AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(12): 29681–29700.
DOI: 10.3934/math.20231519
Received: 21 August 2023
Revised: 19 October 2023
Accepted: 24 October 2023
Published: 01 november 2023

Research article

Fixed point theorems for generalized contractions in $\mathfrak{F}$-bipolar metric spaces with applications

Amer Hassan Albargi*

Department of Mathematics, Faculty of Sciences, King Abdulaziz University, P.O.Box 80203, Jeddah 21589, Saudi Arabia

* **Correspondence:** Email: aalbarqi@kau.edu.sa.

Abstract: The major objectives of this research article are to introduce the notion of (α, ψ)-contraction in the context of $\mathfrak{F}$-bipolar metric space and establish fixed point theorems. In this way, coupled fixed point results are obtained by applying the leading theorems. Some non-trivial examples are also furnished to show the validity of established results. As applications of the main result, we investigate the solution of an integral equation and a homotopy problem.

Keywords: fixed point; (α, ψ)-contractions; $\mathfrak{F}$-bipolar metric space; integral equation; homotopy
Mathematics Subject Classification: 46S40, 47H10, 54H25

1. Introduction

Fixed point theory is one of the well-known and illustrious theories in functional analysis and has broad applications in different fields. The Banach contraction principle [1] is the first theorem in this theory in which a complete metric space (CMS) plays an important role that was naturally accomplished by M. Frechet [2] in 1906. After that, various researchers have extended the notion of the metric by either changing the domain and range or weakening the metric axioms of it. The famous extension of metric space is b-metric space which has been done by Bakhtin [3] that was formally defined by Czerwik [4] in 1993. Another generalization of metric space is rectangular metric space, which was given by Branciari [5]. Recently, Jleli et al. [6] introduced a new metric space, which is known as an $\mathfrak{F}$-metric space ($\mathfrak{F}$-MS) and proved the Banach contraction principle in this context. The notion of $\mathfrak{F}$-MS is an interesting extension of the metric space, b-metric space and rectangular metric space. Later on, Hussain et al. [7] used the notion of $\mathfrak{F}$-MS and established some theorems for (α, ψ)-contractions.

In these generalizations of metric spaces, we consider the distance between points of a single set. Thus, a question arises generally that how distance between elements of two different sets can be explored? These problems of finding distance can be discussed in different fields. On the other hand, Mutlu et al. [8] introduced the concept of bipolar metric space (bip MS) to solve such problems. Also, this new notion of bip MS leads to the evolution and advancement of fixed point theorems in this theory. In due course, Mutlu et al. [9] established coupled fixed point results in the framework of bip MS. Kishore et al. [10] extended the concept of the coupled fixed point to common coupled fixed point and presented an application of it. Kishore et al. [11] genaralized the contractive inequality of Mutlu et al. [9] and presented some contemporary coupled fixed point theorems. Rao et al. [12] proved a common coupled fixed point result for Geraghty type contraction and applied their result to the homotopy theory. Gürdal et al. [13] utilized the notion of bip MS to obtain fixed point theorems for (α, ψ)-contractions. Later on, Kishore et al. [14] proved some related fixed point results to a hybrid pair of mappings in bipolar metric space and investigated the solution of an integral equation. Moreover, various significant problems have been solved for the existence of a fixed point of single-valued and set-valued mappings in the context of bip MS (see [15–19]) and references therein). In the recent past, Rawat et al. [20] unified the above two novel notions, specifically $\mathfrak{F}$-MS and bip MS and established the notion of $\mathfrak{F}$-bipolar metric space ($\mathfrak{F}$-bip MS) and obtained a fixed point result in the background of $\mathfrak{F}$-MS. Very recently, Alamri [21] obtained fixed point theorems for Reich and Fisher type contractions in the framework of $\mathfrak{F}$-bip MS.

In the present research work, we introduce the notion of (α, ψ)-contraction in the context of $\mathfrak{F}$-bip MS and establish fixed point results for covariant and contravariant mappings. In this way, we derive coupled fixed point results by applying the leading theorems. Some non-trivial examples are also furnished to show the validity of established results. The solution of integral equation and homotopy problem is investigated as applications of the main theorem.

