AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(12): 29817–29848.
DOI: 10.3934/math.20231525
Received: 28 September 2023
Revised: 28 October 2023
Accepted: 30 October 2023
Published: 02 November 2023

Research article

Distance measures of picture fuzzy sets and interval-valued picture fuzzy sets with their applications

Sijia Zhu[1,*] **and Zhe Liu**[2,*]

[1] Cw Chu College, Jiangsu Normal University, Xuzhou, 221116, China

[2] School of Computer Sciences, Universiti Sains Malaysia, Penang, 11800, Malaysia

* **Correspondence:** Email: sijiazhu371@gmail.com, zheliu@ieee.org.

Abstract: Picture fuzzy sets (PFSs) are a versatile generalization of fuzzy sets and intuitionistic fuzzy sets (IFSs), providing a robust framework for modeling imprecise, uncertain, and inconsistent information across various fields. As an advanced extension of PFSs, interval-valued picture fuzzy sets (IvPFSs) offer superior capabilities for handling incomplete and indeterminate information in various practical applications. Distance measures have always been an important topic in fuzzy sets and their variants. Some existing distance measures for PFSs have shown limitations and may yield counterintuitive results under certain conditions. Furthermore, there are currently few studies on distance measures for IvPFSs. To solve these problems, in this paper we devised a series of novel distance measures between PFSs and IvPFSs inspired by the Hellinger distance. Specifically, all the distance measures were divided into two parts: One considered the positive membership degree, neutral membership degree and negative membership degree, and the other added the refusal membership degree. Moreover, the proposed distance measures met some important properties, including boundedness, non-degeneracy, symmetry, and consistency, but also showed superiority compared to the existing measures, as confirmed through numerical comparisons. Finally, the proposed distance measures were validated in pattern recognition and medical diagnosis applications, indicating that the proposed distance measures can deliver credible, reasonable results, particularly in similar cases.

Keywords: picture fuzzy sets; interval-valued picture fuzzy sets; Hellinger distance; pattern recognition; medical diagnosis
Mathematics Subject Classification: 03E72, 28E10, 90B50

1. Introduction

In real life, we are often faced with factors of uncertainty and imprecision, which is especially obvious in decision-making [21,35,61]. How to address the uncertain and imprecise information across diverse applications has garnered significant attention over the past decade [33, 34, 36, 37, 62]. Over time, numerous foundational theories have been thoroughly explored, for example, fuzzy sets [44, 67, 68], evidence theory [40, 65, 69], R-numbers [50], rough sets [13, 48] and Z-numbers [5]. Zadeh [67] introduced fuzzy sets to deal with uncertain information. Since that pioneering work, fuzzy sets have gained significant interest from researchers and have been applied across various fields [29, 56, 60, 63]. To address uncertain information with greater efficacy, Atanassov [4] suggested intuitionistic fuzzy sets (IFSs), which comprise membership, non-membership, and hesitancy degrees. IFSs offer a precise and adaptable means to represent uncertainty and ambiguity, garnering considerable interest in various areas [7, 20, 64]. Within the framework of IFSs, every element is assigned both a membership and a non-membership value. Later, Atanassov [3] introduced the concept of interval-valued intuitionistic fuzzy sets (IvIFSs) to enhance IFSs. IvIFSs employ interval-valued defined by lower and upper bounds of membership and non-membership degrees to represent uncertain and imprecise information.

Recently, Cuong and Kreinovich [9] introduced picture fuzzy sets (PFSs) with neutral membership. PFSs depend on four interrelated dimensions: Positive membership, negative membership, neutral membership, and refusal membership degrees. A significant advantage of PFSs is the incorporation of a "neutrality" degree, enhancing the depth of the framework for intricate decision-making in fields such as medical diagnosis, personnel selection, and social choices [2, 16, 51], where a "maybe" or "neutral" position holds relevance. Presently, research on PFSs is advancing across various domains [17, 27, 28]. Arya et al. [2] introduced innovative aggregation operators for PFSs, grounded in fundamental mathematical procedures. These operators provided significant advantages when addressing practical real-world challenges [23]. Ganie et al. [14] proposed novel correlation coefficients for PFSs, showcasing their practical applications. Ali et al. [1] utilized Aczel-Alsina operational laws to develop power aggregation operators through complex picture fuzzy sets (CPFSs). This demonstrated their applicability through a decision-making methodology, a multi-attribute decision-making algorithm, and a real-world illustration. Sindhu et al. [52] introduced the aggregation operators to select the investment based on bipolar PFSs. Signh et al. [53] integrated quality functions deployment with PFSs to propose a multi-criteria group decision-making method. Additionally, Cuong et al. [10] introduced the concept of IvPFSs. Mhamood [41] later examined the interval-valued picture fuzzy frank averaging operator to find the interrelationships among any number of IvPFSs. Khalil [24] proposed some new operations and relative decision-making problems. IvPFSs provide a more apt representation of fuzzy information than IFSs, IvIFSs, and PFSs.

PFSs and IvPFSs are extensions of classical fuzzy sets designed to manage uncertainties in data. While both sets aim to address uncertainties, IvPFSs extend the framework by incorporating interval values, allowing for a more nuanced analysis of uncertainties. This aspect of IvPFSs, being a further generalization of interval-valued fuzzy sets, positions them as an advanced form of fuzzy sets optimized for handling uncertainties during data analysis. Also, IvPFSs have a broader range of applications than PFSs, as they can be applied to scenarios with elements having a specific range of fluctuating values, rather than being limited to data with elements of fixed numerical values like in PFSs. Specifically, when the interval values of IvPFSs have identical lower and upper bounds, IvPFSs equal to PFSs.

The study of distance and similarity measures has been pivotal in fuzzy sets and their variants, garnering significant interest from researchers [11,12,25,26,45]. There are many works on distance and similarity measures for IFSs and IvIFSs [8, 18, 22, 42, 46, 47, 49, 57, 66]. For example, Hatzimichailidis et al. [18] developed a distance measure for IFSs that harnesses matrix norms and fuzzy implications. Hwang et al. [22] introduced novel similarity measures for IFSs, drawing inspiration from the Jaccard index. Ye [66] introduced cosine similarity measures tailored for IvIFSs and applied them to address multiattribute decision making issues. Liu et al. [30] put forth an ordered weighted cosine similarity measure for IvIFSs, which they subsequently employed to tackle investment decision-making challenges. Recently, some distance or similarity measures tailored for PFSs have been crafted over time [15, 38, 43]. For instance, Dinh and Thao [58] introduced several distance and dissimilarity measures among PFSs, subsequently applying them to areas like pattern recognition and multi-attribute decision-making. Singh and Mishra [54] introduced several parameterized distance measures, encompassing the normalized Hamming, Euclidean, and Hausdorff distances as specific instances. Son [55] demonstrated the relevance of distance measures in clustering analysis. Wei et al. [59] introduced a cosine similarity measure tailored for PFSs, broadening its utility in multi-attribute decision-making contexts. Liu and Zeng [31] delved into various distance measures such as picture fuzzy weighted distance, ordered weighted distance, and hybrid weighted distance, refining them for multi-attribute group decision-making. Although some studies have been on distance or similarity measures, research specifically focused on IvPFSs remains limited. Cao [6] proposed a similarity measure between IvPFSs based on a pyramidal center of gravity. Liu [32] introduced some novel similarity measures based on cosine and cotangent functions.

The motivation for this paper primarily arises from two aspects. On the one hand, there are significant flaws and gaps in the current distance measures for PFSs: Many existing distance measures do not fully meet all axiomatic properties, and some existing distance measures may produce inconsistent or counterintuitive results when calculating the difference between PFSs. On the other hand, there exists a substantial void in the research areas concerning distance measures for IvPFSs, with only a few papers available to explore them. Given these circumstances, this study attempts to bridge these gaps by presenting a range of distance measures for PFSs and IvPFSs. The key contributions are fourfold:

- We introduce eight novel distance measures for PFSs and another eight for IvPFSs, drawing inspiration from the Hellinger distance.
- We demonstrate that the proposed measures to meet the properties of the axiomatic definition of the distance measure.
- The proposed distance measures can adeptly address and rectify the counterintuitive outcomes observed with some existing measures in certain cases.
- The efficacy of the proposed distance measures and related measures is validated in pattern classification and medical diagnosis, underscoring their advantages.

This paper is structured as follows: Section 2 presents foundational concepts. In Sections 3 and 4, we introduce a set of innovative distance measures for PFSs and IvPFSs, drawing on the Hellinger distance and accompanied by their formal justifications. Section 5 offers a comparative analysis between existing and our proposed measures through diverse cases. Applications to classification challenges and the medical diagnosis are explored in Sections 6 and 7. Section 8 introduces the advantages of the work. Finally, Section 9 makes a conclusion.

2. Preliminaries

This section will introduce relevant definitions of fuzzy sets and distance measures.

2.1. Fuzzy set

Definition 1. *Let $X = \{x_1, x_2, \cdots, x_n\}$ be a universe of discourse (UOD). A fuzzy set (FS) in X is defined as follows:*

$$\mathcal{E} = \{\langle x, \varUpsilon_{\mathcal{E}}(x)\rangle | x \in X\}$$

where $\varUpsilon_{\mathcal{E}}(x) \in [0, 1]$ expresses the positive membership. For each $x \in X$, we have:

$$0 \leq \varUpsilon_{\mathcal{E}}(x)(x) \leq 1, \forall x \in X$$

and

$$\varPhi_{\mathcal{E}}(x) = 1 - \xi_{\mathcal{E}}(x)$$

where $\varPhi_{\mathcal{E}}(x) : X \to [0, 1]$ indicates the negative membership associated with $x \in X$.

2.2. Intuitionistic fuzzy set

Definition 2. *[4] An intuitionistic fuzzy set (IFS) in X is defined as follows:*

$$\mathcal{E} = \{\langle x, \varUpsilon_{\mathcal{E}}(x), \varPhi_{\mathcal{E}}(x)\rangle | x \in X\}$$

where $\varUpsilon_{\mathcal{E}}(x), \varPhi_{\mathcal{E}}(x) : X \to [0, 1]$ expresses the positive membership and the negative membership. For each $x \in X$, we have:

$$0 \leq \varUpsilon_{\mathcal{E}}(x), \varPhi_{\mathcal{E}}(x) \leq 1$$

and

$$\varPsi_{\mathcal{E}}(x) = 1 - \varUpsilon_{\mathcal{E}}(x) - \varPhi_{\mathcal{E}}(x)$$

where $\varPsi_{\mathcal{E}}(x) : X \to [0, 1]$ indicates the neutral membership associated with $x \in X$.

2.3. Interval-valued intuitionistic fuzzy set

Definition 3. *[3] An interval-valued intuitionistic fuzzy set (IvIFS) in X is defined as follows:*

$$\mathcal{E} = \{\langle x, \varUpsilon_{\mathcal{E}}(x), \varPhi_{\mathcal{E}}(x)\rangle | x \in X\}$$

where $\varUpsilon_{\mathcal{E}}(x) = \left[\varUpsilon^L_{\mathcal{E}}(x), \varUpsilon^U_{\mathcal{E}}(x)\right] = \left[\varPhi^L_{\mathcal{E}}(x), \varPhi^U_{\mathcal{E}}(x)\right]$. These intervals signify the positive and negative membership degrees of an element. For all $x \in X$,

$$0 \leq \Upsilon_{\mathcal{E}}(x), \varPhi_{\mathcal{E}}(x) \leq 1$$

and

$$\varPsi_{\mathcal{E}}(x) = 1 - \varUpsilon_{\mathcal{E}}(x) - \varPhi_{\mathcal{E}}(x)$$

where $\varPsi_{\mathcal{E}}(x) = \left[\varPsi^L_{\mathcal{E}}(x), \varPsi^U_{\mathcal{E}}(x)\right]$ represents neutral membership in intervals of $x \in X$.

2.4. Picture fuzzy set

Definition 4. *[9] A picture fuzzy set (PFS) in X is defined as follows:*

$$\mathcal{E} = \{\langle x, \Upsilon_{\mathcal{E}}(x), \Phi_{\mathcal{E}}(x), \Psi_{\mathcal{E}}(x)\rangle | x \in X\}$$

where $\Upsilon_{\mathcal{E}}(x), \Phi_{\mathcal{E}}(x), \Psi_{\mathcal{E}}(x) : X \rightarrow [0, 1]$. *For each* $x \in X$*, we have:*

$$\Upsilon_{\mathcal{E}}(x), \Phi_{\mathcal{E}}(x), \Psi_{\mathcal{E}}(x) \in [0, 1]$$

and

$$\Omega_{\mathcal{E}}(x) = 1 - \Upsilon_{\mathcal{E}}(x) - \Phi_{\mathcal{E}}(x) - \Psi_{\mathcal{E}}(x)$$

where $\Omega_{\mathcal{E}}(x) : X \rightarrow [0, 1]$ *represents refusal membership degree of* $x \in X$*.*

2.5. Interval-valued picture fuzzy set

Definition 5. *[10] An interval-valued picture fuzzy set (IvPFS) in X is defined as follows:*

$$\mathcal{E} = \{\langle x, \Upsilon_{\mathcal{E}}(x), \Phi_{\mathcal{E}}(x), \Psi_{\mathcal{E}}(x)\rangle | x \in X\}$$

where $\Upsilon_{\mathcal{E}}(x) = \left[\Upsilon^L_{\mathcal{E}}(x), \Upsilon^U_{\mathcal{E}}(x)\right], \Phi_{\mathcal{E}}(x) = \left[\Phi^L_{\mathcal{E}}(x), \Phi^U_{\mathcal{E}}(x)\right], \Psi_{\mathcal{E}}(x) = \left[\Psi^L_{\mathcal{E}}(x), \Psi^U_{\mathcal{E}}(x)\right]$*. These intervals signify the positive, negative, and neutral membership degrees of an element. For all* $x \in X$*,*

$$0 \leq \Upsilon_{\mathcal{E}}(x), \Phi_{\mathcal{E}}(x), \Psi_{\mathcal{E}}(x) \leq 1$$

and

$$\Omega_{\mathcal{E}}(x) = 1 - \Upsilon_{\mathcal{E}}(x) - \Phi_{\mathcal{E}}(x) - \Psi_{\mathcal{E}}(x)$$

where $\Omega_{\mathcal{E}}(x) = \left[\Omega^L_{\mathcal{E}}(x), \Omega^U_{\mathcal{E}}(x)\right]$ *represents refusal membership in intervals of* $x \in X$*.*

2.6. The relationship of different fuzzy sets

For every x in set X, we define $\mathcal{E} = \{\langle x, \Upsilon_{\mathcal{E}}(x), \Phi_{\mathcal{E}}(x), \Psi_{\mathcal{E}}(x)\rangle | x \in X\}$.

(1) In PFSs, when $\Psi_{\mathcal{E}}(x) = 0$, PFSs reduce to IFSs.

(2) Regarding IvIFSs, if $\Upsilon_{\mathcal{E}}(x) = \Upsilon^L_{\mathcal{E}}(x) = \Upsilon^U_{\mathcal{E}}(x), \Phi_{\mathcal{E}}(x) = \Phi^L_{\mathcal{E}}(x) = \Phi^U_{\mathcal{E}}(x)$, then IvIFSs simplify to IFSs.

(3) In the scenario of IvPFSs, if $\Upsilon_{\mathcal{E}}(x) = \Upsilon^L_{\mathcal{E}}(x) = \Upsilon^U_{\mathcal{E}}(x), \Phi_{\mathcal{E}}(x) = \Phi^L_{\mathcal{E}}(x) = \Phi^U_{\mathcal{E}}(x), \Psi_{\mathcal{E}}(x) = \Psi^L_{\mathcal{E}}(x) = \Psi^U_{\mathcal{E}}(x)$, then IvPFSs are equivalent to PFSs.

Consequently, IFSs are a particular instance of PFSs and IvIFSs, while PFSs are a specialized form of IvPFSs. The theory of fuzzy sets continually transitions from specialization to generalization.

2.7. Hellinger distance

Definition 6. *[19]] The Hellinger distance is calculated based on the shape of probability density functions or probability mass functions to measure the distance between two distributions. For two probability distributions* $\mathcal{P}$ *and* $\mathcal{Q}$*, the Hellinger distance can be computed as follows:*

$$\mathbb{D}(\mathcal{P}, \mathcal{Q}) = \frac{1}{\sqrt{2}} \sqrt{\sum_{i=1}^{n} (\sqrt{p_i} - \sqrt{q_i})^2} \tag{2.1}$$

where p_i and q_i represent the probability density of the two distributions at a specific event x.

The Hellinger distance possesses the following characteristics:

(1) Its values range between 0 and 1, where 0 signifies complete similarity between two distributions, and 1 indicates complete dissimilarity.

(2) When two distributions are very similar, the Hellinger distance approaches 0.

(3) The Hellinger distance is symmetric, meaning $\mathbb{D}(\mathcal{P},\mathcal{Q}) = \mathbb{D}(\mathcal{Q},\mathcal{P})$. Compared to other distance metrics, such as Kullback-Leibler (KL) divergence or total variation distance, the Hellinger distance is more robust to outliers and, in certain cases, more accessible to compute. It finds widespread application in probability distribution comparisons and model fitting.

2.8. The existing distance measures and similarity measures for PFSs and IvPFSs

Table 1 shows a series of existing distance measures utilized for PFSs.

Table 1. Existing distance measures of PFSs.

