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Research article

Lie symmetry analysis, conservation laws and diverse solutions of a new extended (2+1)-dimensional Ito equation

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Abstract: In this paper, a new class of extended (2+1)-dimensional Ito equations is investigated for its group invariant solutions. The Lie symmetry method is employed to transform the nonlinear Ito equation into an ordinary differential equation. The general solution of the solvable linear differential equation with different parameters is obtained, and the plot of the solvable linear differential equation is given. A power series solution for the equation is then derived. Furthermore, a conservation law for the equation is constructed by utilizing a new Ibragimov conservation theorem.

Keywords: the Ito equation; conservation laws; particular solutions; power series solutions; Lie symmetry method
Mathematics Subject Classification: 76M60

1. Introduction

Nonlinear partial differential equations (NPDEs) are widely used to describe nonlinear phenomena in various disciplines, such as mechanics, control processes, ecological systems, economic systems, chemical cycle systems, and epidemiology [1–5]. The study of NPDEs is an important branch of modern mathematics, both in theory and practical applications. Finding exact solutions to NPDEs has been a central topic in mathematics and physics. In recent decades, mathematicians have developed several effective methods for finding structural solutions of NPDEs, including the Lie symmetry method [6, 7], Homogeneous equilibrium method [8–10], Darboux transformation [11], Bäcklund transformation [12], F-expansion method [13], Tanh method [14, 15], etc. Among these methods, Lie symmetry analysis is one of the most classical methods. It utilizes a set of one-parameter transformations in the space of independent and dependent variables to keep the NPDE unchanged. Lie symmetry methods have been widely applied to solve problems in mathematical physics, nonlinear science and engineering physics [16, 17, 34]. It is well known that the famous (1+1)-dimensional the Ito equation is

$$u_{tt} + 6u_{xx}u_t + 6u_x u_{xt} + u_{xxt} = 0. \tag{1.1}$$

The above equation was firstly proposed by Ito, and its bilinear Bäcklund transform, Lax pair and multi-soliton solutions were obtained. The rolling behavior of a ship in regular sea is usually predicted by the Ito equation, and the interaction between two internal long waves can also be described by the Ito equation. Due to the typical nature of the Ito equation as a soliton equation, a great deal of research has been done concerning it. Ma and Li investigated the evolution and degradation of the torsional attractor solutions of the (2+1)-dimensional Hirota-satsuma-Ito class equations using symbolic computation and Hirota bilinear equations [18]. Based on an extended homoclinic test and bilinear method, Li and Zhao studied the exact soliton solutions of the (2+1)-dimensional Ito equation and explicit solutions such as trigonometric function solution, soliton solution, double periodic wave solution and periodic solitary wave solution are obtained [19]. Based on a multidimensional Bernhard Riemann ξ function, Tian and Zhang used a clear and direct method to explicitly construct the periodic solutions of the (1+1)-dimensional and the (2+1)-dimensional Ito equations [20]. Then, using Hirota's bilinear method and the positive quadratic function, Tian and Li obtained some global solutions of the (2+1)-dimensional Ito equation [21].

Recently, Ma and Wu obtained the local interaction solution of the Ito equation with free parameters in the (2+1)-dimensional Ito equation by Hirota bilinear transformation [22]. In 2022, based on previous studies [23–25], Wazwaz extended the equation to (3+1)-dimensional Ito equation, proved its complete integrability by Painleve analysis and derived the multi-soliton solution [26] using the simplified Hirota method. Based on the traditional (2+1)-dimensional Ito equation,

$$v_{tt} + v_{xxxt} + 3\left(2v_x v_t + v v_{xt}\right) + 3v_{xx}\int_{-\infty}^{x} v_t dx' + \alpha v_{yt} + \beta v_{xt} = 0.$$

Let $v = u_x$, then the equation can be transformed to the following new form which is studied in this paper

$$u_{xtt} + u_{xxxxt} + 3(2u_{xx}u_{xt} + u_x u_{xxt}) + 3u_{xxx}u_t + \alpha u_{xyt} + \beta u_{xxt} = 0, \tag{1.2}$$

where α and β are real parameters. The Ito equation is completely integrable and possesses many conservation laws. If $\beta = \alpha = 0$, the above equation becomes (1+1)-dimensional. In this paper, Lie symmetry analysis method is used to study the Ito equation, some special solutions are found, their plots are drawn and dynamical behaviors are studied. The arrangement of this paper is shown in Figure1 below.

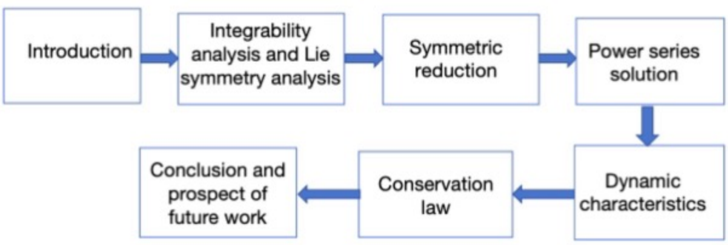


Figure 1. The arrangement of the sections of this paper.

2. Integrability analysis and Lie symmetry analysis of the Ito equation

2.1. Integrability analysis of the Ito equation

Painlevé analysis [27] is a useful method for examining the complete integrability for NPDEs. We first assume that the solution of the equation is explored as

$$u(x,y,t)=\sum_{k=0}^{\infty}u_k(x,y,t)\psi(x,y,t)^{k-\mu},$$

which is a Laurent series with respect to a singular manifold $\psi(x,y,t)$. Following the Painlevé analysis gives resonances at k =1, 4 and 6. As a result, we obtain expressions for u_2, u_3 and u_5,

$$\begin{aligned}
u_2 &= \beta\psi_t+\frac{\alpha^2}{2\beta}t+\alpha^2u_1,\\
u_3 &= u_1\psi_t+\frac{\alpha}{2\beta}t^2+\alpha^2\psi_x,\\
u_5 &= u_1\psi_t+\alpha t\left(u_1u_4+u_1^2\right)+\frac{\beta}{36\alpha^3}\psi_{yt}+u_4^2\psi_{xx}.
\end{aligned}$$

We found that u_1, u_4 and u_6 are left as arbitrary functions and this confirms that Eq (1.2) passes the Painlevé test that confirms the complete integrability for Eq (1.2).

2.2. Lie symmetry analysis of the Ito equation

The one parameter Lie group of infinitesimal transformation of the Ito equation in x, y, t and u is given by

$$\begin{aligned}
\tilde{x} &\to x+\varepsilon\xi(x,y,t,u)+o\left(\varepsilon^2\right),\\
\tilde{y} &\to y+\varepsilon\phi(x,y,t,u)+o\left(\varepsilon^2\right),\\
\tilde{t} &\to t+\varepsilon\tau(x,y,t,u)+o\left(\varepsilon^2\right),\\
\tilde{u} &\to u+\varepsilon\eta(x,y,t,u)+o\left(\varepsilon^2\right),
\end{aligned}$$

where ε is the group parameter. In particular, ξ, ϕ, τ and η are the infinitesimal operators of the transformation for the independent and dependent variables, which will be determined later. The vector field associated with the above mentioned group of transformations is given as

$$V=\xi(x,y,t,u)\frac{\partial}{\partial x}+\phi(x,y,t,u)\frac{\partial}{\partial y}+\tau(x,y,t,u)\frac{\partial}{\partial t}+\eta(x,y,z,t,u)\frac{\partial}{\partial u}. \tag{2.1}$$

