dm}\cdot(s+\chi v_+)$ if $0<m<1$. By referring to the global estimate (2.20), one has

$$\int_0^{\infty}\left|\frac{d}{dt}\int_{-\infty}^{\infty}\pi_z^2(z,t)dz\right|\le C\int_0^{\infty}\int_{-\infty}^{\infty}(\pi_{zz}^2+\pi_z^2+\rho_z^2)\le CN^2(0).\tag{3.49}$$

From (3.48) and (3.49), we get

$$\int_{-\infty}^{\infty}\pi_z^2(z,t)dz\to0\text{ as }t\to+\infty.$$

Moreover, by dealing with the Cauchy-Schwarz inequality, we further have

$$\begin{aligned}
\pi_z^2(z,t)&=2\int_{-\infty}^{z}\pi_z\pi_{zz}(y,t)dy\\
&\le2\left(\int_{-\infty}^{\infty}\pi_z^2(y,t)dy\right)^{\frac12}\left(\int_{-\infty}^{\infty}\pi_{zz}^2(y,t)dy\right)^{\frac12}\\
&\le C\left(\int_{-\infty}^{\infty}\pi_z^2(y,t)dy\right)^{\frac12}\to0\text{ as }t\to+\infty.
\end{aligned}$$

Applying the same argument to ρ_z yields

$$\sup_{z\in\mathbb{R}}|\rho_z(z,t)|\to0\text{ as }t\to+\infty.\tag{3.50}$$

Hence (2.21) is proved. □

3.2. Proof of main results

Now, the main results become the focus of our paper. By employing the transformation (2.16), Theorem 1 is based on Theorem 2.

Proof of Theorem 4. The stability of u has been established in Theorem 1. It only needs to transfer the results of v into c. In view of the transformations (1.4) and (2.16), we have

$$\frac{c(x,t)}{C(x-st)} = e^{\int_{-\infty}^{x}(V(y-st)-v(y,t))dy} = e^{\rho(x,t)}.$$

By Cauchy-Schwarz inequality, the global estimate (2.20) and (3.50), we get

$$\sup_{x\in\mathbb{R}} \rho^2(x,t) = 2\sup_{x\in\mathbb{R}} \int_{-\infty}^{x} \rho\rho_y(y,t)dy \le 2\Big(\int_{\mathbb{R}} \rho^2(y,t)dy\Big)^{1/2}\Big(\int_{\mathbb{R}} \rho_y^2(y,t)dy\Big)^{1/2} \to 0 \text{ as } t\to\infty.$$

Then for all $x \in \mathbb{R}$,

$$\begin{aligned} |c(x,t) - C(x-st)| &= |C(x-st)e^{\rho(x,t)} - C(x-st)| \\ &= C(x-st)|1-e^{\rho(x,t)}| \\ &\le C|1-e^{\rho(x,t)}| \\ &\to 0 \text{ as } t\to\infty. \end{aligned}$$

Hence, the proof is finished. □

Use of AI tools declaration

The author declare no use of Artificial Intelligence (AI) tools in the creation of this paper.

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Conflict of interest

The author declare no conflict of interest in this paper.

References

1. M. Burger, M. Di Francesco, Y. Dolak-Strub, The Keller-Segel model for chemotaxis with prevention of overcrowding: linear vs. nonlinear diffusion, *SIAM J. Math. Anal.*, **38** (2006), 1288–1315. http://dx.doi.org/10.1137/050637923
2. S. Choi, Y. Kim, Chemotactic traveling waves with compact support, *J. Math. Anal. Appl.*, **488** (2020), 124090. http://dx.doi.org/10.1016/j.jmaa.2020.124090
3. C. Deng, T. Li, Well-posedness of a 3D parabolic-hyperbolic Keller-Segel system in the Sobolev space framework, *J. Differ. Equations*, **257** (2014), 1311–1332. http://dx.doi.org/10.1016/j.jde.2014.05.014

4. M. Ghani, Analysis of degenerate Burgers' equations involving small perturbation and large wave amplitude, *Math. Method. Appl. Sci.*, **46** (2023), 13781–13796. http://dx.doi.org/10.1002/mma.9289

5. M. Ghani, Asymptotic stability of singular traveling waves to degenerate advection-diffusion equations under small perturbation, *Differ. Equ. Dyn. Syst.*, in press. http://dx.doi.org/10.1007/s12591-022-00602-1

6. M. Ghani, J. Li, K. Zhang, Asymptotic stability of traveling fronts to a chemotaxis model with nonlinear diffusion, *Discrete Cont. Dyn. Syst.-B*, **26** (2021), 6253–6265. http://dx.doi.org/10.3934/dcdsb.2021017

7. M. Ghani, Nurwidiyanto, Traveling fronts of viscous Burgers' equations with the nonlinear degenerate viscosity, *Math. Sci.*, in press. http://dx.doi.org/10.1007/s40096-023-00519-y

8. T. Hillen, K. Painter, Global existence for a parabolic chemotaxis model with prevention of overcrowding, *Adv. Appl. Math.*, **26** (2001), 280–301. http://dx.doi.org/10.1006/aama.2001.0721

9. D. Horstmann, From 1970 until present: the Keller-Segel model in chemotaxis and its consequences: I, *Jahresber. Deutsch. Math.-Verein.*, **105** (2003), 103–165.