Ref.	Distance Measures
Dutta [12]	$\mathbb{D}^1_{Du}(\mathcal{E},\mathcal{F}) = \frac{1}{2}\sum_{i=1}^{n}\begin{pmatrix} \lvert\Upsilon_{\mathcal{E}}(x_i) - \Upsilon_{\mathcal{F}}(x_i)\rvert + \lvert\Phi_{\mathcal{E}}(x_i) - \Phi_{\mathcal{F}}(x_i)\rvert \\ +\lvert\Psi_{\mathcal{E}}(x_i) - \Psi_{\mathcal{F}}(x_i)\rvert + \lvert\Omega_{\mathcal{E}}(x_i) - \Omega_{\mathcal{F}}(x_i)\rvert \end{pmatrix}$
Dutta [12]	$\mathbb{D}^2_{Du}(\mathcal{E},\mathcal{F}) = \frac{1}{2n}\sum_{i=1}^{n}\begin{pmatrix} \lvert\Upsilon_{\mathcal{E}}(x_i) - \Upsilon_{\mathcal{F}}(x_i)\rvert + \lvert\Phi_{\mathcal{E}}(x_i) - \Phi_{\mathcal{F}}(x_i)\rvert \\ +\lvert\Psi_{\mathcal{E}}(x_i) - \Psi_{\mathcal{F}}(x_i)\rvert + \lvert\Omega_{\mathcal{E}}(x_i) - \Omega_{\mathcal{F}}(x_i)\rvert \end{pmatrix}$
Dutta [12]	$\mathbb{D}^3_{Du}(\mathcal{E},\mathcal{F}) = \frac{1}{2}\sum_{i=1}^{n}\begin{pmatrix} \lvert\Upsilon_{\mathcal{E}}(x_i) - \Upsilon_{\mathcal{F}}(x_i)\rvert^2 + \lvert\Phi_{\mathcal{E}}(x_i) - \Phi_{\mathcal{F}}(x_i)\rvert^2 \\ +\lvert\Psi_{\mathcal{E}}(x_i) - \Psi_{\mathcal{F}}(x_i)\rvert^2 + \lvert\Omega_{\mathcal{E}}(x_i) - \Omega_{\mathcal{F}}(x_i)\rvert^2 \end{pmatrix}$
Dutta [12]	$\mathbb{D}^4_{Du}(\mathcal{E},\mathcal{F}) = \frac{1}{2n}\sum_{i=1}^{n}\begin{pmatrix} \lvert\Upsilon_{\mathcal{E}}(x_i) - \Upsilon_{\mathcal{F}}(x_i)\rvert^2 + \lvert\Phi_{\mathcal{E}}(x_i) - \Phi_{\mathcal{F}}(x_i)\rvert^2 \\ +\lvert\Psi_{\mathcal{E}}(x_i) - \Psi_{\mathcal{F}}(x_i)\rvert^2 + \lvert\Omega_{\mathcal{E}}(x_i) - \Omega_{\mathcal{F}}(x_i)\rvert^2 \end{pmatrix}$
Dinh and Thao [58]	$\mathbb{D}^1_{DT}(\mathcal{E},\mathcal{F}) = \frac{1}{3n}\sum_{i=1}^{n}\begin{pmatrix} \lvert\Upsilon_{\mathcal{E}}(x_i) - \Upsilon_{\mathcal{F}}(x_i)\rvert + \lvert\Phi_{\mathcal{E}}(x_i) - \Phi_{\mathcal{F}}(x_i)\rvert \\ +\lvert\Psi_{\mathcal{E}}(x_i) - \Psi_{\mathcal{F}}(x_i)\rvert \end{pmatrix}$
Dinh and Thao [58]	$\mathbb{D}^2_{DT}(\mathcal{E},\mathcal{F}) = \frac{1}{n}\sqrt{\sum_{i=1}^{n}\begin{pmatrix} \lvert\Upsilon_{\mathcal{E}}(x_i) - \Upsilon_{\mathcal{F}}(x_i)\rvert^2 + \lvert\Phi_{\mathcal{E}}(x_i) - \Phi_{\mathcal{F}}(x_i)\rvert^2 \\ +\lvert\Psi_{\mathcal{E}}(x_i) - \Psi_{\mathcal{F}}(x_i)\rvert^2 \end{pmatrix}}$
Dinh and Thao [58]	$\mathbb{D}^3_{DT}(\mathcal{E},\mathcal{F}) = \frac{1}{n}\sum_{i=1}^{n}\max\begin{pmatrix} \lvert\Upsilon_{\mathcal{E}}(x_i) - \Upsilon_{\mathcal{F}}(x_i)\rvert, \lvert\Phi_{\mathcal{E}}(x_i) - \Phi_{\mathcal{F}}(x_i)\rvert, \\ \lvert\Psi_{\mathcal{E}}(x_i) - \Psi_{\mathcal{F}}(x_i)\rvert \end{pmatrix}$
Dinh and Thao [58]	$\mathbb{D}^4_{DT}(\mathcal{E},\mathcal{F}) = \frac{1}{n}\sqrt{\sum_{i=1}^{n}\max\begin{pmatrix} \lvert\Upsilon_{\mathcal{E}}(x_i) - \Upsilon_{\mathcal{F}}(x_i)\rvert^2, \lvert\Phi_{\mathcal{E}}(x_i) - \Phi_{\mathcal{F}}(x_i)\rvert^2, \\ \lvert\Psi_{\mathcal{E}}(x_i) - \Psi_{\mathcal{F}}(x_i)\rvert^2 \end{pmatrix}}$
Singh et al [54]	$\mathbb{D}^1_{SM}(\mathcal{E},\mathcal{F}) = \frac{1}{4n}\sum_{i=1}^{n}\begin{pmatrix} \lvert\Upsilon_{\mathcal{E}}(x_i) - \Upsilon_{\mathcal{F}}(x_i)\rvert + \lvert\Phi_{\mathcal{E}}(x_i) - \Phi_{\mathcal{F}}(x_i)\rvert \\ +\lvert\Psi_{\mathcal{E}}(x_i) - \Psi_{\mathcal{F}}(x_i)\rvert + \lvert\Omega_{\mathcal{E}}(x_i) - \Omega_{\mathcal{F}}(x_i)\rvert \end{pmatrix}$
Singh et al [54]	$\mathbb{D}^2_{SM}(\mathcal{E},\mathcal{F}) = \sqrt{\frac{1}{4n}\sum_{i=1}^{n}\begin{pmatrix} \lvert\Upsilon_{\mathcal{E}}(x_i) - \Upsilon_{\mathcal{F}}(x_i)\rvert^2 + \lvert\Phi_{\mathcal{E}}(x_i) - \Phi_{\mathcal{F}}(x_i)\rvert^2 \\ +\lvert\Psi_{\mathcal{E}}(x_i) - \Psi_{\mathcal{F}}(x_i)\rvert^2 + \lvert\Omega_{\mathcal{E}}(x_i) - \Omega_{\mathcal{F}}(x_i)\rvert^2 \end{pmatrix}}$
Singh et al [54]	$\mathbb{D}^3_{SM}(\mathcal{E},\mathcal{F}) = \frac{1}{4n}\sum_{i=1}^{n}\max\begin{pmatrix} \lvert\Upsilon_{\mathcal{E}}(x_i) - \Upsilon_{\mathcal{F}}(x_i)\rvert, \lvert\Phi_{\mathcal{E}}(x_i) - \Phi_{\mathcal{F}}(x_i)\rvert, \\ \lvert\Psi_{\mathcal{E}}(x_i) - \Psi_{\mathcal{F}}(x_i)\rvert, \lvert\Omega_{\mathcal{E}}(x_i) - \Omega_{\mathcal{F}}(x_i)\rvert \end{pmatrix}$
Singh et al [54]	$\mathbb{D}^4_{SM}(\mathcal{E},\mathcal{F}) = \sqrt{\frac{1}{4n}\sum_{i=1}^{n}\max\begin{pmatrix} \lvert\Upsilon_{\mathcal{E}}(x_i) - \Upsilon_{\mathcal{F}}(x_i)\rvert^2, \lvert\Phi_{\mathcal{E}}(x_i) - \Phi_{\mathcal{F}}(x_i)\rvert^2, \\ \lvert\Psi_{\mathcal{E}}(x_i) - \Psi_{\mathcal{F}}(x_i)\rvert^2, \lvert\Omega_{\mathcal{E}}(x_i) - \Omega_{\mathcal{F}}(x_i)\rvert^2 \end{pmatrix}}$

Based on the principle of maximizing similarity measures and minimizing distance measures, a higher similarity measure suggests a lower distance measure, so we can define it as follows:

Let two IvPFSs $\mathcal{E}$ and $\mathcal{F}$ in UOD X, and we have:

$$\mathbb{D}(\mathcal{E},\mathcal{F}) = 1 - \mathbb{S}(\mathcal{E},\mathcal{F}). \tag{2.2}$$

Table 2 shows a series of existing similarity/distance measures for IvPFSs.

Table 2. Existing similarity measures of IvPFSs.

Ref.	Similarity Measures
Liu et al. [32]	$\mathbb{D}^1_{Cs}(\mathcal{E},\mathcal{F}) = 1 - \sum_{i=1}^n \cos\left\{\frac{\pi}{2}\begin{bmatrix} \lvert \Upsilon^L_{\mathcal{E}}(x_i) - \Upsilon^L_{\mathcal{F}}(x_i)\rvert \vee \lvert \Upsilon^U_{\mathcal{E}}(x_i) - \Upsilon^U_{\mathcal{F}}(x_i)\rvert \\ \vee\lvert \Phi^L_{\mathcal{E}}(x_i) - \Phi^L_{\mathcal{F}}(x_i)\rvert \vee \lvert \Phi^U_{\mathcal{E}}(x_i) - \Phi^U_{\mathcal{F}}(x_i)\rvert \\ \vee\lvert \Psi^L_{\mathcal{E}}(x_i) - \Psi^L_{\mathcal{F}}(x_i)\rvert \vee \lvert \Psi^U_{\mathcal{E}}(x_i) - \Psi^U_{\mathcal{F}}(x_i)\rvert \end{bmatrix}\right\}$
Liu et al. [32]	$\mathbb{D}^2_{Cs}(\mathcal{E},\mathcal{F}) = 1 - \sum_{i=1}^n \cos\left\{\frac{\pi}{4}\begin{bmatrix} \lvert \Upsilon^L_{\mathcal{E}}(x_i) - \Upsilon^L_{\mathcal{F}}(x_i)\rvert + \lvert \Upsilon^U_{\mathcal{E}}(x_i) - \Upsilon^U_{\mathcal{F}}(x_i)\rvert \\ +\lvert \Phi^L_{\mathcal{E}}(x_i) - \Phi^L_{\mathcal{F}}(x_i)\rvert + \lvert \Phi^U_{\mathcal{E}}(x_i) - \Phi^U_{\mathcal{F}}(x_i)\rvert \\ +\lvert \Psi^L_{\mathcal{E}}(x_i) - \Psi^L_{\mathcal{F}}(x_i)\rvert + \lvert \Psi^U_{\mathcal{E}}(x_i) - \Psi^U_{\mathcal{F}}(x_i)\rvert \end{bmatrix}\right\}$
Liu et al. [32]	$\mathbb{D}^3_{Cs}(\mathcal{E},\mathcal{F}) = 1 - \sum_{i=1}^n \cos\left\{\frac{\pi}{2}\begin{bmatrix} \lvert \Upsilon^L_{\mathcal{E}}(x_i) - \Upsilon^L_{\mathcal{F}}(x_i)\rvert \vee \lvert \Upsilon^U_{\mathcal{E}}(x_i) - \Upsilon^U_{\mathcal{F}}(x_i)\rvert \\ \vee\lvert \Phi^L_{\mathcal{E}}(x_i) - \Phi^L_{\mathcal{F}}(x_i)\rvert \vee \lvert \Phi^U_{\mathcal{E}}(x_i) - \Phi^U_{\mathcal{F}}(x_i)\rvert \\ \vee\lvert \Psi^L_{\mathcal{E}}(x_i) - \Psi^L_{\mathcal{F}}(x_i)\rvert \vee \lvert \Psi^U_{\mathcal{E}}(x_i) - \Psi^U_{\mathcal{F}}(x_i)\rvert \\ \vee\lvert \Omega^L_{\mathcal{E}}(x_i) - \Omega^L_{\mathcal{F}}(x_i)\rvert \vee \lvert \Omega^U_{\mathcal{E}}(x_i) - \Omega^U_{\mathcal{F}}(x_i)\rvert \end{bmatrix}\right\}$
Liu et al. [32]	$\mathbb{D}^4_{Cs}(\mathcal{E},\mathcal{F}) = 1 - \sum_{i=1}^n \cos\left\{\frac{\pi}{4}\begin{bmatrix} \lvert \Upsilon^L_{\mathcal{E}}(x_i) - \Upsilon^L_{\mathcal{F}}(x_i)\rvert + \lvert \Upsilon^U_{\mathcal{E}}(x_i) - \Upsilon^U_{\mathcal{F}}(x_i)\rvert \\ +\lvert \Phi^L_{\mathcal{E}}(x_i) - \Phi^L_{\mathcal{F}}(x_i)\rvert + \lvert \Phi^U_{\mathcal{E}}(x_i) - \Phi^U_{\mathcal{F}}(x_i)\rvert \\ +\lvert \Psi^L_{\mathcal{E}}(x_i) - \Psi^L_{\mathcal{F}}(x_i)\rvert + \lvert \Psi^U_{\mathcal{E}}(x_i) - \Psi^U_{\mathcal{F}}(x_i)\rvert \\ +\lvert \Omega^L_{\mathcal{E}}(x_i) - \Omega^L_{\mathcal{F}}(x_i)\rvert + \lvert \Omega^U_{\mathcal{E}}(x_i) - \Omega^U_{\mathcal{F}}(x_i)\rvert \end{bmatrix}\right\}$
Liu et al. [32]	$\mathbb{D}^1_{Ct}(\mathcal{E},\mathcal{F}) = 1 - \sum_{i=1}^n \cot\left\{\frac{\pi}{4} + \frac{\pi}{4}\begin{bmatrix} \lvert \Upsilon^L_{\mathcal{E}}(x_i) - \Upsilon^L_{\mathcal{F}}(x_i)\rvert \vee \lvert \Upsilon^U_{\mathcal{E}}(x_i) - \Upsilon^U_{\mathcal{F}}(x_i)\rvert \\ \vee\lvert \Phi^L_{\mathcal{E}}(x_i) - \Phi^L_{\mathcal{F}}(x_i)\rvert \vee \lvert \Phi^U_{\mathcal{E}}(x_i) - \Phi^U_{\mathcal{F}}(x_i)\rvert \\ \vee\lvert \Psi^L_{\mathcal{E}}(x_i) - \Psi^L_{\mathcal{F}}(x_i)\rvert \vee \lvert \Psi^U_{\mathcal{E}}(x_i) - \Psi^U_{\mathcal{F}}(x_i)\rvert \end{bmatrix}\right\}$
Liu et al. [32]	$\mathbb{D}^2_{Ct}(\mathcal{E},\mathcal{F}) = 1 - \sum_{i=1}^n \cot\left\{\frac{\pi}{4} + \frac{\pi}{4}\begin{bmatrix} \lvert \Upsilon^L_{\mathcal{E}}(x_i) - \Upsilon^L_{\mathcal{F}}(x_i)\rvert \vee \lvert \Upsilon^U_{\mathcal{E}}(x_i) - \Upsilon^U_{\mathcal{F}}(x_i)\rvert \\ \vee\lvert \Phi^L_{\mathcal{E}}(x_i) - \Phi^L_{\mathcal{F}}(x_i)\rvert \vee \lvert \Phi^U_{\mathcal{E}}(x_i) - \Phi^U_{\mathcal{F}}(x_i)\rvert \\ \vee\lvert \Psi^L_{\mathcal{E}}(x_i) - \Psi^L_{\mathcal{F}}(x_i)\rvert \vee \lvert \Psi^U_{\mathcal{E}}(x_i) - \Psi^U_{\mathcal{F}}(x_i)\rvert \\ \vee\lvert \Omega^L_{\mathcal{E}}(x_i) - \Omega^L_{\mathcal{F}}(x_i)\rvert \vee \lvert \Omega^U_{\mathcal{E}}(x_i) - \Omega^U_{\mathcal{F}}(x_i)\rvert \end{bmatrix}\right\}$

3. New distance measures for PFSs

In this section, we will propose new distance measures for PFSs based on the Hellinger distance in three and four dimensions.

Definition 7. *Suppose* $X = \{x_1, x_2, ..., x_n\}$ *is a UOD. For two PFSs* $\mathcal{E} = \{\langle x, \Upsilon_{\mathcal{E}}(x), \Phi_{\mathcal{E}}(x), \Psi_{\mathcal{E}}(x)\rangle \mid x \in X\}$ *and* $\mathcal{F} = \{\langle x, \Upsilon_{\mathcal{F}}(x), \Phi_{\mathcal{F}}(x), \Psi_{\mathcal{F}}(x)\rangle \mid x \in X\}$*. The proposed four PFSs distance measures based on the Hellinger distance in three dimensions are defined as follows:*

$$\mathbb{D}^1_A(\mathcal{E},\mathcal{F}) = \frac{1}{2n}\sum_{i=1}^{n}\left(\left|\sqrt{\Upsilon_{\mathcal{E}}(x_i)} - \sqrt{\Upsilon_{\mathcal{F}}(x_i)}\right| + \left|\sqrt{\Phi_{\mathcal{E}}(x_i)} - \sqrt{\Phi_{\mathcal{F}}(x_i)}\right| + \left|\sqrt{\Psi_{\mathcal{E}}(x_i)} - \sqrt{\Psi_{\mathcal{F}}(x_i)}\right|\right) \tag{3.1}$$

$$\mathbb{D}_B^1(\mathcal{E},\mathcal{F}) = \left\{\frac{1}{2n}\sum_{i=1}^{n}\left[\left(\sqrt{\Upsilon_{\mathcal{E}}(x_i)} - \sqrt{\Upsilon_{\mathcal{F}}(x_i)}\right)^2 + \left(\sqrt{\Phi_{\mathcal{E}}(x_i)} - \sqrt{\Phi_{\mathcal{F}}(x_i)}\right)^2 + \left(\sqrt{\Psi_{\mathcal{E}}(x_i)} - \sqrt{\Psi_{\mathcal{F}}(x_i)}\right)^2\right]\right\}^{\frac{1}{2}} \tag{3.2}$$

$$\mathbb{D}_C^1(\mathcal{E},\mathcal{F}) = \frac{1}{n}\sum_{i=1}^{n}\max\left(|\sqrt{\Upsilon_{\mathcal{E}}(x_i)} - \sqrt{\Upsilon_{\mathcal{F}}(x_i)}|, |\sqrt{\Phi_{\mathcal{E}}(x_i)} - \sqrt{\Phi_{\mathcal{F}}(x_i)}|, |\sqrt{\Psi_{\mathcal{E}}(x_i)} - \sqrt{\Psi_{\mathcal{F}}(x_i)}|\right) \tag{3.3}$$

$$\mathbb{D}_D^1(\mathcal{E},\mathcal{F}) = \frac{1}{n}\sum_{i=1}^{n}\max\left(\left(\sqrt{\Upsilon_{\mathcal{E}}(x_i)} - \sqrt{\Upsilon_{\mathcal{F}}(x_i)}\right)^2, \left(\sqrt{\Phi_{\mathcal{E}}(x_i)} - \sqrt{\Phi_{\mathcal{F}}(x_i)}\right)^2, \left(\sqrt{\Psi_{\mathcal{E}}(x_i)} - \sqrt{\Psi_{\mathcal{F}}(x_i)}\right)^2\right). \tag{3.4}$$

Property 1. *The following properties are derived from the $\mathbb{D}^1(\mathcal{E},\mathcal{F})$ definition.*

(1) $0 \le \mathbb{D}^1(\mathcal{E},\mathcal{F}) \le 1$.

