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Research article

Higher-order uniform accurate numerical scheme for two-dimensional nonlinear fractional Hadamard integral equations

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Abstract: In this paper, we consider a higher-order numerical scheme for two-dimensional nonlinear fractional Hadamard integral equations with uniform accuracy. First, the high-order numerical scheme is constructed by using piecewise biquadratic logarithmic interpolations to approximate an integral function based on the idea of the modified block-by-block method. Secondly, for $0 < \gamma, \lambda < 1$, the convergence of the high order numerical scheme has the optimal convergence order of $O(\Delta_s^{4-\gamma} + \Delta_t^{4-\lambda})$. Finally, two numerical examples are used for experimental testing to support the theoretical findings.

Keywords: fractional Hadamard integral equations; higher-order uniform accurate numerical scheme; error estimations; optimal convergence order
Mathematics Subject Classification: 65R20, 65D30, 65L12

1. Introduction

In the last few decades, fractional differential equations have become an active research topic due to their applications in many fields, such as computer science, biology, mechanics and nonlinear fractional-order Lorenz system [1]. To date, many researchers have conducted in-depth research on several fractional order derivatives and integrals, such as Caputo, Riemann-Liouville, Riesz and so on. In [2], they develop an extension of a quadrature method for solving a class of ψ-fractional differential equations by converting them to an equivalent linear Volterra integral equation as follows:

$$y(t) = g(t) + \int_a^t \frac{y(\tau)(\psi(t) - \psi(\tau))^{\alpha-1}}{\Gamma(\alpha)}\psi'(\tau)d\tau, \quad 0 < \alpha < 1. \tag{1.1}$$

The Caputo-Hadamard fractional integrals are the special forms for $\psi(t) = log(t)$ in (1.1). However, the Caputo-Hadamard fractional integrals are also very important for simulating different physical problems [3, 4]. Recently, Caputo-Hadamard fractional differential integral equations have made a breakthrough in [5]. Therefore, in practice, the Caputo-Hadamard fractional integral equation is also worth further research. The purpose of this project was to construct an in-depth theory and application of the high-precision algorithm for Caputo-Hadamard fractional integral system on a uniform grid. To this end, we will study the following 2D nonlinear Caputo-Hadamard integral equation:

$$f(s,t) = g(s,t) + \int_a^s \int_c^t \frac{K(s,t,\tau,\omega,f(\tau,\omega))}{(\log s - \log \tau)^{\gamma}(\log t - \log \omega)^{\lambda}} \frac{d\omega}{\omega}\frac{d\tau}{\tau}, \quad (s,t) \in \Theta, 0 < \gamma, \lambda < 1, \tag{1.2}$$

where $K(s,t,\tau,\omega,f(\tau,\omega))$, $g(s,t)$ are known functions and $f(s,t)$ is an unknown function in $\Theta = [a,b]\times[c,d]$, $\Omega = \Theta \times \Theta \times R$. We assume that the solution of (1.2) is smooth and $K(s,t,\tau,\omega,f(\tau,\omega)$ satisfies the Lipschitz condition about the fifth variable.

$$|K(s,t,\tau,\omega,f_1(\tau,\omega)) - K(s,t,\tau,\omega,f_2(\tau,\omega))| \leq L|f_1(\tau,\omega) - f_2(\tau,\omega)|, L > 0. \tag{1.3}$$

In [6], a solution of the Caputo-Hadamard fractional differential equation based on a hierarchical grid is presented. In [7], the solution's existence and uniqueness of fractional Caputo-Hadamard type equations are given. In an existing reference [8], we consider the logarithmic decay and regularity of solutions of Caputo-Hadamard fractional diffusion equations. In [9], a new stable and high order uniform accuracy solution method is given for 2D nonlinear integral equations. In [10], the fractional stochastic differential equations solution's existence and uniqueness was proved by the fixed point method in the Caputo-Hadamard sense. In [11], they present three types of variable fractional orders in the Caputo-Hadamard sense. In [12], numerical methods for the initial singularity fractional equations were investigated by using the modified Laplace transform and FFT in the Caputo-Hadamard sense. In [13], they used the incomplete Gamma function to construct the time discretization scheme for Caputo-Hadamard fractional differential equations. In [14], a local discontinuous Galerkin algorithm based on the Gronwall inequality is proposed for fractional differential equations in the sense of Caputo-Hadamard. In [15], a kind of graded grid algorithm in the sense of Caputo-Hadamard is studied. In [16], they established a method for solving Caputo-Hadamard integrals and a class of fractional derivatives by using the logarithmic transformation of Jacobian polynomials. The calculation methods for three fractional differentials in the sense of Caputo-Hadamard have been given in [17]. In [18], they use uncertain fractional derivatives to discuss the price of European options. The research of Caputo-Hadamard equation can be found in [19], the applicant uses the spectrum method to give an algorithm for sets of Caputo-Hadamard fractional PDE. In [20], a numerical method for time-space fractional discrete systems in the sense of Caputo-Hadamard is given. For more information, please refer to [21–23].

In this paper, we will use a uniform mesh to solve 2D nonlinear fractional Hadamard integral equations based on the idea of [9], achieving uniform accuracy by using piecewise biquadratic logarithmic interpolations. For $0 < \gamma, \lambda < 1$, we obtain an optimal convergence order of $O(\Delta_s^{4-\gamma} + \Delta_t^{4-\lambda})$ for a sufficiently smooth solution and a generalized nonlinear kernel function by using the Gronwall inequality based on the convergence analysis of a new technique.

The framework of this article is as follows. In Section 2, we give a high precision numerical method. The truncation error of the high order numerical method is mainly discussed in Section 3. In Section 4, the convergence of the high order numerical scheme is analyzed. In Section 5, two numerical experiments are presented to support our theory and demonstrate the efficiency of higher-order numerical methods. The final section summarizes the main contribution of the work and the future research work.

2. Higher-order numerical scheme using two-dimensional nonlinear fractional Hadamard integral equations

Now, we can obtain an approximate evaluation of the Caputo Hadamard integral equations in two dimensions. In order to construct a numerical scheme for (1.2), we partition the domain Θ into $2Y \times 2X$ subdomains of equal size $\Delta_s = \frac{b-a}{2Y}$ and $\Delta_t = \frac{d-c}{2X}$, where Y and X are positive integers. Assuming $s_n = a + n\Delta_s$ and $t_m = c + m\Delta_t$, $n = 0, 1, 2, \cdots, 2Y$, $m = 0, 1, 2, \cdots, 2X$ with $a, c \geq 1$, where $a = s_0 < s_1 < \cdots < s_{2Y} = b$ and $c = t_0 < t_1 < \cdots < t_{2X} = d$ are respective partitions of $[a, b]$ and $[c, d]$. And we use f_n^m to represent a numerical scheme for (1.2) at a point (s_n, t_m), and we let $K_n^m(\tau, \omega, f(\tau, \omega)) = K(s_n, t_m, \tau, \omega, f(\tau, \omega))$ and $g_n^m = g(s_n, t_m)$.

In this work, we present a high-order method for solving nonlinear fractional Hadamard integral equations. The basis functions of our approach are determined by the quadratic logarithmic interpolations of polynomials at points s_n, s_{n+1}, s_{n+2} and t_m, t_{m+1}, t_{m+2}, and they are assumed to be $\varphi_k^n(\tau), k = 0, 1, 2; n \in \mathbb{N}$ and $\phi_k^m(\omega), k = 0, 1, 2; m \in \mathbb{N}$, respectively, and they are defined as follows:

$$
\begin{aligned}
\varphi_0^n(\tau) &= \frac{(\log\tau - \log s_{n+1})(\log\tau - \log s_{n+2})}{(\log s_n - \log s_{n+1})(\log s_n - \log s_{n+2})},\\
\varphi_1^n(\tau) &= \frac{(\log\tau - \log s_{n})(\log\tau - \log s_{n+2})}{(\log s_{n+1} - \log s_{n})(\log s_{n+1} - \log s_{n+2})},\\
\varphi_2^n(\tau) &= \frac{(\log\tau - \log s_{n})(\log\tau - \log s_{n+1})}{(\log s_{n+2} - \log s_{n})(\log s_{n+2} - \log s_{n+1})};\\
\phi_0^m(\omega) &= \frac{(\log\omega - \log t_{m+1})(\log\omega - \log t_{m+2})}{(\log t_m - \log t_{m+1})(\log t_m - \log t_{m+2})},\\
\phi_1^m(\omega) &= \frac{(\log\omega - \log t_{m})(\log\omega - \log t_{m+2})}{(\log t_{m+1} - \log t_{m})(\log t_{m+1} - \log t_{m+2})},\\
\phi_2^m(\omega) &= \frac{(\log\omega - \log t_{m})(\log\omega - \log t_{m+1})}{(\log t_{m+2} - \log t_{m})(\log t_{m+2} - \log t_{m+1})}.
\end{aligned}
$$

In what follows, we construct the numerical scheme approximation to $f(s, t)$ at a point (s_1, t_1):

$$
\begin{aligned}
f(s_1, t_1) =& g_1^1 + \int_a^{s_1}\int_c^{t_1} \frac{K_1^1(\tau, \omega, f(\tau, \omega))}{(\log s_1 - \log\tau)^{\gamma}(\log t_1 - \log\omega)^{\lambda}} \frac{d\omega}{\omega}\frac{d\tau}{\tau}\\
\approx& g_1^1 + \int_a^{s_1}\int_c^{t_1} \frac{1}{(\log s_1 - \log\tau)^{\gamma}(\log t_1 - \log\omega)^{\lambda}} \sum_{n=0}^{2}\sum_{m=0}^{2} \varphi_n^0(\tau)\phi_m^0(\omega) K_1^1(s_n, t_m, f_n^m) \frac{d\omega}{\omega}\frac{d\tau}{\tau}\\
=& g_1^1 + \sum_{n=0}^{2}\sum_{m=0}^{2} T_1^{n,0}\hat{T}_1^{m,0} K_1^1(s_n, t_m, f_n^m),
\end{aligned} \tag{2.1}
$$

with

$$
T_1^{n,0} = \int_a^{s_1} \frac{\varphi_n^0(\tau)}{(\log s_1 - \log\tau)^{\gamma}}\frac{d\tau}{\tau}, n = 0, 1, 2; \hat{T}_1^{m,0} = \int_c^{t_1} \frac{\phi_m^0(\omega)}{(\log t_1 - \log\omega)^{\lambda}}\frac{d\omega}{\omega}, m = 0, 1, 2. \tag{2.2}
$$

Similarly, we compute $f(s_2,t_1), f(s_1,t_2)$ and $f(s_2,t_2)$ to obtain the following approximate values:

$$
\begin{aligned}
f(s_2,t_1) &= g_2^1 + \int_a^{s_2}\int_c^{t_1} \frac{K_2^1(\tau,\omega,f(\tau,\omega))}{(\log s_2-\log\tau)^\gamma(\log t_1-\log\omega)^\lambda}\frac{d\omega}{\omega}\frac{d\tau}{\tau} \\
&\approx g_2^1 + \sum_{n=0}^{2}\sum_{m=0}^{2} T_2^{n,0}\hat{T}_1^{m,0}K_2^1(s_n,t_m,f_n^m), \qquad (2.3)\\
f(s_1,t_2) &= g_1^2 + \int_a^{s_1}\int_c^{t_2} \frac{K_1^2(\tau,\omega,f(\tau,\omega))}{(\log s_1-\log\tau)^\gamma(\log t_2-\log\omega)^\lambda}\frac{d\omega}{\omega}\frac{d\tau}{\tau} \\
&\approx g_1^2 + \sum_{n=0}^{2}\sum_{m=0}^{2} T_1^{n,0}\hat{T}_2^{m,0}K_1^2(s_n,t_m,f_n^m), \qquad (2.4)\\
f(s_2,t_2) &= g_2^2 + \int_a^{s_2}\int_c^{t_2} \frac{K_2^2(\tau,\omega,f(\tau,\omega))}{(\log s_2-\log\tau)^\gamma(\log t_2-\log\omega)^\lambda}\frac{d\omega}{\omega}\frac{d\tau}{\tau} \\
&\approx g_2^2 + \sum_{n=0}^{2}\sum_{m=0}^{2} T_2^{n,0}\hat{T}_2^{m,0}K_2^2(s_n,t_m,f_n^m), \qquad (2.5)
\end{aligned}
$$

where

$$
T_2^{n,0} = \int_a^{s_2} \frac{\varphi_n^0(\tau)}{(\log s_2-\log\tau)^\gamma}\frac{d\tau}{\tau}, n=0,1,2; \hat{T}_2^{m,0} = \int_c^{t_2}\frac{\phi_m^0(\omega)}{(\log t_2-\log\omega)^\lambda}\frac{d\omega}{\omega}, m=0,1,2. \qquad (2.6)
$$

We need to compute f_1^1 through the use of (2.1) by using the values of K at s_1,s_2 and t_1,t_2. Particularly the dependence of f_1^1 on K_2^1, K_1^2 and K_2^2 means that (2.1) and (2.3)–(2.5) must be couple solved with the scheme.

Next, we estimate $f(s_{2y+l},t_r), y=1,\cdots,Y-1$ and $f(s_l,t_{2x+r}), l,r=1,2, x=1,\cdots,X-1$. Assuming that $f_n^r, n=0,1,\cdots,2y$ and $f_l^m, m=0,1,\cdots,2x$ are known, we have $f(s_{2y+1},t_1)$:

$$
\begin{aligned}
f(s_{2y+1},t_1) =& g_{2y+1}^1 + \int_a^{s_1}\int_c^{t_1} \frac{K_{2y+1}^1(\tau,\omega,f(\tau,\omega))}{\left(\log s_{2y+1}-\log s\right)^\gamma(\log t_1-\log\omega)^\lambda}\frac{d\omega}{\omega}\frac{d\tau}{\tau}\\
&+\sum_{n=1}^{y}\int_{s_{2n-1}}^{s_{2n+1}}\int_c^{t_1} \frac{K_{2y+1}^1(\tau,\omega,f(\tau,\omega))}{\left(\log s_{2y+1}-\log\tau\right)^\gamma(\log t_1-\log\omega)^\lambda}\frac{d\omega}{\omega}\frac{d\tau}{\tau}\\
\approx& g_{2y+1}^1 + \sum_{k=0}^{2}\sum_{q=0}^{2}\int_a^{s_1}\int_c^{t_1}\frac{\left(\log s_{2y+1}-\log\tau\right)^{-\gamma}}{(\log t_1-\log\omega)^\lambda}\\
&\times\varphi_k^0(\tau)\phi_q^0(\omega)K_{2y+1}^1\left(s_k,t_q,f_k^q\right)\frac{d\omega}{\omega}\frac{d\tau}{\tau}\\
&+\sum_{n=1}^{y}\sum_{k=0}^{2}\sum_{q=0}^{2}\int_{s_{2n-1}}^{s_{2n+1}}\int_c^{t_1}\frac{\left(\log s_{2y+1}-\log\tau\right)^{-\gamma}}{(\log t_1-\log\omega)^\lambda}\\
&\times\varphi_k^{2n-1}(\tau)\phi_q^0(\omega)K_{2y+1}^1\left(s_{2n-1+k},t_q,f_{2n-1+k}^q\right)\frac{d\omega}{\omega}\frac{d\tau}{\tau}\\
=& g_{2y+1}^1 + \sum_{k=0}^{2}\sum_{q=0}^{2}T_{2y+1}^{k,0}\hat{T}_1^{q,0}K_{2y+1}^1\left(s_k,t_q,f_k^q\right)
\end{aligned}
$$

$$+\sum_{n=1}^{y}\sum_{k=0}^{2}\sum_{q=0}^{2}T_{2y+1}^{k,n}\hat{T}_{1}^{q,0}K_{2y+1}^{1}\left(s_{2n-1+k},t_q,f_{2n-1+k}^{q}\right), \tag{2.7}$$

where $\hat{T}_1^{q,0}$ is given by (2.2) and

$$T_{2y+1}^{k,0}=\int_a^{s_1}\frac{\varphi_k^0(\tau)}{\left(\log s_{2y+1}-\log\tau\right)^{\gamma}}\frac{d\tau}{\tau},k=0,1,2, \tag{2.8}$$

$$T_{2y+1}^{k,n}=\int_{s_{2n-1}}^{s_{2n+1}}\frac{\varphi_k^{2n-1}(\tau)}{\left(\log s_{2y+1}-\log\tau\right)^{\gamma}}\frac{d\tau}{\tau},k=0,1,2;n=1,2,\cdots,Y. \tag{2.9}$$

For $f(s_{2y+2},t_1)$ and $f(s_{2y+l},t_2), l=1,2$, we use the following approximate estimates:

$$\begin{aligned}
f\left(s_{2y+2},t_1\right)=&g_{2y+2}^{1}+\sum_{n=0}^{y}\int_{s_{2n}}^{s_{2n+2}}\int_{c}^{t_1}\frac{K_{2y+2}^{1}(\tau,\omega,f(\tau,\omega))}{\left(\log s_{2y+2}-\log\tau\right)^{\gamma}(\log t_1-\log\omega)^{\lambda}}\frac{d\omega}{\omega}\frac{d\tau}{\tau}\\
\approx&g_{2y+2}^{1}+\sum_{n=0}^{y}\sum_{k=0}^{2}\sum_{q=0}^{2}\int_{s_{2n}}^{s_{2n+2}}\int_{c}^{t_1}\frac{\left(\log s_{2y+2}-\log\tau\right)^{-\gamma}}{(\log t_1-\log\omega)^{\lambda}}\\
&\times\varphi_k^{2n}(\tau)\phi_q^{0}(\omega)K_{2y+2}^{1}\left(s_{2n+k},t_q,f_{2n+k}^{q}\right)\frac{d\omega}{\omega}\frac{d\tau}{\tau}\\
=&g_{2y+2}^{1}+\sum_{n=0}^{y}\sum_{k=0}^{2}\sum_{q=0}^{2}T_{2y+2}^{k,n}\hat{T}_{1}^{q,0}K_{2y+2}^{1}\left(s_{2n+k},t_q,f_{2n+k}^{q}\right),
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
f\left(s_{2y+1},t_2\right)=&g_{2y+1}^{2}+\int_{a}^{s_1}\int_{c}^{t_2}\frac{K_{2y+1}^{2}(\tau,\omega,f(\tau,\omega))}{\left(\log s_{2y+1}-\log\tau\right)^{\gamma}(\log t_2-\log\omega)^{\lambda}}\frac{d\omega}{\omega}\frac{d\tau}{\tau}\\
&+\sum_{n=1}^{y}\int_{s_{2n-1}}^{s_{2n+1}}\int_{c}^{t_2}\frac{K_{2y+1}^{2}(\tau,\omega,f(\tau,\omega))}{\left(\log s_{2y+1}-\log\tau\right)^{\gamma}(\log t_2-\log\omega)^{\lambda}}\frac{d\omega}{\omega}\frac{d\tau}{\tau}\\
\approx&g_{2y+1}^{2}+\sum_{k=0}^{2}\sum_{q=0}^{2}\int_{a}^{s_1}\int_{c}^{t_2}\frac{\left(\log s_{2y+1}-\log\tau\right)^{-\gamma}}{(\log t_2-\log\omega)^{\lambda}}\\
&\times\varphi_k^{0}(\tau)\phi_q^{0}(\omega)K_{2y+1}^{2}\left(s_k,t_q,f_k^{q}\right)\frac{d\omega}{\omega}\frac{d\tau}{\tau}\\
&+\sum_{n=1}^{y}\sum_{k=0}^{2}\sum_{q=0}^{2}\int_{s_{2n-1}}^{s_{2n+1}}\int_{c}^{t_2}\frac{\left(\log s_{2y+1}-\log\tau\right)^{-\gamma}}{(\log t_2-\log\omega)^{\lambda}}\\
&\times\varphi_k^{2n-1}(\tau)\phi_q^{0}(\omega)K_{2y+1}^{2}\left(s_{2n-1+k},t_q,f_{2n-1+k}^{q}\right)\frac{d\omega}{\omega}\frac{d\tau}{\tau}\\
=&g_{2y+1}^{2}+\sum_{k=0}^{2}\sum_{q=0}^{2}T_{2y+1}^{k,0}\hat{T}_{2}^{q,0}K_{2y+1}^{2}\left(s_k,t_q,f_k^{q}\right)\\
&+\sum_{n=1}^{y}\sum_{k=0}^{2}\sum_{q=0}^{2}T_{2y+1}^{k,n}\hat{T}_{2}^{q,0}K_{2y+1}^{2}\left(s_{2n-1+k},t_q,f_{2n-1+k}^{q}\right),
\end{aligned} \tag{2.11}$$

$$
\begin{aligned}
f\left(s_{2y+2},t_2\right)=&g_{2y+2}^2+\sum_{n=0}^{y}\int_{s_{2n}}^{s_{2n+2}}\int_c^{t_2}\frac{K_{2y+2}^2(\tau,\omega,f(\tau,\omega))}{\left(\log s_{2y+2}-\log\tau\right)^{\gamma}\left(\log t_2-\log\omega\right)^{\lambda}}\frac{d\omega}{\omega}\frac{d\tau}{\tau}\\
\approx&g_{2y+2}^2+\sum_{n=0}^{y}\sum_{k=0}^{2}\sum_{q=0}^{2}\int_{s_{2n}}^{s_{2n+2}}\int_c^{t_2}\frac{\left(\log s_{2y+2}-\log\tau\right)^{-\gamma}}{\left(\log t_2-\log\omega\right)^{\lambda}}\\
&\times\varphi_k^{2n}(\tau)\phi_q^0(\omega)K_{2y+2}^2\left(s_{2n+k},t_q,f_{2n+k}^q\right)\frac{d\omega}{\omega}\frac{d\tau}{\tau}\\
=&g_{2y+2}^2+\sum_{n=0}^{y}\sum_{k=0}^{2}\sum_{q=0}^{2}T_{2y+2}^{k,n}\hat{T}_2^{q,0}K_{2y+2}^2\left(s_{2n+k},t_q,f_{2n+k}^q\right),
\end{aligned}\tag{2.12}
$$

where $\hat{T}_1^{q,0},\hat{T}_2^{q,0},T_{2y+1}^{k,0},T_{2y+1}^{k,n}$ are defined by (2.2), (2.6), (2.8) and (2.9), respectively, and $T_{2y+2}^{k,n}$ is defined as follows:

$$
T_{2y+2}^{k,n}=\int_{s_{2n}}^{s_{2n+2}}(\log s_{2y+2}-\log\tau)^{-\gamma}\varphi_k^{2n}(\tau)\frac{d\tau}{\tau},k=0,1,2;n=0,1,\cdots,y.\tag{2.13}
$$