The symmetry group of Eq (1.2) will be generated by the vector field of the Eq (2.1). Using the invariant condition

$$Pr^5V(\Delta)\big|_{\Delta=0}=0, \tag{2.2}$$

where $\Delta = u_{xtt}+u_{xxxxt}+3\left(2u_{xx}u_{xt}+u_xu_{xxt}\right)+3u_{xxx}u_t+\alpha u_{xyt}+\beta u_{xxt}$. The five-order prolongation for Eq (1.2) is

$$Pr^5 v = v + \eta^x \frac{\partial}{\partial u_x} + \eta^t \frac{\partial}{\partial u_t} + \eta^{xx} \frac{\partial}{\partial u_{xx}} + \eta^{xt} \frac{\partial}{\partial u_{xt}} + \eta^{xxx} \frac{\partial}{\partial u_{xxx}} + \eta^{xtt} \frac{\partial}{\partial u_{xtt}}$$
$$+ \eta^{xxt} \frac{\partial}{\partial u_{xxt}} + \eta^{xyt} \frac{\partial}{\partial u_{xyt}} + \eta^{xxxxt} \frac{\partial}{\partial u_{xxxxt}},$$

where

$$\begin{aligned}
\eta^x &= D_x\left(\eta - \xi u_x - \phi u_y - \tau u_t\right) + \xi u_{xx} + \phi u_{xy} + \tau u_{xt},\\
\eta^t &= D_t\left(\eta - \xi u_x - \phi u_y - \tau u_t\right) + \xi u_{xt} + \phi u_{yt} + \tau u_{tt},\\
\eta^{xx} &= D_x^2\left(\eta - \xi u_x - \phi u_y - \tau u_t\right) + \xi u_{xxx} + \phi u_{xxy} + \tau u_{xx},\\
\eta^{xt} &= D_x D_t\left(\eta - \xi u_x - \phi u_y - \tau u_t\right) + \xi u_{xxt} + \phi u_{xyt} + \tau u_{xtt},\\
\eta^{xxx} &= D_x^3\left(\eta - \xi u_x - \phi u_y - \tau u_t\right) + \xi u_{xxx} + \phi u_{xxy} + \tau u_{xxt},\\
\eta^{xtt} &= D_x D_t^2\left(\eta - \xi u_x - \phi u_y - \tau u_t\right) + \xi u_{xxtt} + \phi u_{xytt} + \tau u_{xttt},\\
\eta^{xxt} &= D_x^2 D_t\left(\eta - \xi u_x - \phi u_y - \tau u_t\right) + \xi u_{xxxt} + \phi u_{xxyt} + \tau u_{xtt},\\
\eta^{xyt} &= D_x D_y D_t\left(\eta - \xi u_x - \phi u_y - \tau u_t\right) + \xi u_{xxyt} + \phi u_{xyyt} + \tau u_{xytt},\\
\eta^{xxxxt} &= D_x^4 D_t\left(\eta - \xi u_x - \phi u_y - \tau u_t\right) + \xi u_{xxxxxt} + \phi u_{xxxxyt} + \tau u_{xxxxtt}.
\end{aligned}$$

Substituting these extensions in Eq (2.2), we get the determining equations:

$$\begin{aligned}
&\eta_x = -\frac{2}{9}\beta\phi_y + \frac{\alpha}{3}\xi_y, \eta_t = 0, \eta_u = -\frac{1}{3}\phi_y,\\
&\tau_x = \tau_u = 0, \tau_t = -\alpha\left(\tau_y\right) + \phi_y,\\
&\xi_u = \xi_t = 0, \xi_x = \frac{1}{3}\phi_y,\\
&\phi_u = \phi_x = \phi_t = 0.
\end{aligned}$$

By using the software Maple to solve those equations, we get the following system of equations:

$$\xi = \frac{\left(F_{1y}\right)x}{3} + F_3(y), \quad \phi = F_1(y), \quad \tau = \frac{F_2\left(\frac{\alpha t - y}{\alpha}\right)\alpha + F_1(y)}{\alpha},$$
$$\eta = \frac{\alpha\left(F_{1yy}\right)x^2}{18} + \frac{(-4\beta x - 6u)\left(F_{1y}\right)}{18} + \frac{x\alpha\left(F_{3y}\right)}{3} + F_4(y),$$

where F_1, F_3 and F_4 are arbitrary functions of y and F_2 is a arbitrary function of $\frac{\alpha t - y}{\alpha}$. In 2022, Kumar discussed a case where F_2 is a constant [28]. In this paper, we discuss the more general case. So we have

$$F_1 = C_1 y + C_8, \quad F_2 = C_2\left(\frac{\alpha t - y}{\alpha}\right) + C_5, \quad F_3 = C_3 y + C_6, \quad F_4 = C_4 y + C_7,$$

where C_1, C_2, C_3, C_4, C_5, C_6 and C_7 are arbitrary constants. So the vector fields of equation (1.2) are

$$\begin{aligned}
&V_1 = \frac{\partial}{\partial x}, \quad V_2 = \frac{\partial}{\partial t}, \quad V_3 = \frac{\partial}{\partial u}, \quad V_4 = y\frac{\partial}{\partial u}, \quad V_5 = \left(t - \frac{y}{\alpha}\right)\frac{\partial}{\partial t},\\
&V_6 = \frac{1}{\alpha}\frac{\partial}{\partial t} + \frac{\partial}{\partial y}, \qquad V_7 = \frac{\alpha}{3}x\frac{\partial}{\partial u} + y\frac{\partial}{\partial x},\\
&V_8 = \frac{1}{18}(-4\beta x - 6u)\frac{\partial}{\partial u} + \frac{y}{\alpha}\frac{\partial}{\partial t} + \frac{x}{3}\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.
\end{aligned}$$

Thus, all the infinitesimals of Eq (1.2) can be written as the linear combination of the vectors V_i as

$$V = p_1V_1 + p_2V_2 + p_3V_3 + p_4V_7 + p_5V_5 + p_6V_6 + p_7V_7 + p_8V_8.$$

The commutator relations of Lie algebra between the vector fields can be represented in Table 1.

To compute adjoint representations of symmetry operators for Eq (1.2), we use the Lie series [29,30]

$$Ad\left(\exp\left(\vartheta\mathbb{V}_i\right)\right)\mathbb{V}_j = \sum_{n=0}^{\infty}\frac{\vartheta^n}{n!}\left(ad\mathbb{V}_i\right)^n\left(\mathbb{V}_j\right) = \mathbb{V}_j - \vartheta\left[\mathbb{V}_i, \mathbb{V}_j\right] + \frac{1}{2}\vartheta^2\left[\mathbb{V}_i, \left[\mathbb{V}_i, \mathbb{V}_j\right]\right] - \cdots .$$

The full adjoint representation table entries are tabulated in Tables 2 and 3.

With the assistance of Tables 1, 2 and 3, by carefully applying adjoint maps, we discuss useful linear combinations of vector fields for the considered equation, which are taken as follows:

$$\text{(i)}V_1 + V_3 + V_6, \qquad \text{(ii)}V_1 + V_4 + V_6, \qquad \text{(iii)}V_2 + V_7, \qquad \text{(iv)}V_8.$$

Table 1. Commutator table.