(2) $\mathbb{D}^1(\mathcal{E},\mathcal{F}) = 0$ *if, and only if,* $\mathcal{E} = \mathcal{F}$.

(3) $\mathbb{D}^1(\mathcal{E},\mathcal{F}) = \mathbb{D}^1(\mathcal{F},\mathcal{E})$.

(4) If $\mathcal{E} \subseteq \mathcal{F} \subseteq \mathcal{G}$, *then* $\mathbb{D}^1(\mathcal{E},\mathcal{F}) \le \mathbb{D}^1(\mathcal{E},\mathcal{G})$ *and* $\mathbb{D}^1(\mathcal{F},\mathcal{G}) \le \mathbb{D}^1(\mathcal{E},\mathcal{G})$.

Proof. (1) Take $\mathbb{D}_A^1$ as an example.

As for

$$\left|\sqrt{\Upsilon_{\mathcal{E}}(x_i)} - \sqrt{\Upsilon_{\mathcal{F}}(x_i)}\right| \geqslant 0,$$

$$\left|\sqrt{\Phi_{\mathcal{E}}(x_i)} - \sqrt{\Phi_{\mathcal{F}}(x_i)}\right| \geqslant 0,$$

$$\left|\sqrt{\Psi_{\mathcal{E}}(x_i)} - \sqrt{\Psi_{\mathcal{F}}(x_i)}\right| \geqslant 0,$$

we have

$$\mathbb{D}_A^1(\mathcal{E},\mathcal{F}) = \frac{1}{2}\left(\left|\sqrt{\Upsilon_{\mathcal{E}}(x_i)} - \sqrt{\Upsilon_{\mathcal{F}}(x_i)}\right| + \left|\sqrt{\Phi_{\mathcal{E}}(x_i)} - \sqrt{\Phi_{\mathcal{F}}(x_i)}\right| + \left|\sqrt{\Psi_{\mathcal{E}}(x_i)} - \sqrt{\Psi_{\mathcal{F}}(x_i)}\right|\right) \geqslant 0$$

and

$$\begin{aligned}
\mathbb{D}_A^1(\mathcal{E},\mathcal{F}) &= \frac{1}{2}\left(\left|\sqrt{\Upsilon_{\mathcal{E}}(x_i)} - \sqrt{\Upsilon_{\mathcal{F}}(x_i)}\right| + \left|\sqrt{\Phi_{\mathcal{E}}(x_i)} - \sqrt{\Phi_{\mathcal{F}}(x_i)}\right| + \left|\sqrt{\Psi_{\mathcal{E}}(x_i)} - \sqrt{\Psi_{\mathcal{F}}(x_i)}\right|\right) \\
&\le \frac{1}{2}\left(\Upsilon_{\mathcal{E}}(x_i) + \Upsilon_{\mathcal{F}}(x_i) + \Phi_{\mathcal{E}}(x_i) + \Phi_{\mathcal{F}}(x_i) + \Psi_{\mathcal{E}}(x_i) + \Psi_{\mathcal{F}}(x_i)\right) \\
&\le \frac{1}{2}\left(\Upsilon_{\mathcal{E}}(x_i) + \Phi_{\mathcal{E}}(x_i) + \Psi_{\mathcal{E}}(x_i)\right) + \frac{1}{2}\left(\Upsilon_{\mathcal{F}}(x_i) + \Phi_{\mathcal{F}}(x_i) + \Psi_{\mathcal{F}}(x_i)\right) \\
&\le 1.
\end{aligned}$$

Therefore, we can prove that:

$$0 \leqslant \mathbb{D}_A^1(\mathcal{E},\mathcal{F}) \leqslant 1$$

which proves that $\mathbb{D}_A^1(\mathcal{E},\mathcal{F})$ satisfies boundedness. □

Proof. (2) Consider $\mathbb{D}_B^1$ for illustration.

Given $\mathcal{E} = \mathcal{F}$, we have $\Upsilon_{\mathcal{E}}(x_i) = \Upsilon_{\mathcal{F}}(x_i), \Phi_{\mathcal{E}}(x_i) = \Phi_{\mathcal{F}}(x_i), \Psi_{\mathcal{E}}(x_i) = \Psi_{\mathcal{F}}(x_i)$.

Therefore, we can obtain

$$|\sqrt{\Upsilon_{\mathcal{E}}(x_i)} - \sqrt{\Upsilon_{\mathcal{F}}(x_i)}| = |\sqrt{\Phi_{\mathcal{E}}(x_i)} - \sqrt{\Phi_{\mathcal{F}}(x_i)}| = |\sqrt{\Psi_{\mathcal{E}}(x_i)} - \sqrt{\Psi_{\mathcal{F}}(x_i)}| = 0$$

$$\mathbb{D}_B^1(\mathcal{E},\mathcal{F}) = \left\{\frac{1}{2n}\sum_{i=1}^{n}\left[\left(\sqrt{\Upsilon_{\mathcal{E}}(x_i)} - \sqrt{\Upsilon_{\mathcal{F}}(x_i)}\right)^2 + \left(\sqrt{\Phi_{\mathcal{E}}(x_i)} - \sqrt{\Phi_{\mathcal{F}}(x_i)}\right)^2 + \left(\sqrt{\Psi_{\mathcal{E}}(x_i)} - \sqrt{\Psi_{\mathcal{F}}(x_i)}\right)^2\right]\right\}^{\frac{1}{2}} = 0.$$

Similarly, if $\mathbb{D}_B^1(\mathcal{E},\mathcal{F}) = 0$, we can obtain

$$\Upsilon_{\mathcal{E}}(x_i) = \Upsilon_{\mathcal{F}}(x_i), \Phi_{\mathcal{E}}(x_i) = \Phi_{\mathcal{F}}(x_i), \Psi_{\mathcal{E}}(x_i) = \Psi_{\mathcal{F}}(x_i).$$

We can infer that $\mathcal{E} = \mathcal{F}$. Therefore, we can prove that $\mathbb{D}_B^1(\mathcal{E},\mathcal{F}) = 0$ if, and only if, $\mathcal{E} = \mathcal{F}$. □

Proof. (3) Let us use $\mathbb{D}_C^1$ as a case in point.

We have

$$|\sqrt{\Upsilon_{\mathcal{E}}(x_i)} - \sqrt{\Upsilon_{\mathcal{F}}(x_i)}| = |\sqrt{\Upsilon_{\mathcal{F}}(x_i)} - \sqrt{\Upsilon_{\mathcal{E}}(x_i)}|$$
$$|\sqrt{\Phi_{\mathcal{E}}(x_i)} - \sqrt{\Phi_{\mathcal{F}}(x_i)}| = |\sqrt{\Phi_{\mathcal{F}}(x_i)} - \sqrt{\Phi_{\mathcal{E}}(x_i)}|$$
$$|\sqrt{\Psi_{\mathcal{E}}(x_i)} - \sqrt{\Psi_{\mathcal{F}}(x_i)}| = |\sqrt{\Psi_{\mathcal{F}}(x_i)} - \sqrt{\Psi_{\mathcal{E}}(x_i)}|.$$

From this, we can infer that

$$\begin{aligned}&\max\left(|\sqrt{\Upsilon_{\mathcal{E}}(x_i)} - \sqrt{\Upsilon_{\mathcal{F}}(x_i)}|, |\sqrt{\Phi_{\mathcal{E}}(x_i)} - \sqrt{\Phi_{\mathcal{F}}(x_i)}|, |\sqrt{\Psi_{\mathcal{E}}(x_i)} - \sqrt{\Psi_{\mathcal{F}}(x_i)}|\right)\\ =& \max\left(|\sqrt{\Upsilon_{\mathcal{F}}(x_i)} - \sqrt{\Upsilon_{\mathcal{E}}(x_i)}|, |\sqrt{\Phi_{\mathcal{F}}(x_i)} - \sqrt{\Phi_{\mathcal{E}}(x_i)}|, |\sqrt{\Psi_{\mathcal{F}}(x_i)} - \sqrt{\Psi_{\mathcal{E}}(x_i)}|\right).\end{aligned}$$

Therefore, we can prove that at this point $\mathbb{D}_C^1(\mathcal{E},\mathcal{F}) = \mathbb{D}_C^1(\mathcal{F},\mathcal{E})$. □

Proof. (4) Using $\mathbb{D}_D^1$ as a point of example.

If $\mathcal{E} \subseteq \mathcal{F} \subseteq \mathcal{G}$, we have $\Upsilon_{\mathcal{E}}(\mathrm{x_i}) \leqslant \Upsilon_{\mathcal{F}}(\mathrm{x_i}) \leqslant \Upsilon_{\mathcal{G}}(\mathrm{x_i}), \Phi_{\mathcal{E}}(\mathrm{x_i}) \leqslant \Phi_{\mathcal{F}}(\mathrm{x_i}) \leqslant \Phi_{\mathcal{G}}(\mathrm{x_i})$, $\Psi_{\mathcal{G}}(\mathrm{x_i}) \leqslant \Psi_{\mathcal{F}}(\mathrm{x_i}) \leqslant \Psi_{\mathcal{E}}(\mathrm{x_i})$.

Hence, we can derive the subsequent equations

$$\begin{aligned}\mathbb{D}_D^1(\mathcal{E},\mathcal{F}) &= \frac{1}{n}\sum_{i=1}^{n}\max\left((\Upsilon_{\mathcal{E}}(x_i) - \Upsilon_{\mathcal{F}}(x_i))^2, (\Phi_{\mathcal{E}}(x_i) - \Phi_{\mathcal{F}}(x_i))^2, (\Psi_{\mathcal{E}}(x_i) - \Psi_{\mathcal{F}}(x_i))^2\right)\\ &\leqslant \frac{1}{n}\sum_{i=1}^{n}\max\left((\Upsilon_{\mathcal{E}}(x_i) - \Upsilon_{\mathcal{G}}(x_i))^2, (\Phi_{\mathcal{E}}(x_i) - \Phi_{\mathcal{G}}(x_i))^2, (\Psi_{\mathcal{E}}(x_i) - \Psi_{\mathcal{G}}(x_i))^2\right)\\ &= \mathbb{D}_D^1(\mathcal{E},\mathcal{G})\\ \mathbb{D}_D^1(\mathcal{F},\mathcal{G}) &= \frac{1}{n}\sum_{i=1}^{n}\max\left((\Upsilon_{\mathcal{F}}(x_i) - \Upsilon_{\mathcal{G}}(x_i))^2, (\Phi_{\mathcal{F}}(x_i) - \Phi_{\mathcal{G}}(x_i))^2, (\Psi_{\mathcal{F}}(x_i) - \Psi_{\mathcal{G}}(x_i))^2\right)\\ &\leqslant \frac{1}{n}\sum_{i=1}^{n}\max\left((\Upsilon_{\mathcal{E}}(x_i) - \Upsilon_{\mathcal{G}}(x_i))^2, (\Phi_{\mathcal{E}}(x_i) - \Phi_{\mathcal{G}}(x_i))^2, (\Psi_{\mathcal{E}}(x_i) - \Psi_{\mathcal{G}}(x_i))^2\right)\\ &= \mathbb{D}_D^1(\mathcal{E},\mathcal{G}).\end{aligned}$$

Therefore,we can prove that if $\mathcal{E} \subseteq \mathcal{F} \subseteq \mathcal{G}$, then $\mathbb{D}_D^1(\mathcal{E},\mathcal{F}) \leq \mathbb{D}_D^1(\mathcal{E},\mathcal{G})$ and $\mathbb{D}_D^1(\mathcal{F},\mathcal{G}) \leq \mathbb{D}_D^1(\mathcal{E},\mathcal{G})$. □

Definition 8. *The proposed four PFS distance measures based on the Hellinger distance in four dimensions are defined as follows:*

$$\mathbb{D}_A^2(\mathcal{E},\mathcal{F}) = \frac{1}{2n}\sum_{i=1}^{n}\left(\begin{array}{c} |\sqrt{\Upsilon_{\mathcal{E}}(x_i)} - \sqrt{\Upsilon_{\mathcal{F}}(x_i)}| + |\sqrt{\Phi_{\mathcal{E}}(x_i)} - \sqrt{\Phi_{\mathcal{F}}(x_i)}| \\ +|\sqrt{\Psi_{\mathcal{E}}(x_i)} - \sqrt{\Psi_{\mathcal{F}}(x_i)}| + |\sqrt{\Omega_{\mathcal{E}}(x_i)} - \sqrt{\Omega_{\mathcal{F}}(x_i)}| \end{array}\right) \tag{3.5}$$

$$\mathbb{D}_B^2(\mathcal{E},\mathcal{F}) = \left\{\frac{1}{2n}\sum_{i=1}^{n}\left[\begin{array}{c} \left(\sqrt{\Upsilon_{\mathcal{E}}(x_i)} - \sqrt{\Upsilon_{\mathcal{F}}(x_i)}\right)^2 + \left(\sqrt{\Phi_{\mathcal{E}}(x_i)} - \sqrt{\Phi_{\mathcal{F}}(x_i)}\right)^2 \\ +\left(\sqrt{\Psi_{\mathcal{E}}(x_i)} - \sqrt{\Psi_{\mathcal{F}}(x_i)}\right)^2 + \left(\sqrt{\Omega_{\mathcal{E}}(x_i)} - \sqrt{\Omega_{\mathcal{F}}(x_i)}\right)^2 \end{array}\right]\right\}^{\frac{1}{2}} \tag{3.6}$$

$$\mathbb{D}_C^2(\mathcal{E},\mathcal{F}) = \frac{1}{n}\sum_{i=1}^{n}\max\left(\begin{array}{c} |\sqrt{\Upsilon_{\mathcal{E}}(x_i)} - \sqrt{\Upsilon_{\mathcal{F}}(x_i)}|, |\sqrt{\Phi_{\mathcal{E}}(x_i)} - \sqrt{\Phi_{\mathcal{F}}(x_i)}|, \\ |\sqrt{\Psi_{\mathcal{E}}(x_i)} - \sqrt{\Psi_{\mathcal{F}}(x_i)}|, |\sqrt{\Omega_{\mathcal{E}}(x_i)} - \sqrt{\Omega_{\mathcal{F}}(x_i)}| \end{array}\right) \tag{3.7}$$

$$\mathbb{D}_D^2(\mathcal{E},\mathcal{F}) = \frac{1}{n}\sum_{i=1}^{n}\max\left(\begin{array}{c} \left(\sqrt{\Upsilon_{\mathcal{E}}(x_i)} - \sqrt{\Upsilon_{\mathcal{F}}(x_i)}\right)^2, \left(\sqrt{\Phi_{\mathcal{E}}(x_i)} - \sqrt{\Phi_{\mathcal{F}}(x_i)}\right)^2 \\ \left(\sqrt{\Psi_{\mathcal{E}}(x_i)} - \sqrt{\Psi_{\mathcal{F}}(x_i)}\right)^2, \left(\sqrt{\Omega_{\mathcal{E}}(x_i)} - \sqrt{\Omega_{\mathcal{F}}(x_i)}\right)^2 \end{array}\right). \tag{3.8}$$

Property 2. *The following properties are derived from the $\mathbb{D}^2(\mathcal{E},\mathcal{F})$ definition. Proofs (1)–(4) is similar with $\mathbb{D}^1$.*

(1) $0 \le \mathbb{D}^2(\mathcal{E},\mathcal{F}) \le 1$.

(2) $\mathbb{D}^2(\mathcal{E},\mathcal{F}) = 0$ *if, and only if,* $\mathcal{E} = \mathcal{F}$.

(3) $\mathbb{D}^2(\mathcal{E},\mathcal{F}) = \mathbb{D}^2(\mathcal{F},\mathcal{E})$.

(4) If $\mathcal{E} \subseteq \mathcal{F} \subseteq \mathcal{G}$, *then* $\mathbb{D}^2(\mathcal{E},\mathcal{F}) \le \mathbb{D}^2(\mathcal{E},\mathcal{G})$ *and* $\mathbb{D}^2(\mathcal{F},\mathcal{G}) \le \mathbb{D}^2(\mathcal{E},\mathcal{G})$.

4. New distance measures for IvPFSs

In this section, we will propose new distance measures for IvPFSs based on Hellinger distance in three and four dimensions.