In the same way, we estimate $f(s_l,t_{2x+r})$ for $l,r=1,2$ as follows:

$$
\begin{aligned}
f(s_1,t_{2x+1})=&g_1^{2x+1}+\int_a^{s_1}\int_c^{t_1}\frac{K_1^{2x+1}(\tau,\omega,f(\tau,\omega))}{(\log s_1-\log\tau)^{\gamma}(\log t_{2x+1}-\log\omega)^{\lambda}}\frac{d\omega}{\omega}\frac{d\tau}{\tau}\\
&+\sum_{m=1}^{x}\int_a^{s_1}\int_{t_{2m-1}}^{t_{2m+1}}\frac{K_1^{2x+1}(\tau,\omega,f(\tau,\omega))}{(\log s_1-\log\tau)^{\gamma}(\log t_{2x+1}-\log\omega)^{\lambda}}\frac{d\omega}{\omega}\frac{d\tau}{\tau}\\
\approx&g_1^{2x+1}+\sum_{k=0}^{2}\sum_{q=0}^{2}T_1^{k,0}\hat{T}_{2x+1}^{q,0}K_1^{2x+1}(s_k,t_q,f_k^q)\\
&+\sum_{m=1}^{x}\sum_{k=0}^{2}\sum_{q=0}^{2}T_1^{k,0}\hat{T}_{2x+1}^{q,m}K_1^{2x+1}(s_k,t_{2m-1+q},f_k^{2m-1+q}),
\end{aligned}\tag{2.14}
$$

$$
\begin{aligned}
f(s_2,t_{2x+1})=&g_2^{2x+1}+\int_a^{s_2}\int_c^{t_1}\frac{K_2^{2x+1}(\tau,\omega,f(\tau,\omega))}{(\log s_2-\log\tau)^{\gamma}(\log t_{2x+1}-\log\omega)^{\lambda}}\frac{d\omega}{\omega}\frac{d\tau}{\tau}\\
&+\sum_{m=1}^{x}\int_a^{s_2}\int_{t_{2m-1}}^{t_{2m+1}}\frac{K_2^{2x+1}(\tau,\omega,f(\tau,\omega))}{(\log s_2-\log\tau)^{\gamma}(\log t_{2x+1}-\log\omega)^{\lambda}}\frac{d\omega}{\omega}\frac{d\tau}{\tau}\\
\approx&g_2^{2x+1}+\sum_{k=0}^{2}\sum_{q=0}^{2}T_2^{k,0}\hat{T}_{2x+1}^{q,0}K_2^{2x+1}(s_k,t_q,f_k^q)\\
&+\sum_{m=1}^{x}\sum_{k=0}^{2}\sum_{q=0}^{2}T_2^{k,0}\hat{T}_{2x+1}^{q,m}K_2^{2x+1}(s_k,t_{2m-1+q},f_k^{2m-1+q}),
\end{aligned}\tag{2.15}
$$

$$
\begin{aligned}
f(s_1,t_{2x+2})=&g_1^{2x+2}+\sum_{m=0}^{x}\int_a^{s_1}\int_{t_{2m}}^{t_{2m+2}}\frac{K_1^{2x+2}(\tau,\omega,f(\tau,\omega))}{(\log s_1-\log\tau)^{\gamma}(\log t_{2x+2}-\log\omega)^{\lambda}}\frac{d\omega}{\omega}\frac{d\tau}{\tau}\\
\approx&g_1^{2x+2}+\sum_{m=0}^{x}\sum_{k=0}^{2}\sum_{q=0}^{2}T_1^{k,0}\hat{T}_{2x+2}^{q,m}K_1^{2x+2}(s_k,t_{2m+q},f_k^{2m+q}),
\end{aligned}\tag{2.16}
$$

$$f(s_2,t_{2x+2}) = g_2^{2x+2} + \sum_{m=0}^{x}\int_a^{s_2}\int_{t_{2m}}^{t_{2m+2}} \frac{K_2^{2x+2}(\tau,\omega,f(\tau,\omega))}{(\log s_2-\log\tau)^{\gamma}(\log t_{2x+2}-\log\omega)^{\lambda}}\frac{d\omega}{\omega}\frac{d\tau}{\tau}$$
$$\approx g_2^{2x+2} + \sum_{m=0}^{x}\sum_{k=0}^{2}\sum_{q=0}^{2} T_2^{k,0}\hat{T}_{2x+2}^{q,m}K_2^{2x+2}(s_k,t_{2m+q},f_k^{2m+q}), \tag{2.17}$$

where $T_2^{k,0}$ and $T_1^{k,0}$ are given by (2.6) and (2.2), respectively, and

$$\hat{T}_{2x+1}^{q,0} = \int_c^{t_1}(\log t_{2x+1}-\log\omega)^{-\lambda}\phi_q^0(\omega)\frac{d\omega}{\omega}, q=0,1,2, \tag{2.18}$$
$$\hat{T}_{2x+1}^{q,m} = \int_{t_{2m-1}}^{t_{2m+1}}(\log t_{2x+1}-\log\omega)^{-\lambda}\phi_q^{2m-1}(\omega)\frac{d\omega}{\omega}, q=0,1,2; m=1,2,\cdots,x, \tag{2.19}$$
$$\hat{T}_{2x+2}^{q,m} = \int_{t_{2m}}^{t_{2m+2}}(\log t_{2x+2}-\log\omega)^{-\lambda}\phi_q^{2m}(\omega)\frac{d\omega}{\omega}, q=0,1,2; m=0,1,\cdots,x. \tag{2.20}$$

Similarly, when $f_n^m, f_n^{2x+1}, f_n^{2x+2}, f_{2y+1}^m$ and $f_{2y+2}^m, n=0,1,\cdots,2y; m=0,1,\cdots,2x; y=1,\cdots,Y-1; x=1,\cdots,X-1$ are already known, we will construct the high order scheme for $f(s_{2y+p},t_{2x+q}), p,q=1,2$ as follows.

For $f(s_{2y+1},t_{2x+1})$, we have

$$\begin{aligned} f(s_{2y+1},t_{2x+1}) &= g_{2y+1}^{2x+1} + \int_a^{s_1}\int_c^{t_1}\frac{K_{2y+1}^{2x+1}(\tau,\omega,f(\tau,\omega))}{(\log s_{2y+1}-\log\tau)^{\gamma}(\log t_{2x+1}-\log\omega)^{\lambda}}\frac{d\omega}{\omega}\frac{d\tau}{\tau} \\ &+ \sum_{m=1}^{x}\int_a^{s_1}\int_{t_{2m-1}}^{t_{2m+1}}\frac{K_{2y+1}^{2x+1}(\tau,\omega,f(\tau,\omega))}{(\log s_{2y+1}-\log\tau)^{\gamma}(\log t_{2x+1}-\log\omega)^{\lambda}}\frac{d\omega}{\omega}\frac{d\tau}{\tau} \\ &+ \sum_{n=1}^{y}\int_{s_{2n-1}}^{s_{2n+1}}\int_c^{t_1}\frac{K_{2y+1}^{2x+1}(\tau,\omega,f(\tau,\omega))}{(\log s_{2y+1}-\log\tau)^{\gamma}(\log t_{2x+1}-\log\omega)^{\lambda}}\frac{d\omega}{\omega}\frac{d\tau}{\tau} \\ &+ \sum_{n=1}^{y}\sum_{m=1}^{x}\int_{s_{2n-1}}^{s_{2n+1}}\int_{t_{2m-1}}^{t_{2m+1}}\frac{K_{2y+1}^{2x+1}(\tau,\omega,f(\tau,\omega))}{(\log s_{2y+1}-\log\tau)^{\gamma}(\log t_{2x+1}-\log\omega)^{\lambda}}\frac{d\omega}{\omega}\frac{d\tau}{\tau} \\ &\doteq g_{2y+1}^{2x+1} + F_1 + F_2 + F_3 + F_4. \end{aligned} \tag{2.21}$$

For F_1, by using biquadratic interpolation one can obtain

$$\begin{aligned} F_1 &= \int_a^{s_1}\int_c^{t_1}\frac{K_{2y+1}^{2x+1}(\tau,\omega,f(\tau,\omega))}{(\log s_{2y+1}-\log\tau)^{\gamma}(\log t_{2x+1}-\log\omega)^{\lambda}}\frac{d\omega}{\omega}\frac{d\tau}{\tau} \\ &\approx \int_a^{s_1}\int_c^{t_1}\frac{(\log s_{2y+1}-\log\tau)^{-\gamma}}{(\log t_{2x+1}-\log\omega)^{\lambda}}\sum_{k=0}^{2}\sum_{q=0}^{2}\varphi_k^0(\tau)\phi_q^0(\omega)K_{2y+1}^{2x+1}(s_k,t_q,f_k^q)\frac{d\omega}{\omega}\frac{d\tau}{\tau} \\ &= \sum_{k=0}^{2}\sum_{q=0}^{2} T_{2y+1}^{k,0}\hat{T}_{2x+1}^{q,0}K_{2y+1}^{2x+1}(s_k,t_q,f_k^q), \end{aligned} \tag{2.22}$$

where $T_{2y+1}^{k,0}$ and $\hat{T}_{2x+1}^{q,0}$ are defined by (2.8) and (2.18), respectively.

For F_2, through direct calculation, it can be concluded that

$$\begin{aligned}
F_2 &= \sum_{m=1}^{x} \int_a^{s_1} \int_{t_{2m-1}}^{t_{2m+1}} \frac{K_{2y+1}^{2x+1}(\tau,\omega,f(\tau,\omega))}{(\log s_{2y+1}-\log\tau)^{\gamma}(\log t_{2x+1}-\log\omega)^{\lambda}} \frac{d\omega}{\omega}\frac{d\tau}{\tau} \\
&\approx \sum_{m=1}^{x} \int_a^{s_1} \int_{t_{2m-1}}^{t_{2m+1}} \frac{(\log s_{2y+1}-\log\tau)^{-\gamma}}{(\log t_{2x+1}-\log\omega)^{\lambda}} \sum_{k=0}^{2}\sum_{q=0}^{2} \varphi_k^0(\tau)\phi_q^{2m-1}(\omega) K_{2y+1}^{2x+1}(s_k,t_{2m-1+q},f_k^{2m-1+q}) \frac{d\omega}{\omega}\frac{d\tau}{\tau} \\
&= \sum_{m=1}^{x}\sum_{k=0}^{2}\sum_{q=0}^{2} T_{2y+1}^{k,0}\hat{T}_{2x+1}^{q,m} K_{2y+1}^{2x+1}(s_k,t_{2m-1+q},f_k^{2m-1+q}),
\end{aligned} \tag{2.23}$$

where $T_{2y+1}^{k,0}$ and $\hat{T}_{2x+1}^{q,m}$ are given by (2.8) and (2.19), respectively.

For F_3, one can obtain that

$$\begin{aligned}
F_3 &= \sum_{n=1}^{y} \int_{s_{2n-1}}^{s_{2n+1}} \int_c^{t_1} \frac{K_{2y+1}^{2x+1}(\tau,\omega,f(\tau,\omega))}{(\log s_{2y+1}-\log\tau)^{\gamma}(\log t_{2x+1}-\log\omega)^{\lambda}} \frac{d\omega}{\omega}\frac{d\tau}{\tau} \\
&\approx \sum_{n=1}^{y}\sum_{k=0}^{2}\sum_{q=0}^{2} \int_{s_{2n-1}}^{s_{2n+1}} \int_c^{t_1} \frac{(\log s_{2y+1}-\log\tau)^{-\gamma}}{(\log t_{2x+1}-\log\omega)^{\lambda}} \\
&\quad \times \varphi_k^{2n-1}(\tau)\phi_q^{0}(\omega) K_{2y+1}^{2x+1}(s_{2n-1+k},t_q,f_{2n-1+k}^{q}) \frac{d\omega}{\omega}\frac{d\tau}{\tau} \\
&= \sum_{n=1}^{y}\sum_{k=0}^{2}\sum_{q=0}^{2} T_{2y+1}^{k,n}\hat{T}_{2x+1}^{q,0} K_{2y+1}^{2x+1}(s_{2n-1+k},t_q,f_{2n-1+k}^{q}),
\end{aligned} \tag{2.24}$$

where $T_{2y+1}^{k,n}$ and $\hat{T}_{2x+1}^{q,0}$ are given by (2.9) and (2.18), respectively.

Similarly, for F_4, we have

$$\begin{aligned}
F_4 &= \sum_{n=1}^{y}\sum_{m=1}^{x} \int_{s_{2n-1}}^{s_{2n+1}} \int_{t_{2m-1}}^{t_{2m+1}} \frac{K_{2y+1}^{2x+1}(\tau,\omega,f(\tau,\omega))}{(\log s_{2y+1}-\log\tau)^{\gamma}(\log t_{2x+1}-\log\omega)^{\lambda}} \frac{d\omega}{\omega}\frac{d\tau}{\tau} \\
&\approx \sum_{n=1}^{y}\sum_{m=1}^{x}\sum_{k=0}^{2}\sum_{q=0}^{2} \int_{s_{2n-1}}^{s_{2n+1}} \int_{t_{2m-1}}^{t_{2m+1}} \frac{(\log s_{2y+1}-\log\tau)^{-\gamma}}{(\log t_{2x+1}-\log\omega)^{\lambda}} \\
&\quad \times \varphi_k^{2n-1}(\tau)\phi_q^{2m-1}(\omega) K_{2y+1}^{2x+1}(s_{2n-1+k},t_{2m-1+q},f_{2n-1+k}^{2m-1+q}) \frac{d\omega}{\omega}\frac{d\tau}{\tau} \\
&= \sum_{n=1}^{y}\sum_{m=1}^{x}\sum_{k=0}^{2}\sum_{q=0}^{2} T_{2y+1}^{k,n}\hat{T}_{2x+1}^{q,m} K_{2y+1}^{2x+1}(s_{2n-1+k},t_{2m-1+q},f_{2n-1+k}^{2m-1+q}),
\end{aligned} \tag{2.25}$$

where $T_{2y+1}^{k,n}$ and $\hat{T}_{2x+1}^{q,m}$ are given by (2.9) and (2.19), respectively.

Bringing (2.22)–(2.25) into (2.21), one can obtain

$$\begin{aligned}
f_{2y+1}^{2x+1} \quad &= \quad g_{2y+1}^{2x+1} + \sum_{k=0}^{2}\sum_{q=0}^{2} T_{2y+1}^{k,0}\hat{T}_{2x+1}^{q,0} K_{2y+1}^{2x+1}(s_k,t_q,f_k^{q}) \\
&\quad + \sum_{m=1}^{x}\sum_{k=0}^{2}\sum_{q=0}^{2} T_{2y+1}^{k,0}\hat{T}_{2x+1}^{q,m} K_{2y+1}^{2x+1}(s_k,t_{2m-1+q},f_k^{2m-1+q})
\end{aligned}$$

$$+\sum_{n=1}^{y}\sum_{k=0}^{2}\sum_{q=0}^{2}T_{2y+1}^{k,n}\hat{T}_{2x+1}^{q,0}K_{2y+1}^{2x+1}(s_{2n-1+k},t_q,f_{2n-1+k}^{q})$$
$$+\sum_{n=1}^{y}\sum_{m=1}^{x}\sum_{k=0}^{2}\sum_{q=0}^{2}T_{2y+1}^{k,n}\hat{T}_{2x+1}^{q,m}K_{2y+1}^{2x+1}(s_{2n-1+k},t_{2m-1+q},f_{2n-1+k}^{2m-1+q}). \quad (2.26)$$

Furthermore, we will construct a high order numerical scheme for $f(s_{2y+2},t_{2x+1})$. By dividing the integral domain into subdomains and using the piecewise biquadratic interpolation method, we calculate $f(s_{2y+2},t_{2x+1})$ as follows

$$\begin{aligned} f(s_{2y+2},t_{2x+1}) &= g_{2y+2}^{2x+1}+\sum_{n=0}^{y}\int_{s_{2n}}^{s_{2n+2}}\int_{c}^{t_1}\frac{K_{2y+2}^{2x+1}(\tau,\omega,f(\tau,\omega))}{(\log s_{2y+2}-\log\tau)^{\gamma}(\log t_{2x+1}-\log\omega)^{\lambda}}\frac{d\omega}{\omega}\frac{d\tau}{\tau}\\ &\quad+\sum_{n=0}^{y}\sum_{m=1}^{x}\int_{s_{2n}}^{s_{2n+2}}\int_{t_{2m-1}}^{t_{2m+1}}\frac{K_{2y+2}^{2x+1}(\tau,\omega,f(\tau,\omega))}{(\log s_{2y+2}-\log\tau)^{\gamma}(\log t_{2x+1}-\log\omega)^{\lambda}}\frac{d\omega}{\omega}\frac{d\tau}{\tau}\\ &\approx g_{2y+2}^{2x+1}+\sum_{n=0}^{y}\sum_{k=0}^{2}\sum_{q=0}^{2}T_{2y+2}^{k,n}\hat{T}_{2x+1}^{q,0}K_{2y+2}^{2x+1}(s_{2n+k},t_q,f_{2n+k}^{q}) \quad (2.27)\\ &\quad+\sum_{n=0}^{y}\sum_{m=1}^{x}\sum_{k=0}^{2}\sum_{q=0}^{2}T_{2y+2}^{k,n}\hat{T}_{2x+1}^{q,m}K_{2y+2}^{2x+1}(s_{2n+k},t_{2m-1+q},f_{2n+k}^{2m-1+q}),\end{aligned}$$

where $T_{2y+2}^{k,n}$ is defined by (2.13) and $\hat{T}_{2x+1}^{q,0}$ and $\hat{T}_{2x+1}^{q,m}$ are given by (2.18) and (2.19), respectively.

Therefore, we can obtain an approximation of $f(s_{2y+1},t_{2x+2})$ and $f(s_{2y+2},t_{2x+2})$ as follows

$$\begin{aligned} f(s_{2y+1},t_{2x+2}) &= g_{2y+1}^{2x+2}+\sum_{m=0}^{x}\int_{a}^{s_1}\int_{t_{2m}}^{t_{2m+2}}\frac{K_{2y+1}^{2x+2}(\tau,\omega,f(\tau,\omega))}{(\log s_{2y+1}-\log\tau)^{\gamma}(\log t_{2x+2}-\log\omega)^{\lambda}}\frac{d\omega}{\omega}\frac{d\tau}{\tau}\\ &\quad+\sum_{n=1}^{y}\sum_{m=0}^{x}\int_{s_{2n-1}}^{s_{2n+1}}\int_{t_{2m}}^{t_{2m+2}}\frac{K_{2y+1}^{2x+2}(\tau,\omega,f(\tau,\omega))}{(\log s_{2y+1}-\log\tau)^{\gamma}(\log t_{2x+2}-\log\omega)^{\lambda}}\frac{d\omega}{\omega}\frac{d\tau}{\tau}\\ &\approx g_{2y+1}^{2x+2}+\sum_{m=0}^{x}\sum_{k=0}^{2}\sum_{q=0}^{2}T_{2y+1}^{k,0}\hat{T}_{2x+2}^{q,m}K_{2y+1}^{2x+2}(s_k,t_{2m+q},f_{k}^{2m+q}) \quad (2.28)\\ &\quad+\sum_{n=1}^{y}\sum_{m=0}^{x}\sum_{k=0}^{2}\sum_{q=0}^{2}T_{2y+1}^{k,n}\hat{T}_{2x+2}^{q,m}K_{2y+1}^{2x+2}(s_{2n-1+k},t_{2m+q},f_{2n-1+k}^{2m+q}),\\ f(s_{2y+2},t_{2x+2}) &= g_{2y+2}^{2x+2}+\sum_{n=0}^{y}\sum_{m=0}^{x}\int_{s_{2n}}^{s_{2n+2}}\int_{t_{2m}}^{t_{2m+2}}\frac{K_{2y+2}^{2x+2}(\tau,\omega,f(\tau,\omega))}{(\log s_{2y+2}-\log\tau)^{\gamma}(\log t_{2x+2}-\log\omega)^{\lambda}}\frac{d\omega}{\omega}\frac{d\tau}{\tau}\\ &\approx g_{2y+2}^{2x+2}+\sum_{n=0}^{y}\sum_{m=0}^{x}\sum_{k=0}^{2}\sum_{q=0}^{2}T_{2y+2}^{k,n}\hat{T}_{2x+2}^{q,m}K_{2y+2}^{2x+2}(s_{2n+k},t_{2m+q},f_{2n+k}^{2m+q}), \quad (2.29)\end{aligned}$$

where $T_{2y+1}^{k,0}$, $T_{2y+1}^{k,n}$, $T_{2y+2}^{k,n}$ and $\hat{T}_{2x+2}^{q,m}$ are given by (2.8), (2.9), (2.13) and (2.20), respectively.