*	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8
V_1	0	0	0	0	0	0	0	$\frac{1}{3}V_1$
V_2	0	0	0	0	0	V_2	0	0
V_3	0	0	0	0	0	0	0	$-\frac{1}{3}V_3$
V_4	0	0	0	0	$-V_3$	0	0	$-\frac{4}{3}V_4$
V_5	0	0	0	V_3	0	0	V_1	V_5
V_6	0	0	$-V_2$	0	0	0	0	0
V_7	0	0	0	0	$-V_1$	0	0	$-\frac{2}{3}V_7 - \frac{2}{9}\beta V_4$
V_8	$-\frac{1}{3}V_1$	0	$\frac{1}{3}V_3$	$\frac{4}{3}V_4$	$-V_5$	0	$\frac{2}{3}V_7$	0

Table 2. Adjoint table.

Ad.	V_1	V_2	V_3	V_4
V_1	V_1	V_2	V_3	V_4
V_2	V_1	V_2	V_3	V_4
V_3	V_1	V_2	V_3	V_4
V_4	V_1	V_2	V_3	V_4
V_5	V_1	V_2	V_3	$V_4 - \varepsilon V_3$
V_6	V_1	$V_2 + V_2 ln(\varepsilon + 1)$	V_3	V_4
V_7	V_1	V_2	V_3	V_4
V_8	$V_1 - V_1 ln(1 - \frac{1}{3}\varepsilon)$	V_2	$V_3 - V_3 ln(\frac{1}{3}\varepsilon + 1)$	V_8

Table 3. Adjoint table.

Ad.	V_5	V_6	V_7	V_8
V_1	V_5	V_6	V_7	$V_8 - \frac{1}{3}\varepsilon V_1$
V_2	V_5	$V_6 - \varepsilon V_2$	V_7	V_8
V_3	V_5	V_6	V_7	$V_8 + \frac{1}{3}\varepsilon V_3$
V_4	$V_5 + \varepsilon V_3$	V_6	V_7	$V_8 + \frac{4}{3}\varepsilon V_4$
V_5	V_5	V_6	$V_7 - \varepsilon V_1$	$V_8 - \varepsilon V_5$
V_6	V_5	V_6	V_7	V_8
V_7	$V_5 + \varepsilon V_1$	V_6	V_7	$(1 + \frac{2}{3}\varepsilon)V_7 + \frac{2}{9}\varepsilon V_4$
V_8	εV_5	V_6	$V_7 - V_7 ln(\frac{2}{3}\varepsilon + 1)$	V_8

3. Similarity reduction of the Ito equation

In this section, we obtain numerous closed-form invariant solutions for equation (1.2) utilizing the Lie symmetry analysis. Two stages of symmetry reducations will be taken with the aid of invariant (or similarity) functions. We firstly solve the associated Lagrange's characteristic system given by

$$\frac{dx}{\xi} = \frac{dy}{\phi} = \frac{dt}{\tau} = \frac{du}{\eta},$$

which leads to similarity functions.

3.1. Symmetry reduction for $V_1 + V_3 + V_6 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{1}{\alpha}\frac{\partial}{\partial t} + \frac{\partial}{\partial u}$

We use the symmetry $V_1 + V_3 + V_6$ to reduce the Ito equation (1.2) to a nonlinear partial differential equation (NODE). The characteristic equations of $V_1 + V_3 + V_6$ give the invariants

$$T = -\alpha t + y, \quad X = -\alpha t + x, \quad V = -\alpha t + u. \tag{3.1}$$

Replacing Eq (1.2) with Eq (3.1), we obtain the reduced (1+1)-dimensional equation:

$$\begin{aligned}&(\alpha - \beta)V_{XXT} + (\alpha - \beta)V_{XXX} - V_{XXXXX} - V_{XXXXT} - 6V_{XX}V_{XT} - 6V_{XX}^2 \\ &- 6V_X V_{XXX} - 3V_X V_{XXT} - 3V_T V_{XXX} = 0.\end{aligned} \tag{3.2}$$

Then we apply the classical symmetry again, the vector fields of Eq (3.2) are

$$\bar{V}_1 = \frac{1}{3}[(-2T + 2X)\alpha + (2T - 2X)\beta - 3V]\frac{\partial}{\partial V} + T\frac{\partial}{\partial T} + X\frac{\partial}{\partial X},$$
$$\bar{V}_2 = \frac{\partial}{\partial T}, \qquad \bar{V}_3 = \frac{\partial}{\partial V}, \qquad \bar{V}_4 = T\frac{\partial}{\partial T} + T\frac{\partial}{\partial X}.$$

3.1.1. For $\bar{V}_1$

We obtain the invariants

$$w = \frac{X}{T}, R = \left[V + \left(\frac{\alpha}{3} - \frac{\beta}{3}\right)T - \left(\frac{\alpha}{3} - \frac{\beta}{3}\right)X\right]T.$$

By substituting group invariant solution, we have the reduced equation

$$6(w-1)R_w R_{www} + 6(w-1)R_{ww}^2 + (w-1)R_{wwwww} + 21R_w R_{ww} + 3R_{www}R_{ww} = 0. \tag{3.3}$$

The exact solutions of equation (3.3) have been found in Section 4.1.

3.1.2. For $\bar{V}_2 + \bar{V}_3 + \bar{V}_4$

We obtain the invariants

$$w = -T + \ln(T+1) + X, R = V - \ln(T+1),$$

which reduce Eq (3.2) to a NODE

$$(\alpha - \beta)R_{www} - R_{wwwww} - 6R_{ww}^2 - 6R_w R_{www} = 0.$$

Obviously, the general solution is not easy to find so we obtain a particular solution

$$u = -\tan\left(\frac{\alpha x - y + \ln y}{2} + 1\right) + 3 + \beta \ln y. \tag{3.4}$$

3.2. Symmetry reduction for $V_1 + V_4 + V_6 = y\frac{\partial}{\partial u} + \frac{1}{\alpha}\frac{\partial}{\partial t} + \frac{\partial}{\partial y} + \frac{\partial}{\partial x}$

We use the symmetry $V_1 + V_4 + V_6$ to reduce equation (1.2) to a NODE. The characteristic equations give the similar invariants

$$X = -\alpha t + x, \quad T = -\alpha t + y, \quad V = u + \frac{1}{2}\alpha^2 t^2 - \alpha t y.$$

By substituting the group invariant solution, we have the reduced equation

$$\begin{aligned}&(\alpha - \beta)V_{XXT} + (\alpha - \beta + 3T)V_{XXX} - V_{XXXXX} - V_{XXXXT}\\&- 6V_{XX}^2 - 6V_{XX}V_{XT} - 3V_X V_{XXT} - 6V_X V_{XX} - 3V_T V_{XXX} = 0.\end{aligned} \tag{3.5}$$

In order to simplify the equation further, we use Maple to analyze Eq (3.5) for the second Lie symmetry have the vector fields,

$$\bar{V}_1 = \left[9T^2 + (4\beta - 4\alpha)(T - X) - 6V\right]\frac{\partial}{\partial V} + 6T\frac{\partial}{\partial T} + 6X\frac{\partial}{\partial X}, \quad \bar{V}_3 = \frac{\partial}{\partial V},$$
$$\bar{V}_2 = T\frac{\partial}{\partial V} + \frac{\partial}{\partial T}, \qquad \bar{V}_4 = T^2\frac{\partial}{\partial V} + T\frac{\partial}{\partial T} + T\frac{\partial}{\partial X}.$$

3.2.1. For $\bar{V}_1$

We obtain the similarity variable in the same way as follows:

$$w = \frac{X}{T}, \quad R = \left[V - \frac{T^2}{2} + \left(\frac{\alpha}{3} - \frac{\beta}{3}\right)(T - X)\right]T.$$

Then we obtain the following reduced equation:

$$(w-1)R^{(5)} + 5R^{(4)} + 6(w-1)R_{ww}^2 + 21R_wR_{ww} + 6(w-1)R_wR_{www} + 3RR_{www} = 0. \tag{3.6}$$

The exact solutions of equation (3.6) have been found in Section 4.2.