Definition 9. *Consider* $X = \{x_1, x_{2,}..., x_n\}$ *as an UOD for two PFSs* $\mathcal{E} = \{\langle x, \left[\Upsilon_{\mathcal{E}}^L(x), \Upsilon_{\mathcal{E}}^U(x)\right], \Phi_{\mathcal{E}}(x) = \left[\Phi_{\mathcal{E}}^L(x), \Phi_{\mathcal{E}}^U(x)\right], \Psi_{\mathcal{E}}(x) = \left[\Psi_{\mathcal{E}}^L(x), \Psi_{\mathcal{E}}^U(x)\right]\rangle | x \in X\}$ *and* $\mathcal{F} = \{\langle x, \left[\Upsilon_{\mathcal{F}}^L(x), \Upsilon_{\mathcal{F}}^U(x)\right], \Phi_{\mathcal{F}}(x) = \left[\Phi_{\mathcal{F}}^L(x), \Phi_{\mathcal{F}}^U(x)\right], \Psi_{\mathcal{F}}(x) = \left[\Psi_{\mathcal{F}}^L(x), \Psi_{\mathcal{F}}^U(x)\right]\rangle | x \in X\}$. *The proposed four IvPFSs distance measures based on the Hellinger distance in three dimensions are defined as follows:*

$$\mathbb{D}_A^3(\mathcal{E},\mathcal{F}) = \frac{1}{4n}\sum_{i=1}^{n}\left(\begin{array}{c} |\sqrt{\Upsilon_{\mathcal{E}}^L(x_i)} - \sqrt{\Upsilon_{\mathcal{F}}^L(x_i)}| + |\sqrt{\Upsilon_{\mathcal{E}}^U(x_i)} - \sqrt{\Upsilon_{\mathcal{F}}^U(x_i)}| \\ +|\sqrt{\Phi_{\mathcal{E}}^L(x_i)} - \sqrt{\Phi_{\mathcal{F}}^L(x_i)}| + |\sqrt{\Phi_{\mathcal{E}}^U(x_i)} - \sqrt{\Phi_{\mathcal{F}}^U(x_i)}| \\ +|\sqrt{\Psi_{\mathcal{E}}^L(x_i)} - \sqrt{\Psi_{\mathcal{F}}^L(x_i)}| + |\sqrt{\Psi_{\mathcal{E}}^U(x_i)} - \sqrt{\Psi_{\mathcal{F}}^U(x_i)}| \end{array}\right) \tag{4.1}$$

$$\mathbb{D}_B^3(\mathcal{E},\mathcal{F}) = \left\{\frac{1}{4n}\sum_{i=1}^{n}\left[\begin{array}{c} \left(\sqrt{\Upsilon_{\mathcal{E}}^L(x_i)} - \sqrt{\Upsilon_{\mathcal{F}}^L(x_i)}\right)^2 + \left(\sqrt{\Upsilon_{\mathcal{E}}^U(x_i)} - \sqrt{\Upsilon_{\mathcal{F}}^U(x_i)}\right)^2 \\ +\left(\sqrt{\Phi_{\mathcal{E}}^L(x_i)} - \sqrt{\Phi_{\mathcal{F}}^L(x_i)}\right)^2 + \left(\sqrt{\Phi_{\mathcal{E}}^U(x_i)} - \sqrt{\Phi_{\mathcal{F}}^U(x_i)}\right)^2 \\ +\left(\sqrt{\Psi_{\mathcal{E}}^L(x_i)} - \sqrt{\Psi_{\mathcal{F}}^L(x_i)}\right)^2 + \left(\sqrt{\Psi_{\mathcal{E}}^U(x_i)} - \sqrt{\Psi_{\mathcal{F}}^U(x_i)}\right)^2 \end{array}\right]\right\}^{\frac{1}{2}} \tag{4.2}$$

$$\mathbb{D}_C^3(\mathcal{E},\mathcal{F})=\frac{1}{2n}\sum_{i=1}^{n}\max\begin{pmatrix}|\sqrt{\Upsilon_{\mathcal{E}}^L(x_i)}-\sqrt{\Upsilon_{\mathcal{F}}^L(x_i)}|,|\sqrt{\Upsilon_{\mathcal{E}}^U(x_i)}-\sqrt{\Upsilon_{\mathcal{F}}^U(x_i)}|\\|\sqrt{\Phi_{\mathcal{E}}^L(x_i)}-\sqrt{\Phi_{\mathcal{F}}^L(x_i)}|,|\sqrt{\Phi_{\mathcal{E}}^U(x_i)}-\sqrt{\Phi_{\mathcal{F}}^U(x_i)}|\\|\sqrt{\Psi_{\mathcal{E}}^L(x_i)}-\sqrt{\Psi_{\mathcal{F}}^L(x_i)}|,|\sqrt{\Psi_{\mathcal{E}}^U(x_i)}-\sqrt{\Psi_{\mathcal{F}}^U(x_i)}|\end{pmatrix} \tag{4.3}$$

$$\mathbb{D}_D^3(\mathcal{E},\mathcal{F})=\frac{1}{2n}\sum_{i=1}^{n}\max\begin{pmatrix}\left(\sqrt{\Upsilon_{\mathcal{E}}^L(x_i)}-\sqrt{\Upsilon_{\mathcal{F}}^L(x_i)}\right)^2,\left(\sqrt{\Upsilon_{\mathcal{E}}^U(x_i)}-\sqrt{\Upsilon_{\mathcal{F}}^U(x_i)}\right)^2\\\left(\sqrt{\Phi_{\mathcal{E}}^L(x_i)}-\sqrt{\Phi_{\mathcal{F}}^L(x_i)}\right)^2,\left(\sqrt{\Phi_{\mathcal{E}}^U(x_i)}-\sqrt{\Phi_{\mathcal{F}}^U(x_i)}\right)^2\\\left(\sqrt{\Psi_{\mathcal{E}}^L(x_i)}-\sqrt{\Psi_{\mathcal{F}}^L(x_i)}\right)^2,\left(\sqrt{\Psi_{\mathcal{E}}^U(x_i)}-\sqrt{\Psi_{\mathcal{F}}^U(x_i)}\right)^2\end{pmatrix}. \tag{4.4}$$

Property 3. *The following properties are derived from the $\mathbb{D}^3(\mathcal{E},\mathcal{F})$ definition.*

(1) $0\le\mathbb{D}^3(\mathcal{E},\mathcal{F})\le 1$.

(2) $\mathbb{D}^3(\mathcal{E},\mathcal{F})=0$ *if, and only if,* $\mathcal{E}=\mathcal{F}$.

(3) $\mathbb{D}^3(\mathcal{E},\mathcal{F})=\mathbb{D}^3(\mathcal{F},\mathcal{E})$.

(4) If $\mathcal{E}\subseteq\mathcal{F}\subseteq\mathcal{G}$, *then* $\mathbb{D}^3(\mathcal{E},\mathcal{F})\le\mathbb{D}^3(\mathcal{E},\mathcal{G})$ *and* $\mathbb{D}^3(\mathcal{F},\mathcal{G})\le\mathbb{D}_A^3(\mathcal{E},\mathcal{G})$.

Proof. (1) Take $\mathbb{D}_A^3$ as an example.

As for

$$|\sqrt{\Upsilon_{\mathcal{E}}^L(x_i)}-\sqrt{\Upsilon_{\mathcal{F}}^L(x_i)}|\geqslant 0,|\sqrt{\Upsilon_{\mathcal{E}}^U(x_i)}-\sqrt{\Upsilon_{\mathcal{F}}^U(x_i)}|\geqslant 0,$$

$$|\sqrt{\Phi_{\mathcal{E}}^L(x_i)}-\sqrt{\Phi_{\mathcal{F}}^L(x_i)}|\geqslant 0,|\sqrt{\Phi_{\mathcal{E}}^U(x_i)}-\sqrt{\Phi_{\mathcal{F}}^U(x_i)}|\geqslant 0,$$

$$|\sqrt{\Psi_{\mathcal{E}}^L(x_i)}-\sqrt{\Psi_{\mathcal{F}}^L(x_i)}|\geqslant 0,|\sqrt{\Psi_{\mathcal{E}}^U(x_i)}-\sqrt{\Psi_{\mathcal{F}}^U(x_i)}|\geqslant 0.$$

we have

$$\mathbb{D}_A^3(\mathcal{E},\mathcal{F})=\frac{1}{4n}\sum_{i=1}^{n}\begin{pmatrix}|\sqrt{\Upsilon_{\mathcal{E}}^L(x_i)}-\sqrt{\Upsilon_{\mathcal{F}}^L(x_i)}|+|\sqrt{\Upsilon_{\mathcal{E}}^U(x_i)}-\sqrt{\Upsilon_{\mathcal{F}}^U(x_i)}|\\+|\sqrt{\Phi_{\mathcal{E}}^L(x_i)}-\sqrt{\Phi_{\mathcal{F}}^L(x_i)}|+|\sqrt{\Phi_{\mathcal{E}}^U(x_i)}-\sqrt{\Phi_{\mathcal{F}}^U(x_i)}|\\+|\sqrt{\Psi_{\mathcal{E}}^L(x_i)}-\sqrt{\Psi_{\mathcal{F}}^L(x_i)}|+|\sqrt{\Psi_{\mathcal{E}}^U(x_i)}-\sqrt{\Psi_{\mathcal{F}}^U(x_i)}|\end{pmatrix}\geqslant 0$$

and

$$\begin{aligned}\mathbb{D}_A^3(\mathcal{E},\mathcal{F})&=\frac{1}{4n}\sum_{i=1}^{n}\begin{pmatrix}|\sqrt{\Upsilon_{\mathcal{E}}^L(x_i)}-\sqrt{\Upsilon_{\mathcal{F}}^L(x_i)}|+|\sqrt{\Upsilon_{\mathcal{E}}^U(x_i)}-\sqrt{\Upsilon_{\mathcal{F}}^U(x_i)}|\\+|\sqrt{\Phi_{\mathcal{E}}^L(x_i)}-\sqrt{\Phi_{\mathcal{F}}^L(x_i)}|+|\sqrt{\Phi_{\mathcal{E}}^U(x_i)}-\sqrt{\Phi_{\mathcal{F}}^U(x_i)}|\\+|\sqrt{\Psi_{\mathcal{E}}^L(x_i)}-\sqrt{\Psi_{\mathcal{F}}^L(x_i)}|+|\sqrt{\Psi_{\mathcal{E}}^U(x_i)}-\sqrt{\Psi_{\mathcal{F}}^U(x_i)}|\end{pmatrix}\\&\le\frac{1}{4n}\sum_{i=1}^{n}\begin{pmatrix}\sqrt{\Upsilon_{\mathcal{E}}^L(x_i)}+\sqrt{\Upsilon_{\mathcal{F}}^L(x_i)}+\sqrt{\Upsilon_{\mathcal{E}}^U(x_i)}+\sqrt{\Upsilon_{\mathcal{F}}^U(x_i)}\\+\sqrt{\Phi_{\mathcal{E}}^L(x_i)}+\sqrt{\Phi_{\mathcal{F}}^L(x_i)}+\sqrt{\Phi_{\mathcal{E}}^U(x_i)}+\sqrt{\Phi_{\mathcal{F}}^U(x_i)}\\+\sqrt{\Psi_{\mathcal{E}}^L(x_i)}+\sqrt{\Psi_{\mathcal{F}}^L(x_i)}+\sqrt{\Psi_{\mathcal{E}}^U(x_i)}+\sqrt{\Psi_{\mathcal{F}}^U(x_i)}\end{pmatrix}\end{aligned}$$

$$\le \frac{1}{4n}\sum_{i=1}^{n}4$$
$$\le 1.$$

As $\Upsilon_{\mathcal{E}}(x_i)$, $\Phi_{\mathcal{E}}(x_i)$, $\Psi_{\mathcal{E}}(x_i)$, $\Upsilon_{\mathcal{F}}(x_i)$, $\Phi_{\mathcal{F}}(x_i)$, $\Psi_{\mathcal{F}}(x_i)$ of both IvPFSs belong to [0,1] it is clear that $\mathbb{D}_A^3(\mathcal{E},\mathcal{F})$ belongs to [0,1]. □

Proof. (2) Consider the case of $\mathbb{D}_B^3$.

Given $\mathcal{E}=\mathcal{F}$, we have

$$\Upsilon^L_{\mathcal{E}}(x_i)=\Upsilon^L_{\mathcal{F}}(x_i), \Upsilon^U_{\mathcal{E}}(x_i)=\Upsilon^U_{\mathcal{F}}(x_i)$$
$$\Phi^L_{\mathcal{E}}(x_i)=\Phi^L_{\mathcal{F}}(x_i), \Phi^U_{\mathcal{E}}(x_i)=\Phi^U_{\mathcal{F}}(x_i)$$
$$\Psi^L_{\mathcal{E}}(x_i)=\Psi^L_{\mathcal{F}}(x_i), \Psi^U_{\mathcal{E}}(x_i)=\Psi^U_{\mathcal{F}}(x_i).$$

Consequently, we can acquire

$$\left(\sqrt{\Upsilon^L_{\mathcal{E}}(x_i)}-\sqrt{\Upsilon^L_{\mathcal{F}}(x_i)}\right)^2=\left(\sqrt{\Upsilon^U_{\mathcal{E}}(x_i)}-\sqrt{\Upsilon^U_{\mathcal{F}}(x_i)}\right)^2=\left(\sqrt{\Phi^L_{\mathcal{E}}(x_i)}-\sqrt{\Phi^L_{\mathcal{F}}(x_i)}\right)^2$$
$$=\left(\sqrt{\Phi^U_{\mathcal{E}}(x_i)}-\sqrt{\Phi^U_{\mathcal{F}}(x_i)}\right)^2=\left(\sqrt{\Psi^L_{\mathcal{E}}(x_i)}-\sqrt{\Psi^L_{\mathcal{F}}(x_i)}\right)^2=\left(\sqrt{\Psi^U_{\mathcal{E}}(x_i)}-\sqrt{\Psi^U_{\mathcal{F}}(x_i)}\right)^2=0.$$

Similarly, if $\mathbb{D}_B^3(\mathcal{E},\mathcal{F})=0$,we can acquire

$$\Upsilon^L_{\mathcal{E}}(x_i)=\Upsilon^L_{\mathcal{F}}(x_i), \Upsilon^U_{\mathcal{E}}(x_i)=\Upsilon^U_{\mathcal{F}}(x_i)$$
$$\Phi^L_{\mathcal{E}}(x_i)=\Phi^L_{\mathcal{F}}(x_i), \Phi^U_{\mathcal{E}}(x_i)=\Phi^U_{\mathcal{F}}(x_i)$$
$$\Psi^L_{\mathcal{E}}(x_i)=\Psi^L_{\mathcal{F}}(x_i), \Psi^U_{\mathcal{E}}(x_i)=\Psi^U_{\mathcal{F}}(x_i).$$

From this, we can infer that $\mathcal{E}=\mathcal{F}$.

Therefore, we can prove that at this point $\mathbb{D}_B^3(\mathcal{E},\mathcal{F})=0$ if, and only if, $\mathcal{E}=\mathcal{F}$. □

Proof. (3) Let us examine $\mathbb{D}_C^3$ for illustration.

We have

$$\left|\sqrt{\Upsilon^L_{\mathcal{E}}(x_i)}-\sqrt{\Upsilon^L_{\mathcal{F}}(x_i)}\right|=\left|\sqrt{\Upsilon^L_{\mathcal{F}}(x_i)}-\sqrt{\Upsilon^L_{\mathcal{E}}(x_i)}\right|,\left|\sqrt{\Upsilon^U_{\mathcal{E}}(x_i)}-\sqrt{\Upsilon^U_{\mathcal{F}}(x_i)}\right|=\left|\sqrt{\Upsilon^U_{\mathcal{F}}(x_i)}-\sqrt{\Upsilon^U_{\mathcal{E}}(x_i)}\right|$$
$$\left|\sqrt{\Phi^L_{\mathcal{E}}(x_i)}-\sqrt{\Phi^L_{\mathcal{F}}(x_i)}\right|=\left|\sqrt{\Phi^L_{\mathcal{F}}(x_i)}-\sqrt{\Phi^L_{\mathcal{E}}(x_i)}\right|,\left|\sqrt{\Phi^U_{\mathcal{E}}(x_i)}-\sqrt{\Phi^U_{\mathcal{F}}(x_i)}\right|=\left|\sqrt{\Phi^U_{\mathcal{F}}(x_i)}-\sqrt{\Phi^U_{\mathcal{E}}(x_i)}\right|.$$
$$\left|\sqrt{\Psi^L_{\mathcal{E}}(x_i)}-\sqrt{\Psi^L_{\mathcal{F}}(x_i)}\right|=\left|\sqrt{\Psi^L_{\mathcal{F}}(x_i)}-\sqrt{\Psi^L_{\mathcal{E}}(x_i)}\right|,\left|\sqrt{\Psi^U_{\mathcal{E}}(x_i)}-\sqrt{\Psi^U_{\mathcal{F}}(x_i)}\right|=\left|\sqrt{\Psi^U_{\mathcal{F}}(x_i)}-\sqrt{\Psi^U_{\mathcal{E}}(x_i)}\right|$$

From this, we can infer

$$\frac{1}{2n}\sum_{i=1}^{n}\max\begin{pmatrix}|\sqrt{\Upsilon^L_{\mathcal{E}}(x_i)}-\sqrt{\Upsilon^L_{\mathcal{F}}(x_i)}|,\\|\sqrt{\Upsilon^U_{\mathcal{E}}(x_i)}-\sqrt{\Upsilon^U_{\mathcal{F}}(x_i)}|,\\|\sqrt{\Phi^L_{\mathcal{E}}(x_i)}-\sqrt{\Phi^L_{\mathcal{F}}(x_i)}|,\\|\sqrt{\Phi^U_{\mathcal{E}}(x_i)}-\sqrt{\Phi^U_{\mathcal{F}}(x_i)}|,\\|\sqrt{\Psi^L_{\mathcal{E}}(x_i)}-\sqrt{\Psi^L_{\mathcal{F}}(x_i)}|,\\|\sqrt{\Psi^U_{\mathcal{E}}(x_i)}-\sqrt{\Psi^U_{\mathcal{F}}(x_i)}|\end{pmatrix}=\frac{1}{2n}\sum_{i=1}^{n}\max\begin{pmatrix}|\sqrt{\Upsilon^L_{\mathcal{F}}(x_i)}-\sqrt{\Upsilon^L_{\mathcal{E}}(x_i)}|,\\|\sqrt{\Upsilon^U_{\mathcal{F}}(x_i)}-\sqrt{\Upsilon^U_{\mathcal{E}}(x_i)}|,\\|\sqrt{\Phi^L_{\mathcal{F}}(x_i)}-\sqrt{\Phi^L_{\mathcal{E}}(x_i)}|,\\|\sqrt{\Phi^U_{\mathcal{F}}(x_i)}-\sqrt{\Phi^U_{\mathcal{E}}(x_i)}|,\\|\sqrt{\Psi^L_{\mathcal{F}}(x_i)}-\sqrt{\Psi^L_{\mathcal{E}}(x_i)}|,\\|\sqrt{\Psi^U_{\mathcal{F}}(x_i)}-\sqrt{\Psi^U_{\mathcal{E}}(x_i)}|\end{pmatrix}.$$

Therefore, we can prove that $\mathbb{D}_C^3(\mathcal{E},\mathcal{F})=\mathbb{D}_C^3(\mathcal{F},\mathcal{E})$. □

Proof. (4) As a representative example, we can look at $\mathbb{D}_D^3$.