In summary, we combine (2.1), (2.3)–(2.5), (2.7), (2.10)–(2.12), (2.14)-(2.17) and (2.26)–(2.29) to obtain the high-order numerical scheme of (1.2) as follows

$$
\begin{aligned}
f_{\bar{n}}^{\bar{m}} &= g_{\bar{n}}^{\bar{m}} + \sum_{n=0}^{2}\sum_{m=0}^{2} T_{\bar{n}}^{n,0}\hat{T}_{\bar{m}}^{m,0}K_{\bar{n}}^{\bar{m}}(s_n, t_m, f_n^m), \bar{n}, \bar{m} = 1, 2,\\
f_{2y+1}^{\bar{m}} &= g_{2y+1}^{\bar{m}} + \sum_{k=0}^{2}\sum_{q=0}^{2} T_{2y+1}^{k,0}\hat{T}_{\bar{m}}^{q,0}K_{2y+1}^{\bar{m}}(s_k, t_q, f_k^q)\\
&\quad + \sum_{n=1}^{y}\sum_{k=0}^{2}\sum_{q=0}^{2} T_{2y+1}^{k,n}\hat{T}_{\bar{m}}^{q,0}K_{2y+1}^{\bar{m}}(s_{2n-1+k}, t_q, f_{2n-1+k}^q), \bar{m} = 1, 2,\\
f_{2y+2}^{\bar{m}} &= g_{2y+2}^{\bar{m}} + \sum_{n=0}^{y}\sum_{k=0}^{2}\sum_{q=0}^{2} T_{2y+2}^{k,n}\hat{T}_{\bar{m}}^{q,0}K_{2y+2}^{\bar{m}}(s_{2n+k}, t_q, f_{2n+k}^q), \bar{m} = 1, 2,\\
f_{\bar{n}}^{2x+1} &= g_{\bar{n}}^{2x+1} + \sum_{k=0}^{2}\sum_{q=0}^{2} T_{\bar{n}}^{k,0}\hat{T}_{2x+1}^{q,0}K_{\bar{n}}^{2x+1}(s_k, t_q, f_k^q)\\
&\quad + \sum_{m=1}^{x}\sum_{k=0}^{2}\sum_{q=0}^{2} T_{\bar{n}}^{k,0}\hat{T}_{2x+1}^{q,m}K_{\bar{n}}^{2x+1}(s_k, t_{2m-1+q}, f_k^{2m-1+q}), \bar{n} = 1, 2,\\
f_{\bar{n}}^{2x+2} &= g_{\bar{n}}^{2x+2} + \sum_{m=0}^{x}\sum_{k=0}^{2}\sum_{q=0}^{2} T_{\bar{n}}^{k,0}\hat{T}_{2x+2}^{q,m}K_{\bar{n}}^{2x+2}(s_k, t_{2m+q}, f_k^{2m+q}), \bar{n} = 1, 2,\\
f_{2y+1}^{2x+1} &= g_{2y+1}^{2x+1} + \sum_{k=0}^{2}\sum_{q=0}^{2} T_{2y+1}^{k,0}\hat{T}_{2x+1}^{q,0}K_{2y+1}^{2x+1}(s_k, t_q, f_k^q)\\
&\quad + \sum_{m=1}^{x}\sum_{k=0}^{2}\sum_{q=0}^{2} T_{2y+1}^{k,0}\hat{T}_{2x+1}^{q,m}K_{2y+1}^{2x+1}(s_k, t_{2m-1+q}, f_k^{2m-1+q}) \qquad (2.30)\\
&\quad + \sum_{n=1}^{y}\sum_{k=0}^{2}\sum_{q=0}^{2} T_{2y+1}^{k,n}\hat{T}_{2x+1}^{q,0}K_{2y+1}^{2x+1}(s_{2n-1+k}, t_q, f_{2n-1+k}^q)\\
&\quad + \sum_{n=1}^{y}\sum_{m=1}^{x}\sum_{k=0}^{2}\sum_{q=0}^{2} T_{2y+1}^{k,n}\hat{T}_{2x+1}^{q,m}K_{2y+1}^{2x+1}(s_{2n-1+k}, t_{2m-1+q}, f_{2n-1+k}^{2m-1+q}),\\
f_{2y+2}^{2x+1} &= g_{2y+2}^{2x+1} + \sum_{n=0}^{y}\sum_{k=0}^{2}\sum_{q=0}^{2} T_{2y+2}^{k,n}\hat{T}_{2x+1}^{q,0}K_{2y+2}^{2x+1}(s_{2n+k}, t_q, f_{2n+k}^q)\\
&\quad + \sum_{n=0}^{y}\sum_{m=1}^{x}\sum_{k=0}^{2}\sum_{q=0}^{2} T_{2y+2}^{k,n}\hat{T}_{2x+1}^{q,m}K_{2y+2}^{2x+1}(s_{2n+k}, t_{2m-1+q}, f_{2n+k}^{2m-1+q}),\\
f_{2y+1}^{2x+2} &= g_{2y+1}^{2x+2} + \sum_{m=0}^{x}\sum_{k=0}^{2}\sum_{q=0}^{2} T_{2y+1}^{k,0}\hat{T}_{2x+2}^{q,m}K_{2y+1}^{2x+2}(s_k, t_{2m+q}, f_k^{2m+q})\\
&\quad + \sum_{n=1}^{y}\sum_{m=0}^{x}\sum_{k=0}^{2}\sum_{q=0}^{2} T_{2y+1}^{k,n}\hat{T}_{2x+2}^{q,m}K_{2y+1}^{2x+2}(s_{2n-1+k}, t_{2m+q}, f_{2n-1+k}^{2m+q}),
\end{aligned}
$$

$$f_{2y+2}^{2x+2} = g_{2y+2}^{2x+2} + \sum_{n=0}^{y}\sum_{m=0}^{x}\sum_{k=0}^{2}\sum_{q=0}^{2} T_{2y+2}^{k,n}\hat{T}_{2x+2}^{q,m}K_{2y+2}^{2x+2}(s_{2n+k}, t_{2m+q}, f_{2n+k}^{2m+q}).$$

3. Estimation of the truncation errors

We will introduce several lemmas that will be used in convergence analysis. In the this paper, we assume that C is a constant; the value of C may vary at different positions and it is independent of the discrete step size.

Lemma 1. *(Discrete Gronwall Inequality [24]) Assume that* $0 < \gamma < 1$ *and* $b > a > 0$*, and*

$$a_{n,y} = \begin{cases} \left(\log \frac{s_y}{a} - \log \frac{s_n}{a}\right)^{-\gamma} \log \frac{s_{n+1}}{s_n}, & n = 1, 2, \cdots, y-1, \\ 0, & n \geq y. \end{cases}$$

Let $\sum_{n=p}^{y} a_{n,y} |e_n| = 0$ *for* $p > y \geq 1$*. If*

$$\left|e_y\right| \leq A \sum_{n=1}^{y-1} a_{n,y} |e_n| + |\eta_0|, \quad y = 1, 2, \cdots, k,$$

then

$$|e_k| \leq C |\eta_0|, \quad k = 1, 2, \cdots,$$

where A and C are positive constants.

Lemma 2. *[25] For* $g > f > 0$*, the following holds*

$$\int_f^g \left(\log \frac{g}{s}\right)^{\gamma-1} \left(\log \frac{s}{f}\right)^{\lambda-1} \frac{ds}{s} = \frac{\Gamma(\gamma)\Gamma(\lambda)}{\Gamma(\gamma+\lambda)} \left(\log \frac{g}{f}\right)^{\gamma+\lambda-1},$$

where $\gamma > 0, \lambda > 0$.

Based on the idea of [25], we can prove the following Lemma 3–Lemma 5.

Lemma 3. *It holds that*

$$\int_{s_n}^{s_r} \left(\log \frac{b}{s}\right)^{-\gamma} \frac{ds}{s} \leq \left(\log \frac{b}{s_r}\right)^{-\gamma} \log \frac{s_r}{s_n},$$

where $r > n$ *and b is a positive constant.*

Lemma 4. *Let* $n > m$ *and* $p > q$*; then,*

$$(\log s_n - \log s_m) \leq \left(\frac{n-m}{p-q} + 1\right)\left(\log s_p - \log s_q\right).$$

Lemma 5. *Assuming that p and q are positive integers and* $p, q = 0, 1, 2, \cdots, 2y+1$*, when* $p > q$ *is satisfied, the following conclusion holds:*

$$\log s_p - \log s_q \leq (p-q)\Delta_s, p, q = 0, 1, 2, \cdots, 2y+1. \tag{3.1}$$

In order to estimate the truncation error for (2.30), we introduce the definition of truncation error at a point (s_n, t_m):

$$r_n^m := f(s_n, t_m) - \bar{f}_n^m, \tag{3.2}$$

where $\bar{f}_n^m$ is used as an approximation of $f(s_n, t_m)$ in order to obtain a precise solution when substituted into (2.30). Specifically, $\bar{f}_{2y+1}^{2x+1}$ is defined by

$$\begin{aligned}
\bar{f}_{2y+1}^{2x+1} \quad = \quad & g_{2y+1}^{2x+1} + \sum_{k=0}^{2}\sum_{q=0}^{2} T_{2y+1}^{k,0}\hat{T}_{2x+1}^{q,0}K_{2y+1}^{2x+1}(s_k, t_q, f(s_k, t_q)) \\
& + \sum_{m=1}^{x}\sum_{k=0}^{2}\sum_{q=0}^{2} T_{2y+1}^{k,0}\hat{T}_{2x+1}^{q,m}K_{2y+1}^{2x+1}(s_k, t_{2m-1+q}, f(s_k, t_{2m-1+q})) \\
& + \sum_{n=1}^{y}\sum_{k=0}^{2}\sum_{q=0}^{2} T_{2y+1}^{k,n}\hat{T}_{2x+1}^{q,0}K_{2y+1}^{2x+1}(s_{2n-1+k}, t_q, f(s_{2n-1+k}, t_q)) \\
& + \sum_{n=1}^{y}\sum_{m=1}^{x}\sum_{k=0}^{2}\sum_{q=0}^{2} T_{2y+1}^{k,n}\hat{T}_{2x+1}^{q,m}K_{2y+1}^{2x+1}(s_{2n-1+k}, t_{2m-1+q}, f(s_{2n-1+k}, t_{2m-1+q})).
\end{aligned} \tag{3.3}$$

Lemma 6. *Let r_n^m represent the definition of truncation error in* (3.2). *If $K(\cdot,\cdot,\cdot,\cdot, f(\cdot,\cdot)) \in C^4([a, b] \times [c, d])$, then we have*

$$|r_n^m| \le C(\Delta_s^{4-\gamma} + \Delta_t^{4-\lambda}),$$

where C is only related to $\gamma, \lambda, G_1, G_2$, and the respective definitions of G_1, G_2 are as follows:

$$G_1 = \max_{\substack{s,\tau\in[a,b] \\ t,\omega\in[c,d]}} (|\partial_\tau^3 K(s, t, \tau, \omega, f(\tau, \omega))|, |\partial_\omega^3 K(s, t, \tau, \omega, f(\tau, \omega))|), \tag{3.4}$$

$$G_2 = \max_{\substack{s,\tau\in[a,b] \\ t,\omega\in[c,d]}} (|\partial_\tau^4 K(s, t, \tau, \omega, f(\tau, \omega))|, |\partial_\omega^4 K(s, t, \tau, \omega, f(\tau, \omega))|). \tag{3.5}$$

Proof. Let us first estimate the truncation error r_{2y+1}^{2x+1}. From (3.2), it can be seen that the truncation error has already been defined at the point (s_{2y+1}, t_{2x+1}). By combining (2.26), (3.2) and (3.3), it can be obtained that

$$\begin{aligned}
& r_{2y+1}^{2x+1} = f\left(s_{2y+1}, t_{2x+1}\right) - \bar{f}_{2y+1}^{2x+1} \\
= \quad & \int_a^{s_1}\int_c^{t_1} (\log s_{2y+1} - \log\tau)^{-\gamma}(\log t_{2x+1} - \log\omega)^{-\lambda}[K_{2y+1}^{2x+1}(\tau, \omega, f(\tau, \omega)) \\
& - \sum_{k=0}^{2}\sum_{q=0}^{2} \varphi_k^0(\tau)\phi_q^0(\omega)K_{2y+1}^{2x+1}(s_k, t_q, f(s_k, t_q))]\frac{d\omega}{\omega}\frac{d\tau}{\tau} \\
& + \sum_{m=1}^{x}\int_a^{s_1}\int_{t_{2m-1}}^{t_{2m+1}} (\log s_{2y+1} - \log\tau)^{-\gamma}(\log t_{2x+1} - \log\omega)^{-\lambda}[K_{2y+1}^{2x+1}(\tau, \omega, f(\tau, \omega))
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{k=0}^{2}\sum_{q=0}^{2}\varphi_k^0(\tau)\phi_q^{2m-1}(\omega)K_{2y+1}^{2x+1}(s_k,t_{2m-1+q},f(s_k,t_{2m-1+q}))]\frac{d\omega}{\omega}\frac{d\tau}{\tau}\\
&+\sum_{n=1}^{y}\int_{s_{2n-1}}^{s_{2n+1}}\int_{c}^{t_1}(\log s_{2y+1}-\log\tau)^{-\gamma}(\log t_{2x+1}-\log\omega)^{-\lambda}[K_{2y+1}^{2x+1}(\tau,\omega,f(\tau,\omega))\\
&-\sum_{k=0}^{2}\sum_{q=0}^{2}\varphi_k^{2n-1}(\tau)\phi_q^{0}(\omega)K_{2y+1}^{2x+1}(s_{2n-1+k},t_q,f(s_{2n-1+k},t_q))]\frac{d\omega}{\omega}\frac{d\tau}{\tau}\\
&+\sum_{n=1}^{y}\sum_{m=1}^{x}\int_{s_{2n-1}}^{s_{2n+1}}\int_{t_{2m-1}}^{t_{2m+1}}(\log s_{2y+1}-\log\tau)^{-\gamma}(\log t_{2x+1}-\log\omega)^{-\lambda}[K_{2y+1}^{2x+1}(\tau,\omega,f(\tau,\omega))\\
&-\sum_{k=0}^{2}\sum_{q=0}^{2}\varphi_k^{2n-1}(\tau)\phi_q^{2m-1}(\omega)K_{2y+1}^{2x+1}(s_{2n-1+k},t_{2m-1+q},f(s_{2n-1+k},t_{2m-1+q}))]\frac{d\omega}{\omega}\frac{d\tau}{\tau}\\
=\;&\int_{a}^{s_1}\int_{c}^{t_1}\frac{(\log s_{2y+1}-\log\tau)^{-\gamma}}{(\log t_{2x+1}-\log\omega)^{\lambda}}R_1\frac{d\omega}{\omega}\frac{d\tau}{\tau}+\sum_{m=1}^{x}\int_{a}^{s_1}\int_{t_{2m-1}}^{t_{2m+1}}\frac{(\log s_{2y+1}-\log\tau)^{-\gamma}}{(\log t_{2x+1}-\log\omega)^{\lambda}}R_2\frac{d\omega}{\omega}\frac{d\tau}{\tau}\\
&+\sum_{n=1}^{y}\int_{s_{2n-1}}^{s_{2n+1}}\int_{c}^{t_1}\frac{(\log s_{2y+1}-\log\tau)^{-\gamma}}{(\log t_{2x+1}-\log\omega)^{\lambda}}R_3\frac{d\omega}{\omega}\frac{d\tau}{\tau}\\
&+\sum_{n=1}^{y}\sum_{m=1}^{x}\int_{s_{2n-1}}^{s_{2n+1}}\int_{t_{2m-1}}^{t_{2m+1}}\frac{(\log s_{2y+1}-\log\tau)^{-\gamma}}{(\log t_{2x+1}-\log\omega)^{\lambda}}R_4\frac{d\omega}{\omega}\frac{d\tau}{\tau}\\
\doteq\;& r_{2y+1}^{2x+1,(1)}+r_{2y+1}^{2x+1,(2)}+r_{2y+1}^{2x+1,(3)}+r_{2y+1}^{2x+1,(4)}. \qquad (3.6)
\end{aligned}
$$

We can obtain the following from Taylor's theorem for all $(\tau,\omega)\in[a,s_1]\times[c,t_1]$:

$$
\begin{aligned}
R_1 \;=\;& \frac{1}{3!}\partial_\tau^3K_{2y+1}^{2x+1}(\xi_1(\tau),\omega,f(\xi_1(\tau),\omega))\prod_{k=0}^{2}(\log\tau-\log s_k)\\
&+\sum_{k=0}^{2}\frac{\varphi_k^0(\tau)}{3!}\partial_\omega^3K_{2y+1}^{2x+1}(s_k,\eta_1(\omega),f(s_k,\eta_1(\omega)))\prod_{q=0}^{2}(\log\omega-\log t_q),
\end{aligned}
$$

where $(\xi_1(\tau),\eta_1(\omega))\in[a,s_1]\times[c,t_1]$. For all $(\tau,\omega)\in[a,s_1]\times[t_{2m-1},t_{2m+1}]$ with $(\xi_2(\tau),\eta_m(\omega))\in[a,s_1]\times[t_{2m-1},t_{2m+1}]$, then

$$
\begin{aligned}
R_2 \;=\;& \frac{1}{3!}\partial_\tau^3K_{2y+1}^{2x+1}(\xi_2(\tau),\omega,f(\xi_2(\tau),\omega))\prod_{k=0}^{2}(\log\tau-\log s_k)\\
&+\sum_{k=0}^{2}\frac{\varphi_k^0(\tau)}{3!}\partial_\omega^3K_{2y+1}^{2x+1}(s_k,\eta_m(\omega),f(s_k,\eta_m(\omega)))\prod_{q=0}^{2}(\log\omega-\log t_{2m-1+q}),
\end{aligned}
$$

and for all $(\tau,\omega)\in[s_{2n-1},s_{2n+1}]\times[c,t_1]$, there exists $(\xi_n(\tau),\eta_2(\omega))\in[s_{2n-1},s_{2n+1}]\times[c,t_1]$, such that

$$
R_3 \;=\; \frac{1}{3!}\partial_\tau^3K_{2y+1}^{2x+1}(\xi_n(\tau),\omega,f(\xi_n(\tau),\omega))\prod_{k=0}^{2}(\log\tau-\log s_{2n-1+k})
$$

$$+\sum_{k=0}^{2}\frac{\varphi_k^{2n-1}(\tau)}{3!}\partial_\omega^3 K_{2y+1}^{2x+1}(s_{2n-1+k},\eta_2(\omega),f(s_{2n-1+k},\eta_2(\omega)))\prod_{q=0}^{2}(\log\omega-\log t_q).$$

In the same way, for all $(\tau,\omega)\in[s_{2n-1},s_{2n+1}]\times[t_{2m-1},t_{2m+1}]$, there exists $(\xi_{n1}(\tau),\eta_{m2}(\omega))\in[s_{2n-1},s_{2n+1}]\times[t_{2m-1},t_{2m+1}]$, such that

$$\begin{aligned}R_4 &= \frac{1}{3!}\partial_\tau^3 K_{2y+1}^{2x+1}(\xi_{n1}(\tau),\omega,f(\xi_{n1}(\tau),\omega))\prod_{k=0}^{2}(\log\tau-\log s_{2n-1+k})\\&\quad+\sum_{k=0}^{2}\frac{\varphi_k^{2n-1}(\tau)}{3!}\partial_\omega^3 K_{2y+1}^{2x+1}(s_{2n-1+k},\eta_{m2}(\omega),f(s_{2n-1+k},\eta_{m2}(\omega)))\prod_{q=0}^{2}(\log\omega-\log t_{2m-1+q}).\end{aligned}$$

So, we can obtain the value of $r_{2y+1}^{2x+1,(1)}$ as follows

$$\begin{aligned}&|r_{2y+1}^{2x+1,(1)}|\le\Big|\int_a^{s_1}\int_c^{t_1}\frac{\partial_\tau^3 K_{2y+1}^{2x+1}(\xi_1(\tau),\omega,f(\xi_1(\tau),\omega))\prod_{k=0}^{2}(\log\tau-\log s_k)}{3!(\log s_{2y+1}-\log\tau)^\gamma(\log t_{2x+1}-\log\omega)^\lambda}\frac{d\omega}{\omega}\frac{d\tau}{\tau}\Big|\\&+\Big|\int_a^{s_1}\int_c^{t_1}\sum_{k=0}^{2}\varphi_k^0(\tau)\frac{\partial_\omega^3 K_{2y+1}^{2x+1}(s_k,\eta_1(\omega),f(s_k,\eta_1(\omega)))}{3!(\log s_{2y+1}-\log\tau)^\gamma(\log t_{2x+1}-\log\omega)^\lambda}\prod_{q=0}^{2}(\log\omega-\log t_q)\frac{d\omega}{\omega}\frac{d\tau}{\tau}\Big|\\&\doteq R_1^{(1)}+R_1^{(2)}.\end{aligned}\tag{3.7}$$

For the convenience of estimating of each item on the right side of (3.7), according to Lemma 3 and Lemma 5, let us first estimate the following:

$$\begin{aligned}&\int_a^{s_1}\left(\log s_{2y+1}-\log\tau\right)^{-\gamma}\frac{d\tau}{\tau}=\int_a^{s_1}\left(\log\frac{s_{2y+1}}{\tau}\right)^{-\gamma}\frac{d\tau}{\tau}\le\left(\log\frac{s_{2y+1}}{s_1}\right)^{-\gamma}\log\frac{s_1}{s_0},\\\le&\ \left(\log s_{2y+1}-\log s_1\right)^{-\gamma}\Delta_s\le(\log s_2-\log s_1)^{-\gamma}\Delta_s\\\le&\ \left(\log\frac{s_2}{s_1}\right)^{-\gamma}\Delta_s=\left(\log(1+\frac{\Delta_s}{s_1})\right)^{-\gamma}\Delta_s\\\le&\ \left(\frac{\Delta_s}{s_1}-\frac{1}{2}\left(\frac{\Delta_s}{s_1}\right)^2\right)^{-\gamma}\Delta_s\le\left(\frac{1}{2}\frac{\Delta_s}{s_1}\right)^{-\gamma}\Delta_s=2^\gamma s_1^\gamma\Delta_s^{-\gamma}\Delta_s\le 2b\Delta_s^{1-\gamma}.\end{aligned}\tag{3.8}$$

Similarly, we have

$$\int_c^{t_1}(\log t_{2x+1}-\log\omega)^{-\lambda}\frac{d\omega}{\omega}\le 2d\Delta_t^{1-\lambda}.\tag{3.9}$$

Then we have

$$\begin{aligned}R_1^{(1)}&\le\int_a^{s_1}\int_c^{t_1}\Big|\frac{\partial_\tau^3 K_{2y+1}^{2x+1}(\xi_1(\tau),\omega,f(\xi_1(\tau),\omega))\prod_{k=0}^{2}(\log\tau-\log s_k)}{3!(\log s_{2y+1}-\log\tau)^\gamma(\log t_{2x+1}-\log\omega)^\lambda}\Big|\frac{d\omega}{\omega}\frac{d\tau}{\tau}\\&\le G_1\Big|\log\frac{\delta}{s_0}\log\frac{\delta}{s_1}\log\frac{\delta}{s_2}\Big|\int_a^{s_1}\int_c^{t_1}(\log s_{2y+1}-\log\tau)^{-\gamma}(\log t_{2x+1}-\log\omega)^{-\lambda}\frac{d\omega}{\omega}\frac{d\tau}{\tau}\end{aligned}$$

$$\leq \ 4bdG_1\Delta_s^3\Delta_s^{1-\gamma}\Delta_t^{1-\lambda} = 4bdG_1\Delta_s^{4-\gamma}\Delta_t^{1-\lambda}, \tag{3.10}$$

$$R_1^{(2)} \ \leq \ G_1\Big|\log\frac{\rho}{t_0}\log\frac{\rho}{t_1}\log\frac{\rho}{t_2}\Big|\int_a^{s_1}\int_c^{t_1}(\log s_{2y+1}-\log\tau)^{-\gamma}(\log t_{2x+1}-\log\omega)^{-\lambda}\frac{d\omega}{\omega}\frac{d\tau}{\tau}$$
$$\leq \ 4bdG_1\Delta_s^{1-\gamma}\Delta_t^{4-\lambda}, \tag{3.11}$$

where $a < \delta < s_1$ and $c < \rho < t_1$; G_1 is defined by (3.4).