2. Preliminaries

The well-known Banach contraction principle [1] is given in this way.

Theorem 2.1. *[1] Let T be self mapping on CMS (X, d). If there exists a nonnegative constant $\lambda \in [0, 1)$ such that*

$$d(Tx, Ty) \leq \lambda d(x, y),$$

for all $x, y \in X$, then T has a unique fixed point.

Jleli et al. [6] introduced an engrossing generalization of a metric space as follows:

Let $\mathcal{F}$ be the family of functions $f : (0, +\infty) \to \mathbb{R}$ satisfying the following assertions:

($\mathcal{F}_1$) f is non-decreasing,
($\mathcal{F}_2$) for each sequence $\{t_i\} \subseteq \mathbb{R}^+$, $\lim_{i\to\infty} \alpha_i = 0$ if and only if $\lim_{i\to\infty} f(\alpha_i) = -\infty$.

Definition 2.1. *[6]* Let $X \neq \emptyset$ and let $d : X\times X \to [0, +\infty)$. Assume that there exist $(f, \pi) \in \mathcal{F}\times[0, +\infty)$ such that for all $(x, y) \in X\times X$,

(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,

(iii) for all $p \geq 2$ and $(u_i)_{i=1}^{p} \subset X$ along with $(u_1, u_p) = (x, y)$, we have

$$d(x,y) > 0 \Rightarrow f(d(x,y)) \leq f\left(\sum_{i=1}^{p-1} d\left(u_i, u_{i+1}\right)\right) + \pi,$$

where $p \in \mathbb{N}$. Then d is said to be an $\mathfrak{F}$- metric on X and (X, d) is said to be an $\mathfrak{F}$-MS.

Example 2.1. *[6]* Let X=$\mathbb{R}$, $f(t) = \ln(t)$ and $\pi = \ln(3)$. Define $d : X{\times}X \to [0, +\infty)$ by

$$d(x,y) = \begin{cases} (x-y)^2 & \text{if } (x,y) \in [0,3] \times [0,3] \\ |x-y| & \text{if } (x,y) \notin [0,3] \times [0,3] \end{cases}$$

then (X,d) is an $\mathfrak{F}$-MS.

Mutlu et al. [8] introduced the notion of bip MS in such manner.

Definition 2.2. *[8]* Let $X \neq \emptyset$, $Y \neq \emptyset$ and let $d : X \times Y \to [0, +\infty)$ be a function satisfying the following conditions

(bi_1) $d(x, y) = 0$ if and only if $x = y$,
(bi_2) $d(x, y) = d(y, x)$, if $x, y \in X \cap Y$,
(bi_3) $d(x, y) \leq d(x, y^/) + d(x^/, y^/) + d(x^/, y)$,

for all $(x, y), (x^/, y^/) \in X \times Y$. Then (X,Y, d) is called a bip MS.

Example 2.2. *[8]* Let X and Y be the class of all compact and singleton subsets of $\mathbb{R}$ accordingly. Define $d : X \times Y \to [0, +\infty)$ by

$$d(x, \Xi) = |x - \inf(\Xi)| + \left|x - \sup(\Xi)\right|$$

for $\{x\} \subseteq X$ and $\Xi \subseteq Y$, then (X,Y, d) is a complete bip MS.

Definition 2.3. *[8]* Let (X_1, Y_1, d_1) and (X_2, Y_2, d_2) be two bip MSs. A mapping $T : (X, Y, d) \rightrightarrows (X, Y, d)$ is said to be a covariant mapping, if $T(X_1) \subseteq X_2$ and $T(Y_1) \subseteq Y_2$. Similarly, a mapping $T : (X_1, Y_1) \rightleftarrows (X_2, Y_2)$ is said to be a contravariant mapping, if $T(X_1) \subseteq Y_2$ and $T(X_2) \subseteq Y_1$.

We symbolize the covariant mappings as $T : (X_1, Y_1) \rightrightarrows (X_2, Y_2)$ and contravariant mappings as $T : (X_1, Y_1) \rightleftarrows (X_2, Y_2)$ to generate the distinction between the mappings.