10. H. Jin, J. Li, Z. Wang, Asymptotic stability of traveling waves of a chemotaxis model with singular sensitivity, *J. Differ. Equations*, **255** (2013) 193–219. http://dx.doi.org/10.1016/j.jde.2013.04.002

11. Y. Kalinin, L. Jiang, Y. Tu, M. Wu, Logarithmic sensing in Escherichia coli bacterial chemotaxis, *Biophys. J.*, **96** (2009), 2439–2448. http://dx.doi.org/10.1016/j.bpj.2008.10.027

12. S. Kawashima, A. Matsumura, Stability of shock profiles in viscoelasticity with non-convex constitutive relations, *Commun. Pur. Appl. Math.*, **47** (1994), 1547–1569. http://dx.doi.org/10.1002/cpa.3160471202

13. E. Keller, L. Segel, Traveling bands of chemotactic bacteria: a theoretical analysis, *J. Theor. Biol.*, **30** (1971), 235–248. http://dx.doi.org/10.1016/0022-5193(71)90051-8

14. D. Li, R. Pan, K. Zhao, Quantitative decay of a hybrid type chemotaxis model with large data, *Nonlinearity*, **28** (2015), 2181. http://dx.doi.org/10.1088/0951-7715/28/7/2181

15. J. Li, Z. Wang, Convergence to traveling waves of a singular PDE-ODE hybrid chemotaxis system in the half space, *J. Differ. Equations*, **268** (2020), 6940–6970. http://dx.doi.org/10.1016/j.jde.2019.11.076

16. T. Li, R. Pan, K. Zhao, Global dynamics of a hyperbolic-parabolic model arising from chemotaxis, *SIAM J. Appl. Math.*, **72** (2012), 417–443. http://dx.doi.org/10.1137/110829453

17. T. Li, Z. Wang, Nonlinear stability of traveling waves to a hyperbolic-parabolic system modeling chemotaxis, *SIAM J. Appl. Math.*, **70** (2010), 1522–1541. http://dx.doi.org/10.1137/09075161X

18. V. Martinez, Z. Wang, K. Zhao, Asymptotic and viscous stability of large-amplitude solutions of a hyperbolic system arising from biology, *Indiana Univ. Math. J.*, **67** (2018), 1383–1424. http://dx.doi.org/10.1512/iumj.2018.67.7394

19. A. Matsumura, K. Nishihara, On the stability of travelling wave solutions of a one dimensional model system for compressible viscous gas, *Japan J. Appl. Math.*, **2** (1985), 17–25. http://dx.doi.org/10.1007/BF03167036

20. T. Nishida, Nonlinear hyperbolic equations and related topics in fluid dynamics, *Publ. Math. D'Orsay*, **78** (1978), 46–53.
21. M. Olson, R. Ford, J. Smith, E. Fernandez, Quantification of bacterial chemotaxis in porous media using magnetic resonance imaging, *Environ. Sci. Technol.*, **38** (2004), 3864–3870. http://dx.doi.org/10.1021/es035236s
22. H. Othmer, A. Stevens, Aggregation, blowup, and collapse: the ABCs of taxis in reinforced random walks, *SIAM J. Appl. Math.*, **57** (1997), 1044–1081. http://dx.doi.org/10.1137/S0036139995288976
23. B. Sleeman, H. Levine, A system of reaction diffusion equations arising in the theory of reinforced random walks, *SIAM J. Appl. Math.*, **57** (1997), 683–730. http://dx.doi.org/10.1137/S0036139995291106
24. Y. Tao, M. Winkler, Global existence and boundedness in a Keller-Segel-Stokes model with arbitrary porous medium diffusion, *Discrete Cont. Dyn.-A*, **32** (2012), 1901–1914. http://dx.doi.org/10.3934/dcds.2012.32.1901
25. F. Valdaes-Parada, M. Porter, K. Narayanaswamy, R. Ford, B. Wood, Upscaling microbial chemotaxis in porous media, *Adv. Water Resour.*, **32** (2009), 1413–1428. http://dx.doi.org/10.1016/j.advwatres.2009.06.010
26. Z. Wang, Mathematics of traveling waves in chemotaxis: a review paper, *Discrete Cont. Dyn. Syst.-B*, **18** (2013), 601–641. http://dx.doi.org/10.3934/dcdsb.2013.18.601
27. Z. Wang, T. Hillen, Shock formation in a chemotaxis model, *Math. Method. Appl. Sci.*, **31** (2008), 45–70. http://dx.doi.org/10.1002/mma.898
28. Z. Wang, Z. Xiang, P. Yu, Asymptotic dynamics on a singular chemotaxis system modeling onset of tumor angiogenesis, *J. Differ. Equations*, **260** (2016), 2225–2258. http://dx.doi.org/10.1016/j.jde.2015.09.063
29. Y. Yang, H. Chen, W. Liu, On existence of global solutions and blow-up to a system of the reaction-diffusion equations modelling chemotaxis, *SIAM J. Math. Anal.*, **33** (2001), 763–785. http://dx.doi.org/10.1137/S0036141000337796
30. Y. Yang, H. Chen, W. Liu, B. Sleeman, The solvability of some chemotaxis systems, *J. Differ. Equations*, **212** (2005), 432–451. http://dx.doi.org/10.1016/j.jde.2005.01.002
31. M. Zhang, C. Zhu, Global existence of solutions to a hyperbolic-parabolic system, *Proc. Amer. Math. Soc.*, **135** (2007), 1017–1027.