By combining (3.10) and (3.11), it is easy to conclude that

$$|r_{2y+1}^{2x+1,(1)}| \leq 4bdG_1(\Delta_s^{4-\gamma}\Delta_t^{1-\lambda} + \Delta_s^{1-\gamma}\Delta_t^{4-\lambda}). \tag{3.12}$$

Similarly, for $r_{2y+1}^{2x+1,(2)}$, we use the same technique to obtain

$$|r_{2y+1}^{2x+1,(2)}| \leq \sum_{m=1}^{x}\Big|\int_a^{s_1}\int_{t_{2m-1}}^{t_{2m+1}}\frac{\partial_\tau^3 K_{2y+1}^{2x+1}(\xi_2(\tau),\omega,f(\xi_2(\tau),\omega))}{3!(\log s_{2y+1}-\log\tau)^{\gamma}(\log t_{2x+1}-\log\omega)^{\lambda}}\prod_{k=0}^{2}(\log\tau-\log s_k)\frac{d\omega}{\omega}\frac{d\tau}{\tau}\Big|$$
$$+\sum_{m=1}^{x}\Big|\int_a^{s_1}\int_{t_{2m-1}}^{t_{2m+1}}\frac{\sum\limits_{k=0}^{2}\varphi_k^0(\tau)\partial_\omega^3 K_{2y+1}^{2x+1}(s_k,\eta_m(\omega),f(s_k,\eta_m(\omega)))\prod\limits_{q=0}^{2}(\log\omega-\log t_{2m-1+q})}{3!(\log s_{2y+1}-\log\tau)^{\gamma}(\log t_{2x+1}-\log\omega)^{\lambda}}\frac{d\omega}{\omega}\frac{d\tau}{\tau}\Big|$$
$$\doteq R_2^{(1)} + R_2^{(2)}. \tag{3.13}$$

For $R_2^{(1)}$ of (3.13), based on (3.8) and Lemma 5 we can see that

$$R_2^{(1)} \leq G_1\sum_{m=1}^{x}\int_a^{s_1}\int_{t_{2m-1}}^{t_{2m+1}}\Big|(\log s_{2y+1}-\log\tau)^{-\gamma}(\log t_{2x+1}-\log\omega)^{-\lambda}\log\frac{\tau}{s_0}\log\frac{\tau}{s_1}\log\frac{\tau}{s_2}\Big|\frac{d\omega}{\omega}\frac{d\tau}{\tau}$$
$$\leq G_1\Big|\log\frac{\delta}{s_0}\log\frac{\delta}{s_1}\log\frac{\delta}{s_2}\Big|\sum_{m=1}^{x}\int_a^{s_1}\int_{t_{2m-1}}^{t_{2m+1}}(\log s_{2y+1}-\log\tau)^{-\gamma}(\log t_{2x+1}-\log\omega)^{-\lambda}\frac{d\omega}{\omega}\frac{d\tau}{\tau}$$
$$\leq 2bG_1\Delta_s^3\Delta_s^{1-\gamma}\int_{t_1}^{t_{2x+1}}(\log t_{2x+1}-\log\omega)^{-\lambda}\frac{d\omega}{\omega}$$
$$= 2bG_1\Delta_s^{4-\gamma}\frac{-(\log t_{2x+1}-\log\omega)^{1-\lambda}}{1-\lambda}\Big|_{t_1}^{t_{2x+1}}$$
$$\leq 2bG_1\Delta_s^{4-\gamma}\frac{(2x\Delta t)^{1-\lambda}}{1-\lambda} \leq \frac{2bG_1 d^{1-\lambda}}{1-\lambda}\Delta_s^{4-\gamma}. \tag{3.14}$$

For $R_2^{(2)}$, we decompose it into the following estimates

$$R_2^{(2)} \leq \sum_{m=1}^{x}\Big|\int_a^{s_1}\int_{t_{2m-1}}^{t_{2m+1}}\frac{\sum\limits_{k=0}^{2}\varphi_k^0(\tau)\partial_\omega^3 K_{2y+1}^{2x+1}(s_k,\eta_m(\tilde\omega_m),f(s_k,\eta_m(\tilde\omega_m)))\prod\limits_{q=0}^{2}(\log\omega-\log t_{2m-1+q})}{3!(\log s_{2y+1}-\log\tau)^{\gamma}(\log t_{2x+1}-\log\omega)^{\lambda}}\frac{d\omega}{\omega}\frac{d\tau}{\tau}\Big|$$
$$+\sum_{m=1}^{x}\Big|\int_a^{s_1}\int_{t_{2m-1}}^{t_{2m+1}}\frac{(\log s_{2y+1}-\log\tau)^{-\gamma}}{(\log t_{2x+1}-\log\omega)^{\lambda}}\sum_{k=0}^{2}\varphi_{k,0}(\tau)[\frac{1}{3!}\partial_\omega^3 K_{2y+1}^{2x+1}(s_k,\eta_m(\omega),f(s_k,\eta_m(\omega)))$$
$$-\frac{1}{3!}\partial_\omega^3 K_{2y+1}^{2x+1}(s_k,\eta_m(\tilde\omega_m),f(s_k,\eta_m(\tilde\omega_m)))]\prod_{q=0}^{2}(\log\omega-\log t_{2m-1+q})\frac{d\omega}{\omega}\frac{d\tau}{\tau}\Big|$$

$$\doteq D_1 + D_2, \tag{3.15}$$

where $\tilde{\omega}_m = t_{2m}$. Based on $\varphi_k^n(\tau), \tau \in (s_n, s_{n+2}), k = 0, 1, 2; n \in \mathbb{N}$, we derive $|\varphi_k^n(\tau)| \leq 1$ by using (3.8); so, we have the following inequality

$$\begin{aligned}
D_1 &\leq G_1 \sum_{m=1}^{x} \Big| \int_a^{s_1} \int_{t_{2m-1}}^{t_{2m+1}} \frac{(\log s_{2y+1} - \log \tau)^{-\gamma}}{(\log t_{2x+1} - \log \omega)^{\lambda}} \sum_{k=0}^{2} \varphi_k^0(\tau) \prod_{q=0}^{2} (\log \omega - \log t_{2m-1+q}) \Big| \frac{d\omega}{\omega} \frac{d\tau}{\tau} \\
&= G_1 \int_a^{s_1} \Big| \frac{\sum_{k=0}^{2} \varphi_k^0(\tau)}{(\log s_{2y+1} - \log \tau)^{\gamma}} \Big| \frac{d\tau}{\tau} \times \sum_{m=1}^{x} \int_{t_{2m-1}}^{t_{2m+1}} \Big| \frac{\prod_{q=0}^{2} (\log \omega - \log t_{2m-1+q})}{(\log t_{2x+1} - \log \omega)^{\lambda}} \Big| \frac{d\omega}{\omega} \\
&\leq 2b \cdot 3G_1 \Delta_s^{1-\gamma} \sum_{m=1}^{x} \Big[\int_{t_{2m-1}}^{t_{2m}} \Big| \frac{\prod_{q=0}^{2} (\log \omega - \log t_{2m-1+q})}{(\log t_{2x+1} - \log \omega)^{\lambda}} \Big| \frac{d\omega}{\omega} + \int_{t_{2m}}^{t_{2m+1}} \Big| \frac{\prod_{q=0}^{2} (\log \omega - \log t_{2m-1+q})}{(\log t_{2x+1} - \log \omega)^{\lambda}} \Big| \frac{d\omega}{\omega} \Big] \\
&\leq 6bG_1 \Delta_s^{1-\gamma} \sum_{m=1}^{x-2} \Big[\int_{t_{2m-1}}^{t_{2m}} \Big| \frac{\prod_{q=0}^{2} (\log \omega - \log t_{2m-1+q})}{(\log t_{2x+1} - \log \omega)^{\lambda}} \Big| \frac{d\omega}{\omega} + \int_{t_{2m}}^{t_{2m+1}} \Big| \frac{\prod_{q=0}^{2} (\log \omega - \log t_{2m-1+q})}{(\log t_{2x+1} - \log \omega)^{\lambda}} \Big| \frac{d\omega}{\omega} \Big] \\
&\quad + 6bG_1 \Delta_s^{1-\gamma} \Big(\int_{t_{2x-3}}^{t_{2x-1}} \Big| \frac{\prod_{q=0}^{2} (\log \omega - \log t_{2x-3+q})}{(\log t_{2x+1} - \log \omega)^{\lambda}} \Big| \frac{d\omega}{\omega} + \Big| \int_{t_{2x-1}}^{t_{2x+1}} \frac{\prod_{q=0}^{2} (\log \omega - \log t_{2x-1+q})}{(\log t_{2x+1} - \log \omega)^{\lambda}} \Big| \frac{d\omega}{\omega} \Big) \\
&\doteq 6bG_1 \Delta_s^{1-\gamma} D_{11} + 6bG_1 \Delta_s^{1-\gamma} D_{12}.
\end{aligned} \tag{3.16}$$

For D_{11}, we can obtain that

$$\begin{aligned}
D_{11} &= \sum_{m=1}^{x-2} \Big| (\log t_{2x+1} - \log \hat{\omega}_m)^{-\lambda} \int_{t_{2m-1}}^{t_{2m}} \prod_{q=0}^{2} (\log \omega - \log t_{2m-1+q}) \frac{d\omega}{\omega} \\
&\quad + (\log t_{2x+1} - \log \bar{\omega}_m)^{-\lambda} \int_{t_{2m}}^{t_{2m+1}} \prod_{q=0}^{2} (\log \omega - \log t_{2m-1+q}) \frac{d\omega}{\omega} \Big| \\
&= \sum_{m=1}^{x-2} \Big| \frac{1}{4} (\log t_{2x+1} - \log \hat{\omega}_m)^{-\lambda} \Delta_t^4 - \frac{1}{4} (\log t_{2x+1} - \log \bar{\omega}_m)^{-\lambda} \Delta_t^4 \Big| \\
&= \frac{1}{4} \Delta_t^4 \sum_{m=1}^{x-2} \Big| (\log t_{2x+1} - \log \hat{\omega}_m)^{-\lambda} - (\log t_{2x+1} - \log \bar{\omega}_m)^{-\lambda} \Big| \\
&= \frac{1}{4} \Delta_t^4 \sum_{m=1}^{x-2} \Big| -\lambda (\log t_{2x+1} - \log \omega_m)^{-\lambda-1} (\log \bar{\omega_m} - \log \hat{\omega_m}) \Big| \\
&\leq \frac{\lambda}{4} \Delta_t^4 \sum_{m=1}^{x-2} |2(\log t_{2x+1} - \log t_{2m+1})^{-\lambda-1} \Delta_t| \\
&\leq \frac{\lambda}{4} \Delta_t^4 \int_{t_3}^{t_{2x-1}} (\log t_{2x+1} - \log \omega)^{-\lambda-1} \frac{d\omega}{\omega}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\Delta_t^4\Big[(\log t_{2x+1}-\log t_{2x-1})^{-\lambda}-(\log t_{2x+1}-\log t_3)^{-\lambda}\Big]\\
&\leq \frac{1}{4}\Delta_t^4\Big[(\log t_{2x+1}-\log t_{2x-1})^{-\lambda}+(\log t_{2x+1}-\log t_3)^{-\lambda}\Big]\\
&\leq \frac{1}{2}\Delta_t^4(\log t_{2x+1}-\log t_{2x-1})^{-\lambda}=\frac{1}{2}\Delta_t^4\Big[\log\Big(1+\frac{2\Delta_t}{t_{2x-1}}\Big)\Big]^{-\lambda}\\
&\leq \frac{1}{2}\Delta_t^4\Big(\frac{2\Delta_t}{t_{2x+1}}-\frac{1}{2}\Big(\frac{2\Delta_t}{t_{2x-1}}\Big)^2\Big)^{-\lambda}=\frac{1}{2}\Delta_t^4\Big(\frac{2\Delta_t}{t_{2x-1}}\Big(1-\frac{\Delta_t}{t_{2x-1}}\Big)\Big)^{-\lambda}\\
&\leq \frac{1}{2}\Delta_t^4\Big(\frac{4}{3}\frac{\Delta_t}{t_{2x-1}}\Big)^{-\lambda}=\frac{1}{2}\Delta_t^4\cdot\frac{3}{4}t_{2x-1}^{\lambda}\Delta_t^{-\lambda}\leq\frac{3}{8}d\Delta_t^{4-\lambda},
\end{aligned}
\tag{3.17}
$$

where $\log\hat{\omega}_m \leq log\omega_m \leq \log\bar{\omega}_m$, $\log t_{2m-1} \leq \log\hat{\omega}_m \leq \log t_{2m} \leq \log\bar{\omega}_m \leq \log t_{2m+1}$. Furthermore, $2x-1 \geq 3$; then, x satisfies that $x \geq 2$, so we know that $t_{2x-1} \geq t_3 = c+3\Delta_t$; then, $\frac{\Delta_t}{t_{2x-1}} \leq \frac{1}{3}$, $1-\frac{\Delta_t}{t_{2x-1}} \geq \frac{2}{3}$, and we have that $\frac{2\Delta_t}{t_{2x-1}}\Big(1-\frac{\Delta_t}{t_{2x-1}}\Big) \geq \frac{4}{3}\frac{\Delta_t}{t_{2x-1}}$.

For D_{12}, one can similarly obtain that

$$
\begin{aligned}
D_{12} &\leq \Delta_t^3\Big(\int_{t_{2x-3}}^{t_{2x-1}}(\log t_{2x+1}-\log\omega)^{-\lambda}\frac{d\omega}{\omega}+\int_{t_{2x-1}}^{t_{2x+1}}(\log t_{2x+1}-\log\omega)^{-\lambda}\frac{d\omega}{\omega}\Big),\\
&= \Delta_t^3\int_{t_{2x-3}}^{t_{2x+1}}(\log t_{2x+1}-\log\omega)^{-\lambda}\frac{d\omega}{\omega}=\frac{4^{1-\lambda}}{1-\lambda}\Delta_t^{4-\lambda}.
\end{aligned}
\tag{3.18}
$$

We substitute (3.17) and (3.18) into (3.16) to obtain D_1:

$$
D_1 \leq 6b\Big(\frac{3}{8}d+\frac{4^{1-\lambda}}{1-\lambda}\Big)G_1\Delta_s^{1-\gamma}\Delta_t^{4-\lambda}. \tag{3.19}
$$

So, we get D_2:

$$
\begin{aligned}
D_2 &\leq G_2\Delta_t\int_a^{s_1}\frac{|\sum_{k=0}^{2}\varphi_k^0(\tau)|}{3!(\log s_{2y+1}-\log\tau)^{\gamma}}\frac{d\tau}{\tau}\times\sum_{m=1}^{x}\int_{t_{2m-1}}^{t_{2m+1}}\frac{|\prod_{q=0}^{2}(\log\omega-\log t_{2m-1+q})|}{(\log t_{2x+1}-\log\omega)^{\lambda}}\frac{d\omega}{\omega}\\
&\leq G_2\Delta_s^{1-\gamma}\Delta_t^4\sum_{m=1}^{x}\int_{t_{2m-1}}^{t_{2m+1}}(\log t_{2x+1}-\log\omega)^{-\lambda}\frac{d\omega}{\omega}\leq\frac{G_2d^{1-\lambda}}{1-\lambda}\Delta_s^{1-\gamma}\Delta_t^4,
\end{aligned}
\tag{3.20}
$$

where G_2 is defined by (3.5).

According to (3.19) and (3.20), (3.15) becomes

$$
R_2^{(2)} \leq 6b\Big(\frac{3}{8}d+\frac{4^{1-\lambda}}{1-\lambda}\Big)G_1\Delta_s^{1-\gamma}\Delta_t^{4-\lambda}+\frac{G_2d^{1-\lambda}}{1-\lambda}\Delta_s^{1-\gamma}\Delta_t^4. \tag{3.21}
$$

By combining (3.14) and (3.21) with (3.13), we can obtain the result of $r_{2y+1}^{2x+1,(2)}$ as follows:

$$
|r_{2y+1}^{2x+1,(2)}| \leq \frac{2bG_1d^{1-\lambda}}{1-\lambda}\Delta_s^{4-\gamma}+6b\Big(\frac{3}{8}d+\frac{4^{1-\lambda}}{1-\lambda}\Big)G_1\Delta_s^{1-\gamma}\Delta_t^{4-\lambda}+\frac{G_2d^{1-\lambda}}{1-\lambda}\Delta_s^{1-\gamma}\Delta_t^4. \tag{3.22}
$$

Next, we estimate $r_{2y+1}^{2x+1,(3)}$ and obtain

$$
|r_{2y+1}^{2x+1,(3)}| \leq \Big|\sum_{n=1}^{y}\int_{s_{2n-1}}^{s_{2n+1}}\int_c^{t_1}\frac{\partial_\tau^3K_{2y+1}^{2x+1}(\xi_n(\tau),\omega,f(\xi_n(\tau),\omega))}{3!(\log s_{2y+1}-\log\tau)^{\gamma}(\log t_{2x+1}-\log\omega)^{\lambda}}\prod_{k=0}^{2}(\log\tau-\log s_{2n-1+k})\frac{d\omega}{\omega}\frac{d\tau}{\tau}\Big|
$$

$$+\Big|\sum_{n=1}^{y}\int_{s_{2n-1}}^{s_{2n+1}}\int_{c}^{t_1}\sum_{k=0}^{2}\varphi_k^{2n-1}(\tau)\frac{\partial_\omega^3 K_{2y+1}^{2x+1}(s_{2n-1+k},\eta_2(\omega),f(s_{2n-1+k},\eta_2(\omega)))}{3!(\log s_{2y+1}-\log\tau)^{\gamma}(\log t_{2x+1}-\log\omega)^{\lambda}}\prod_{q=0}^{2}(\log\omega-\log t_q)\frac{d\omega}{\omega}\frac{d\tau}{\tau}\Big|$$
$$\doteq R_3^{(1)}+R_3^{(2)}.$$

For $R_3^{(1)}$, one can obtain that

$$\begin{aligned}
R_3^{(1)}\le&\sum_{n=1}^{y}\int_{s_{2n-1}}^{s_{2n+1}}\int_{c}^{t_1}\Big|\frac{\partial_\tau^3 K_{2y+1}^{2x+1}(\xi_n(\tilde\tau_n),\omega,f(\xi_n(\tilde\tau_n),\omega))}{3!(\log s_{2y+1}-\log\tau)^{\gamma}(\log t_{2x+1}-\log\omega)^{\lambda}}\prod_{k=0}^{2}(\log\tau-\log s_{2n-1+k})\Big|\frac{d\omega}{\omega}\frac{d\tau}{\tau}\\
&+\sum_{n=1}^{y}\int_{s_{2n-1}}^{s_{2n+1}}\int_{c}^{t_1}\Big|\frac{(\log s_{2y+1}-\log\tau)^{-\gamma}}{3!(\log t_{2x+1}-\log\omega)^{\lambda}}\prod_{k=0}^{2}(\log\tau-\log s_{2n-1+k})\\
&\times(\partial_\tau^3 K_{2y+1}^{2x+1}(\xi_n(\tau),\omega,f(\xi_n(\tau),\omega))-\partial_\tau^3 K_{2y+1}^{2x+1}(\xi_n(\tilde\tau_n),\omega,f(\xi_n(\tilde\tau_n),\omega)))\Big|\frac{d\omega}{\omega}\frac{d\tau}{\tau}\\
\doteq&B_1+B_2,
\end{aligned}$$

where $\tilde\tau_n=s_{2n}$.