3.2.2. For $\bar{V}_2$

We obtain the similarity variable in the same way as follows:

$$w = X, \qquad R = V - \frac{T^2}{2}.$$

Then we get following reduced equation:

$$(\alpha - \beta + 3T)R_{www} - R_{wwwww} - 6R_{ww}^2 - 6R_wR_{www} - 3TR_{www} = 0.$$

By simple calculation, we find a solution for equation (1.2)

$$u = -\frac{1}{2}\tanh\left(\frac{\alpha t - x}{2}\right)^2 + \frac{1}{2}y^2 + \frac{1}{2}. \tag{3.7}$$

3.2.3. For $\bar{V}_2 + \bar{V}_3 + \bar{V}_4$

We obtain the similar variable invariants by solving the characteristic equation,

$$w = X + \ln(T+1) - T, \quad R = V - \frac{T^2}{2} - \ln(T+1).$$

Then we have the reduced equation

$$(\alpha - \beta)R_{www} - R_{wwwww} - 6R_w^2 - 6R_wR_{www} = 0.$$

By simple calculation, we can obtain the group invariant solution of the equation (1.2)

$$u = -\tan\left(\frac{x - y + \ln(-\alpha t + y + 1)}{2} + \frac{1}{2}\right) + \frac{1}{2} + \frac{1}{2}y^2 + \ln(-\alpha t + y + 1). \tag{3.8}$$

3.3. Symmetry reduction for $V_2 + V_7 = \frac{\alpha}{3}x\frac{\partial}{\partial u} + y\frac{\partial}{\partial x} + \frac{\partial}{\partial t}$

Solving the characteristic equation, we obtain the similarity variables and function

$$T = -ty + x, \quad Y = y, \quad V = u + \frac{\alpha}{6}yt^2 - \frac{\alpha}{3}tx.$$

Now, substituting the values of the similarity variables in Eq (1.2), we obtain the reduced (1+1)-dimensional equation as follows:

$$\left(Y^2 + \alpha T - \beta Y\right) V_{TTT} - YV_{TTTTT} - 6YV_{TT}^2 - \alpha V_{TT} - \alpha Y V_{TTY} = 0. \tag{3.9}$$

Again applying similarity transformation method (STM) on Eq (3.9), we get the following results:

$$\bar{V}_1 = 3V\frac{\partial}{\partial V} + \left(3T + \frac{Y(3\beta - 5Y)}{\alpha}\right)\frac{\partial}{dT} + 9Y\frac{\partial}{dY}, \quad \bar{V}_2 = \frac{1}{Y}\frac{\partial}{\partial T}.$$

3.3.1. For $\bar{V}_1$

We obtain the similarity variable as follows:

$$w = \frac{6\alpha T + 2Y^2 - 3\beta Y}{6\alpha Y^{\frac{1}{3}}}, R = \frac{V}{Y^{\frac{1}{3}}}.$$

Now, we have the following equation by substituting:

$$4\alpha w R_{www} - 3R_{wwwww} - 18R_{ww}^2 - 2\alpha R_{ww} = 0. \tag{3.10}$$

The exact solutions of equation (3.10) have been found in Section 4.

3.4. Symmetry reduction for $V_8 = \frac{1}{18}(-4\beta x - 6u)\frac{\partial}{\partial u} + \frac{y}{\alpha}\frac{\partial}{\partial t} + \frac{x}{3}\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$

The characteristic equations of V_8 give the similar invariants

$$X = xy^{-\frac{1}{3}}, \quad T = -\alpha t + y, \quad V = \left(u + \frac{\beta x}{3}\right)y^{\frac{1}{3}}, \tag{3.11}$$

we put Eq (3.11) into Eq (1.2) and get the following reduced equation:

$$-3V_{XXXXT} - 18V_{XX}V_{XT} - 9V_{XXT}V_X - 9V_{XXX}V_T + XV_{XXT} + 2\alpha V_{XT} = 0.$$

Since the above equation contains one dependent and two independent variables, we again apply the STM. Thus the Lie algebra of the above equation is spanned by the following vector fields

$$\bar{V}_1 = \frac{1}{9}X\frac{\partial}{\partial V} + \frac{\partial}{\partial X}, \qquad \bar{V}_2 = \frac{\partial}{\partial V}, \qquad \bar{V}_3 = T\frac{\partial}{\partial T}.$$

3.4.1. For $\bar{V}_1 + \bar{V}_3$

We obtain the similarity variable as follows:

$$w = -\ln T + X, \quad R = V + \frac{\ln^2 T}{18} - \frac{X \ln T}{9}.$$

By substituting, we obtain the reduced equation as:

$$-3R^{(5)} + 18\left(R_{ww}\right)^2 - 2(\alpha + 1)R_{ww} + 18R_w R_{www} - 2wR_{www} + \frac{2\alpha}{9} = 0. \tag{3.12}$$

The exact solutions of equation (3.12) have been found in Section 4.

3.4.2. For $\bar{V}_2 + \bar{V}_3$

We obtain the similarity variable as follows:

$$w = X, \quad R = V - \ln T.$$

Similarly, we obtain the reduced equation by substituting:

$$-9\frac{1}{T}R_{www} = 0.$$

By simple calculation, we obtain a solution of equation (1.2)

$$u = -\frac{\beta}{3}x + y^{-\frac{1}{3}}\ln(-\alpha t + y) + a_0 y^{-\frac{1}{3}} + a_1 x y^{-\frac{2}{3}} + a_2 x^2 y^{-1}. \tag{3.13}$$

4. Exact explicit solutions of reduction equation

The solutions of nonlinear ordinary differential equations cannot be expressed by elementary functions. So we assume that the power series solution of equation (1.2) is

$$R(w) = \sum_{n=0}^{\infty} a_n w^n. \tag{4.1}$$

So

$$\begin{aligned}
R' &= \sum_{n=0}^{\infty}(n+1)a_{n+1}w^n,\\
R'' &= \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}w^n,\\
(R'')^2 &= \sum_{n=0}^{\infty}\sum_{i=0}^{n}(i+1)(i+2)(n+1)(n+2)a_{n+2}a_{i+2}w^{n+i},\\
R''' &= \sum_{n=0}^{\infty}(n+1)(n+2)(n+3)a_{n+3}w^n,\\
R^{(4)} &= \sum_{n=0}^{\infty}(n+1)(n+2)(n+3)(n+4)a_{n+4}w^n,\\
R^{(5)} &= \sum_{n=0}^{\infty}(n+1)(n+2)(n+3)(n+4)(n+5)a_{n+5}w^n.
\end{aligned} \tag{4.2}$$

4.1. The power series solution of Eq (3.3)

Substituting Eq (4.1) and Eq (4.2) into Eq (3.3) and comparing coefficients, we have

$$a_5 = \frac{7}{20}a_1a_2 + \frac{3}{10}a_2a_3 - \frac{3}{10}a_1a_3 - \frac{1}{5}a_2^2.$$