Rawat et al. [20] unified the two novel notions, $\mathfrak{F}$-MS and bip MS and introduced the notion of $\mathfrak{F}$-bip MS in this way.

Definition 2.4. *[20]* Let $X \neq \emptyset$, $Y \neq \emptyset$ and $d : X \times Y \to [0, +\infty)$. Suppose that there exist $(f, \pi) \in \mathcal{F} \times [0, +\infty)$ such that for all $(x, y) \in X \times Y$,

(D_1) $d(x, y) = 0$ if and only if $x = y$,
(D_2) $d(x, y) = d(y, x)$, if $x, y \in X \cap Y$,
(D_3) for all $p \geq 2$ and $(u_i)_{i=1}^{p} \subset X$ and $(v_i)_{i=1}^{p} \subset Y$ along with $(u_1, v_p) = (x, y)$, we have

$$d(x,y) > 0 \Rightarrow f(d(x,y)) \leq f\left(\sum_{i=1}^{p-1} d\left(u_{i+1}, v_i\right) + \sum_{i=1}^{p} d\left(u_i, v_i\right)\right) + \pi,$$

where $p \in \mathbb{N}$. Then d is said to be an $\mathfrak{F}$-bip metric on the pair (X, Y) and (X, Y, d) is said to be an $\mathfrak{F}$-bip MS.

Example 2.3. Let $X=\{1,2\}$ and $Y=\{2,7\}$. Define $d: X\times Y\to[0,+\infty)$ by

$$d(1,2)=6,\ d(1,7)=10, d(2,7)=2,\ d(2,2)=0,$$

then d satisfies all the conditions of an $\mathfrak{F}$-bip metric with $\alpha=0$ and $f(\mathrm{t})=\ln t$, for $t>0$. Thus (X,Y,d) is an $\mathfrak{F}$-bip MS but not bip MS.

Remark 2.1. *[20]* Taking $Y=X$, $p=2i$, $u_J=u_{2J-1}$ and $v_J=u_{2J}$ in the above definition (2.4), we obtain a sequence $\left(u_J\right)_{J=1}^{2i}\in X$ with $(u_1,u_{2i})=(x,y)$ such that condition (iii) of Definition 2.1 holds. Thus every $\mathfrak{F}$-MS is an $\mathfrak{F}$-bip MS but every $\mathfrak{F}$-bip MS is not an $\mathfrak{F}$-MS.

Definition 2.5. *[20]* Let (X,Y,d) be an $\mathfrak{F}$-bip MS.

(i) A point $x\in X\cup Y$ is called a right point if $x\in Y$ and a left point if $x\in X$. And x is called a central point if it is both left and right point.

(ii) (x_i) on X is called a left sequence and (y_i) on Y is considered as a right sequence.

(iii) (x_i) is called to converge to a point x, iff x is a left point, (x_i) is a right sequence and $\lim_{i\to\infty} d(x,x_i)=0$, or x is a right point, (x_i) is a left sequence and $\lim_{i\to\infty} d(x_i,x)=0$. A bisequence (x_i,y_i) on (X,Y,d) is a sequence on the set $X\times Y$. If the sequences (x_i) and (y_i) are convergent, then the bisequence (x_i,y_i) is also convergent, and if (x_i) and (y_i) converge to a common element, then the bisequence (x_i,y_i) is said to be biconvergent.

(iv) A bisequence (x_i,y_i) in an $\mathfrak{F}$-bip MS (X,Y,d) is said to be a Cauchy bisequence, if for each $\epsilon>0$, there exist $i_0\in\mathbb{N}$, such that $d\left(x_i,y_p\right)<\epsilon$, for all $i,p\geq i_0$.

Definition 2.6. *[20]* If every Cauchy bisequence in $\mathfrak{F}$-bip MS (X,Y,d) is convergent, then (X,Y,d) is complete.

On the other hand, Samet et al. [22] introduced the notions α-admissible mapping and (α,ψ)-contraction in this manner.