For B_1, we obtain

$$\begin{aligned}
B_1\ \le&\ 2dG_1\Delta_t^{1-\lambda}\sum_{n=1}^{m}\Big|\int_{s_{2n-1}}^{s_{2n}}\frac{\prod_{k=0}^{2}(\log\tau-\log s_{2n-1+k})}{(\log s_{2y+1}-\log\tau)^{\gamma}}\frac{d\tau}{\tau}+\int_{s_{2n}}^{s_{2n+1}}\frac{\prod_{k=0}^{2}(\log\tau-\log s_{2n-1+k})}{(\log s_{2y+1}-\log\tau)^{\gamma}}\frac{d\tau}{\tau}\Big|\\
\le&\ 2dG_1\Delta_t^{1-\lambda}\sum_{n=1}^{y-2}\Big|\int_{s_{2n-1}}^{s_{2n}}\frac{\prod_{k=0}^{2}(\log\tau-\log s_{2n-1+k})}{(\log s_{2y+1}-\log\tau)^{\gamma}}\frac{d\tau}{\tau}+\int_{s_{2n}}^{s_{2n+1}}\frac{\prod_{k=0}^{2}(\log\tau-\log s_{2n-1+k})}{(\log s_{2y+1}-\log\tau)^{\gamma}}\frac{d\tau}{\tau}\Big|\\
&+2dG_1\Delta_t^{1-\lambda}\Big(\Big|\int_{s_{2y-3}}^{s_{2y-1}}\frac{\prod_{k=0}^{2}(\log\tau-\log s_{2y-3+k})}{(\log s_{2y+1}-\log\tau)^{\gamma}}\frac{d\tau}{\tau}\Big|+\Big|\int_{s_{2y-1}}^{s_{2y+1}}\frac{\prod_{k=0}^{2}(\log\tau-\log s_{2y-1+k})}{(\log s_{2y+1}-\log\tau)^{\gamma}}\frac{d\tau}{\tau}\Big|\Big)\\
\doteq&\ 2dG_1\Delta_t^{1-\lambda}B_{11}+2dG_1\Delta_t^{1-\lambda}B_{12}.
\end{aligned}$$

According to (3.17) and (3.18), we can use the same method to obtain the forms of B_{11} and B_{12} as follows

$$B_{11}\le\frac{3}{8}b\Delta_s^{4-\gamma},\quad B_{12}\le\frac{4^{1-\gamma}}{1-\gamma}\Delta_s^{4-\gamma}.$$

So, B_1 can be directly obtained

$$B_1\le 2d\Big(\frac{3}{8}b+\frac{4^{1-\gamma}}{1-\gamma}\Big)G_1\Delta_s^{4-\gamma}\Delta_t^{1-\lambda}.\tag{3.23}$$

For B_2, similar to D_2, we can also directly obtain

$$B_2\le G_2\Delta_s^4\sum_{n=1}^{y}\int_{s_{2n-1}}^{s_{2n+1}}\int_{c}^{t_1}\frac{(\log s_{2y+1}-\log\tau)^{-\gamma}}{(\log t_{2x+1}-\log\omega)^{\lambda}}\frac{d\omega}{\omega}\frac{d\tau}{\tau}\le\frac{G_2 b^{1-\gamma}}{1-\gamma}\Delta_s^4\Delta_t^{1-\gamma}.\tag{3.24}$$

According to (3.23) and (3.24), we have

$$R_3^{(1)} \leq 2d\Big(\frac{3}{8}b + \frac{4^{1-\gamma}}{1-\gamma}\Big)G_1\Delta_s^{4-\gamma}\Delta_t^{1-\lambda} + \frac{G_2 b^{1-\gamma}}{1-\gamma}\Delta_s^4\Delta_t^{1-\gamma}. \tag{3.25}$$

Next, let us estimate $R_3^{(2)}$ as follows

$$\begin{aligned}
R_3^{(2)} &\leq G_1\sum_{n=1}^{y}\int_{s_{2n-1}}^{s_{2n+1}}\Big|\frac{\sum\limits_{k=0}^{2}\varphi_k^{2n-1}(\tau)}{3!(\log s_{2y+1}-\log\tau)^{\gamma}}\frac{d\tau}{\tau}\Big|\cdot\int_c^{t_1}\Big|\frac{\prod\limits_{q=0}^{2}(\log\omega-\log t_q)}{(\log t_{2x+1}-\log\omega)^{\lambda}}\frac{d\omega}{\omega}\Big| \\
&\leq 3G_1\Delta_t^3\sum_{n=1}^{y}\int_{s_{2n-1}}^{s_{2n+1}}\int_c^{t_1}\frac{(\log s_{2y+1}-\log\tau)^{-\gamma}}{(\log t_{2x+1}-\log\omega)^{\lambda}}\frac{d\omega}{\omega}\frac{d\tau}{\tau} \\
&\leq 3G_1\Delta_t^3\cdot 2d\Delta t^{1-\lambda}\int_{s_1}^{s_{2y+1}}(\log s_{2y+1}-\log\tau)^{-\gamma}\frac{d\tau}{\tau} \\
&= 3G_1\Delta_t^3\cdot 2d\Delta t^{1-\lambda}\cdot\frac{1}{1-\gamma}(\log s_{2y+1}-\log s_1)^{1-\gamma} \\
&\leq 3G_1\Delta_t^3\cdot 2d\Delta t^{1-\lambda}\cdot\frac{1}{1-\gamma}(2y\Delta s)^{1-\gamma}\leq\frac{6dG_1b^{1-\gamma}}{1-\gamma}\Delta_t^{4-\lambda}.
\end{aligned}\tag{3.26}$$

So according to (3.25) and (3.26), we can obtain $r_{2y+1}^{2x+1,(3)}$ as follows

$$|r_{2y+1}^{2x+1,(3)}| \leq 2d\Big(\frac{3}{8}b + \frac{4^{1-\gamma}}{1-\gamma}\Big)G_1\Delta_s^{4-\gamma}\Delta_t^{1-\lambda} + \frac{G_2 b^{1-\gamma}}{1-\gamma}\Delta_s^4\Delta_t^{1-\gamma} + \frac{6dG_1b^{1-\gamma}}{1-\gamma}\Delta_t^{4-\lambda}. \tag{3.27}$$

We estimate $r_{2y+1}^{2x+1,(4)}$ and obtain the following form

$$\begin{aligned}
&|r_{2y+1}^{2x+1,(4)}| \leq \\
&\sum_{n=1}^{y}\sum_{m=1}^{x}\Big|\int_{s_{2n-1}}^{s_{2n+1}}\int_{t_{2m-1}}^{t_{2m+1}}\frac{\partial_\tau^3 K_{2y+1}^{2x+1}(\xi_{n1}(\tau),\omega,f(\xi_{n1}(\tau),\omega))}{3!(\log s_{2y+1}-\log\tau)^{\gamma}(\log t_{2x+1}-\log\omega)^{\lambda}}\prod_{k=0}^{2}(\log\tau-\log s_{2n-1+k})\frac{d\omega}{\omega}\frac{d\tau}{\tau}\Big| \\
&\quad+\sum_{n=1}^{y}\sum_{m=1}^{x}\Big|\int_{s_{2n-1}}^{s_{2n+1}}\int_{t_{2m-1}}^{t_{2m+1}}\frac{(\log s_{2y+1}-\log\tau)^{-\gamma}}{3!(\log t_{2x+1}-\log\omega)^{\lambda}}\sum_{k=0}^{2}\varphi_k^{2n-1}(\tau) \\
&\quad\times\partial_\omega^3 K_{2y+1}^{2x+1}(s_{2n-1+k},\eta_{m2}(\omega),f(s_{2n-1+k},\eta_{m2}(\omega)))\prod_{q=0}^{2}(\log\omega-\log t_{2m-1+q})\frac{d\omega}{\omega}\frac{d\tau}{\tau}\Big| \\
&\quad\doteq R_4^{(1)}+R_4^{(2)}.
\end{aligned}$$

We conducted detailed estimates for $R_4^{(1)}$ and $R_4^{(2)}$ respectively, and obtained $R_4^{(1)}$

$$\begin{aligned}
R_4^{(1)} \leq\ &\sum_{n=1}^{y}\sum_{m=1}^{x}\Big|\int_{s_{2n-1}}^{s_{2n+1}}\int_{t_{2m-1}}^{t_{2m+1}}\frac{\partial_\tau^3 K_{2y+1}^{2x+1}(\xi_{n1}(\tilde\tau_n),\omega,f(\xi_{n1}(\tilde\tau_n),\omega))\prod\limits_{k=0}^{2}(\log\tau-\log s_{2n-1+k})}{3!(\log s_{2y+1}-\log\tau)^{\gamma}(\log t_{2x+1}-\log\omega)^{\lambda}}\frac{d\omega}{\omega}\frac{d\tau}{\tau}\Big| \\
&+\sum_{n=1}^{y}\sum_{m=1}^{x}\Big|\int_{s_{2n-1}}^{s_{2n+1}}\int_{t_{2m-1}}^{t_{2m+1}}\frac{(\log s_{2y+1}-\log\tau)^{-\gamma}}{3!(\log t_{2x+1}-\log\omega)^{\lambda}}\prod_{k=0}^{2}(\log\tau-\log s_{2n-1+k})
\end{aligned}$$

$$\times(\partial_\tau^3 K_{2y+1}^{2x+1}(\xi_{n1}(\tau),\omega,f(\xi_{n1}(\tau),\omega))-\partial_\tau^3 K_{2y+1}^{2n+1}(\xi_{n1}(\tilde{\tau}_n),\omega,f(\xi_{n1}(\tilde{\tau}_n),\omega)))\frac{d\omega}{\omega}\frac{d\tau}{\tau}\Big|$$
$$\doteq N_1+N_2,$$

where $\tilde{\tau}_n=s_{2n}$; using the same processing method as B_1, we can obtain N_1 as follows:

$$\begin{aligned}
N_1 &\le G_1\sum_{n=1}^{y}\sum_{m=1}^{x}\Big|\int_{s_{2n-1}}^{s_{2n+1}}\int_{t_{2m-1}}^{t_{2m+1}}\frac{(\log s_{2y+1}-\log\tau)^{-\gamma}}{(\log t_{2x+1}-\log\omega)^{\lambda}}\prod_{k=0}^{2}(\log\tau-\log s_{2n-1+k})\frac{d\omega}{\omega}\frac{d\tau}{\tau}\Big|\\
&= G_1\sum_{m=1}^{x}\int_{t_{2m-1}}^{t_{2m+1}}(\log t_{2n+1}-\log\omega)^{-\lambda}\frac{d\omega}{\omega}\sum_{n=1}^{y}\Big|\int_{s_{2n-1}}^{s_{2n+1}}(\log s_{2y+1}-\log\tau)^{-\gamma}\prod_{k=0}^{2}(\log\tau-\log s_{2n-1+k})\frac{d\tau}{\tau}\Big|\\
&\le \frac{d^{1-\lambda}G_1}{1-\lambda}\sum_{n=1}^{y}\Big|\int_{s_{2n-1}}^{s_{2n}}\frac{\prod_{k=0}^{2}(\log\tau-\log s_{2n-1+k})}{(\log s_{2y+1}-\log\tau)^{\gamma}}\frac{d\tau}{\tau}+\int_{s_{2n}}^{s_{2n+1}}\frac{\prod_{k=0}^{2}(\log\tau-\log s_{2n-1+k})}{(\log s_{2y+1}-\log\tau)^{\gamma}}\frac{d\tau}{\tau}\Big|\\
&\le \frac{d^{1-\lambda}}{1-\lambda}\Big(\frac{3}{8}b+\frac{4^{1-\gamma}}{1-\gamma}\Big)G_1\Delta_s^{4-\gamma}.
\end{aligned}$$

Therefore, for N_2, it can be obtained that

$$N_2\le G_2\Delta_s^4\sum_{n=1}^{y}\sum_{m=1}^{x}\int_{s_{2n-1}}^{s_{2n+1}}\int_{t_{2m-1}}^{t_{2m+1}}\frac{(\log s_{2y+1}-\log\tau)^{-\gamma}}{(\log t_{2x+1}-\log\omega)^{\lambda}}\frac{d\omega}{\omega}\frac{d\tau}{\tau}\le\frac{b^{1-\gamma}d^{1-\lambda}}{(1-\gamma)(1-\lambda)}G_2\Delta_s^4.$$

Finally, Let us estimate $R_4^{(2)}$ again

$$\begin{aligned}
R_4^{(2)} \le{}& \sum_{n=1}^{y}\sum_{m=1}^{x}\Big|\int_{s_{2n-1}}^{s_{2n+1}}\int_{t_{2m-1}}^{t_{2m+1}}\frac{(\log s_{2y+1}-\log\tau)^{-\gamma}}{3!(\log t_{2x+1}-\log\omega)^{\lambda}}\sum_{k=0}^{2}\varphi_k^{2n-1}(\tau)\\
&\times\partial_\omega^3 K_{2y+1}^{2x+1}(s_{2n-1+k},\eta_{m2}(\tilde{\omega}_m)),f(s_{2n-1+k},\eta_{m2}(\tilde{\omega}_m))\prod_{q=0}^{2}(\log\omega-\log t_{2m-1+q})\frac{d\omega}{\omega}\frac{d\tau}{\tau}\Big|\\
&+\sum_{n=1}^{y}\sum_{m=1}^{x}\Big|\int_{s_{2n-1}}^{s_{2n+1}}\int_{t_{2m-1}}^{t_{2m+1}}\sum_{k=0}^{2}\varphi_{k,2n-1}(\tau)\Big[\frac{\partial_\omega^3 K_{2y+1}^{2x+1}(s_{2n-1+k},\eta_{m2}(\omega)),f(s_{2n-1+k},\eta_{m2}(\omega))}{3!(\log s_{2y+1}-\log\tau)^{\gamma}(\log t_{2x+1}-\log\omega)^{\lambda}}\\
&-\frac{\partial_\omega^3 K_{2y+1}^{2x+1}(s_{2n-1+k},\eta_{m2}(\tilde{\omega}_m)),f(s_{2n-1+k},\eta_{m2}(\tilde{\omega}_m))}{3!(\log s_{2y+1}-\log\tau)^{\gamma}(\log t_{2x+1}-\log\omega)^{\lambda}}\Big]\prod_{q=0}^{2}(\log\omega-\log t_{2m-1+q})\frac{d\omega}{\omega}\frac{d\tau}{\tau}\Big|,
\end{aligned}$$

where $\tilde{\omega}_m$=t_{2m}. V_1 and V_2 are denoted as the two terms at the right end of the above formula. Refer to D_1 and use the same method to obtain V_1:

$$\begin{aligned}
V_1 &\le G_1\sum_{n=1}^{y}\Big|\int_{s_{2n-1}}^{s_{2n+1}}\frac{\sum_{k=0}^{2}\varphi_k^{2n-1}(\tau)}{3!(\log s_{2y+1}-\log\tau)^{\gamma}}\frac{d\tau}{\tau}\Big|\times\sum_{m=1}^{x}\Big|\int_{t_{2m-1}}^{t_{2m+1}}\frac{\prod_{q=0}^{2}(\log\omega-\log t_{2m-1+q})}{(\log t_{2x+1}-\log\omega)^{\lambda}}\frac{d\omega}{\omega}\Big|\\
&\le \frac{G_1b^{1-\gamma}}{1-\gamma}\sum_{m=1}^{x}\Big|\int_{t_{2m-1}}^{t_{2m}}\frac{\prod_{q=0}^{2}(\log\omega-\log t_{2m-1+q})}{(\log t_{2x+1}-\log\omega)^{\lambda}}\frac{d\omega}{\omega}+\int_{t_{2m}}^{t_{2m+1}}\frac{\prod_{q=0}^{2}(\log\omega-\log t_{2m-1+q})}{(\log t_{2x+1}-\log\omega)^{\lambda}}\frac{d\omega}{\omega}\Big|
\end{aligned}$$

$$\leq \frac{G_1 b^{1-\gamma}}{1-\gamma}\Big(\frac{3}{8}d + \frac{4^{1-\lambda}}{1-\lambda}\Big)\Delta_t^{4-\lambda},$$

$$V_2 \leq G_2\Delta_t^4 \sum_{n=1}^{y}\Big|\int_{s_{2n-1}}^{s_{2n+1}} \frac{\sum_{k=0}^{2}\varphi_k^{2n-1}(\tau)}{3!(\log s_{2y+1}-\log\tau)^{\gamma}}\frac{d\tau}{\tau}\Big| \times \sum_{m=1}^{x}\int_{t_{2m-1}}^{t_{2m+1}} (\log t_{2x+1}-\log\omega)^{-\lambda}\frac{d\omega}{\omega}$$
$$\leq \frac{G_2 b^{1-\gamma} d^{1-\lambda}}{(1-\gamma)(1-\lambda)}\Delta_t^4.$$

So, $r_{2y+1}^{2x+1,(4)}$ is obtained as follows

$$|r_{2y+1}^{2x+1,(4)}| \leq G_1\Big[\Big(\frac{3}{8}b + \frac{4^{1-\gamma}}{1-\gamma}\Big)\frac{d^{1-\lambda}}{1-\lambda}\Delta_s^{4-\gamma} + \Big(\frac{3}{8}d + \frac{4^{1-\lambda}}{1-\lambda}\Big)\frac{b^{1-\gamma}}{1-\gamma}\Delta_t^{4-\lambda}\Big] + \frac{b^{1-\gamma}d^{1-\lambda}G_2}{(1-\gamma)(1-\lambda)}(\Delta_s^4 + \Delta_t^4). \tag{3.28}$$

We then substitute (3.12) and (3.22)–(3.28) into (3.6) and calculate the form of r_{2y+1}^{2x+1} as follows

$$|r_{2y+1}^{2x+1}| \leq C(\Delta_s^{4-\gamma} + \Delta_t^{4-\lambda}).$$

The constant C only relies on G_1, G_2, γ and λ.
Similar to r_{2y+1}^{2x+1}, we can prove that

$$|r_{2y+l}^{2x+m}| \leq C(\Delta_s^{4-\gamma} + \Delta_t^{4-\lambda}), l = 1, 2; m = 1, 2.$$

Therefore, we conclude that the truncation error r_n^m satisfies the following conditions:

$$|r_n^m| \leq C(\Delta_s^{4-\gamma} + \Delta_t^{4-\lambda}), n = 1, 2, \cdots, 2Y; m = 1, 2, \cdots, 2X. \tag{3.29}$$

The proof is thus completed. □

4. Convergence analysis

In this section, to simplify the symbols for convergence analysis, we introduce the following coefficients to rewrite the numerical scheme; carefully observing (2.9), (2.13), (2.19) and (2.20), it is obvious that $T_{2y+1}^{k,n} = T_{2y+2}^{k,n}, k = 0, 1, 2, n = 1, 2, \cdots, y$ and $\hat{T}_{2x+1}^{q,m} = \hat{T}_{2x+2}^{q,m}, q = 0, 1, 2, m = 1, 2, \cdots, x$. Therefore, we can use the following equivalent form to recalculate the numerical scheme (2.30):

$$f_1^1 = g_1^1 + \sum_{n=0}^{2}\sum_{m=0}^{2}\hat{Q}_n\hat{W}_m K_1^1(s_n, t_m, f_n^m), \quad f_2^1 = g_2^1 + \sum_{n=0}^{2}\sum_{m=0}^{2}\tilde{Q}_n\hat{W}_m K_2^1(s_n, t_m, f_n^m),$$

$$f_1^2 = g_1^2 + \sum_{n=0}^{2}\sum_{m=0}^{2}\hat{Q}_n\tilde{W}_m K_1^2(s_n, t_m, f_n^m), \quad f_2^2 = g_2^2 + \sum_{n=0}^{2}\sum_{m=0}^{2}\tilde{Q}_n\tilde{W}_m K_2^2(s_n, t_m, f_n^m),$$

$$f_{2y+1}^1 = g_{2y+1}^1 + \sum_{n=0}^{2}\sum_{m=0}^{2}\bar{Q}_n^y\hat{W}_m K_{2y+1}^1(s_n, t_m, f_n^m) + \sum_{n=3}^{2y+1}\sum_{m=0}^{2} Q_{n+1}^y\hat{W}_m K_{2y+1}^1(s_n, t_m, f_n^m),$$

$$
\begin{aligned}
f_{2y+2}^{1} &= g_{2y+2}^{1} + \sum_{n=0}^{2y+2}\sum_{m=0}^{2} Q_n^y \hat{W}_m K_{2y+2}^{1}(s_n, t_m, f_n^m),\\
f_{2y+1}^{2} &= g_{2y+1}^{2} + \sum_{n=0}^{2}\sum_{m=0}^{2} \bar{Q}_n^y \tilde{W}_m K_{2y+1}^{2}(s_n, t_m, f_n^m) + \sum_{n=3}^{2y+1}\sum_{m=0}^{2} Q_{n+1}^y \tilde{W}_m K_{2y+1}^{2}(s_n, t_m, f_n^m),\\
f_{2y+2}^{2} &= g_{2y+2}^{2} + \sum_{n=0}^{2y+2}\sum_{m=0}^{2} Q_n^y \tilde{W}_m K_{2y+2}^{2}(s_n, t_m, f_n^m),\\
f_{1}^{2x+1} &= g_{1}^{2x+1} + \sum_{n=0}^{2}\sum_{m=0}^{2} \hat{Q}_n \bar{W}_m^x K_{1}^{2x+1}(s_n, t_m, f_n^m) + \sum_{n=0}^{2}\sum_{m=3}^{2x+1} \hat{Q}_n W_{m+1}^x K_{1}^{2x+1}(s_n, t_m, f_n^m),\\
f_{2}^{2x+1} &= g_{2}^{2x+1} + \sum_{n=0}^{2}\sum_{m=0}^{2} \tilde{Q}_n \bar{W}_m^x K_{2}^{2x+1}(s_n, t_m, f_n^m) + \sum_{n=0}^{2}\sum_{m=3}^{2x+1} \tilde{Q}_n W_{m+1}^x K_{2}^{2x+1}(s_n, t_m, f_n^m),\\
f_{1}^{2x+2} &= g_{1}^{2x+2} + \sum_{n=0}^{2}\sum_{m=0}^{2x+2} \hat{Q}_n W_m^x K_{1}^{2x+2}(s_n, t_m, f_n^m),\\
f_{2}^{2x+2} &= g_{2}^{2x+2} + \sum_{n=0}^{2}\sum_{m=0}^{2x+2} \tilde{Q}_n W_m^x K_{2}^{2x+2}(s_n, t_m, f_n^m),\\
f_{2y+1}^{2x+1} &= g_{2y+1}^{2x+1} + \sum_{n=0}^{2}\sum_{m=0}^{2} \bar{Q}_n^y \bar{W}_m^x K_{2y+1}^{2x+1}(s_n, t_m, f_n^m)\\
&\quad + \sum_{n=3}^{2y+1}\sum_{m=0}^{2} Q_{n+1}^y \bar{W}_m^x K_{2y+1}^{2x+1}(s_n, t_m, f_n^m) + \sum_{n=0}^{2}\sum_{m=3}^{2x+1} \bar{Q}_n^y W_{m+1}^x K_{2y+1}^{2x+1}(s_n, t_m, f_n^m)\\
&\quad + \sum_{n=3}^{2y+1}\sum_{m=3}^{2x+1} Q_{n+1}^y W_{m+1}^x K_{2y+1}^{2x+1}(s_n, t_m, f_n^m),\\
f_{2y+2}^{2x+1} &= g_{2y+2}^{2x+1} + \sum_{n=0}^{2y+2}\sum_{m=0}^{2} Q_n^y \bar{W}_m^x K_{2y+2}^{2x+1}(s_n, t_m, f_n^m) + \sum_{n=0}^{2y+2}\sum_{m=3}^{2x+1} Q_n^y W_{m+1}^x K_{2y+2}^{2x+1}(s_n, t_m, f_n^m),\\
f_{2y+1}^{2x+2} &= g_{2y+1}^{2x+2} + \sum_{n=0}^{2}\sum_{m=0}^{2x+2} \bar{Q}_n^y W_m^x K_{2y+1}^{2x+2}(s_n, t_m, f_n^m) + \sum_{n=3}^{2y+1}\sum_{m=0}^{2x+2} Q_{n+1}^y W_m^x K_{2y+1}^{2x+2}(s_n, t_m, f_n^m),\\
f_{2y+2}^{2x+2} &= g_{2y+2}^{2x+2} + \sum_{n=0}^{2y+2}\sum_{m=0}^{2x+2} Q_n^y W_m^x K_{2y+2}^{2x+2}(s_n, t_m, f_n^m),
\end{aligned}
\tag{4.1}
$$