If $\mathcal{E}\subseteq\mathcal{F}\subseteq\mathcal{G}$, we have

$$\Upsilon_{\mathcal{E}}^L(x_i)\leqslant\Upsilon_{\mathcal{F}}^L(x_i)\leqslant\Upsilon_{\mathcal{G}}^L(x_i),\Upsilon_{\mathcal{E}}^U(x_i)\leqslant\Upsilon_{\mathcal{F}}^U(x_i)\leqslant\Upsilon_{\mathcal{G}}^U(x_i).$$

Similarly,

$$\Phi_{\mathcal{E}}^L(x_i)\leqslant\Phi_{\mathcal{F}}^L(x_i)\leqslant\Phi_{\mathcal{G}}^L(x_i),\Phi_{\mathcal{E}}^U(x_i)\leqslant\Phi_{\mathcal{F}}^U(x_i)\leqslant\Phi_{\mathcal{G}}^U(x_i)$$
$$\Psi_{\mathcal{E}}^L(x_i)\geqslant\Psi_{\mathcal{F}}^L(x_i)\geqslant\Psi_{\mathcal{G}}^L(x_i),\Psi_{\mathcal{E}}^U(x_i)\geqslant\Psi_{\mathcal{F}}^U(x_i)\geqslant\Psi_{\mathcal{G}}^U(x_i)$$

and we can get the following conclusion:

$$\max\left\{\begin{array}{c}\left(\sqrt{\Upsilon_{\mathcal{E}}^L(x_i)}-\sqrt{\Upsilon_{\mathcal{F}}^L(x_i)}\right)^2,\\\left(\sqrt{\Upsilon_{\mathcal{E}}^U(x_i)}-\sqrt{\Upsilon_{\mathcal{F}}^U(x_i)}\right)^2,\\\left(\sqrt{\Phi_{\mathcal{E}}^L(x_i)}-\sqrt{\Phi_{\mathcal{F}}^L(x_i)}\right)^2,\\\left(\sqrt{\Phi_{\mathcal{E}}^U(x_i)}-\sqrt{\Phi_{\mathcal{F}}^U(x_i)}\right)^2,\\\left(\sqrt{\Psi_{\mathcal{E}}^L(x_i)}-\sqrt{\Psi_{\mathcal{F}}^L(x_i)}\right)^2,\\\left(\sqrt{\Psi_{\mathcal{E}}^U(x_i)}-\sqrt{\Psi_{\mathcal{F}}^U(x_i)}\right)^2\end{array}\right\}\leqslant\max\left\{\begin{array}{c}\left(\sqrt{\Upsilon_{\mathcal{E}}^L(x_i)}-\sqrt{\Upsilon_{\mathcal{G}}^L(x_i)}\right)^2,\\\left(\sqrt{\Upsilon_{\mathcal{E}}^U(x_i)}-\sqrt{\Upsilon_{\mathcal{G}}^U(x_i)}\right)^2,\\\left(\sqrt{\Phi_{\mathcal{E}}^L(x_i)}-\sqrt{\Phi_{\mathcal{G}}^L(x_i)}\right)^2,\\\left(\sqrt{\Phi_{\mathcal{E}}^U(x_i)}-\sqrt{\Phi_{\mathcal{G}}^U(x_i)}\right)^2,\\\left(\sqrt{\Psi_{\mathcal{E}}^L(x_i)}-\sqrt{\Psi_{\mathcal{G}}^L(x_i)}\right)^2,\\\left(\sqrt{\Psi_{\mathcal{E}}^U(x_i)}-\sqrt{\Psi_{\mathcal{G}}^U(x_i)}\right)^2\end{array}\right\}$$

$$\max\left\{\begin{array}{c}\left(\sqrt{\Upsilon_{\mathcal{F}}^L(x_i)}-\sqrt{\Upsilon_{\mathcal{G}}^L(x_i)}\right)^2,\\\left(\sqrt{\Upsilon_{\mathcal{F}}^U(x_i)}-\sqrt{\Upsilon_{\mathcal{G}}^U(x_i)}\right)^2,\\\left(\sqrt{\Phi_{\mathcal{F}}^L(x_i)}-\sqrt{\Phi_{\mathcal{G}}^L(x_i)}\right)^2,\\\left(\sqrt{\Phi_{\mathcal{F}}^U(x_i)}-\sqrt{\Phi_{\mathcal{G}}^U(x_i)}\right)^2,\\\left(\sqrt{\Psi_{\mathcal{F}}^L(x_i)}-\sqrt{\Psi_{\mathcal{G}}^L(x_i)}\right)^2,\\\left(\sqrt{\Psi_{\mathcal{F}}^U(x_i)}-\sqrt{\Psi_{\mathcal{G}}^U(x_i)}\right)^2\end{array}\right\}\leqslant\max\left\{\begin{array}{c}\left(\sqrt{\Upsilon_{\mathcal{E}}^L(x_i)}-\sqrt{\Upsilon_{\mathcal{G}}^L(x_i)}\right)^2,\\\left(\sqrt{\Upsilon_{\mathcal{E}}^U(x_i)}-\sqrt{\Upsilon_{\mathcal{G}}^U(x_i)}\right)^2,\\\left(\sqrt{\Phi_{\mathcal{E}}^L(x_i)}-\sqrt{\Phi_{\mathcal{G}}^L(x_i)}\right)^2,\\\left(\sqrt{\Phi_{\mathcal{E}}^U(x_i)}-\sqrt{\Phi_{\mathcal{G}}^U(x_i)}\right)^2,\\\left(\sqrt{\Psi_{\mathcal{E}}^L(x_i)}-\sqrt{\Psi_{\mathcal{G}}^L(x_i)}\right)^2,\\\left(\sqrt{\Psi_{\mathcal{E}}^U(x_i)}-\sqrt{\Psi_{\mathcal{G}}^U(x_i)}\right)^2\end{array}\right\}.$$

Thus, $\mathbb{D}_D^3(\mathcal{E},\mathcal{F})\leq\mathbb{D}_D^3(\mathcal{E},\mathcal{G})$. Similarly, we can prove $\mathbb{D}_D^3(\mathcal{F},\mathcal{G})\leq\mathbb{D}_D^3(\mathcal{E},\mathcal{G})$. □

Definition 10. *The proposed four IvPFSs distance measures based on the Hellinger distance in four dimensions are defined as follows:*

$$\mathbb{D}_A^4(\mathcal{E},\mathcal{F})=\frac{1}{4n}\sum_{i=1}^{n}\left(\begin{array}{c}|\sqrt{\Upsilon_{\mathcal{E}}^L(x_i)}-\sqrt{\Upsilon_{\mathcal{F}}^L(x_i)}|+|\sqrt{\Upsilon_{\mathcal{E}}^U(x_i)}-\sqrt{\Upsilon_{\mathcal{F}}^U(x_i)}|\\+|\sqrt{\Phi_{\mathcal{E}}^L(x_i)}-\sqrt{\Phi_{\mathcal{F}}^L(x_i)}|+|\sqrt{\Phi_{\mathcal{E}}^U(x_i)}-\sqrt{\Phi_{\mathcal{F}}^U(x_i)}|\\+|\sqrt{\Psi_{\mathcal{E}}^L(x_i)}-\sqrt{\Psi_{\mathcal{F}}^L(x_i)}|+|\sqrt{\Psi_{\mathcal{E}}^U(x_i)}-\sqrt{\Psi_{\mathcal{F}}^U(x_i)}|\\+|\sqrt{\Omega_{\mathcal{E}}^L(x_i)}-\sqrt{\Omega_{\mathcal{F}}^L(x_i)}|+|\sqrt{\Omega_{\mathcal{E}}^U(x_i)}-\sqrt{\Omega_{\mathcal{F}}^U(x_i)}|\end{array}\right)\tag{4.5}$$

$$\mathbb{D}_B^4(\mathcal{E},\mathcal{F}) = \left\{ \frac{1}{4n}\sum_{i=1}^{n} \left[\begin{array}{c} \left(\sqrt{\Upsilon_{\mathcal{E}}^L(x_i)} - \sqrt{\Upsilon_{\mathcal{F}}^L(x_i)}\right)^2 + \left(\sqrt{\Upsilon_{\mathcal{E}}^U(x_i)} - \sqrt{\Upsilon_{\mathcal{F}}^U(x_i)}\right)^2 \\ + \left(\sqrt{\Phi_{\mathcal{E}}^L(x_i)} - \sqrt{\Phi_{\mathcal{F}}^L(x_i)}\right)^2 + \left(\sqrt{\Phi_{\mathcal{E}}^U(x_i)} - \sqrt{\Phi_{\mathcal{F}}^U(x_i)}\right)^2 \\ + \left(\sqrt{\Psi_{\mathcal{E}}^L(x_i)} - \sqrt{\Psi_{\mathcal{F}}^L(x_i)}\right)^2 + \left(\sqrt{\Psi_{\mathcal{E}}^U(x_i)} - \sqrt{\Psi_{\mathcal{F}}^U(x_i)}\right)^2 \\ + \left(\sqrt{\Omega_{\mathcal{E}}^L(x_i)} - \sqrt{\Omega_{\mathcal{F}}^L(x_i)}\right)^2 + \left(\sqrt{\Omega_{\mathcal{E}}^U(x_i)} - \sqrt{\Omega_{\mathcal{F}}^U(x_i)}\right)^2 \end{array} \right] \right\}^{\frac{1}{2}} \tag{4.6}$$

$$\mathbb{D}_C^4(\mathcal{E},\mathcal{F}) = \frac{1}{2n}\sum_{i=1}^{n} \max \left(\begin{array}{c} |\sqrt{\Upsilon_{\mathcal{E}}^L(x_i)} - \sqrt{\Upsilon_{\mathcal{F}}^L(x_i)}|, |\sqrt{\Upsilon_{\mathcal{E}}^U(x_i)} - \sqrt{\Upsilon_{\mathcal{F}}^U(x_i)}| \\ |\sqrt{\Phi_{\mathcal{E}}^L(x_i)} - \sqrt{\Phi_{\mathcal{F}}^L(x_i)}|, |\sqrt{\Phi_{\mathcal{E}}^U(x_i)} - \sqrt{\Phi_{\mathcal{F}}^U(x_i)}| \\ |\sqrt{\Psi_{\mathcal{E}}^L(x_i)} - \sqrt{\Psi_{\mathcal{F}}^L(x_i)}|, |\sqrt{\Psi_{\mathcal{E}}^U(x_i)} - \sqrt{\Psi_{\mathcal{F}}^U(x_i)}| \\ |\sqrt{\Omega_{\mathcal{E}}^L(x_i)} - \sqrt{\Omega_{\mathcal{F}}^L(x_i)}|, |\sqrt{\Omega_{\mathcal{E}}^U(x_i)} - \sqrt{\Omega_{\mathcal{F}}^U(x_i)}| \end{array} \right) \tag{4.7}$$

$$\mathbb{D}_D^4(\mathcal{E},\mathcal{F}) = \frac{1}{2n}\sum_{i=1}^{n} \max \left(\begin{array}{c} \left(\sqrt{\Upsilon_{\mathcal{E}}^L(x_i)} - \sqrt{\Upsilon_{\mathcal{F}}^L(x_i)}\right)^2, \left(\sqrt{\Upsilon_{\mathcal{E}}^U(x_i)} - \sqrt{\Upsilon_{\mathcal{F}}^U(x_i)}\right)^2 \\ \left(\sqrt{\Phi_{\mathcal{E}}^L(x_i)} - \sqrt{\Phi_{\mathcal{F}}^L(x_i)}\right)^2, \left(\sqrt{\Phi_{\mathcal{E}}^U(x_i)} - \sqrt{\Phi_{\mathcal{F}}^U(x_i)}\right)^2 \\ \left(\sqrt{\Psi_{\mathcal{E}}^L(x_i)} - \sqrt{\Psi_{\mathcal{F}}^L(x_i)}\right)^2, \left(\sqrt{\Psi_{\mathcal{E}}^U(x_i)} - \sqrt{\Psi_{\mathcal{F}}^U(x_i)}\right)^2 \\ \left(\sqrt{\Omega_{\mathcal{E}}^L(x_i)} - \sqrt{\Omega_{\mathcal{F}}^L(x_i)}\right)^2, \left(\sqrt{\Omega_{\mathcal{E}}^U(x_i)} - \sqrt{\Omega_{\mathcal{F}}^U(x_i)}\right)^2 \end{array} \right). \tag{4.8}$$

Property 4. *The following properties are derived from the $\mathbb{D}^4(\mathcal{E},\mathcal{F})$ definition. Proof (1–4) is similar with $\mathbb{D}^3$.*

(1) $0 \le \mathbb{D}^4(\mathcal{E},\mathcal{F}) \le 1$.

(2) $\mathbb{D}^4(\mathcal{E},\mathcal{F}) = 0$ *if, and only if,* $\mathcal{E} = \mathcal{F}$.

(3) $\mathbb{D}^4(\mathcal{E},\mathcal{F}) = \mathbb{D}^4(\mathcal{F},\mathcal{E})$.

(4) If $\mathcal{E} \subseteq \mathcal{F} \subseteq \mathcal{G}$, *then* $\mathbb{D}^4(\mathcal{E},\mathcal{F}) \le \mathbb{D}^4(\mathcal{E},\mathcal{G})$ *and* $\mathbb{D}^4(\mathcal{F},\mathcal{G}) \le \mathbb{D}^4(\mathcal{E},\mathcal{G})$.

5. Numerical comparisons

5.1. Numerical examples of PFSs

In this section, we use three numerical examples to demonstrate that the proposed distance measures not only meet the required properties, but also exhibit superiority compared to existing distance measures.

Example 1. *Let three PFSs $\mathcal{E},\mathcal{F},\mathcal{G}$ in the UOD $X = \{x_1, x_2\}$,*

$$\begin{aligned} \mathcal{E} &= \{\langle x_1, 0.3, 0.2, 0.3\rangle, \langle x_2, 0.4, 0.3, 0.1\rangle\} \\ \mathcal{F} &= \{\langle x_1, 0.3, 0.2, 0.3\rangle, \langle x_2, 0.4, 0.3, 0.1\rangle\} \\ \mathcal{G} &= \{\langle x_1, 0.15, 0.25, 0.2\rangle, \langle x_2, 0.25, 0.35, 0.1\rangle\} \end{aligned}$$

we can arrive at the result:

$$\mathbb{D}_A^1(\mathcal{E},\mathcal{F}) = \mathbb{D}_A^1(\mathcal{F},\mathcal{E}) = 0.0000, \mathbb{D}_A^1(\mathcal{E},\mathcal{G}) = \mathbb{D}_A^1(\mathcal{G},\mathcal{E}) = 0.1225$$
$$\mathbb{D}_B^1(\mathcal{E},\mathcal{F}) = \mathbb{D}_B^1(\mathcal{F},\mathcal{E}) = 0.0000, \mathbb{D}_B^1(\mathcal{E},\mathcal{G}) = \mathbb{D}_B^1(\mathcal{G},\mathcal{E}) = 0.1205$$
$$\mathbb{D}_C^1(\mathcal{E},\mathcal{F}) = \mathbb{D}_C^1(\mathcal{F},\mathcal{E}) = 0.0000, \mathbb{D}_C^1(\mathcal{E},\mathcal{G}) = \mathbb{D}_C^1(\mathcal{G},\mathcal{E}) = 0.1464$$
$$\mathbb{D}_D^1(\mathcal{E},\mathcal{F}) = \mathbb{D}_D^1(\mathcal{F},\mathcal{E}) = 0.0000, \mathbb{D}_D^1(\mathcal{E},\mathcal{G}) = \mathbb{D}_D^1(\mathcal{G},\mathcal{E}) = 0.0216.$$

Similarly, we can calculate $\mathbb{D}_A^2, \mathbb{D}_B^2, \mathbb{D}_C^2, \mathbb{D}_D^2$. It is clear that the distance measure $\mathbb{D}_A^1$, $\mathbb{D}_A^2$, $\mathbb{D}_B^1$, $\mathbb{D}_B^2$, $\mathbb{D}_C^1$, $\mathbb{D}_C^2$, $\mathbb{D}_D^1$, $\mathbb{D}_D^2$ satisfies the (2) (3) property.