Consider $n \geq 1$, we get

$$a_{n+5} = \frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)}\Bigg[6\sum_{i=0}^{n}\sum_{j=0}^{i}(j+1)(i+1)(i+2)(i+j)a_{i+2}a_{i+1}$$
$$-6\sum_{i=0}^{n}\sum_{j=0}^{i}(j+1)(i+1)(i+2)(ia_{j+1}a_{i+3}+3a_{j+1}a_{i+3}+ja_{i+2}a_{j+2}+2a_{i+2}a_{j+2})$$
$$+\sum_{i=0}^{n}i(i+1)(i+2)(i+3)(i+4)a_{i+4}+3\sum_{i=0}^{n}\sum_{j=0}^{i}(j+1)(j+2)(i+1)[(i+2)(i+3)a_{i+3}$$
$$+7a_{i+1}]a_{j+2}\Bigg].$$

From the above recurrence formula, the coefficients of the power series solution can be determined by a_1, a_2, a_3 and a_4, then the power series solution of Eq (3.3) can be written as:

$$R(w) = a_0 + a_1 w + a_2 w^2 + a_3 w^3 + a_4 w^4 + a_5 w^5 + \sum_{n=1}^{\infty} a_{n+5} w^{n+5}.$$

Thus, we obtain the exact power series solution of Eq (1.2) as follows:

$$\begin{aligned} u =& \frac{1}{(-\alpha t+y)}\Bigg[a_0 + a_1\left(\frac{-\alpha t+x}{-\alpha t+y}\right) + a_2\left(\frac{-\alpha t+x}{-\alpha t+y}\right)^2 + a_3\left(\frac{-\alpha t+x}{-\alpha t+y}\right)^3 + a_4\left(\frac{-\alpha t+x}{-\alpha t+y}\right)^4 \\ &+a_5\left(\frac{-\alpha t+x}{-\alpha t+y}\right)^5 + \sum_{n=1}^{\infty} a_{n+5}\left(\frac{-\alpha t+x}{\alpha t+y}\right)^{n+5}\Bigg] + \alpha t - \frac{1}{3}(\alpha-\beta)(y-x). \end{aligned} \tag{4.3}$$

Next, the convergence of the solution is considered

$$a_{n+5} \leq \left|\frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)}\right|\Bigg[\sum_{i=0}^{n}(i+1)(i+2)(i+3)(i+4)(i+5)\,|a_{i+4}|$$
$$+\sum_{i=0}^{n}\sum_{j=0}^{i}(i+1)(i+2)(j+1)(6i+21)\,\left|a_{j+1}\right|\,|a_{i+2}| + \sum_{i=0}^{n}\sum_{j=0}^{i}(i+1)(i+2)(i+3)$$
$$|3a_j - 6a_{j+1}(j+1)|\,|a_{i+3}| + 6\sum_{i=0}^{n}\sum_{j=0}^{i}(j+1)(j+2)(i+1)|ia_{i+1} - ia_{i+2} - 2a_{i+2}||a_{j+2}|$$
$$\leq M[\sum_{i=0}^{n}|a_{i+4}| + \sum_{i=0}^{n}\sum_{j=0}^{i}|a_{i+2}||a_{j+1}|$$

where $M \geq 1$. Then we set up a new power series of the form $P(z) = \sum_{n=0}^{\infty} b_n z^n$, where $b_0 = |a_0|$, $b_1 = |a_1|$, $b_2 = |a_2|$, $b_3 = |a_3|$, $b_4 = |a_4|$ and

$$b_{n+5} = M\left[\sum_{i=0}^{n} b_{i+4} + \sum_{i=0}^{n}\sum_{j=0}^{i} b_{i+2}b_{j+1}\right], n = 0, 1, 2\cdots.$$

Obviously, we can get $|a_n| \leq b_n$, $n = 0, 1, 2, 3, \cdots$. In other words, the constructed power series $P(z)$ is a superior series of $R(w)$. So

$$P(z) = b_1 z + b_2 z^2 + b_3 z^3 + b_4 z^4 + b_5 z^5 + M \sum_{n=1}^{N_i} \sum_{i=0}^{n} [b_{i+4} + \sum_{j=0}^{i} b_{i+2} b_{j+2} +] z^{n+5}.$$

Now, we construct an implicit function of z

$$F(z, P) = P - b_0 - b_2 z - b_2 z^2 - b_3 z^3 - b_4 z^4 - b_5 z^5 - M z^5 \left(P \pm P' P''\right)$$

F is analytic in $(0, b_0)$, so $F(0, b_0)$ and $\frac{\partial}{\partial b} F\,(0, b_0) \neq 0$. According to the implicit function theorem, we prove that the solution of this equation is convergent.

4.2. The power series solution of Eq (3.6)

For Eq (3.6), we get in the same way

$$\begin{aligned}
a_5 &= a_4 - \frac{1}{20}\left(4a_4^2 - 7a_1 a_2 + 3a_0 a_3 + 6a_1 a_3\right),\\
a_{n+5} &= \frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)} \left[\sum_{i=0}^{n} (i+1)(i+2)(i+3)(i+4)(i+5) a_{i+4}.\right.\\
&+ \sum_{i=0}^{n} \sum_{j=0}^{i} (i+1)(i+2)(j+1) a_{j+1} a_{i+2} (6i+21) + \sum_{i=0}^{n} \sum_{j=0}^{i} (i+1)(i+2)(i+3) a_{i+3}\\
&\left.\left[3a_j - 6a_{j+1}(j+1)\right] + 6 \sum_{i=0}^{n} \sum_{j=0}^{i} (j+1)(j+2)(i+1) a_{j+2} \left(i a_{i+1} - i a_{i+2} - 2a_{i+2}\right)\right].
\end{aligned}$$

Thus, the particular solution of Eq (1.2) is

$$\begin{aligned}
u =& \frac{1}{-\alpha t + y}[a_0 + a_1(\frac{-\alpha t + x}{-\alpha t + y}) + a_2(\frac{-\alpha t + x}{-\alpha t + y})^2 + a_3(\frac{-\alpha t + x}{-\alpha t + y})^3 + a_4(\frac{-\alpha t + x}{-\alpha t + y})^4 + a_5(\frac{-\alpha t + x}{-\alpha t + y})^5\\
&+ \sum_{n=1}^{\infty} a_{n+5}(\frac{-\alpha t + x}{-\alpha t + y})^{n+5}] - \frac{1}{2}\alpha^2 t^2 + \alpha t y + \frac{(-\alpha t + y)^2}{2} - \frac{1}{3}(\alpha - \beta)(y - x).
\end{aligned} \tag{4.4}$$

Similar to the convergence analysis method of solution (4.3), we can also obtain the convergence of solution (4.4). Similar to the above method, we also obtain the power series solutions for Eq (3.10) and Eq (3.12). The results will not be repeated.