where $y = 1,\ 2,\ \cdots,\ Y-1;\ x = 1,\ 2,\ \cdots,\ X-1$, and

$$
\begin{aligned}
&\hat{Q}_n = T_1^{n,0}; \tilde{Q}_n = T_2^{n,0},\ n = 0, 1, 2;\\
&\bar{Q}_0^y = T_{2y+1}^{0,0}, \bar{Q}_1^y = T_{2y+1}^{1,0} + T_{2y+1}^{0,1}, \bar{Q}_2^y = T_{2y+1}^{2,0} + T_{2y+1}^{1,1},\\
&\bar{Q}_{2r+1}^y = T_{2y+1}^{2,r} + T_{2y+1}^{0,r+1}, r = 1, \cdots, y-1;\\
&\bar{Q}_{2y+1}^y = T_{2y+1}^{2,y}, \bar{Q}_{2r}^y = T_{2y+1}^{1,r}, r = 2, \cdots, y;\\
&Q_0^y = T_{2y+2}^{0,0}, Q_{2n+1}^y = T_{2y+2}^{1,n}, n = 0, 1, \cdots, y; Q_{2n}^y = T_{2y+2}^{2,n-1} + T_{2y+2}^{0,n}, n = 1, 2, \cdots, y;
\end{aligned}
\tag{4.2}
$$

$$
\begin{aligned}
&Q^{y}_{2y+2} = T^{2,y}_{2y+2}; \hat{W}_m = \hat{T}^{m,0}_{1}, \tilde{W}_m = \hat{T}^{m,0}_{2},\ m = 0,1,2;\\
&\bar{W}^x_0 = \hat{T}^{0,0}_{2x+1}, \bar{W}^x_1 = \hat{T}^{1,0}_{2x+1} + \hat{T}^{0,1}_{2x+1}, \bar{W}^x_2 = \hat{T}^{2,0}_{2x+1} + \hat{T}^{1,1}_{2x+1};\\
&W^x_0 = \hat{T}^{0,0}_{2x+2}, W^x_{2m+1} = \hat{T}^{1,m}_{2x+2}, m = 0,1,\cdots,x;\\
&W^x_{2m} = \hat{T}^{2,m-1}_{2x+2} + \hat{T}^{0,m}_{2x+2}, m = 1,2,\cdots,x; W^x_{2x+2} = \hat{T}^{2,x}_{2x+2}.\\
&\bar{W}^x_m = \hat{T}^{0,0}_{2x+1}, m = 0, \bar{W}^x_m = \hat{T}^{1,0}_{2x+1} + \hat{T}^{0,1}_{2x+1}, m = 1, \bar{W}^x_m = \hat{T}^{2,0}_{2x+1} + \hat{T}^{1,1}_{2x+1}, m = 2,\\
&\bar{W}^x_m = \hat{T}^{2,k}_{2x+1} + \hat{T}^{0,k+1}_{2x+1}, m = 2k+1, k = 1,\cdots,x-1;\\
&\bar{W}^x_m = \hat{T}^{2,x}_{2x+1}, m = 2x+1, \bar{W}^x_m = \hat{T}^{1,k}_{2x+1}, m = 2k, k = 2,\cdots,x.
\end{aligned}
$$

Next, we propose Lemma 7.

Lemma 7. *The coefficients of $\bar{Q}^y_n, n = 0,1,\cdots,2y+1,$ and $\bar{W}^x_m, m = 0,1,\cdots,2x+1,$ are shown in (4.2), and they satisfy*

$$\left|\bar{Q}^y_n\right| \le C\left(\log\frac{s_{2y+1}}{s_n}\right)^{-\gamma}\log\frac{s_{n+1}}{s_n}, n = 0,1,\cdots,2y, \tag{4.3}$$

$$\left|\bar{Q}^y_{2y+1}\right| \le \frac{2^{1-\gamma}(12-2\gamma)\Gamma(1-\gamma)}{\Gamma(4-\gamma)a^{1-\gamma}}\Delta_s^{1-\gamma}, \tag{4.4}$$

$$\left|\bar{W}^x_m\right| \le C\left(\log\frac{t_{2x+1}}{t_m}\right)^{-\lambda}\log\frac{t_{m+1}}{t_m}, m = 0,1,\cdots,2x, \tag{4.5}$$

$$\left|\bar{W}^x_{2x+1}\right| \le \frac{2^{1-\lambda}(12-2\lambda)\Gamma(1-\lambda)}{\Gamma(4-\lambda)c^{1-\lambda}}\Delta_t^{1-\lambda}. \tag{4.6}$$

Proof. This proves the estimate of (4.3) for $n = 0$; using Lemma 3, we have

$$
\begin{aligned}
\left|\bar{Q}^y_0\right| &= \left|T^{0,0}_{2y+1}\right| = \left|\int_a^{s_1}\left(\log\frac{s_{2y+1}}{\tau}\right)^{-\gamma}\varphi^0_0(\tau)\frac{d\tau}{\tau}\right|\\
&\le \int_a^{s_1}\left(\log\frac{s_{2y+1}}{\tau}\right)^{-\gamma}\left|\varphi^0_0(\tau)\right|\frac{d\tau}{\tau} \le \int_a^{s_1}\left(\log\frac{s_{2y+1}}{\tau}\right)^{-\gamma}\frac{d\tau}{\tau}\\
&\le \left(\log\frac{s_{2y+1}}{s_1}\right)^{-\gamma}\log\frac{s_1}{a} = \frac{\left(\log\frac{s_{2y+1}}{a}\right)^{\gamma}}{\left(\log\frac{s_{2y+1}}{s_1}\right)^{\gamma}}\left(\log\frac{s_{2y+1}}{a}\right)^{-\gamma}\log\frac{s_1}{a}\\
&= \left(\frac{\frac{1}{\xi_1}(s_{2y+1}-a)}{\frac{1}{\xi_2}(s_{2y+1}-s_1)}\right)^{\gamma}\left(\log\frac{s_{2y+1}}{a}\right)^{-\gamma}\log\frac{s_1}{a}\\
&= \left(\frac{\xi_2}{\xi_1}\right)^{\gamma}\left(\frac{2y+1}{2y}\right)^{\gamma}\left(\log\frac{s_{2y+1}}{a}\right)^{-\gamma}\log\frac{s_1}{a}\\
&\le 2^{\gamma}\left(\frac{b}{a}\right)^{\gamma}\left(\log\frac{s_{2y+1}}{a}\right)^{-\gamma}\log\frac{s_1}{a} \le C\left(\log\frac{s_{2y+1}}{a}\right)^{-\gamma}\log\frac{s_1}{a},
\end{aligned}
\tag{4.7}
$$

where $a < \xi_1 < s_{2y+1} < b, a < s_1 < \xi_2 < s_{2y+1} < b$ and C is independent on Δ_t.

For $n = 1$, using Lemma 3 and Lemma 7, we have

$$\left|\bar{Q}^y_1\right| = \left|T^{1,0}_{2y+1} + T^{0,1}_{2y+1}\right| \le \left|T^{1,0}_{2y+1}\right| + \left|T^{0,1}_{2y+1}\right|$$

$$
\begin{aligned}
&= \left|\int_a^{s_1}\left(\log\frac{s_{2y+1}}{\tau}\right)^{-\gamma}\varphi_1^0(\tau)\frac{d\tau}{\tau}\right| + \left|\int_{s_1}^{s_3}\left(\log\frac{s_{2y+1}}{\tau}\right)^{-\gamma}\varphi_0^1(\tau)\frac{d\tau}{\tau}\right| \\
&\leq \int_a^{s_1}\left(\log\frac{s_{2y+1}}{\tau}\right)^{-\gamma}\frac{d\tau}{\tau} + \int_{s_1}^{s_3}\left(\log\frac{s_{2y+1}}{\tau}\right)^{-\gamma}\frac{d\tau}{\tau} \\
&\leq \left(\log\frac{s_{2y+1}}{s_1}\right)^{-\gamma}\log\frac{s_1}{a} + \left(\log\frac{s_{2y+1}}{s_3}\right)^{-\gamma}\log\frac{s_3}{s_1} \qquad (4.8)\\
&= \left(\log\frac{s_{2y+1}}{s_1}\right)^{-\gamma}\log\frac{s_2}{s_1}\frac{\log\frac{s_1}{a}}{\log\frac{s_2}{s_1}} + \left(\frac{\log\frac{s_{2y+1}}{s_1}}{\log\frac{s_{2y+1}}{s_3}}\right)^{\gamma}\left(\log\frac{s_{2y+1}}{s_1}\right)^{-\gamma}\log\frac{s_2}{s_1}\frac{\log\frac{s_3}{s_1}}{\log\frac{s_2}{s_1}} \\
&\leq 2\left(\log\frac{s_{2y+1}}{s_1}\right)^{-\gamma}\log\frac{s_2}{s_1} + 3\left(\frac{2y}{2y-2}+1\right)^{\gamma}\left(\log\frac{s_{2y+1}}{s_1}\right)^{-\gamma}\log\frac{s_2}{s_1} \\
&\leq C\left(\log\frac{s_{2y+1}}{s_1}\right)^{-\gamma}\log\frac{s_2}{s_1}.
\end{aligned}
$$

Next, for $n = 2$, we obtain

$$
\left|\bar{Q}_2^y\right| = \left|T_{2y+1}^{2,0} + T_{2y+1}^{1,1}\right|.
$$

On one side, using Lemma 3, we have

$$
\begin{aligned}
\left|T_{2y+1}^{2,0}\right| &= \left|\int_a^{s_1}\left(\log\frac{s_{2y+1}}{\tau}\right)^{-\gamma}\varphi_2^0(\tau)\frac{d\tau}{\tau}\right| \leq \int_a^{s_1}\left(\log\frac{s_{2y+1}}{\tau}\right)^{-\gamma}\left|\frac{\log\frac{\tau}{a}\log\frac{\tau}{s_1}}{\log\frac{s_2}{a}\log\frac{s_2}{s_1}}\right|\frac{d\tau}{\tau} \\
&\leq 2\int_a^{s_1}\left(\log\frac{s_{2y+1}}{\tau}\right)^{-\gamma}\frac{d\tau}{\tau} \leq 2\left(\log\frac{s_{2y+1}}{s_1}\right)^{-\gamma}\log\frac{s_1}{a} \\
&= 2\left(\frac{\log\frac{s_{2y+1}}{s_2}}{\log\frac{s_{2y+1}}{s_1}}\right)^{\gamma}\left(\log\frac{s_{2y+1}}{s_2}\right)^{-\gamma}\log\frac{s_3}{s_2}\frac{\log\frac{s_1}{a}}{\log\frac{s_3}{s_2}} \\
&\leq 4\left(\frac{2y-1}{2y}+1\right)^{\gamma}\left(\log\frac{s_{2y+1}}{s_2}\right)^{-\gamma}\log\frac{s_3}{s_2}.
\end{aligned}
$$

On the other side,

$$
\begin{aligned}
\left|T_{2y+1}^{1,1}\right| &= \left|\int_{s_1}^{s_3}\left(\log\frac{s_{2y+1}}{\tau}\right)^{-\gamma}\varphi_1^1(\tau)\frac{d\tau}{\tau}\right| \leq \int_{s_1}^{s_3}\left(\log\frac{s_{2y+1}}{\tau}\right)^{-\gamma}\left|\frac{\log\frac{\tau}{s_1}\log\frac{\tau}{s_3}}{\log\frac{s_2}{s_1}\log\frac{s_2}{s_3}}\right|\frac{d\tau}{\tau} \\
&\leq \frac{\left(\log\frac{s_3}{s_1}\right)^2}{4\log\frac{s_2}{s_1}\log\frac{s_3}{s_2}}\int_{s_1}^{s_3}\left(\log\frac{s_{2y+1}}{\tau}\right)^{-\gamma}\frac{d\tau}{\tau} \leq \frac{9}{4}\left(\log\frac{s_{2y+1}}{s_3}\right)^{-\gamma}\log\frac{s_3}{s_1} \\
&= \frac{9}{4}\left(\frac{\log\frac{s_{2y+1}}{s_2}}{\log\frac{s_{2y+1}}{s_3}}\right)^{\gamma}\left(\log\frac{s_{2y+1}}{s_2}\right)^{-\gamma}\log\frac{s_3}{s_2}\frac{\log\frac{s_3}{s_1}}{\log\frac{s_3}{s_2}} \\
&\leq \frac{27}{4}\left(\frac{2y-1}{2y-2}+1\right)^{\gamma}\left(\log\frac{s_{2y+1}}{s_2}\right)^{-\gamma}\log\frac{s_3}{s_2}.
\end{aligned}
$$

So, we have

$$
\left|\bar{Q}_2^y\right| \leq 4\left(\frac{2y-1}{2y}+1\right)^{\gamma}\left(\log\frac{s_{2y+1}}{s_2}\right)^{-\gamma}\log\frac{s_3}{s_2} + \frac{27}{4}\left(\frac{2y-1}{2y-2}+1\right)^{\gamma}\left(\log\frac{s_{2y+1}}{s_2}\right)^{-\gamma}\log\frac{s_3}{s_2}
$$

$$\leq \quad C\left(\log\frac{s_{2y+1}}{s_2}\right)^{-\gamma}\log\frac{s_3}{s_2}. \tag{4.9}$$

Next, we will estimate $\left|\bar{Q}_n^y\right|, n = 3, \cdots, 2y$. For $n = 2r + 1, r = 1, 2, \cdots, y - 1$, we obtain

$$\begin{aligned}
&\left|\bar{Q}_{2r+1}^y\right| = \left|T_{2y+1}^{2,r} + T_{2y+1}^{0,r+1}\right| \\
&= \left|\int_{s_{2r-1}}^{s_{2r+1}}\left(\log\frac{s_{2y+1}}{\tau}\right)^{-\gamma}\varphi_2^{2r-1}(\tau)\frac{d\tau}{\tau} + \int_{s_{2r+1}}^{s_{2r+3}}\left(\log\frac{s_{2y+1}}{\tau}\right)^{-\gamma}\varphi_0^{2r+1}(\tau)\frac{d\tau}{\tau}\right| \\
&= \int_{s_{2r-1}}^{s_{2r+1}}\left(\log\frac{s_{2y+1}}{\tau}\right)^{-\gamma}\left|\frac{\log\frac{\tau}{s_{2r-1}}\log\frac{\tau}{s_{2r}}}{\log\frac{s_{2r+1}}{s_{2r-1}}\log\frac{s_{2r+1}}{s_{2r}}}\right|\frac{d\tau}{\tau} + \int_{s_{2r+1}}^{s_{2r+3}}\left(\log\frac{s_{2y+1}}{\tau}\right)^{-\gamma}\left|\frac{\log\frac{\tau}{s_{2r+2}}\log\frac{\tau}{s_{2r+3}}}{\log\frac{s_{2r+1}}{s_{2r+2}}\log\frac{s_{2r+1}}{s_{2r+3}}}\right|\frac{d\tau}{\tau} \\
&\leq 3\int_{s_{2r-1}}^{s_{2r+1}}\left(\log\frac{s_{2y+1}}{\tau}\right)^{-\gamma}\frac{d\tau}{\tau} + 3\int_{s_{2r+1}}^{s_{2r+3}}\left(\log\frac{s_{2y+1}}{\tau}\right)^{-\gamma}\frac{d\tau}{\tau} \\
&\leq 3\left(\log\frac{s_{2y+1}}{s_{2r+1}}\right)^{-\gamma}\log\frac{s_{2r+1}}{s_{2r-1}} + 3\left(\log\frac{s_{2y+1}}{s_{2r+3}}\right)^{-\gamma}\log\frac{s_{2r+3}}{s_{2r+1}} \\
&= 3\left(\log\frac{s_{2y+1}}{s_{2r+1}}\right)^{-\gamma}\log\frac{s_{2r+2}}{s_{2r+1}}\frac{\log\frac{s_{2r+1}}{s_{2r-1}}}{\log\frac{s_{2r+2}}{s_{2r+1}}} + 3\left(\frac{\log\frac{s_{2y+1}}{s_{2r+1}}}{\log\frac{s_{2y+1}}{s_{2r+3}}}\right)^{\gamma}\left(\log\frac{s_{2y+1}}{s_{2r+1}}\right)^{-\gamma}\log\frac{s_{2r+2}}{s_{2r+1}}\frac{\log\frac{s_{2r+3}}{s_{2r+1}}}{\log\frac{s_{2r+2}}{s_{2r+1}}} \\
&\leq 9\left(\log\frac{s_{2y+1}}{s_{2r+1}}\right)^{-\gamma}\log\frac{s_{2r+2}}{s_{2r+1}} + 9\left(\frac{2y-2r}{2y-2r-2}+1\right)^{\gamma}\left(\log\frac{s_{2y+1}}{s_{2r+1}}\right)^{-\gamma}\log\frac{s_{2r+2}}{s_{2r+1}} \\
&\leq C\left(\log\frac{s_{2y+1}}{s_{2r+1}}\right)^{-\gamma}\log\frac{s_{2r+2}}{s_{2r+1}}.
\end{aligned}$$

For $n = 2r, r = 2, \cdots, y$, we have

$$\begin{aligned}
\left|\bar{Q}_{2r}^y\right| &= \left|T_{2y+1}^{1,r}\right| = \left|\int_{s_{2r-1}}^{s_{2r+1}}\left(\log\frac{s_{2y+1}}{\tau}\right)^{-\gamma}\varphi_1^{2r-1}(\tau)\frac{d\tau}{\tau}\right| \\
&= \int_{s_{2r-1}}^{s_{2r+1}}\left(\log\frac{s_{2y+1}}{\tau}\right)^{-\gamma}\left|\frac{\log\frac{\tau}{s_{2r-1}}\log\frac{\tau}{s_{2r+1}}}{\log\frac{s_{2r}}{s_{2r-1}}\log\frac{s_{2r}}{s_{2r+1}}}\right|\frac{d\tau}{\tau} \\
&\leq \frac{\left(\log\frac{s_{2r+1}}{s_{2r-1}}\right)^2}{\log\frac{s_{2r}}{s_{2r-1}}\log\frac{s_{2r+1}}{s_{2r}}}\left(\log\frac{s_{2y+1}}{s_{2r+1}}\right)^{-\gamma}\log\frac{s_{2r+1}}{s_{2r-1}} \\
&\leq 5\left(\frac{\log\frac{s_{2y+1}}{s_{2r}}}{\log\frac{s_{2y+1}}{s_{2r+1}}}\right)^{\gamma}\left(\log\frac{s_{2y+1}}{s_{2r}}\right)^{-\gamma}\log\frac{s_{2r+1}}{s_{2r}}\frac{\log\frac{s_{2r+1}}{s_{2r-1}}}{\log\frac{s_{2r+1}}{s_{2r}}} \leq C\left(\log\frac{s_{2y+1}}{s_{2r}}\right)^{-\gamma}\log\frac{s_{2r+1}}{s_{2r}}.
\end{aligned}$$