Example 2. *Given three PFSs $\mathcal{E},\mathcal{F},\mathcal{G}$ in the UOD $X = \{x_1, x_2\}$,*

$$\begin{aligned}\mathcal{E} &= \{\langle x_1, 0.2, 0.3, 0.4\rangle, \langle x_2, 0.1, 0.2, 0.4\rangle\}\\ \mathcal{F} &= \{\langle x_1, 0.3, 0.4, 0.3\rangle, \langle x_2, 0.25, 0.35, 0.2\rangle\}\\ \mathcal{G} &= \{\langle x_1, 0.4, 0.5, 0.1\rangle \langle x_2, 0.3, 0.4, 0.1\rangle\}.\end{aligned}$$

Clearly $\mathcal{E} \subseteq \mathcal{F} \subseteq \mathcal{G}$.

$$\mathbb{D}_A^1(\mathcal{E},\mathcal{F}) = 0.1958, \mathbb{D}_A^1(\mathcal{E},\mathcal{G}) = 0.3485, \mathbb{D}_A^1(\mathcal{F},\mathcal{G}) = 0.1526, \mathbb{D}_A^1(\mathcal{E},\mathcal{G}) = 0.3485$$
$$\mathbb{D}_B^1(\mathcal{E},\mathcal{F}) = 0.1684, \mathbb{D}_B^1(\mathcal{E},\mathcal{G}) = 0.2948, \mathbb{D}_B^1(\mathcal{F},\mathcal{G}) = 0.1479, \mathbb{D}_B^1(\mathcal{E},\mathcal{G}) = 0.2948$$
$$\mathbb{D}_C^1(\mathcal{E},\mathcal{F}) = 0.1429, \mathbb{D}_C^1(\mathcal{E},\mathcal{G}) = 0.3162, \mathbb{D}_C^1(\mathcal{F},\mathcal{G}) = 0.1812, \mathbb{D}_C^1(\mathcal{E},\mathcal{G}) = 0.3162$$
$$\mathbb{D}_D^1(\mathcal{E},\mathcal{F}) = 0.0222, \mathbb{D}_D^1(\mathcal{E},\mathcal{G}) = 0.1000, \mathbb{D}_D^1(\mathcal{F},\mathcal{G}) = 0.0354, \mathbb{D}_D^1(\mathcal{E},\mathcal{G}) = 0.1000.$$

By calculating, we can find

$$\mathbb{D}_A^1(\mathcal{E},\mathcal{F}) \leq \mathbb{D}_A^1(\mathcal{E},\mathcal{G}), \mathbb{D}_A^1(\mathcal{F},\mathcal{G}) \leq \mathbb{D}_A^1(\mathcal{E},\mathcal{G})$$
$$\mathbb{D}_B^1(\mathcal{E},\mathcal{F}) \leq \mathbb{D}_B^1(\mathcal{E},\mathcal{G}), \mathbb{D}_B^1(\mathcal{F},\mathcal{G}) \leq \mathbb{D}_B^1(\mathcal{E},\mathcal{G})$$
$$\mathbb{D}_C^1(\mathcal{E},\mathcal{F}) \leq \mathbb{D}_C^1(\mathcal{E},\mathcal{G}), \mathbb{D}_C^1(\mathcal{F},\mathcal{G}) \leq \mathbb{D}_C^1(\mathcal{E},\mathcal{G})$$
$$\mathbb{D}_D^1(\mathcal{E},\mathcal{F}) \leq \mathbb{D}_D^1(\mathcal{E},\mathcal{G}), \mathbb{D}_D^1(\mathcal{F},\mathcal{G}) \leq \mathbb{D}_D^1(\mathcal{E},\mathcal{G}).$$

Similarly, we can calculate $\mathbb{D}_A^2, \mathbb{D}_B^2, \mathbb{D}_C^2, \mathbb{D}_D^2$.
It is clear that the distance measure $\mathbb{D}_A^1, \mathbb{D}_A^2, \mathbb{D}_B^1, \mathbb{D}_B^2, \mathbb{D}_C^1, \mathbb{D}_C^2, \mathbb{D}_D^1, \mathbb{D}_D^2$ satisfies (4) property.

Example 3. *Given two PFSs $\mathcal{E}$ and $\mathcal{F}$ in UOD X, the specific numerical values are as illustrated in the following Table 3, and the results for different distance measurement methods applied to $\mathcal{E}$ and $\mathcal{F}$ are displayed in Table 4 as shown below.*

Table 3. Two PFSs in six cases under Example 3.

PFS	Case 1	Case 2	Case 3	Case 4	Case 5	Case6
$\mathcal{E}$	<x,0.4,0.4,0.2>	<x,0.1,0.4,0.5>	<x,0.3,0,0.7>	<x,0.3,0,0.7>	<x,0.1,0.2,0.6>	<x,0.1,0.3,0.3>
$\mathcal{F}$	<x,0.2,0.5,0.3>	<x,0.2,0.5,0.3>	<x,0.2,0,0.8>	<x,0.4,0,0.6>	<x,0.2,0.4,0.3>	<x,0.4,0.3,0.1>

Table 4. A comparison of distance measures of PFSs.

	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
$\mathbb{D}_{Du}^1$&$\mathbb{D}_{Du}^2$	0.2	0.2	0.1	0.1	0.3	0.3
$\mathbb{D}_{Du}^3$&$\mathbb{D}_{Du}^4$	0.1732	0.1732	0.1	0.1	0.07	0.07
$\mathbb{D}_{DT}^1$	0.1333	0.1333	0.0667	0.0667	0.3	0.167
$\mathbb{D}_{DT}^2$	0.2449	0.2449	0.1414	0.1414	0.3742	0.3606
$\mathbb{D}_{DT}^3$&$\mathbb{D}_{DT}^4$	0.2	0.2	0.1	0.1	0.3	0.3
$\mathbb{D}_{SM}^1$	0.1	0.1	0.05	0.05	0.15	0.15
$\mathbb{D}_{SM}^2$	0.1225	0.1225	0.0707	0.0707	0.1871	0.1871
$\mathbb{D}_{SM}^3$	0.05	0.05	0.025	0.025	0.1369	0.1369
$\mathbb{D}_{SM}^4$	0.1	0.1	0.05	0.05	0.2739	0.2739
$\mathbb{D}_{A}^1$	0.1802	0.16	0.0791	0.0735	0.2716	0.2739
$\mathbb{D}_{A}^2$	0.1802	0.16	0.0791	0.0735	0.2716	0.3242
$\mathbb{D}_{B}^1$	0.1581	0.1552	0.1158	0.0745	0.2268	0.2771
$\mathbb{D}_{B}^2$	0.1581	0.1552	0.1158	0.0745	0.2268	0.2861
$\mathbb{D}_{C}^1$	0.1853	0.1594	0.1005	0.0848	0.2269	0.3163
$\mathbb{D}_{C}^2$	0.1853	0.1594	0.1005	0.0848	0.1135	0.1582
$\mathbb{D}_{D}^1$	0.0343	0.0254	0.0101	0.0072	0.0515	0.1000
$\mathbb{D}_{D}^2$	0.0343	0.0254	0.0101	0.0072	0.0515	0.1000

The existing distance measure $\mathbb{D}_{Du}^1$,$\mathbb{D}_{Du}^2$, $\mathbb{D}_{Du}^3$, $\mathbb{D}_{Du}^4$, $\mathbb{D}_{DT}^1$, $\mathbb{D}_{DT}^2$, $\mathbb{D}_{DT}^3$, $\mathbb{D}_{DT}^4$, $\mathbb{D}_{SM}^1$, $\mathbb{D}_{SM}^2$, $\mathbb{D}_{SM}^3$, $\mathbb{D}_{SM}^4$ *produced the same results between Cases 1 and 2. In the context of highly similar cases between Cases 3 and 4,* $\mathbb{D}_{Du}^1$,$\mathbb{D}_{Du}^2$, $\mathbb{D}_{Du}^3$, $\mathbb{D}_{Du}^4$, $\mathbb{D}_{DT}^1$, $\mathbb{D}_{DT}^2$, $\mathbb{D}_{DT}^3$, $\mathbb{D}_{DT}^4$, $\mathbb{D}_{SM}^1$, $\mathbb{D}_{SM}^2$, $\mathbb{D}_{SM}^3$ *produce consistent results, failing to effectively distinguish between Cases 3 and 4. Similarly, when calculating for Cases 5 and 6,* $\mathbb{D}_{Du}^1$,$\mathbb{D}_{Du}^2$, $\mathbb{D}_{Du}^3$, $\mathbb{D}_{Du}^4$, $\mathbb{D}_{DT}^1$, $\mathbb{D}_{DT}^3$, $\mathbb{D}_{DT}^4$, $\mathbb{D}_{SM}^1$, $\mathbb{D}_{SM}^2$, $\mathbb{D}_{SM}^3$ *also produce identical results. However, the proposed distance measures demonstrated strong performance. They excelled in calculating distances when dealing with counterintuitive or subtly different dates in Cases 1–6, proving their superiority.*

5.2. Numerical examples of IvPFSs

In this section, we showcase three illustrative examples to highlight how the proposed distance measures adhere to the properties and outperform the existing similarity measures.

Example 4. *Assume three IvPFSs* $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ *as follows:*

$$\begin{aligned}\mathcal{E} &= \{\langle[0.2, 0.3], [0.4, 0.5], [0.1, 0.2]\rangle\}\\ \mathcal{F} &= \{\langle[0.2, 0.3], [0.4, 0.5], [0.1, 0.2]\rangle\}\\ \mathcal{G} &= \{\langle[0.6, 0.7], [0.1, 0.2], [0.1, 0.2]\rangle\}.\end{aligned}$$

We can arrive at the result:

$$\mathbb{D}_A^3(\mathcal{E},\mathcal{F}) = \mathbb{D}_A^3(\mathcal{F},\mathcal{E}) = 0.0000, \mathbb{D}_A^3(\mathcal{E},\mathcal{G}) = \mathbb{D}_A^3(\mathcal{G},\mathcal{E}) = 0.2981$$

$$\mathbb{D}_B^3(\mathcal{E},\mathcal{F}) = \mathbb{D}_B^3(\mathcal{F},\mathcal{E}) = 0.0000, \mathbb{D}_B^3(\mathcal{E},\mathcal{G}) = \mathbb{D}_B^3(\mathcal{G},\mathcal{E}) = 0.2993$$

$$\mathbb{D}_C^3(\mathcal{E},\mathcal{F}) = \mathbb{D}_C^3(\mathcal{F},\mathcal{E}) = 0.0000, \mathbb{D}_C^3(\mathcal{E},\mathcal{G}) = \mathbb{D}_C^3(\mathcal{G},\mathcal{E}) = 0.1637$$

$$\mathbb{D}_D^3(\mathcal{E},\mathcal{F}) = \mathbb{D}_D^3(\mathcal{F},\mathcal{E}) = 0.0000, \mathbb{D}_D^3(\mathcal{E},\mathcal{G}) = \mathbb{D}_D^3(\mathcal{G},\mathcal{E}) = 0.0536.$$

In the same vein, we can work out $\mathbb{D}_A^4, \mathbb{D}_B^4, \mathbb{D}_C^4, \mathbb{D}_D^4$.

It's evident that the distance measure $\mathbb{D}_A^3, \mathbb{D}_A^4, \mathbb{D}_B^3, \mathbb{D}_B^4, \mathbb{D}_C^3, \mathbb{D}_C^4, \mathbb{D}_D^3, \mathbb{D}_D^4$ *adheres to (2) and (3) property.*

Example 5. *Consider the following three IvPFSs* $\mathcal{E}, \mathcal{F}$ *and* $\mathcal{G}$*:*

$$\begin{aligned} \mathcal{E} &= \{\langle [0.10, 0.20], [0.10, 0.20], [0.40, 0.50] \rangle\} \\ \mathcal{F} &= \{\langle [0.15, 0.25], [0.20, 0.30], [0.30, 0.40] \rangle\} \\ \mathcal{G} &= \{\langle [0.20, 0.30], [0.30, 0.40], [0.20, 0.30] \rangle\}. \end{aligned}$$

Clearly, $\mathcal{E} \subseteq \mathcal{F} \subseteq \mathcal{G}$.

$$\mathbb{D}_A^3(\mathcal{E},\mathcal{F}) = 0.1287, \mathbb{D}_A^3(\mathcal{E},\mathcal{G}) = 0.2482, \mathbb{D}_A^3(\mathcal{F},\mathcal{G}) = 0.1195, \mathbb{D}_A^3(\mathcal{E},\mathcal{G}) = 0.2482$$

$$\mathbb{D}_B^3(\mathcal{E},\mathcal{F}) = 0.1094, \mathbb{D}_B^3(\mathcal{E},\mathcal{G}) = 0.2091, \mathbb{D}_B^3(\mathcal{F},\mathcal{G}) = 0.1005, \mathbb{D}_B^3(\mathcal{E},\mathcal{G}) = 0.2091$$

$$\mathbb{D}_C^3(\mathcal{E},\mathcal{F}) = 0.0655, \mathbb{D}_C^3(\mathcal{E},\mathcal{G}) = 0.1157, \mathbb{D}_C^3(\mathcal{F},\mathcal{G}) = 0.0503, \mathbb{D}_C^3(\mathcal{E},\mathcal{G}) = 0.1157$$

$$\mathbb{D}_D^3(\mathcal{E},\mathcal{F}) = 0.0086, \mathbb{D}_D^3(\mathcal{E},\mathcal{G}) = 0.0268, \mathbb{D}_D^3(\mathcal{F},\mathcal{G}) = 0.0051, \mathbb{D}_D^3(\mathcal{E},\mathcal{G}) = 0.0268.$$

By calculating, we can find

$$\mathbb{D}_A^3(\mathcal{E},\mathcal{F}) \leq \mathbb{D}_A^3(\mathcal{E},\mathcal{G}), \mathbb{D}_A^3(\mathcal{F},\mathcal{G}) \leq \mathbb{D}_A^3(\mathcal{E},\mathcal{G})$$

$$\mathbb{D}_B^3(\mathcal{E},\mathcal{F}) \leq \mathbb{D}_B^3(\mathcal{E},\mathcal{G}), \mathbb{D}_B^3(\mathcal{F},\mathcal{G}) \leq \mathbb{D}_B^3(\mathcal{E},\mathcal{G})$$

$$\mathbb{D}_C^3(\mathcal{E},\mathcal{F}) \leq \mathbb{D}_C^3(\mathcal{E},\mathcal{G}), \mathbb{D}_C^3(\mathcal{F},\mathcal{G}) \leq \mathbb{D}_C^3(\mathcal{E},\mathcal{G})$$

$$\mathbb{D}_D^3(\mathcal{E},\mathcal{F}) \leq \mathbb{D}_D^3(\mathcal{E},\mathcal{G}), \mathbb{D}_D^3(\mathcal{F},\mathcal{G}) \leq \mathbb{D}_D^3(\mathcal{E},\mathcal{G}).$$

Similarly, we can calculate $\mathbb{D}_A^4, \mathbb{D}_B^4, \mathbb{D}_C^4, \mathbb{D}_D^4$.

The distance measures denoted as $\mathbb{D}_A^3, \mathbb{D}_A^4,$ $\mathbb{D}_B^3, \mathbb{D}_B^4, \mathbb{D}_C^3, \mathbb{D}_C^4, \mathbb{D}_D^3$ *and* $\mathbb{D}_D^4$ *unequivocally meet the criteria stipulated by the (4) properties.*

Example 6. *Given the IvPFSs* $\mathcal{E}_i$ *and* $\mathcal{F}_i$ *under Case i(i=1,2,3,4), which are shown in Table 5, the results obtained for the four cases are presented in Table 6.*

Table 5. Two IvPFSs $\mathcal{E}_i$ and $\mathcal{F}_i$ under different cases in Example 6.

IvPFSs	Case 1	Case 2
$\mathcal{E}_i$	<(0.3,0.3)(0.3,0.3)(0.3,0.3)>	<(0.2,0.2)(0.2,0.2)(0.2,0.2)>
$\mathcal{F}_i$	<(0.2,0.2)(0.2,0.2)(0.2,0.2)>	<(0.1,0.1)(0.1,0.1)(0.1,0.1)>
IvPFSs	Case3	Case4
$\mathcal{E}_i$	<(0.3,0.8)(0.1,0.1)(0.1,0.1)>	<(0.3,0.8)(0.1,0.1)(0.1,0.1)>
$\mathcal{F}_i$	<(0.1,0.2)(0.1,0.1)(0.1,0.1)>	<(0.1,0.2)(0.1,0.2)(0.1,0.1)>

Table 6. A comparison of different measures of IvPFSs.

	Case 1	Case 2	Case 3	Case 4
$\mathbb{D}_{Cs}^{1}$	0.0122	0.0122	0.1090	0.1090
$\mathbb{D}_{Cs}^{2}$	0.1090	0.1090	0.2929	0.2929
$\mathbb{D}_{Cs}^{3}$	0.1090	0.4122	0.1090	0.1090
$\mathbb{D}_{Cs}^{4}$	0.1090	0.2396	1.0000	1.0000
$\mathbb{D}_{Ct}^{1}$	0.1459	0.1459	0.3872	0.3872
$\mathbb{D}_{Ct}^{2}$	0.3872	0.6751	0.3872	0.3872
$\mathbb{D}_{A}^{3}$	0.0754	0.1965	0.1229	0.1146
$\mathbb{D}_{A}^{4}$	0.1545	0.2986	0.2039	0.2013
$\mathbb{D}_{B}^{3}$	0.1231	0.0258	0.1741	0.1622
$\mathbb{D}_{B}^{4}$	0.2553	0.0467	0.2105	0.2069
$\mathbb{D}_{C}^{3}$	0.0500	0.0655	0.1300	0.1135
$\mathbb{D}_{C}^{4}$	0.1000	0.1021	0.1300	0.1135
$\mathbb{D}_{D}^{3}$	0.0050	0.0086	0.0008	0.0006
$\mathbb{D}_{D}^{4}$	0.0050	0.0086	0.0008	0.0006

Based on the results presented in Table 6, it is apparent that the eight proposed distance measures for IvPFSs can better distinguish different cases, especially in handling counterintuitive datas. In the context of highly similar cases between Case 1 and Case 2, $\mathbb{S}_{Cs}^{1}$, $\mathbb{S}_{Cs}^{2}$ *and* $\mathbb{S}_{Ct}^{1}$ *produce consistent results, failing to effectively distinguish between Case 1 and Case 2. Similarly, when calculating for Case 3 and Case 4,* $\mathbb{S}_{Cs}^{1}$, $\mathbb{S}_{Cs}^{2}$, $\mathbb{S}_{Cs}^{3}$, $\mathbb{S}_{Cs}^{4}$, $\mathbb{S}_{Ct}^{1}$, *and* $\mathbb{S}_{Ct}^{2}$ *also yield the same results, so the proposed distance measures are demonstrated to be superior.*

6. Applications for PFSs

In this section, we will introduce two applications of PFSs, including pattern recognition and medical diagnosis.