5. Dynamical characteristics

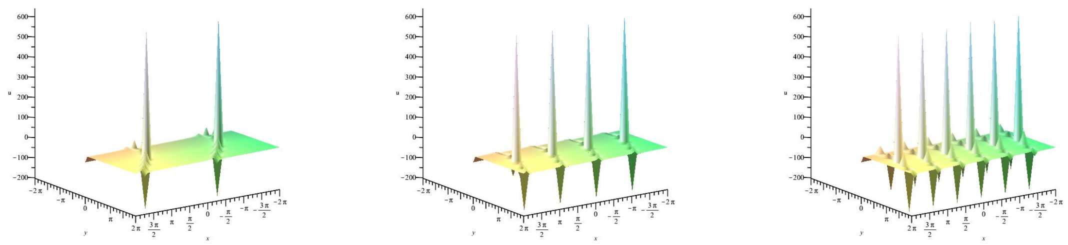
In this section, we will analyze the dynamical behavior of the solutions derived in the previous section with three-dimensional and corresponding contour plots for the (2+1)-dimensional Ito equation when parameters take different values.

5.1. Three-dimensional and contour plots of the solution (3.4)

In 3.1.2, we have obtained a solution (3.4) to the (2+1)-dimensional Ito equation,

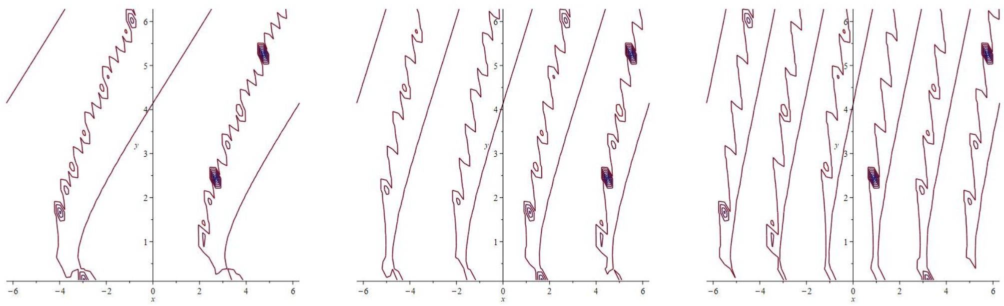
$$u = -\tan\left(\frac{x - y + \ln y}{2} + 1\right) + 3 + \ln y.$$

Clearly, we find that u is independent of time t. Then we have $\beta = 1$ and observe the dynamics of u_1 when α varies. With the help of Maple, the physical properties and characteristics of the solution are clearly depicted in Figures 2 and 3.



(a) $\alpha = 1, \beta = 1$ (b) $\alpha = 2, \beta = 1$ (c) $\alpha = 3, \beta = 1$

Figure 2. Three-dimensional plots of solution (3.4) with arbitrary constants $\alpha = 1, 2, 3$, $\beta = 1$.



(a) $\alpha = 1, \beta = 1$ (b) $\alpha = 2, \beta = 1$ (c) $\alpha = 3, \beta = 1$

Figure 3. 2D-contour plots of solution(3.4) with arbitrary constants $\alpha = 1, 2, 3, \beta = 1$.

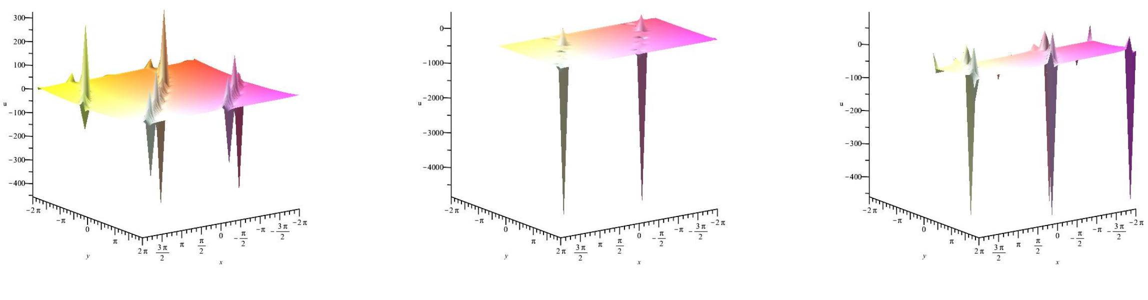
The three-dimensional graphs represent the local structure, while the contour plots show the wave fluctuations. The denser the location in the plot, the greater the fluctuation. We observe the interaction of (3.4) at $\alpha = 1, 2, 3$, and this change alters the speed, amplitude and shape of the wave so that there are more points of aggregation and the image becomes more dense when α increases. In Eq (1.2), parameter α is the coefficient of u_{xyt}, parameter β is the coefficient of u_{xxt}, and we fixed the value of β and took different values for β. We found that the amplitude and shape of the wave did not change significantly, but the motion of the wave accelerated, making the wave in the plot more dense, which can be understood in a physical sense. As the interaction effect of the two internal long waves increases, the rolling behavior of the ship in the common sea area is accelerated.

5.2. *Three-dimensional and contour plots of the solution* (3.8)

In 3.2.3, we have obtained the group invariant solution (3.8) of the Eq (1.2),

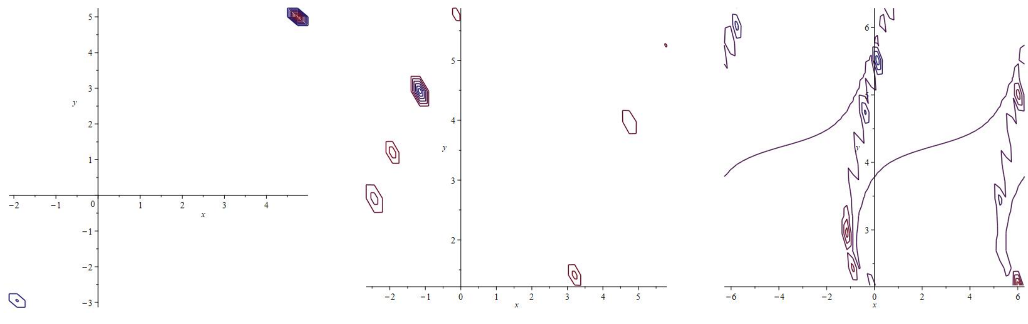
$$u = -\tan\left(\frac{x - y + \ln(-\alpha t + y + 1)}{2} + \frac{1}{2}\right) + \frac{1}{2} + \frac{1}{2}y^2 + \ln(-\alpha t + y + 1).$$

With the help of Maple, the three-dimensional dynamic graphs of the wave married with corresponding contour plots were depicted in Figures 4 and 5, and we directly observe the interaction phenomena of the solutions at times $t = -5, 0, 2$. At first, the distribution of aggregation points is more scattered, then the aggregation points gradually increase and the wave spreads and continues to move. For Eq (3.8), where the interaction parameters of two internal long waves are fixed, which is the values of arbitrary constants α and β, we discuss the dynamic behavior of the wave with respect to t. With the increase of t, the amplitude and shape of a single wave does not change significantly, and the frequency of the wave first decreases and then increases. At $t = 0$, the frequency of the wave is the smallest. Then, we also found that the equilibrium position of the fluctuations increased significantly over time. From a physical point of view, we believe that over time, there is a clear upward trend in the equilibrium position of ships where rolling behavior occurs in common sea areas.



(a) $t = -5, \alpha = 1$ (b) $t = 0, \alpha = 1$ (c) $t = 2, \alpha = 1$

Figure 4. Three-dimensional plots of solution (3.8) at $t = -5, 0, 2$, $\alpha = 1$.



(a) $t = -5, \alpha = 1$ (b) $t = 0, \alpha = 1$ (c) $t = 2, \alpha = 1$

Figure 5. 2D-contour plots of solution (3.8) at $t = -5, 0, 2$, $\alpha = 1$.