Next, let us prove (4.4). According to Lemma 2, we proceed as follows:

$$\begin{aligned}
\bar{Q}_{2y+1}^y &= T_{2y+1}^{2,y} = \int_{s_{2y-1}}^{s_{2y+1}}\left(\log\frac{s_{2y+1}}{\tau}\right)^{-\gamma}\varphi_2^{2y-1}(\tau)\frac{d\tau}{\tau} \\
&= \int_{s_{2y-1}}^{s_{2y+1}}\left(\log\frac{s_{2y+1}}{\tau}\right)^{-\gamma}\frac{\log\frac{\tau}{s_{2y-1}}\log\frac{\tau}{s_{2y}}}{\log\frac{s_{2y+1}}{s_{2y-1}}\log\frac{s_{2y+1}}{s_{2y}}}\frac{d\tau}{\tau} \\
&= \int_{s_{2y-1}}^{s_{2y+1}}\left(\log\frac{s_{2y+1}}{\tau}\right)^{-\gamma}\frac{\log\frac{\tau}{s_{2y-1}}\left(\log\frac{\tau}{s_{2y-1}} + \log\frac{s_{2y-1}}{s_{2y}}\right)}{\log\frac{s_{2y+1}}{s_{2y-1}}\log\frac{s_{2y+1}}{s_{2y}}}\frac{d\tau}{\tau}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\log\frac{s_{2y+1}}{s_{2y-1}}\log\frac{s_{2y+1}}{s_{2y}}}\int_{s_{2y-1}}^{s_{2y+1}}\left(\log\frac{s_{2y+1}}{\tau}\right)^{-\gamma}\left(\log^2\frac{\tau}{s_{2y-1}}+\log\frac{\tau}{s_{2y-1}}\log\frac{s_{2y-1}}{s_{2y}}\right)\frac{d\tau}{\tau}\\
&= \frac{1}{\log\frac{s_{2y+1}}{s_{2y-1}}\log\frac{s_{2y+1}}{s_{2y}}}\left(\frac{\Gamma(1-\gamma)\Gamma(3)}{\Gamma(4-\gamma)}\left(\log\frac{s_{2y+1}}{s_{2y-1}}\right)^{3-\gamma}+\log\frac{s_{2y-1}}{s_{2y}}\frac{\Gamma(1-\gamma)\Gamma(2)}{\Gamma(3-\gamma)}\left(\log\frac{s_{2y+1}}{s_{2y-1}}\right)^{2-\gamma}\right)\\
&= \left(\log\frac{s_{2y+1}}{s_{2y-1}}\right)^{1-\gamma}\left(\frac{2\Gamma(1-\gamma)\log\frac{s_{2y+1}}{s_{2y-1}}}{\Gamma(4-\gamma)\log\frac{s_{2y+1}}{s_{2y}}}+\frac{\Gamma(1-\gamma)\log\frac{s_{2y-1}}{s_{2y}}}{\Gamma(3-\gamma)\log\frac{s_{2y+1}}{s_{2y}}}\right).
\end{aligned}
$$

So we can get

$$
\begin{aligned}
\left|\bar{Q}^y_{2y+1}\right| &\le \left(\log\frac{s_{2y+1}}{s_{2y-1}}\right)^{1-\gamma}\left(\frac{2\Gamma(1-\gamma)\log\frac{s_{2y+1}}{s_{2y-1}}}{\Gamma(4-\gamma)\log\frac{s_{2y+1}}{s_{2y}}}+\frac{\Gamma(1-\gamma)\log\frac{s_{2y}}{s_{2y-1}}}{\Gamma(3-\gamma)\log\frac{s_{2y+1}}{s_{2y}}}\right)\\
&\le \left(\frac{2\Delta s}{a}\right)^{1-\gamma}\left(\frac{6\Gamma(1-\gamma)}{\Gamma(4-\gamma)}+\frac{2\Gamma(1-\gamma)}{\Gamma(3-\gamma)}\right)\\
&= \frac{2^{1-\gamma}(12-2\gamma)\Gamma(1-\gamma)}{\Gamma(4-\gamma)a^{1-\gamma}}\Delta_s^{1-\gamma},
\end{aligned}
$$

where $\left(\log\frac{s_{2y+1}}{s_{2y-1}}\right)^{1-\gamma}=\left(\log\left(1+\frac{2\Delta_s}{s_{2y-1}}\right)\right)^{1-\gamma}\le\left(\frac{2\Delta_s}{s_{2y-1}}\right)^{1-\gamma}\le\left(\frac{2\Delta_s}{a}\right)^{1-\gamma}$. Taking all of the results together, we can obtain the estimates (4.3) and (4.4). □

Since the proof process for $\bar{Q}^y_n$ is the same as that for $\bar{W}^y_m$, we will omit it without further proof.

Lemma 8. *The coefficients of $Q^y_n, n=0,1,\cdots,2y+2$, and $W^x_m, m=0,1,\cdots,2x+2$, satisfy*

$$\left|Q^y_n\right|\le C\left(\log\frac{s_{2y+2}}{s_n}\right)^{-\gamma}\log\frac{s_{n+1}}{s_n}, n=0,1,\cdots,2y, \tag{4.10}$$

$$\left|Q^y_{2y+2}\right|\le\frac{2^{1-\gamma}(12-2\gamma)\Gamma(1-\gamma)}{\Gamma(4-\gamma)a^{1-\gamma}}\Delta_s^{1-\gamma}, \tag{4.11}$$

$$\left|W^x_m\right|\le C\left(\log\frac{t_{2x+2}}{t_m}\right)^{-\lambda}\log\frac{t_{m+1}}{t_m}, m=0,1,\cdots,2x, \tag{4.12}$$

$$\left|W^x_{2x+2}\right|\le\frac{2^{1-\lambda}(12-2\lambda)\Gamma(1-\lambda)}{\Gamma(4-\lambda)c^{1-\lambda}}\Delta_t^{1-\lambda}. \tag{4.13}$$

Proof. The proof process for Lemma 8 can be achieved by referring to the method used in Lemma 7. □

Next, we analyze the convergence by using the scheme (4.1), as in the following Theorem 1.

Theorem 1. *Let u be the solution of (1.2) and f^m_n ($n=0,1,\cdots,2Y$, $m=0,1,\cdots,2X$) be the numerical solution of (1.2) by using scheme* (4.1)*. Assume that $K(\cdot,\cdot,\cdot,\cdot,f(\cdot,\cdot))\in C^4([a,b]\times[c,d]\times R)$ satisfies (1.3) and Δ_s, Δ_t satisfy*

$$\frac{2^{1-\gamma}(12-2\gamma)\Gamma(1-\gamma)}{\Gamma(4-\gamma)a^{1-\gamma}}\Delta_s^{1-\gamma}L<1,\quad \frac{2^{1-\lambda}(12-2\lambda)\Gamma(1-\lambda)}{\Gamma(4-\lambda)c^{1-\lambda}}\Delta_t^{1-\lambda}L<1. \tag{4.14}$$

Then the error is that

$$|f(s_n,t_m)-f_n^m|\le C(\Delta_s^{4-\gamma}+\Delta_t^{4-\lambda}), n=1,2,\cdots,2Y; m=1,2,\cdots,2X, \tag{4.15}$$

where C is constant and only dependent on L, K, b, d, γ and λ.

Proof. Assume that $e_i^m = f(s_n,t_m)-f_n^m, n=0,1,\cdots,2Y; m=0,1,\cdots,2X$. $e_n^0 = e_0^m = 0$ can be readily observed. First, $e_n^m, n,m=1,2$, it can be known that

$$e_1^1 = \sum_{n=0}^{2}\sum_{m=0}^{2}\hat{Q}_n\hat{W}_m[K_1^1(s_n,t_m,f(s_n,t_m))-K_1^1(s_n,t_m,f_n^m)]+r_1^1,$$
$$e_1^2 = \sum_{n=0}^{2}\sum_{m=0}^{2}\hat{Q}_n\tilde{W}_m[K_1^2(s_n,t_m,f(s_n,t_m))-K_1^2(s_n,t_m,f_n^m)]+r_1^2,$$
$$e_2^1 = \sum_{n=0}^{2}\sum_{m=0}^{2}\tilde{Q}_n\hat{W}_m[K_2^1(s_n,t_m,f(s_n,t_m))-K_2^1(s_n,t_m,f_n^m)]+r_2^1,$$
$$e_2^2 = \sum_{n=0}^{2}\sum_{m=0}^{2}\tilde{Q}_n\tilde{W}_m[K_2^2(s_n,t_m,f(s_n,t_m))-K_2^2(s_n,t_m,f_n^m)]+r_2^2.$$

The calculation can prove that $\hat{Q}_n, \tilde{Q}_n, n=0,1,2$, and (1.3) are satisfied by K, and that $\hat{W}_m, \tilde{W}_m, m=0,1,2$, are bounded. So, we come to the following conclusion:

$$|e_1^1|\le CL\sum_{n=0}^{2}\sum_{m=0}^{2}|e_n^m|+|r_1^1|,\quad |e_1^2|\le CL\sum_{n=0}^{2}\sum_{m=0}^{2}|e_n^m|+|r_1^2|,$$
$$|e_2^1|\le CL\sum_{n=0}^{2}\sum_{m=0}^{2}|e_n^m|+|r_2^1|,\quad |e_2^2|\le CL\sum_{n=0}^{2}\sum_{m=0}^{2}|e_n^m|+|r_2^2|.$$

We combine these four inequalities and obtain

$$|e_n^m|\le CL(|r_1^1|+|r_1^2|+|r_2^1|+|r_2^2|), n,m=1,2.$$

Second, $e_n^m, n\ge 3; m=1,2$, satisfies

$$\begin{aligned} e_{2y+1}^1 &= \sum_{n=0}^{2}\sum_{m=0}^{2}\bar{Q}_n^y\hat{W}_m[K_{2y+1}^1(s_n,t_m,f(s_n,t_m))-K_{2y+1}^1(s_n,t_m,f_n^m)] \\ &\quad +\sum_{n=3}^{2y+1}\sum_{m=0}^{2}Q_{n+1}^y\hat{W}_m[K_{2y+1}^1(s_n,t_m,f(s_n,t_m))-K_{2y+1}^1(s_n,t_m,f_n^m)]+r_{2y+1}^1, \end{aligned} \tag{4.16}$$

$$e_{2y+2}^1 = \sum_{n=0}^{2y+2}\sum_{m=0}^{2}Q_n^y\hat{W}_m[K_{2y+2}^1(s_n,t_m,f(s_n,t_m))-K_{2y+2}^1(s_n,t_m,f_n^m)]+r_{2y+2}^1, \tag{4.17}$$

$$e_{2y+1}^2 = \sum_{n=0}^{2}\sum_{m=0}^{2}\bar{Q}_n^y\tilde{W}_m[K_{2y+1}^2(s_n,t_m,f(s_n,t_m))-K_{2y+1}^2(s_n,t_m,f_n^m)]$$

$$+\sum_{n=3}^{2y+1}\sum_{m=0}^{2} Q_{n+1}^{y}\tilde{W}_m[K_{2y+1}^2(s_n,t_m,f(s_n,t_m))-K_{2y+1}^2(s_n,t_m,f_n^m)]+r_{2y+1}^2, \tag{4.18}$$

$$e_{2y+2}^2 = \sum_{n=0}^{2y+2}\sum_{m=0}^{2} Q_n^y\tilde{W}_m[K_{2y+2}^2(s_n,t_m,f(s_n,t_m))-K_{2y+2}^2(s_n,t_m,f_n^m)]+r_{2y+2}^2. \tag{4.19}$$

It can be seen that the above four equations are coupled, so they need to be solved simultaneously. For convenience, we assume that $\|\hat{e}_n\| = \max\{|e_n^1|, |e_n^2|, n = 0, 1, \cdots, 2Y\}$ and $\|\hat{r}_n\| = \max\{|r_y^1|, |r_y^2|, y = 0, 1, \cdots, 2Y\}$. According to (1.3), (4.3) in Lemma 7, (4.10) and (4.11) in Lemma 8, (4.16) and (4.18) becomes

$$\begin{aligned}\left\|\hat{e}_{2y+1}\right\| \le\; & LC\sum_{n=0}^{2}\left(\log\frac{s_{2y+1}}{s_n}\right)^{-\gamma}\log\frac{s_{n+1}}{s_n}\|\hat{e}_n\| + LC\sum_{n=3}^{2y}\left(\log\frac{s_{2y+2}}{s_{n+1}}\right)^{-\gamma}\log\frac{s_{n+2}}{s_{n+1}}\|\hat{e}_n\| \\ & + \frac{2^{1-\gamma}(12-2\gamma)\Gamma(1-\gamma)}{\Gamma(4-\gamma)a^{1-\gamma}}\Delta_s^{1-\gamma}L\left\|\hat{e}_{2y+1}\right\| + |\hat{r}_{2y+1}|. \end{aligned} \tag{4.20}$$

By Lemma 4, and through careful calculation, we have

$$\frac{\left(\log\frac{s_{2y+2}}{s_{n+1}}\right)^{-\gamma}\log\frac{s_{n+2}}{s_{n+1}}}{\left(\log\frac{s_{2y+1}}{s_n}\right)^{-\gamma}\log\frac{s_{n+1}}{s_n}} = \frac{\left(\log\frac{s_{2y+2}}{s_{n+1}}\right)^{\gamma}\log\frac{s_{n+1}}{s_n}}{\left(\log\frac{s_{2y+1}}{s_n}\right)^{\gamma}\log\frac{s_{n+2}}{s_{n+1}}} \le 2^\gamma\cdot 2\le 4.$$

Therefore, (4.20) becomes

$$\left\|\hat{e}_{2y+1}\right\| \le 4LC\sum_{n=0}^{2y}\left(\log\frac{s_{2y+1}}{s_n}\right)^{-\gamma}\log\frac{s_{n+1}}{s_n}\|\hat{e}_n\| + \frac{2^{1-\gamma}(12-2\gamma)\Gamma(1-\gamma)}{\Gamma(4-\gamma)a^{1-\gamma}}\Delta_s^{1-\gamma}L\left\|\hat{e}_{2y+1}\right\| + |\hat{r}_{2y+1}|.$$

Using the above same method, the estimates (4.17) and (4.19) are as shown below

$$\left\|\hat{e}_{2y+2}\right\| \le LC\sum_{n=0}^{2y+1}\left(\log\frac{s_{2y+2}}{s_n}\right)^{-\gamma}\log\frac{s_{n+1}}{s_n}\|\hat{e}_n\| + \frac{2^{1-\gamma}(12-2\gamma)\Gamma(1-\gamma)}{\Gamma(4-\gamma)a^{1-\gamma}}\Delta_s^{1-\gamma}L\left\|\hat{e}_{2y+2}\right\| + \left\|\hat{r}_{2y+2}\right\|.$$

For $\left\|\hat{e}_{2y+1}\right\|$, it is easy to verify that $\|\hat{e}_0\| = 0$; then we have

$$\left(1-\frac{2^{1-\gamma}(12-2\gamma)\Gamma(1-\gamma)}{\Gamma(4-\gamma)a^{1-\gamma}}\Delta_s^{1-\gamma}L\right)\left\|\hat{e}_{2y+1}\right\| \le 4LC\sum_{n=1}^{2y}\left(\log\frac{s_{2y+1}}{s_n}\right)^{-\gamma}\log\frac{s_{n+1}}{s_n}\|\hat{e}_n\| + \left\|\hat{r}_{2y+1}\right\|.$$

For enough small Δ_s, we obtain

$$\left\|\hat{e}_{2y+1}\right\| \le LC\sum_{n=1}^{2y}\left(\log\frac{s_{2y+1}}{s_n}\right)^{-\gamma}\log\frac{s_{n+1}}{s_n}\|\hat{e}_n\| + C\left\|\hat{r}_{2y+1}\right\|. \tag{4.21}$$

Applying Lemma 1 to (4.21) leads to

$$\left\|\hat{e}_{2y+1}\right\| \le C\left\|\hat{r}_{2y+1}\right\|.$$

Using the same argument for $\left\|\hat{e}_{2y+2}\right\|$, we can directly conclude that there are some C that

$$\left\|\hat{e}_{2y+2}\right\| \leq C \left\|\hat{r}_{2y+2}\right\|.$$

Now, based on the above results and Lemma 6, we can reach the following conclusion.

$$|e_n^m| \leq C(\Delta_s^{4-\gamma} + \Delta_t^{4-\lambda}), n \geq 1; m = 1, 2, \tag{4.22}$$

$$|e_n^m| \leq C(\Delta_s^{4-\gamma} + \Delta_t^{4-\lambda}), n = 1, 2; m \geq 3. \tag{4.23}$$

Next, satisfy $e_n^m, n, m \geq 3$ to obtain

$$\begin{aligned}
e_{2y+1}^{2x+1} \ &= \ \sum_{n=0}^{2}\sum_{m=0}^{2} \bar{Q}_n^y \bar{W}_m^x [K_{2y+1}^{2x+1}(s_n, t_m, f(s_n, t_m)) - K_{2y+1}^{2x+1}(s_n, t_m, f_n^m)] \\
&\quad + \sum_{n=3}^{2y+1}\sum_{m=0}^{2} Q_{n+1}^y \bar{W}_m^x [K_{2y+1}^{2x+1}(s_n, t_m, f(s_n, t_m)) - K_{2y+1}^{2x+1}(s_n, t_m, f_n^m)] \\
&\quad + \sum_{n=0}^{2}\sum_{m=3}^{2x+1} \bar{Q}_n^y W_{m+1}^x [K_{2y+1}^{2x+1}(s_n, t_m, f(s_n, t_m)) - K_{2y+1}^{2x+1}(s_n, t_m, f_n^m)] \\
&\quad + \sum_{n=3}^{2y+1}\sum_{m=3}^{2x+1} Q_{n+1}^y W_{m+1}^x [K_{2y+1}^{2x+1}(s_n, t_m, f(s_n, t_m)) - K_{2y+1}^{2x+1}(s_n, t_m, f_n^m)] + r_{2y+1}^{2x+1}, \\
e_{2y+2}^{2x+1} \ &= \ \sum_{n=0}^{2y+2}\sum_{m=0}^{2} Q_n^y \bar{W}_m^x [K_{2y+2}^{2x+1}(s_n, t_m, f(s_n, t_m)) - K_{2y+2}^{2x+1}(s_n, t_m, f_n^m)] \\
&\quad + \sum_{n=0}^{2y+2}\sum_{m=3}^{2x+1} Q_n^y W_{m+1}^x [K_{2y+2}^{2x+1}(s_n, t_m, f(s_n, t_m)) - K_{2y+2}^{2x+1}(s_n, t_m, f_n^m)] + r_{2y+2}^{2x+1}, \\
e_{2y+1}^{2x+2} \ &= \ \sum_{n=0}^{2}\sum_{m=0}^{2x+2} \bar{Q}_n^y W_m^x [K_{2y+1}^{2x+2}(s_n, t_m, f(s_n, t_m)) - K_{2y+1}^{2x+2}(s_n, t_m, f_n^m)] \\
&\quad + \sum_{n=3}^{2y+1}\sum_{m=0}^{2x+2} Q_{n+1}^y W_m^x [K_{2y+1}^{2x+2}(s_n, t_m, f(s_n, t_m)) - K_{2y+1}^{2x+2}(s_n, t_m, f_n^m)] + r_{2y+1}^{2x+2}, \\
e_{2y+2}^{2x+2} \ &= \ \sum_{n=0}^{2y+2}\sum_{m=0}^{2x+2} Q_n^y W_m^x [K_{2y+2}^{2x+2}(s_n, t_m, f(s_n, t_m)) - K_{2y+2}^{2x+2}(s_n, t_m, f_n^m)] + r_{2y+2}^{2x+2},
\end{aligned}$$

where $\bar{Q}_n^y, Q_n^y, \bar{W}_m^x$ and W_m^x satisfy Lemma 6 and Lemma 7.

For e_{2y+1}^{2x+1}, we can obtain that

$$\begin{aligned}
&\left|e_{2y+1}^{2x+1}\right| \leq CL \sum_{n=0}^{2y}\sum_{m=0}^{2x} \left(\log \frac{s_{2y+1}}{s_n}\right)^{-\gamma} \log \frac{s_{n+1}}{s_n} \left(\log \frac{t_{2x+1}}{t_m}\right)^{-\lambda} \log \frac{t_{m+1}}{t_m} |e_n^m| \\
&+ CL \left|Q_{2y+2}^y\right| \sum_{m=0}^{2x} \left(\log \frac{t_{2x+1}}{t_m}\right)^{-\lambda} \log \frac{t_{m+1}}{t_m} |e_{2y+1}^m| + CL \left|W_{2x+2}^x\right| \sum_{n=0}^{2y} \left(\log \frac{s_{2y+1}}{s_n}\right)^{-\gamma} \log \frac{s_{n+1}}{s_n} |e_n^{2x+1}|
\end{aligned}$$

$$+ L\left|Q^y_{2y+2}\right|\left|W^x_{2x+2}\right|\left|e^{2x+1}_{2y+1}\right| + \left|r^{2x+1}_{2y+1}\right|. \tag{4.24}$$

We rearrange it into the following form

$$\begin{aligned}
\left|e^{2x+1}_{2y+1}\right| \leq\ & CL\sum_{n=0}^{2y}\sum_{m=0}^{2x}\left(\log\frac{s_{2y+1}}{s_n}\right)^{-\gamma}\log\frac{s_{n+1}}{s_n}\left(\log\frac{t_{2x+1}}{t_m}\right)^{-\lambda}\log\frac{t_{m+1}}{t_m}\left|e^m_n\right| \\
&+CL\left|Q^y_{2y+2}\right|\sum_{m=0}^{2x}\left(\log\frac{t_{2x+1}}{t_m}\right)^{-\lambda}\log\frac{t_{m+1}}{t_m}\left|e^m_{2y+1}\right| \\
&+CL\left|W^x_{2x+2}\right|\sum_{n=0}^{2y}\left(\log\frac{s_{2y+1}}{s_n}\right)^{-\gamma}\log\frac{s_{n+1}}{s_n}\left|e^{2x+1}_n\right| + C\left|r^{2x+1}_{2y+1}\right|.
\end{aligned}$$