Application 1. *We give four sets in the format of PFS and compute the distances between* $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$, *and* $\mathcal{F}$. *Each set has four elements, and Table 7 presents a comparison between the classification outcomes generated by the proposed distance measures and those produced by existing distance measures.*

$$\mathcal{E}_1=\{(a_1,0.3,0.0,0.4),\quad (a_2,0.7,0.0,0.1),\quad (a_3,0.2,0.0,0.6),\quad (a_4,0.7,0.0,0.1)\}$$
$$\mathcal{E}_2=\{(a_1,0.5,0.0,0.2),\quad (a_2,0.6,0.0,0.1),\quad (a_3,0.2,0.0,0.7),\quad (a_4,0.7,0.0,0.3)\}$$
$$\mathcal{E}_3=\{(a_1,0.5,0.0,0.3),\quad (a_2,0.7,0.0,0.0),\quad (a_3,0.4,0.0,0.5),\quad (a_4,0.7,0.0,0.3)\}$$
$$\mathcal{F}=\{(b_1,0.4,0.0,0.3),\quad (b_2,0.7,0.0,0.1),\quad (b_3,0.3,0.0,0.6),\quad (b_4,0.7,0.0,0.3)\}.$$

Table 7. Classification results on Application 1.

	$(\mathcal{E}_1,\mathcal{F})$	$(\mathcal{E}_2,\mathcal{F})$	$(\mathcal{E}_3,\mathcal{F})$	Result
$\mathbb{D}^1_{Du}$	0.4000	0.3000	0.3000	NaN
$\mathbb{D}^2_{Du}$	0.1000	0.0750	0.0750	NaN
$\mathbb{D}^3_{Du}$	0.0600	0.0300	0.0300	NaN
$\mathbb{D}^4_{Du}$	0.0150	0.0075	0.0075	NaN
$\mathbb{D}^1_{DT}$	0.0417	0.0417	0.0333	NaN
$\mathbb{D}^2_{DT}$	0.0661	0.0559	0.0500	$\mathcal{E}_3$
$\mathbb{D}^3_{DT}$	0.1000	0.0750	0.0750	NaN
$\mathbb{D}^4_{DT}$	0.0612	0.0433	0.0433	NaN
$\mathbb{D}^1_{SM}$	0.0500	0.0375	0.0375	NaN
$\mathbb{D}^2_{SM}$	0.0866	0.0612	0.0612	NaN
$\mathbb{D}^3_{SM}$	0.0675	0.0593	0.0593	NaN
$\mathbb{D}^4_{SM}$	0.0612	0.0433	0.0433	NaN
$\mathbb{D}^1_{A}$	0.0627	0.0500	0.0679	$\mathcal{E}_2$
$\mathbb{D}^2_{A}$	0.1350	0.0625	0.0930	$\mathcal{E}_2$
$\mathbb{D}^1_{B}$	0.0989	0.0647	0.1211	$\mathcal{E}_2$
$\mathbb{D}^2_{B}$	0.1921	0.0738	0.1311	$\mathcal{E}_2$
$\mathbb{D}^1_{C}$	0.1042	0.0658	0.1189	$\mathcal{E}_2$
$\mathbb{D}^2_{C}$	0.1657	0.0754	0.1254	$\mathcal{E}_2$
$\mathbb{D}^1_{D}$	0.0177	0.0060	0.0282	$\mathcal{E}_2$
$\mathbb{D}^2_{D}$	0.0561	0.0076	0.0293	$\mathcal{E}_2$

All the proposed distance measures get the same classification results that the test sample belongs to $\mathcal{E}_2$. However, $\mathbb{D}^1_{Du}$, $\mathbb{D}^2_{Du}$, $\mathbb{D}^3_{Du}$, $\mathbb{D}^4_{Du}$, $\mathbb{D}^1_{DT}$, $\mathbb{D}^3_{DT}$, $\mathbb{D}^4_{DT}$, $\mathbb{D}^1_{SM}$, $\mathbb{D}^2_{SM}$, $\mathbb{D}^3_{SM}$, $\mathbb{D}^4_{SM}$ cannot distinguish the sample D as the results are $(\mathcal{E}_1,\mathcal{F}) = (\mathcal{E}_2,\mathcal{F})$ or $(\mathcal{E}_2,\mathcal{F}) = (\mathcal{E}_3,\mathcal{F})$.

Hence, the proposed distance measures successfully address pattern recognition problems that existing measures fail to resolve, demonstrating their superior performance.

Application 2. *[39] Consider four patients $\mathcal{E},\mathcal{F},\mathcal{G},\mathcal{H}$ and sets a set of patients as $P=\{\mathcal{E},\mathcal{F},\mathcal{G},\mathcal{H}\}$. The set of diagnostic symptoms is $S=\{Temperature, Headache, Stomachpain, Cough, Chestpain\}$. Table 8 outlines the symptoms associated with each patient. Table 9 presents the symptoms related to the various diseases. Each element of the tables are given as PFSs.*

Table 8. Diagnostic criteria for various symptoms of different diseases.

	Temperature	Headache	Stomach pain	Cough	Chest pain
VF	<0.10,0.00,0.00>	<0.20,0.20,0.50>	<0.10,0.25,0.60>	<0.30,0.40,0.20>	<0.20,0.15,0.50>
M	<0.70,0.00,0.00>	<0.20,0.40,0.35>	<0.00,0.40,0.50>	<0.70,0.10,0.00>	<0.10,0.30,0.50>
T	<0.30,0.40,0.30>	<0.60,0.20,0.10>	<0.20,0.30,0.40>	<0.20,0.35,0.30>	<0.10,0.20,0.60>
SP	<0.10,0.30,0.50>	<0.20,0.40,0.30>	<0.80,0.00,0.00>	<0.20,0.40,0.30>	<0.20,0.35,0.30>
CP	<0.10,0.30,0.50>	<0.00,0.50,0.35>	<0.20,0.30,0.50>	<0.20,0.35,0.40>	<0.80,0.00,0.10>

Note: *VF:Viral Fever, M: Malaria, T:Typhoid, SP: Stomach Problem, CP: Chest Problem.*

Table 9. Indicators of various symptoms in four patients.

	Temperature	Headache	Stomach pain	Cough	Chest pain
$\mathcal{E}$	<0.80,0.00,0.10>	<0.60,0.30,0.10>	<0.20,0.40,0.40>	<0.50,0.15,0.10>	<0.10,0.40,0.40>
$\mathcal{F}$	<0.00,0.50,0.40>	<0.40,0.25,0.30>	<0.60,0.20,0.10>	<0.10,0.30,0.60>	<0.10,0.35,0.40>
$\mathcal{G}$	<0.80,0.00,0.10>	<0.80,0.00,0.10>	<0.50,0.40,0.00>	<0.20,0.30,0.40>	<0.00,0.40,0.40>
$\mathcal{H}$	<0.70,0.20,0.10>	<0.40,0.25,0.25>	<0.00,0.40,0.50>	<0.70,0.10,0.15>	<0.10,0.30,0.05>

Based on the governing principle of minimum distance measures, a smaller distance measure signifies a more accurate diagnosis. In Table 10, it is discerned that patient $\mathcal{E}$ diagnoses with Malaria, patient $\mathcal{F}$ faces a stomach problem, patient $\mathcal{G}$ is diagnosed with Typhoid and patient $\mathcal{H}$ suffers from Malaria. In Table 11 and Figure 1, a comparative analysis with existing measures is conducted. It becomes apparent that the distance measure $\mathbb{D}^1_{Du}$ is unable to accurately diagnose patients $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{G}$. Furthermore, $\mathbb{D}^3_{Du}$ encounters limitations in calculating some distances, thereby resulting in an outcome that goes against the desired property. Besides, when analyzing other existing measures, it is observed that the diagnostic outcomes generated by the proposed distance measures are in harmony with the results, demonstrating satisfactory accuracy and reliability. This alignment emphasizes the potential effectiveness and appropriateness of the proposed distance measures in diagnosing the ailments above, thereby contributing to a more precise and trustworthy diagnostic procedure.

Table 10. Disease diagnosis of proposed measures with four patients.

	Patient	VF	M	T	SP	CP	Result
$\mathbb{D}^1_A$	$\mathcal{E}$	0.3105	0.2192	0.2486	0.4933	0.5260	M
	$\mathcal{F}$	0.4076	0.5534	0.2566	0.2410	0.4056	SP
	$\mathcal{G}$	0.4615	0.4872	0.3527	0.4390	0.6332	T
	$\mathcal{H}$	0.3202	0.1560	0.2684	0.4812	0.4902	M
$\mathbb{D}^2_A$	$\mathcal{E}$	0.4420	0.3023	0.3435	0.5839	0.5811	M
	$\mathcal{F}$	0.5210	0.6283	0.3433	0.2949	0.4830	SP
	$\mathcal{G}$	0.5400	0.5458	0.4045	0.4581	0.6943	T
	$\mathcal{H}$	0.3959	0.1977	0.3295	0.5237	0.5420	M
$\mathbb{D}^1_B$	$\mathcal{E}$	0.3074	0.2492	0.2912	0.4692	0.5084	M
	$\mathcal{F}$	0.4145	0.5808	0.2680	0.2444	0.4086	SP
	$\mathcal{G}$	0.4555	0.4997	0.3777	0.4581	0.6356	T
	$\mathcal{H}$	0.3167	0.2259	0.2662	0.5010	0.4656	M
$\mathbb{D}^2_B$	$\mathcal{E}$	0.3879	0.2877	0.3388	0.5012	0.5255	M
	$\mathcal{F}$	0.4728	0.6026	0.3133	0.2689	0.4303	SP
	$\mathcal{G}$	0.4987	0.5093	0.3935	0.4604	0.6458	T
	$\mathcal{H}$	0.3573	0.2406	0.3059	0.5057	0.4784	M
$\mathbb{D}^1_C$	$\mathcal{E}$	0.3203	0.2984	0.2661	0.3993	0.4795	T
	$\mathcal{F}$	0.3765	0.5291	0.2956	0.2613	0.4147	SP
	$\mathcal{G}$	0.4865	0.5209	0.4226	0.4750	0.6236	T
	$\mathcal{H}$	0.3041	0.2040	0.2820	0.4298	0.5135	M
$\mathbb{D}^2_C$	$\mathcal{E}$	0.3475	0.2984	0.3555	0.3993	0.4795	M
	$\mathcal{F}$	0.3765	0.5291	0.3277	0.2792	0.4310	SP
	$\mathcal{G}$	0.4973	0.5209	0.4226	0.4750	0.6334	T
	$\mathcal{H}$	0.3041	0.2040	0.3136	0.4298	0.5135	M
$\mathbb{D}^1_D$	$\mathcal{E}$	0.1255	0.1029	0.1118	0.1932	0.2897	M
	$\mathcal{F}$	0.1787	0.3880	0.1066	0.0806	0.2046	SP
	$\mathcal{G}$	0.2737	0.3000	0.2214	0.2683	0.4872	T
	$\mathcal{H}$	0.1079	0.0769	0.0958	0.2564	0.2714	M
$\mathbb{D}^2_D$	$\mathcal{E}$	0.1446	0.1029	0.1483	0.1932	0.2897	M
	$\mathcal{F}$	0.1787	0.3880	0.1263	0.0903	0.2106	SP
	$\mathcal{G}$	0.2869	0.3000	0.2214	0.2683	0.4886	T
	$\mathcal{H}$	0.1079	0.0769	0.1191	0.2564	0.2714	M

Table 11. Final diagnostics for existing distance measures.

	Patient	VF	M	T	SP	CP	Result
$\mathbb{D}^1_{Du}$	$\mathcal{E}$	\	\	\	\	\	Cannot be diagnosed
	$\mathcal{F}$	\	\	\	\	\	Cannot be diagnosed
	$\mathcal{G}$	\	\	\	\	\	Cannot be diagnosed
	$\mathcal{H}$	\	0.6000	\	\	\	M
$\mathbb{D}^2_{Du}$	$\mathcal{E}$	0.4200	0.2100	0.2800	0.5000	0.5200	M
	$\mathcal{F}$	0.4600	0.5200	0.2900	0.2200	0.4000	SP
	$\mathcal{G}$	0.5000	0.4300	0.3200	0.4000	0.5700	T
	$\mathcal{H}$	0.3600	0.1200	0.3000	0.4800	0.4800	M
$\mathbb{D}^3_{Du}$	$\mathcal{E}$	0.8775	0.2000	0.4100	0.9850	\	M
	$\mathcal{F}$	0.9650	\	0.3275	0.1825	0.6975	SP
	$\mathcal{G}$	\	0.8225	0.4575	0.8175	\	T
	$\mathcal{H}$	0.6700	0.0900	0.3850	\	0.9750	M
$\mathbb{D}^4_{Du}$	$\mathcal{E}$	0.1755	0.0400	0.0820	0.1970	0.2285	M
	$\mathcal{F}$	0.1930	0.2380	0.0655	0.0365	0.1395	SP
	$\mathcal{G}$	0.2420	0.1645	0.0915	0.1635	0.2960	T
	$\mathcal{H}$	0.1340	0.0180	0.0770	0.2085	0.1950	M
$\mathbb{D}^1_{DT}$	$\mathcal{E}$	0.2100	0.1133	0.1667	0.3067	0.3300	M
	$\mathcal{F}$	0.2400	0.3167	0.1700	0.1300	0.2467	SP
	$\mathcal{G}$	0.2733	0.2567	0.1967	0.2567	0.3600	T
	$\mathcal{H}$	0.1833	0.0600	0.1800	0.3000	0.3033	M
$\mathbb{D}^2_{DT}$	$\mathcal{E}$	0.2095	0.174	0.1778	0.2768	0.3002	M
	$\mathcal{F}$	0.2258	0.3032	0.1572	0.1170	0.2340	SP
	$\mathcal{G}$	0.2665	0.2516	0.1889	0.2548	0.3426	T
	$\mathcal{H}$	0.1836	0.0762	0.1697	0.2871	0.2777	M
$\mathbb{D}^3_{DT}$	$\mathcal{E}$	0.3600	0.1800	0.2600	0.4400	0.5000	M
	$\mathcal{F}$	0.3600	0.4400	0.2800	0.2000	0.3800	SP
	$\mathcal{G}$	0.4700	0.3600	0.2800	0.4000	0.5700	T
	$\mathcal{H}$	0.3100	0.1100	0.2800	0.4600	0.4800	M
$\mathbb{D}^4_{DT}$	$\mathcal{E}$	0.1806	0.0959	0.1371	0.2173	0.2458	M
	$\mathcal{F}$	0.1720	0.2245	0.1296	0.0938	0.1887	SP
	$\mathcal{G}$	0.2291	0.1876	0.1414	0.2059	0.2844	T
	$\mathcal{H}$	0.1584	0.0640	0.1414	0.2307	0.2280	M
$\mathbb{D}^1_{SM}$	$\mathcal{E}$	0.2100	0.1050	0.1400	0.2500	0.2600	M
	$\mathcal{F}$	0.2300	0.2600	0.1450	0.1100	0.2000	SP
	$\mathcal{G}$	0.2500	0.2150	0.1600	0.2000	0.2850	T
	$\mathcal{H}$	0.1800	0.0600	0.1500	0.2400	0.2400	M
$\mathbb{D}^2_{SM}$	$\mathcal{E}$	0.2962	0.1414	0.2025	0.3138	0.3380	M
	$\mathcal{F}$	0.3106	0.3450	0.1810	0.1351	0.2641	SP
	$\mathcal{G}$	0.3479	0.2868	0.2139	0.2859	0.3847	T
	$\mathcal{H}$	0.2588	0.9487	0.1962	0.3229	0.3123	M
$\mathbb{D}^3_{SM}$	$\mathcal{E}$	0.1487	0.1080	0.1210	0.1596	0.1698	M
	$\mathcal{F}$	0.1564	0.1575	0.1311	0.1103	0.1498	SP
	$\mathcal{G}$	0.1695	0.1476	0.1275	0.1504	0.1778	T
	$\mathcal{H}$	0.1372	0.0641	0.1275	0.1635	0.1699	M
$\mathbb{D}^4_{SM}$	$\mathcal{E}$	0.2197	0.1140	0.1533	0.2429	0.2748	M
	$\mathcal{F}$	0.2377	0.2510	0.1449	0.1049	0.2110	SP
	$\mathcal{G}$	0.2704	0.2133	0.1581	0.2302	0.3180	T
	$\mathcal{H}$	0.1946	0.0716	0.1581	0.2579	0.2550	M

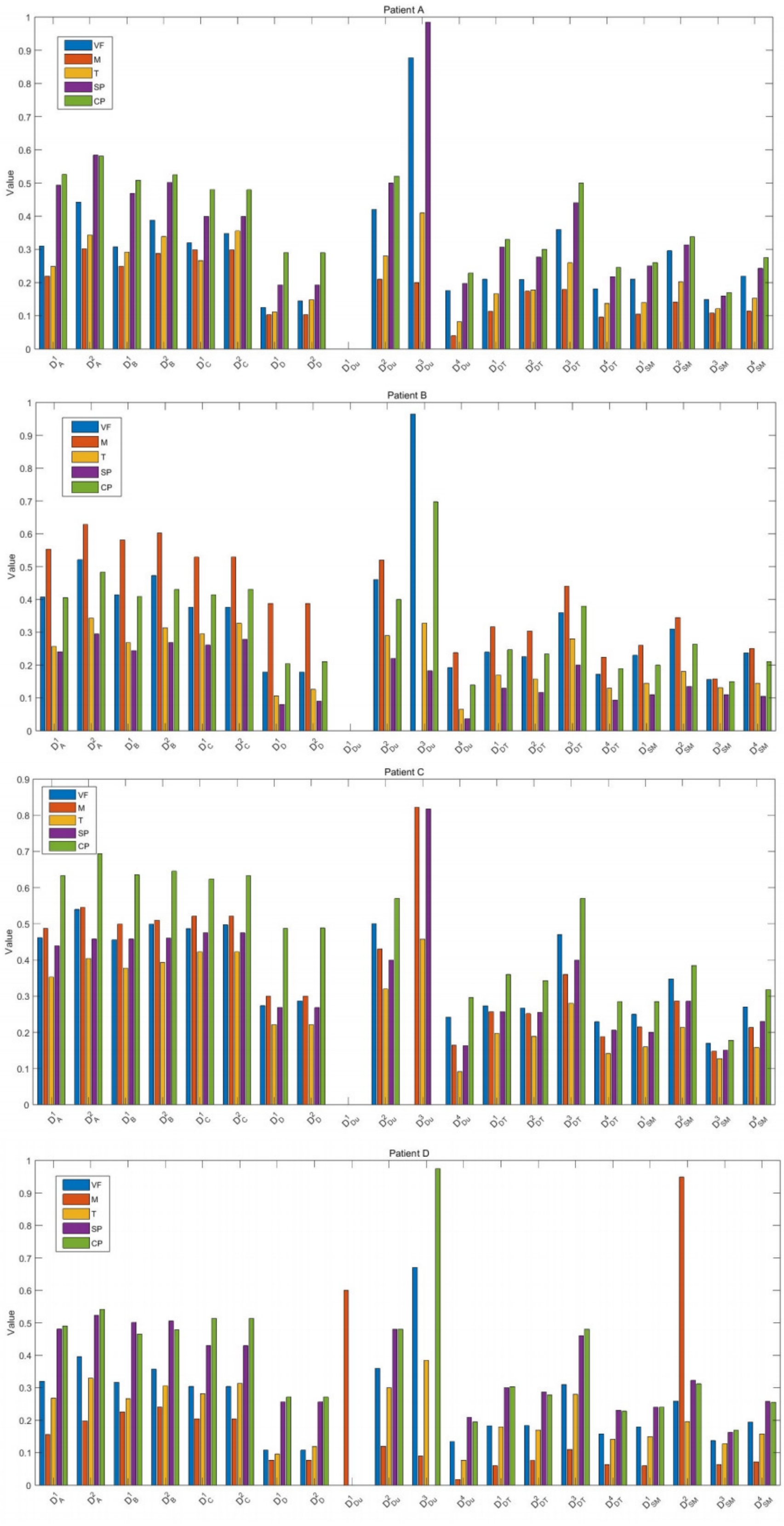

Figure 1. Diagnosis result comparison for patients across different distance measures.