6. Conservation laws of the Ito equation

This section describes a new conservation law proposed by Ibragimov [5, 31–33] for the Ito equation, which is necessary for testing the integrability and the existence and uniqueness of the solutions. The conservation laws hoids that

$$D_x C^x + D_y C^y + D_t C^t = 0,$$

where D_i is the total derivative operator

$$D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + s_i \frac{\partial}{\partial s} + u_{ij} \frac{\partial}{\partial u_j} + s_{ij} \frac{\partial}{\partial s_j} + u_{ijk} \frac{\partial}{\partial u_{jk}} + s_{ijk} \frac{\partial}{\partial S_{jk}} + \cdots\cdots .$$

The Ito equation has the formal Lagrangian

$$L = s(x, y, t) \left[u_{xtt} + u_{xxxt} + 6u_{xx}u_{xt} + 3u_x u_{xxt} + 3u_{xx}u_t + \alpha u_{xyt} + \beta u_{xxt} \right]. \tag{6.1}$$

From (6.1), we have

$$\frac{\partial L}{\partial u_x} = 3su_{xt}, \quad \frac{\partial L}{\partial u_t} = 3su_{xxx}, \quad \frac{\partial L}{\partial u_{tt}} = 6su_{xx}, \quad \frac{\partial L}{\partial u_{xx}} = 6su_{xt},$$
$$\frac{\partial L}{\partial u_{xt}} = s, \quad \frac{\partial L}{\partial u_{xxt}} = (3u_x + \beta)s, \quad \frac{\partial L}{\partial u_{xx}} = 3su_t, \quad \frac{\partial L}{\partial u_{xy}} = \alpha s.$$

The adjoint equation of the Ito equation is expressed as

$$F^* = 3s_x u_{xxt} - 3s_{xx}u_{xt} - s_{xtt} - 3s_{xxt}u_x - 3s_{xx}u_{xxt} - 3s_x u_{xxxt} - \beta s_{xxt} - 3s_{xxx}u_t - \alpha s_{xyt} - s_{xxxxxt}.$$

Since the Ito equation is nonlinear and self-adjoint, let u=s and we have

$$-3u_{xx}u_{xt} - u_{xtt} - 3u_{xx}u_{xxt} - 3u_x u_{xxxt} - \beta u_{xx} - 3u_{xxx}u_t - \alpha u_{xyt} - u_{xxxt} = 0.$$

Therefore, we observe that Eq (1.2) is not fully recovered. Thus, Eq (1.2) is not self-adjoint. Now, the conserved vectors are given by

$$\begin{aligned} C^x = {} & \xi_x L + W[u_{xxt}(3s - 9s_x) + 6u_{xt}(s_x - s_{xx}) - 6su_{xxxt} - 3s_t u_{xx} + (3u_x + \beta)s_{xt} + s_{tt} \\ & + 3s_{xx}u_t + \alpha s_{yt} + s_{xxxt}] + W_x(-3s_t u_x - \beta s_t - 3s_x u_t - s_{xxt}) - \alpha W_y s_t + W_t(3su_{xx} - s_t \\ & - 3u_x s_x - \beta s_x - \alpha s_y - s_{xxx}) + W_{xx}(3su_t + s_{xt}) + W_{xt}(3su_x + \beta s + s_{xx}) + \alpha W_{yt} s \\ & + W_{tt}s - W_{xx}s_t - W_{xx}s_x + W_{xxx}s, \end{aligned}$$

$$C^y = \xi_y L + \alpha W s_{xt} - \alpha W_x s_t - \alpha W_t s_x + W_{xt} s,$$
$$C^t = \tau L + W(s_{xt} + 3u_x s_{xx} + \beta s_{xx} + \alpha s_{xy} + s_{xxxxx}) + W_x(3s_{u_x} - s_t - 3u_x s_x - \beta s_x - \alpha s_y - s_{xxx}) - \alpha W_y s_x - W_t s_x + W_{xx}(3su_x + \beta s + s_{xx}) + W_{xx}s + \alpha W_{xy}s - W_{xx}s_x + W_{xxxx}s.$$

with $W = \eta - \xi u_x - \phi u_y - \tau u_t$. These equations contain arbitrary function $s(x, y, t)$, arbitrary parameters α and β. This leads to the generation of an infinite number of conservation laws by defining their conservation vectors. Therefore, we can write the conservation laws for the vector fields (i)–(iv) in this way.

6.1. For $V_1 + V_3 + V_6 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{1}{\alpha}\frac{\partial}{\partial t} + dy\frac{\partial}{\partial u}$

For vector field $V_1 + V_3 + V_6$, we have $W = dy - u_x - u_y - \frac{1}{\alpha}u_t$.

$$\begin{aligned}
C^x =\ & s(u_{xtt} + u_{xxxxt} + 6u_{xx}u_{xt} + 3u_x u_{xxt} + 3u_{xxx}u_t + \alpha u_{xyt} + \beta u_{xxt}) + (dy - u_x - u_y \\
& - \frac{1}{\alpha}u_t)(3su_{xxt} - 6s_{xx}u_{xt} - 9s_x u_{xxt} - 6su_{xxxt} - 3s_t u_{xx} + 3s_{xt}u_x + \beta s_{xt} + s_{tt} + 3s_{xx}u_t \\
& + 6s_x u_{xt} + \alpha s_{yt} + s_{xxxt}) + (u_{xx} + u_{xy} + \frac{1}{\alpha}u_{xt})(3s_t u_x + \beta s_t + 3s_x u_t + s_{xxt}) - \alpha s_t(d \\
& - u_{xy} - u_{yy} - \frac{1}{\alpha}u_{yt}) + (-u_{xt} - u_{yt} - \frac{1}{\alpha}u_{tt})(3su_{xx} - s_t - 3u_x s_x - \beta s_x - \alpha s_y - s_{xxx}) \\
& - (u_{xx} + u_{xxy} + \frac{1}{\alpha}u_{xxt})(3su_t + s_{xt}) + (-u_{xxt} - u_{xyt} - \frac{1}{\alpha}u_{xtt})(3su_x + \beta s + s_{xx}) \\
& + \alpha s(-u_{xyt} - u_{yyt} - \frac{1}{\alpha}u_{ytt}) + s(-u_{xtt} - u_{ytt} - \frac{1}{\alpha}u_{ttt}) + s_t(u_{xxxx} + u_{xxxy} + \frac{1}{\alpha}u_{xxxt}) \\
& + s_x(u_{xxxt} + u_{xxyt} + \frac{1}{\alpha}u_{xxtt}) - s(u_{xxxxt} + u_{xxxyt} + \frac{1}{\alpha}u_{xxxtt}), \\
C^y =\ & s(u_{xtt} + u_{xxxxt} + 6u_{xx}u_{xt} + 3u_x u_{xxt} + 3u_{xxx}u_t + \alpha u_{xyt} + \beta u_{xxt}) + (dy - u_x - \frac{1}{\alpha}u_t \\
& - u_y)\alpha s_{xt} + \alpha s_t(u_{xx} + u_{xy} + \frac{1}{\alpha}u_{xt}) + \alpha s_x(u_{xt} + u_{yt} + \frac{1}{\alpha}u_{tt}) - s(u_{xxt} + u_{xyt} + \frac{1}{\alpha}u_{xtt}), \\
C^t =\ & \frac{1}{\alpha}s(u_{xtt} + u_{xxxxt} + 6u_{xx}u_{xt} + 3u_x u_{xxt} + 3u_{xx}u_t + \alpha u_{xyt} + \beta u_{xxt}) + (dy - u_x - u_y \\
& - \frac{1}{\alpha}u_t)(s_{xt} + 3u_x s_{xx} + \beta s_{xx} + \alpha s_{xy} + s_{xxxx}) - (u_{xx} + u_{xy} + \frac{1}{\alpha}u_{xt})(3su_{xx} - s_t - 3u_x s_x \\
& - \beta s_x - \alpha s_y - s_{xxx}) - \alpha s_x(d - u_{xy} - u_{yy} - \frac{1}{\alpha}u_{xt}) + s_x(u_{xt} + u_{yt} + \frac{1}{\alpha}u_{tt}) - (u_{xx} + u_{xxy} \\
& + \frac{1}{\alpha}u_{xxt})(3su_x + \beta s + s_{xx}) + s(-u_{xxt} - u_{xyt} - \frac{1}{\alpha}u_{xtt}) + \alpha s(-u_{xxy} - u_{xyy} - \frac{1}{\alpha}u_{xyt}) \\
& + s_x(u_{xxxx} + u_{xxxy} + \frac{1}{\alpha}u_{xxxt}) + s(-u_{xxxxx} - u_{xxxxy} - \frac{1}{\alpha}u_{xxxxt}).
\end{aligned}$$