If we let $\|e_n\| = \max\limits_{0\leq m\leq 2X}|e^m_n|$ and $\|r_n\| = \max\limits_{0\leq m\leq 2X}|r^m_n|$, we can conclude that

$$\begin{aligned}
\left|e^{2x+1}_{2y+1}\right| \leq\ & CL\sum_{n=0}^{2y}\sum_{m=0}^{2x}\left(\log\frac{s_{2y+1}}{s_n}\right)^{-\gamma}\log\frac{s_{n+1}}{s_n}\left(\log\frac{t_{2x+1}}{t_m}\right)^{-\lambda}\log\frac{t_{m+1}}{t_m}\|e_n\| \\
&+CL\left|Q^y_{2y+2}\right|\sum_{m=0}^{2x}\left(\log\frac{t_{2x+1}}{t_m}\right)^{-\lambda}\log\frac{t_{m+1}}{t_m}\left\|e_{2y+1}\right\| \\
&+CL\left|W^x_{2x+2}\right|\sum_{n=0}^{2y}\left(\log\frac{s_{2y+1}}{s_n}\right)^{-\gamma}\log\frac{s_{n+1}}{s_n}\|e_n\| + C\left\|r_{2y+1}\right\| \\
=\ & CL\sum_{n=0}^{2y}\left(\log\frac{s_{2y+1}}{s_n}\right)^{-\gamma}\log\frac{s_{n+1}}{s_n}\|e_n\|\sum_{m=0}^{2x}\left((\log\frac{t_{2x+1}}{t_m})^{-\lambda}\log\frac{t_{m+1}}{t_m}\right) \\
&+CL\left|Q^y_{2y+2}\right|\left\|e_{2y+1}\right\|\sum_{m=0}^{2x}\left((\log\frac{t_{2x+1}}{t_m})^{-\lambda}\log\frac{t_{m+1}}{t_m}\right) \\
&+CL\left|W^x_{2x+2}\right|\sum_{n=0}^{2y}\left(\log\frac{s_{2y+1}}{s_n}\right)^{-\gamma}\log\frac{s_{n+1}}{s_n}\|e_n\| + C\left\|r_{2y+1}\right\| \\
\leq\ & CL\sum_{n=0}^{2y}\left(\log\frac{s_{2y+1}}{s_n}\right)^{-\gamma}\log\frac{s_{n+1}}{s_n}\|e_n\|\int_{t_0}^{t_{2x+1}}(\log t_{2x+1}-\log\omega)^{-\lambda}\frac{d\omega}{\omega} \\
&+CL\left|Q^y_{2y+2}\right|\left\|e_{2y+1}\right\|\int_{t_0}^{t_{2x+1}}(\log t_{2x+1}-\log\omega)^{-\lambda}\frac{d\omega}{\omega} \\
&+CL\left|W^x_{2x+2}\right|\sum_{n=0}^{2y}\left(\log\frac{s_{2y+1}}{s_n}\right)^{-\gamma}\log\frac{s_{n+1}}{s_n}\|e_n\| + C\left\|r_{2y+1}\right\| \\
=\ & CL\frac{t^{1-\lambda}_{2x+1}}{1-\lambda}\sum_{n=0}^{2y}\left(\log\frac{s_{2y+1}}{s_n}\right)^{-\gamma}\log\frac{s_{n+1}}{s_n}\|e_n\| + CL\left|Q^y_{2y+2}\right|\frac{t^{1-\lambda}_{2x+1}}{1-\lambda}\left\|e_{2y+1}\right\| \\
&+CL\left|W^x_{2x+2}\right|\sum_{n=0}^{2y}\left(\log\frac{s_{2y+1}}{s_n}\right)^{-\gamma}\log\frac{s_{n+1}}{s_n}\|e_n\| + C\left\|r_{2y+1}\right\|
\end{aligned}$$

$$\leq CL\frac{d^{1-\lambda}}{1-\lambda}\sum_{n=0}^{2y}\left(\log\frac{s_{2y+1}}{s_n}\right)^{-\gamma}\log\frac{s_{n+1}}{s_n}\|e_n\| + CL\left|Q_{2y+2}^{y}\right|\frac{d^{1-\lambda}}{1-\lambda}\left\|e_{2y+1}\right\|$$
$$+CL\left|W_{2x+2}^{x}\right|\sum_{n=0}^{2y}\left(\log\frac{s_{2y+1}}{s_n}\right)^{-\gamma}\log\frac{s_{n+1}}{s_n}\|e_n\| + C\left\|r_{2y+1}\right\|.$$

Then we have

$$\left\|e_{2y+1}\right\| \leq CL\frac{d^{1-\lambda}}{1-\lambda}\sum_{n=0}^{2y}\left(\log\frac{s_{2y+1}}{s_n}\right)^{-\gamma}\log\frac{s_{n+1}}{s_n}\|e_n\| + CL\left|Q_{2y+2}^{y}\right|\frac{d^{1-\lambda}}{1-\lambda}\left\|e_{2y+1}\right\|$$
$$+CL\left|W_{2x+2}^{x}\right|\sum_{n=0}^{2y}\left(\log\frac{s_{2y+1}}{s_n}\right)^{-\gamma}\log\frac{s_{n+1}}{s_n}\|e_n\| + C\left\|r_{2y+1}\right\|.$$

According to (4.11) in Lemma 8, for enough small Δ_s, we obtain,

$$\left\|e_{2y+1}\right\| \leq CL\frac{d^{1-\lambda}}{1-\lambda}\sum_{n=0}^{2y}\left(\log\frac{s_{2y+1}}{s_n}\right)^{-\gamma}\log\frac{s_{n+1}}{s_n}\|e_n\|$$
$$+CL\left|W_{2x+2}^{x}\right|\sum_{n=0}^{2y}\left(\log\frac{s_{2y+1}}{s_n}\right)^{-\gamma}\log\frac{s_{n+1}}{s_n}\|e_n\| + C\left\|r_{2y+1}\right\|.$$

According to (4.13) in Lemma 8, for enough small Δ_t, let $\Delta_t < d$, so we have

$$\left\|e_{2y+1}\right\| \leq \Big(CL\frac{d^{1-\lambda}}{1-\lambda} + CL\frac{2^{1-\lambda}(12-2\lambda)\Gamma(1-\lambda)}{\Gamma(4-\lambda)c^{1-\lambda}}d^{1-\lambda}\Big)\sum_{n=0}^{2y}\left(\log\frac{s_{2y+1}}{s_n}\right)^{-\gamma}\log\frac{s_{n+1}}{s_n}\|e_n\| + C\left\|r_{2y+1}\right\|.$$

Then, it follows from the Gronwall inequality introduced in Lemma 1 that

$$\left\|e_{2y+1}\right\| \leq C\left\|r_{2y+1}\right\|.$$

From Lemma 6, we can conclude that

$$\left|e_{2y+1}^{2x+1}\right| \leq C(\Delta_s^{4-\gamma} + \Delta_t^{4-\lambda}). \tag{4.25}$$

Similarly, we can also obtain

$$\left|e_{2y+2}^{2x+1}\right| \leq C(\Delta_s^{4-\gamma} + \Delta_t^{4-\lambda}), \left|e_{2y+p}^{2x+2}\right| \leq C(\Delta_s^{4-\gamma} + \Delta_t^{4-\lambda}), p = 1, 2. \tag{4.26}$$

We jointly establish (4.22), (4.23), (4.25) and (4.26); with this, we complete the proof of Theorem 1. □

5. Numerical examples

Now, this section applies the numerical scheme to solve the 2D fractional Caputo-Hadamard integral equation. We present two calculation examples to demonstrate its effectiveness.

Example 5.1 Take into account the subsequent equation, which is linear 2D fractional Caputo-Hadamard integral equations:

$$f(s,t) = g(s,t) + \int_1^s \int_1^t \frac{(st^2 + \log\tau + \log\omega)f(\tau,\omega)}{(\log s - \log\tau)^\gamma(\log t - \log\omega)^\lambda}\frac{d\omega}{\omega}\frac{d\tau}{\tau}, \quad (s,t) \in [1,2]\times[1,2],$$

where $a = 1$, $c = 1$, $b = 2$ and $d = 2$, and

$$\begin{aligned} g(s,t) \;=\; & (\log s)^4(\log t)^4 - 576st^2(\log s)^{5-\gamma}(\log t)^{5-\lambda}\prod_{n=1}^{5}\frac{1}{(n-\gamma)}\prod_{m=1}^{5}\frac{1}{(m-\lambda)} \\ & -2880(\log s)^{6-\gamma}(\log t)^{5-\lambda}\prod_{n=1}^{6}\frac{1}{(n-\gamma)}\prod_{m=1}^{5}\frac{1}{(m-\lambda)} \\ & -2880(\log s)^{5-\gamma}(\log t)^{6-\lambda}\prod_{n=1}^{5}\frac{1}{(n-\gamma)}\prod_{m=1}^{6}\frac{1}{(m-\lambda)}, \end{aligned}$$

where $f(s,t) = (\log s)^4(\log t)^4$ is the exact solution.

We choose separately $\gamma = 0.3, \lambda = 0.6$ and $\gamma = 0.4, \lambda = 0.5$ to conduct experiments. $\Delta_s = \frac{b-a}{2Y}$ is the size of the step taken in the direction of s, while $\Delta_t = \frac{d-c}{2X}$ is the width of the step taken in the direction of t. For errors, we define them as follows

$$e_h^f = \max_{\substack{n=1,\cdots,2Y \\ m=1,\cdots,2X}} \left| f(s_n,t_m) - f_n^m \right|.$$

In this example, we test the convergence order for different values of γ and λ where the convergence order is determined as $\log_2(e_{2h}^f/e_h^f)$. The application of Theorem 1 allows us to determine the theoretical convergence order of the numerical scheme as $O(\Delta_s^{4-\gamma} + \Delta_t^{4-\lambda})$. By considering Δ_s to be significantly smaller than Δ_t, we can deduce that $O(\Delta_s^{4-\gamma} + \Delta_t^{4-\lambda}) = O(\Delta_t^{4-\lambda})$. Tables 1 presents the values of $\gamma = 0.3, \lambda = 0.6, \gamma = 0.4, \lambda = 0.5$ and $Y = 2X$ with the theoretical order $4 - \lambda$. When $\gamma = 0.3$ and $\lambda = 0.6$, the theoretical convergence order is 3.4. Similarly, the theoretical convergence order is 3.5 for $\gamma = 0.4$ and $\lambda = 0.5$.

Table 1. The maximum number of errors and the rate of decay with $Y = 2X$ for Example 5.1.

X	$\gamma = 0.3, \lambda = 0.6$	**Order**	$\gamma = 0.4, \lambda = 0.5$	**Order**
8	$6.8989589227\times10^{-3}$	–	$3.7877332874\times10^{-3}$	–
16	$7.8140473872\times10^{-4}$	3.1422367608	$4.3496713026\times10^{-4}$	3.1223564591
32	$7.9156289920\times10^{-5}$	3.3032941051	$4.3449207934\times10^{-5}$	3.3235046005
64	$7.6518039345\times10^{-6}$	3.3708321819	$4.0890764577\times10^{-6}$	3.4094829336
128	$7.2494161456\times10^{-7}$	3.3998631944	$3.7407272951\times10^{-7}$	3.4503843386

Similarly, we use the proposed numerical scheme in another way to verify the convergence order of the test. We take $X = [Y^{\frac{4-\gamma}{4-\lambda}}]$, where $[\cdot]$ represents rounding, and the theoretical order $O(\Delta_s^{4-\gamma} + \Delta_t^{4-\lambda}) = O(\Delta_t^{4-\gamma})$ of the numerical scheme. In Table 2, when $\gamma = 0.3$ and $\lambda = 0.6$, it is easy to see that the numerical results of the test are close to 3.7, and the test results indicate that when $\gamma = 0.4$ and $\lambda = 0.5$, the numerical values are approximately 3.6. By referring to Tables 1 and 2, it is easy to see that the high-order numerical scheme results in a convergence order of $O(\Delta_s^{4-\gamma} + \Delta_t^{4-\lambda})$.

Table 2. The maximum number of errors and the rate of decay with $X = [Y^{(4-\gamma)/(4-\lambda)}]$ for Example 5.1.

Y	$\alpha = 0.3, \beta = 0.6$	Order	$\alpha = 0.4, \beta = 0.5$	Order
8	$1.8639874340\times10^{-3}$	–	$9.2944718555\times10^{-4}$	–
16	$2.0141512357\times10^{-4}$	3.2101482146	$9.6463652018\times10^{-5}$	3.2683155532
32	$1.8416998390\times10^{-5}$	3.4510621597	$8.9446371042\times10^{-6}$	3.4308905736
64	$1.5670179200\times10^{-6}$	3.5549443663	$7.9219275691\times10^{-7}$	3.4970995357
128	$1.2872757561\times10^{-7}$	3.6056286369	$6.8539526533\times10^{-8}$	3.5308433796

Next, let us draw an error distribution map with $Y = 2X, X = 128$. As shown in Figure 1, the error distribution of $\alpha = 0.3, \beta = 0.6$ is on the left, while the error distribution of $\alpha = 0.4, \beta = 0.5$ is on the right. In addition, we can easily see that the error can be as small as 10^{-7}, indicating that the approximate value is very close to the exact value.

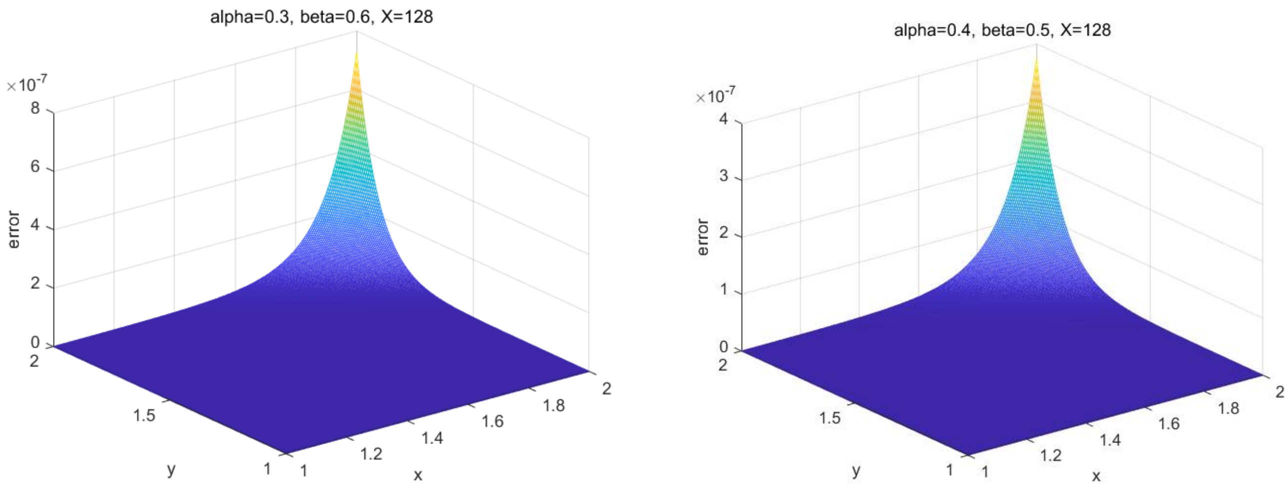

Figure 1. Error distribution of $\alpha = 0.3, \beta = 0.6$ (**left**) and $\alpha = 0.4, \beta = 0.5$ (**right**) for $X = 128$.

Example 5.2 Take into account the 2D nonlinear fractional Caputo-Hadamard integral equation as follows:

$$f(s,t) = g(s,t) + \int_1^s \int_1^t \frac{(st^2 + \log\tau + \log\omega) f^2(\tau,\omega)}{(\log s - \log\tau)^\gamma (\log t - \log\omega)^\lambda} \frac{d\omega}{\omega} \frac{d\tau}{\tau}, (s,t) \in [1,2] \times [1,2],$$

where $a = 1$, $c = 1$, $b = 2$ and $d = 2$, and

$$g(s,t) = (\log s)^3 (\log t)^2 - 17280 st^2 (\log s)^{7-\gamma} (\log t)^{5-\lambda} \prod_{n=1}^{7} \frac{1}{(n-\gamma)} \prod_{m=1}^{5} \frac{1}{(m-\lambda)}$$

$$-120960(\log s)^{8-\gamma}(\log t)^{5-\lambda}\prod_{n=1}^{8}\frac{1}{(n-\gamma)}\prod_{m=1}^{5}\frac{1}{(m-\lambda)}$$
$$-86400(\log s)^{7-\gamma}(\log t)^{6-\lambda}\prod_{n=1}^{7}\frac{1}{(n-\gamma)}\prod_{m=1}^{6}\frac{1}{(m-\lambda)},$$

where $f(s,t) = (\log s)^3(\log t)^2$ is the exact solution.

In Table 3, given $\gamma = 0.3, \lambda = 0.6$ and $\gamma = 0.4, \lambda = 0.5$, our experimental results closely match the expected values of 3.4 and 3.5, respectively. In Table 4, when $\gamma = 0.3$ and $\lambda = 0.6$, the experimental results closely mirror the theoretical value of 3.7, and when $\gamma = 0.4$ and $\lambda = 0.5$, the experimental results align closely with the projected value of 3.6. It is easy to see that the high-order numerical scheme results in a convergence order of $O(\Delta_s^{4-\gamma} + \Delta_t^{4-\lambda})$.

Table 3. The maximum number of errors and the rate of decay with $Y = 2X$ for Example 5.2.

X	$\gamma = 0.3, \lambda = 0.6$	**Order**	$\gamma = 0.4, \lambda = 0.5$	**Order**
8	$5.0197780760\times10^{-6}$	–	$4.4096514739\times10^{-6}$	–
16	$4.9528160060\times10^{-7}$	3.3413026523	$4.2636169692\times10^{-7}$	3.3705148922
32	$4.7538570025\times10^{-8}$	3.3810786144	$3.9488986998\times10^{-8}$	3.4325555722
64	$4.5029294460\times10^{-9}$	3.4001627291	$3.5760421846\times10^{-9}$	3.4650146987
128	$4.2379935672\times10^{-10}$	3.4094105701	$3.1969468739\times10^{-10}$	3.4835970802

Table 4. The maximum number of errors and the rate of decay with $X = [Y^{(4-\gamma)/(4-\lambda)}]$ for Example 5.2.

Y	$\gamma = 0.3, \lambda = 0.6$	**Order**	$\gamma = 0.4, \lambda = 0.5$	**Order**
8	$1.1512351119\times10^{-6}$	–	$1.3725247072\times10^{-6}$	–
16	$9.9372553575\times10^{-8}$	3.4625604385	$1.2830582435\times10^{-7}$	3.4191735526
32	$8.1408764174\times10^{-9}$	3.5341912504	$1.1267508093\times10^{-8}$	3.5093462706
64	$6.5234456725\times10^{-10}$	3.6095914151	$9.7534660903\times10^{-10}$	3.5301096754
128	$5.2919446603\times10^{-11}$	3.6237643174	$8.3465706568\times10^{-11}$	3.5466595342

Finally, let us draw an error distribution map with $Y = 2X, X = 128$. As shown in Figure 2, the error distribution of $\alpha = 0.3, \beta = 0.6$ is on the left, while the error distribution of $\alpha = 0.4, \beta = 0.5$ is on the right. In addition, we can easily see that the error can be as small as 10^{-10}.

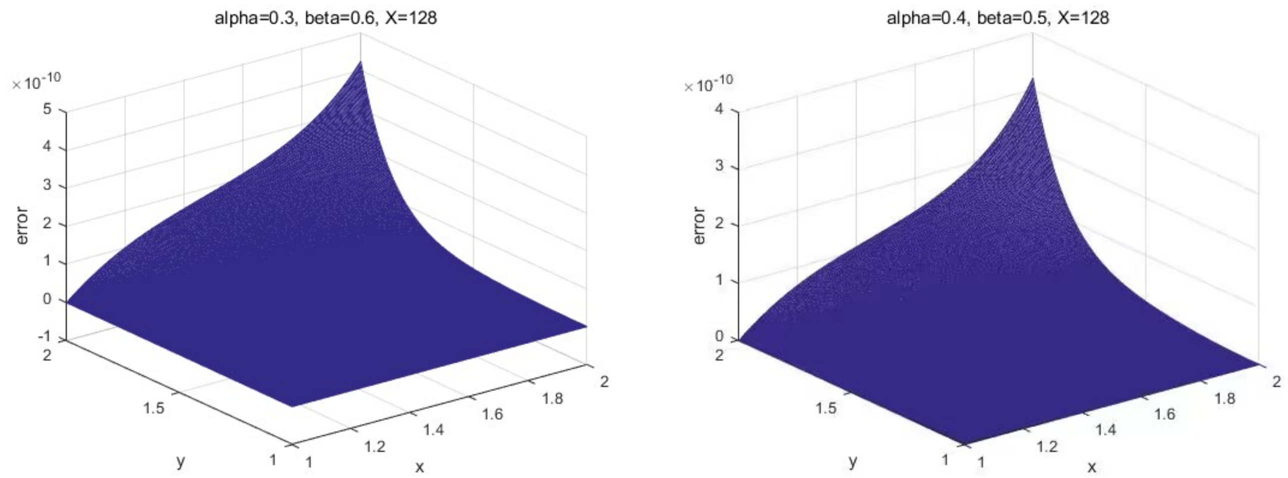

Figure 2. Error distribution of $\alpha = 0.3, \beta = 0.6$ (**left**) and $\alpha = 0.4, \beta = 0.5$ (**right**) for $X = 128$.

6. Concluding remarks

Based on the idea of the modified block-by-block method of [6], we have proposed a uniform accuracy high-order scheme (2.30) to approximate the 2D nonlinear Hadamard Volterra integral equation. A high-order numerical scheme was constructed by using the piecewise biquadratic interpolation. The uniform accuracy high-order scheme in the present article is an explicit and high efficiency scheme except for two boundary layers which are solved by implicit and coupled. Based on the test results of the above example, one can see that the convergence order is $O(\Delta_s^{4-\gamma} + \Delta_t^{4-\lambda})$. We conducted a detailed error analysis of the high-order numerical scheme and confirmed the accuracy of the theoretical analysis through numerical examples and experiments. Through extensive analysis of this work, it can be seen that the numerical scheme (2.30) has the advantages of high accuracy, easy calculation and implementation, and it does not require coupled solutions. In future work, we intend to use a Fourier spectral method to solve such problems. Based on the ideas in [5, 26], we found that high-order numerical non-uniform grids can be used to solve fractional order integral differential equation problems as follows

$$f(s,t) = g(s,t) + \int_s^b \int_t^d \frac{K(s,t,\tau,\omega,f(\tau,\omega))}{(\log s - \log\tau)^\gamma (\log t - \log\omega)^\lambda} \frac{d\omega}{\omega} \frac{d\tau}{\tau}, \quad (s,t) \in \Theta, 0 < \gamma, \lambda < 1.$$

In addition, deep learning can also provide solutions for such problems, so research on this topic is very challenging.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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