7. Applications for IvPFSs

In this section, we explore medical dianoses related to IvPFSs. Through a detailed analysis, we aim to prove the measures proposed in this paper have strong robustness.

Application 3. *Let us assume we have three patients: P_1, P_2, P_3. The patient set can be denoted as $P = \{P_1, P_2, P_3\}$. The symptom set can be articulated as $S = \{S_1.S_2, S_3, S_4, S_5\}$, while the diagnostic set is denoted by $D = \{D_1, D_2, D_3, D_4\}$. The symptoms associated with the patients are outlined in Table 12, while the symptoms linked to the diseases are detailed in Table 13. Each entry in these tables is presented as IvPFSs.We perform a reasoned diagnosis for each patient using the proposed distance measures.*

Table 12. Symptoms characteristic for the patients.

	S_1	S_2	S_3	S_4	S_5
P_1	$\begin{pmatrix} [0.3, 0.3], \\ [0.3, 0.3], \\ [0.3, 0.3] \end{pmatrix}$	$\begin{pmatrix} [0.3, 0.3], \\ [0.3, 0.3], \\ [0.3, 0.3] \end{pmatrix}$	$\begin{pmatrix} [0.4, 0.4], \\ [0.1, 0.1], \\ [0.3, 0.3] \end{pmatrix}$	$\begin{pmatrix} [0.5, 0.5], \\ [0.1, 0.1], \\ [0.4, 0.4] \end{pmatrix}$	$\begin{pmatrix} [0.3, 0.3], \\ [0.2, 0.2], \\ [0.3, 0.3] \end{pmatrix}$
P_2	$\begin{pmatrix} [0.0, 0.0], \\ [0.1, 0.1], \\ [0.8, 0.8] \end{pmatrix}$	$\begin{pmatrix} [0.4, 0.4], \\ [0.1, 0.1], \\ [0.4, 0.4] \end{pmatrix}$	$\begin{pmatrix} [0.6, 0.6], \\ [0.3, 0.3], \\ [0.1, 0.1] \end{pmatrix}$	$\begin{pmatrix} [0.1, 0.1], \\ [0.1, 0.1], \\ [0.7, 0.7] \end{pmatrix}$	$\begin{pmatrix} [0.1, 0.1], \\ [0.0, 0.0], \\ [0.8, 0.8] \end{pmatrix}$
P_3	$\begin{pmatrix} [0.3, 0.3], \\ [0.3, 0.3], \\ [0.3, 0.3] \end{pmatrix}$	$\begin{pmatrix} [0.7, 0.7], \\ [0.2, 0.2], \\ [0.1, 0.1] \end{pmatrix}$	$\begin{pmatrix} [0.0, 0.0], \\ [0.3, 0.3], \\ [0.7, 0.7] \end{pmatrix}$	$\begin{pmatrix} [0.2, 0.2], \\ [0.0, 0.0], \\ [0.7, 0.7] \end{pmatrix}$	$\begin{pmatrix} [0.1, 0.1], \\ [0.0, 0.0], \\ [0.9, 0.9] \end{pmatrix}$

Table 13. Symptoms characteristic for the diagnoses.

	S_1	S_2	S_3	S_4	S_5
D_1	$\begin{pmatrix} [0.5, 0.5], \\ [0.1, 0.1], \\ [0.1, 0.1] \end{pmatrix}$	$\begin{pmatrix} [0.3, 0.3], \\ [0.1, 0.1], \\ [0.3, 0.3] \end{pmatrix}$	$\begin{pmatrix} [0.3, 0.3], \\ [0.1, 0.1], \\ [0.4, 0.4] \end{pmatrix}$	$\begin{pmatrix} [0.4, 0.4], \\ [0.2, 0.2], \\ [0.3, 0.3] \end{pmatrix}$	$\begin{pmatrix} [0.1, 0.1], \\ [0.1, 0.1], \\ [0.5, 0.5] \end{pmatrix}$
D_2	$\begin{pmatrix} [0.4, 0.4], \\ [0.3, 0.3], \\ [0.2, 0.2] \end{pmatrix}$	$\begin{pmatrix} [0.3, 0.3], \\ [0.2, 0.2], \\ [0.5, 0.5] \end{pmatrix}$	$\begin{pmatrix} [0.4, 0.4], \\ [0.1, 0.1], \\ [0.3, 0.3] \end{pmatrix}$	$\begin{pmatrix} [0.5, 0.5], \\ [0.2, 0.2], \\ [0.3, 0.3] \end{pmatrix}$	$\begin{pmatrix} [0.2, 0.2], \\ [0.3, 0.3], \\ [0.4, 0.4] \end{pmatrix}$
D_3	$\begin{pmatrix} [0.3, 0.3], \\ [0.3, 0.3], \\ [0.3, 0.3] \end{pmatrix}$	$\begin{pmatrix} [0.6, 0.6], \\ [0.2, 0.2], \\ [0.1, 0.1] \end{pmatrix}$	$\begin{pmatrix} [0.2, 0.2], \\ [0.1, 0.1], \\ [0.7, 0.7] \end{pmatrix}$	$\begin{pmatrix} [0.2, 0.2], \\ [0.0, 0.0], \\ [0.6, 0.6] \end{pmatrix}$	$\begin{pmatrix} [0.1, 0.1], \\ [0.0, 0.0], \\ [0.9, 0.9] \end{pmatrix}$
D_4	$\begin{pmatrix} [0.1, 0.1], \\ [0.2, 0.2], \\ [0.7, 0.7] \end{pmatrix}$	$\begin{pmatrix} [0.2, 0.2], \\ [0.3, 0.3], \\ [0.4, 0.4] \end{pmatrix}$	$\begin{pmatrix} [0.8, 0.8], \\ [0.2, 0.2], \\ [0.0, 0.0] \end{pmatrix}$	$\begin{pmatrix} [0.2, 0.2], \\ [0.1, 0.1], \\ [0.7, 0.7] \end{pmatrix}$	$\begin{pmatrix} [0.2, 0.2], \\ [0.0, 0.0], \\ [0.7, 0.7] \end{pmatrix}$

From Table 14, it is discernible that patient P_1 suffers from disease D_2, patient P_2 has been diagnosed with disease D_4, and patient P_3 suffers from disease D_3. Table 15 and Figure 2 compare with other existing measures. We can see that the proposed measures and existing measures produce the same diagnostic results, which demonstrates the robustness and reliability of the proposed distance measures. Moreover, the consistency of diagnostic results between the proposed and existing measures emphasizes their potential for seamless integration with current diagnostic frameworks. Additionally, new perspectives or additional insights may be provided, representing a significant step toward improving the accuracy and effectiveness of medical diagnostic procedures.

Table 14. The diagnosis by the proposed distance measures.

	Patient	D_1	D_2	D_3	D_4	Result
$\mathbb{D}_A^3$	P_1	0.0379	0.0201	0.0514	0.0661	D_2
	P_2	0.0694	0.0792	0.0672	0.0356	D_4
	P_3	0.0747	0.0802	0.0161	0.0764	D_3
$\mathbb{D}_A^4$	P_1	0.0618	0.0227	0.0846	0.0904	D_2
	P_2	0.0922	0.1020	0.0672	0.0482	D_4
	P_3	0.1102	0.1018	0.0250	0.1017	D_3
$\mathbb{D}_B^3$	P_1	0.0794	0.0490	0.1247	0.1429	D_2
	P_2	0.1667	0.1703	0.1528	0.0861	D_4
	P_3	0.1522	0.1766	0.0723	0.2024	D_3
$\mathbb{D}_B^4$	P_1	0.1248	0.0523	0.1719	0.1696	D_2
	P_2	0.1871	0.1903	0.1602	0.1068	D_4
	P_3	0.2006	0.1979	0.0870	0.2213	D_3
$\mathbb{D}_C^3$	P_1	0.0231	0.0159	0.0447	0.0548	D_2
	P_2	0.0707	0.0632	0.0548	0.0316	D_4
	P_3	0.0548	0.0632	0.0447	0.0894	D_3
$\mathbb{D}_C^4$	P_1	0.0548	0.0159	0.0447	0.0548	D_2
	P_2	0.0707	0.0632	0.0548	0.0316	D_4
	P_3	0.0548	0.0632	0.0447	0.0894	D_3
$\mathbb{D}_D^3$	P_1	0.0054	0.0025	0.0200	0.0300	D_2
	P_2	0.0500	0.0400	0.0300	0.0100	D_4
	P_3	0.0300	0.0400	0.0200	0.0800	D_3
$\mathbb{D}_D^4$	P_1	0.0300	0.0025	0.0200	0.0300	D_2
	P_2	0.0500	0.0400	0.0300	0.0100	D_4
	P_3	0.0300	0.0400	0.0200	0.0800	D_3

Table 15. The diagnosis by the existing similarity measures.

	Patient	D_1	D_2	D_3	D_4	Result
$\mathbb{S}^1_{Cs}$	P_1	0.0343	0.0172	0.1522	0.1582	D_2
	P_2	0.1935	0.1711	0.1677	0.0270	D_4
	P_3	0.1462	0.1756	0.0147	0.2472	D_3
$\mathbb{S}^2_{Cs}$	P_1	0.1944	0.0796	0.3867	0.4976	D_2
	P_2	0.5862	0.6730	0.5685	0.1104	D_4
	P_3	0.5266	0.6854	0.0431	0.6640	D_3
$\mathbb{S}^3_{Cs}$	P_1	0.0463	0.0172	0.1522	0.1582	D_2
	P_2	0.2008	0.1784	0.1677	0.0270	D_4
	P_3	0.1462	0.1756	0.0147	0.2472	D_3
$\mathbb{S}^4_{Cs}$	P_1	0.3511	0.0960	0.5764	0.6413	D_2
	P_2	0.7764	0.8830	0.5922	0.1342	D_4
	P_3	0.8146	0.8098	0.0578	0.7480	D_3
$\mathbb{S}^1_{Ct}$	P_1	0.2224	0.1422	0.3652	0.4492	D_2
	P_2	0.4341	0.4151	0.3880	0.1969	D_4
	P_3	0.4264	0.4406	0.1131	0.4675	D_3
$\mathbb{S}^2_{Ct}$	P_1	0.2452	0.1422	0.3652	0.4492	D_2
	P_2	0.4596	0.4406	0.3880	0.1969	D_4
	P_3	0.4264	0.4406	0.1131	0.4675	D_3

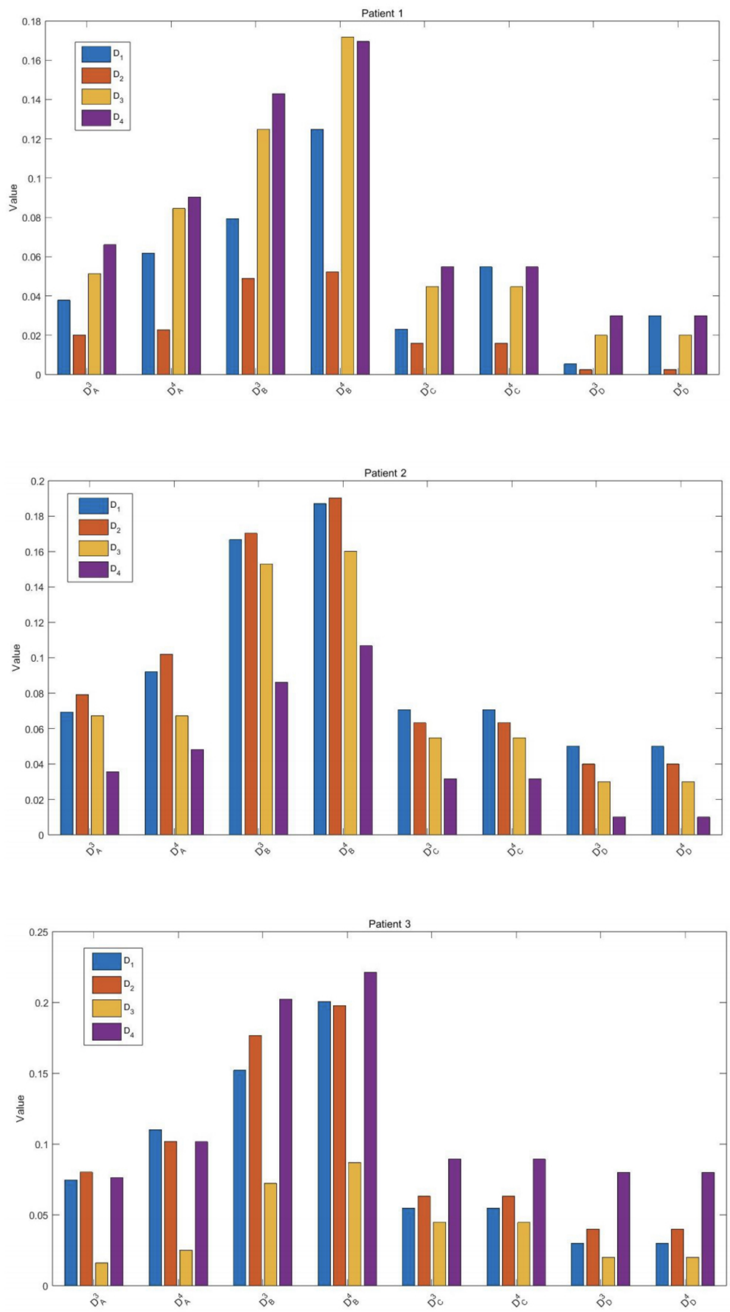

Figure 2. Diagnosis result comparison for patients across different distance measures.

8. Advantages of the work

The proposed distance measures for PFSs and IvPFSs, inspired by Hellinger distance, manifest a series of advantages that significantly contribute to the existing knowledge and practical applications. Below, we delineate the key advantages of our work:

8.1. Advantages based on PFSs

- Compared to IFSs, PFSs introduce a "refusal membership", allowing PFSs to express uncertainty information more comprehensively than IFSs.
- When the "refusal membership" is equal to 0, PFSs equals to IFSs, making PFSs a generalized form of IFSs.
- The distance measures proposed in this paper based on PFSs demonstrates stronger adaptability in applications compared to those grounded in IFSs.
- The distance measures introduced in this paper founded on PFSs can overcome the limitations of existing distance measures, producing superior results.

8.2. Advantages based on IvPFSs

- Compared to IvIFSs, IvPFSs introduce a "refusal membership". Compared to PFSs, IvPFSs have interval membership. These allow IvPFSs to express uncertainty information more comprehensively than IvIFSs and PFSs.
- When the "refusal membership" is equal to 0, IvPFSs equals to IvIFSs, positioning IvPFSs as an extended version of IvIFSs. When IvPFSs have equal intervals, IvPFSs equals to PFSs, further establishing IvPFSs as a more general representation of PFSs.
- The proposed distance measures based on IvPFSs can exhibit enhanced adaptability in practical applications compared to IvIFSs and PFSs.
- The proposed distance measures based on IvPFSs can overcome the limitations of existing distance or similarity measures, producing superior results.

9. Conclusions

In this work, we proposed novel distance measures for PFSs and IvPFSs, leveraging Hellinger distance to overcome limitations in existing measures. These measures adhered to critical properties such as boundedness, non-degeneracy, symmetry, and monotonicity, affirming their theoretical robustness. The newly introduced PFSs distance measures addressed the challenges posed by nuanced data intricacies often overlooked by existing measures, thereby enhancing accuracy and reliability in a picture fuzzy environment. Similarly, the IvPFSs distance measures tackled the heightened uncertainty inherent in IvPFSs, offering improved precision and reliability. Practical applications of these measures in pattern recognition and medical diagnosis have shown promising results, demonstrating their potential in real-world scenarios.

However, PFSs and IvPFSs are not without their shortcomings, as they exhibit certain limitations, such as the inability to handle uncertain information in complex number fields. In light of these limitations, our future work aims to extend the distance measures proposed in this paper to CPFSs and CIvPFSs. This extension is envisioned to broaden the span of applications, thereby fostering a more robust framework for tackling uncertain information in complex number fields. Through these advancements, we aspire to bridge the existing gaps and propel the practical utility of fuzzy sets in many complex scenarios.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare they have no conflict of interest.

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