6.2. For $V_1 + V_4 + V_6 = y\frac{\partial}{\partial u} + \frac{1}{\alpha}\frac{\partial}{\partial t} + \frac{\partial}{\partial y} + \frac{\partial}{\partial x}$

For vector field $V_1 + V_4 + V_6$, we have $W = y - u_x - u_y - \frac{1}{\alpha}u_t$.

$$\begin{aligned}
C^x =\ & s(u_{xtt} + u_{xxxxt} + 6u_{xx}u_{xt} + 3u_xu_{xxt} + 3u_{xxx}u_t + \alpha u_{xyt} + \beta u_{xxt}) + (-u_x - \frac{1}{\alpha}u_t - u_y \\
& + y)(3su_{xxt} - 6s_{xx}u_{xt} - 9s_xu_{xt} - 6su_{xxxt} - 3s_tu_{xx} + 3s_{xt}u_x + \beta s_{xt} + s_{tt} + 3s_{xx}u_t + \alpha s_{yt} \\
& + 6s_xu_{xt} + s_{xxxt}) + (-u_{xx} - u_{xy} - \frac{1}{\alpha}u_{xt})(-3s_tu_x - \beta s_t - 3s_xu_t - s_{xxt}) - \alpha s_t(1 - \frac{1}{\alpha}u_{ty} \\
& - u_{yy} - u_y) - (u_{xt} + u_{yt} + \frac{1}{\alpha}u_{tt})(3su_{xx} - s_t - 3u_xs_x - \beta s_x - \alpha s_y - s_{xxx}) - (u_{xxx} + u_{xxy} \\
& + \frac{1}{\alpha}u_{xt})(3su_t + s_{xt}) + (-u_{xxt} - u_{xyt} - \frac{1}{\alpha}u_{xtt})(3s_x + \beta s + s_{xx}) + s(-u_{xtt} - u_{ytt} - \frac{1}{\alpha}u_{ttt}) \\
& + \alpha s(-u_{yt} - u_{yyt} - \frac{1}{\alpha}u_{ytt}) + s_t(u_{xxxx} + u_{xxy} + \frac{1}{\alpha}u_{xxxt}) + s_x(u_{xxxt} + u_{xxyt} + \frac{1}{\alpha}u_{xxtt}) \\
& - (u_{xxxxt} + u_{xxyt} + \frac{1}{\alpha}u_{xxxtt})s, \\
C^y =\ & s(u_{xtt} + u_{xxxxt} + 6u_{xx}u_{xt} + 3u_xu_{xxt} + 3u_{xxx}u_t + \alpha u_{xyt} + \beta u_{xxt} + \alpha s_{xt}(y - u_x - \frac{1}{\alpha}u_t \\
& - u_y) + \alpha s_t(u_{xx} + u_{xy} + \frac{1}{\alpha}u_{xt}) + \alpha s_x(u_{xt} + u_{yt} + \frac{1}{\alpha}u_{tt}) + s(-u_{xxt} - u_{xyt} - \frac{1}{\alpha}u_{xtt}), \\
C^t =\ & \frac{1}{\alpha}s(u_{xtt} + u_{xxxxt} + 6u_{xx}u_{xt} + 3u_xu_{xxt} + 3u_{xxx}u_t + \alpha u_{xyt} + \beta u_{xxt}) + (-u_x - u_y - \frac{1}{\alpha}u_t \\
& + y)(s_{xt} + 3u_xs_{xx} + \beta s_{xx} + \alpha s_{xy} + s_{xxxx}) + (-u_{xx} - u_{xy} - \frac{1}{\alpha}u_{xt})(3su_{xx} - 3u_xs_x - \beta s_x \\
& - s_t - \alpha s_y - s_{xxx}) - \alpha s_x(1 - u_{xy} - u_{yy} - \frac{1}{\alpha}u_{ty}) + s_x(u_{xt} + u_{yt} + \frac{1}{\alpha}u_{tt}) + (-u_{xxx} - u_{xxy} \\
& - \frac{1}{\alpha}u_{xxt})(3su_x + \beta s + s_{xx}) + s(-u_{xxt} - u_{xyt} - \frac{1}{\alpha}u_{xtt}) + \alpha s(-u_{xxy} - u_{xyy} - \frac{1}{\alpha}u_{xyt}) \\
& + s_x(u_{xxxx} + u_{xxxy} + \frac{1}{\alpha}u_{xxxt}) + s(-u_{xxxxx} - u_{xxxxy} - \frac{1}{\alpha}u_{xxxxt}).
\end{aligned}$$

By repeating the previous calculation method, we calculate the remaining two Lie point symmetric conservation vectors. The results will not be repeated.

7. Conclusions and future work

In this paper, Lie symmetry analysis, the optimal systems and conservation laws of the (2+1)-dimensional Ito equation are given and the power series solutions and some special solutions and their plots are discussed. Because there are arbitrary constants and functions in these solutions, we select several appropriate parameters to draw the Figures 2, 3, 4 and 5, which show what happens when these parameters are different. Next, we obtain some power series solutions for the nonlinear ordinary differential equations. Finally, we establish the conservation laws of the Ito equation.
This paper employs the Lie symmetry analysis method to investigate the problem at hand. The selection of the vector field is restricted to the linear form of the undetermined coefficients. However, exploring other forms of undetermined functions may result in different solutions. Notably, recent studies by

some scholars have shown that the Lie symmetry transformation method can establish exact analytical solutions for parabolic waves, traveling waves, block solitons, multi-solitons, curved multi-solitons and other shapes. These results have significant implications for explaining various mathematical and physical phenomena and enhance the value of this research. Accordingly, we explore this topic further in subsequent work.

Use of AI tools declaration

The author states that he has not used artificial intelligence (AI) tools in the creation of this article.

Author contributions

Conceptualization, Z.Q. and L.L.; methodology, Z.Q. and L.L.; validation, Z.Q. and L.L.; formal analysis, Z.Q. and L.L.; investigation, Z.Q. and L.L.; writing—original draft preparation, Z.Q. and L.L.; writing—review and editing, L.L. All authors have read and agreed to the published version of the manuscript.

Conflict of interest

We declare that we have no competing interests in this paper.

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