Definition 2.7. *[22]* Let Ψ be a family of functions $\psi:[0,+\infty)\to[0,+\infty)$ satisfying the following conditions

(ψ_1) ψ is nondecreasing,

(ψ_2) $\sum\limits_{i=1}^{\infty}\psi^i(t)<+\infty$, for all $t>0$, where ψ^i is the i-th iterate of ψ.

Lemma 2.1. *[22] If $\psi\in\Psi$, then for each $t>0$, $\psi(t)<t$ and $\psi(0)=0$.*

Definition 2.8. *[22]* Let $\alpha: X\times X\to[0,+\infty)$ be any function. A mapping $T: X\to X$ is said to be α-admissible if

$$\alpha(x,y)\geq 1 \text{ implies } \alpha(Tx,Ty)\geq 1,$$

for all $x,y\in X$.

Definition 2.9. *[22]* Let (X,d) be a MS. A mapping $T: X\to X$ is said to be (α,ψ)-contraction if there exist some $\alpha: X\times X\to[0,+\infty)$ and $\psi\in\Psi$ such that

$$\alpha(x,y)\,d(Tx,Ty)\leq\psi(d(x,y)),$$

for all $x,y\in X$.

3. Main results

Throughout this section, we consider (X, Y, d) as complete $\mathfrak{F}$-bip MS.

Definition 3.1. Let $\alpha : X \times Y \rightarrow [0, +\infty)$ be any function. A mapping $T : (X, Y, d) \rightrightarrows (X, Y, d)$ is called a covariant α-admissible if

$$\alpha(x, y) \geq 1 \implies \alpha(Tx, Ty) \geq 1, \tag{3.1}$$

for all $(x, y) \in X \times Y$.

Example 3.1. Let $X = [0, +\infty)$ and $Y = (-\infty, 0]$ and $\alpha : X \times Y \rightarrow [0, +\infty)$ is defined as

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

A covariant mapping $T : (X, Y, d) \rightrightarrows (X, Y, d)$ defined by $T(x) = x$ is covariant α-admissible.

Definition 3.2. Let $\alpha : X \times Y \rightarrow [0, +\infty)$ be any function. A mapping $T : (X, Y, d) \rightleftarrows (X, Y, d)$ is called a contravariant α-admissible if there exist some $\alpha : X \times Y \rightarrow [0, +\infty)$ such that

$$\alpha(x, y) \geq 1 \implies \alpha(Ty, Tx) \geq 1, \tag{3.2}$$

for all $(x, y) \in X \times Y$.

Example 3.2. Let $X = [0, +\infty)$ and $Y = (-\infty, 0]$ and $\alpha : X \times Y \rightarrow [0, +\infty)$ is defined as

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

A contravariant mapping $T : (X, Y, d) \rightleftarrows (X, Y, d)$ defined by $T(x) = -x$ is contravariant α-admissible.

Definition 3.3. A mapping $T : (X, Y, d) \rightrightarrows (X, Y, d)$ is called a covariant (α, ψ)-contraction if T is covariant mapping and there exist some $\alpha : X \times Y \rightarrow [0, +\infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y)\, d(Tx, Ty) \leq \psi(d(x, y)), \tag{3.3}$$

for all $(x, y) \in X \times Y$.

Example 3.3. Let $X = \mathbb{N} \cup \{0\}$ and $Y = \frac{1}{n} \cup \{0\}$ for $n \in \mathbb{N}$. Define $d : X \times Y \rightarrow [0, +\infty)$ as

$$d(x, y) = \begin{cases} 2, & \text{if } (x, y) = (2, 1) \\ |x - y|, & \text{otherwise.} \end{cases}$$

Then (X, Y, d) is complete $\mathfrak{F}$-bip MS for $f(t) = \ln(t)$ and $\pi > 2$. Define the covariant mapping $T : X \cup Y \rightarrow X \cup Y$ by

$$T(x) = \begin{cases} 0, & \text{if } x \in X - \{0, 1\} \\ 1, & \text{if } x \in Y. \end{cases}$$

and $\alpha : X \times Y \rightarrow [0, +\infty)$ by

$$\alpha(x, y) = 1$$

for all $(x, y) \in X \times Y$. Then the covariant mapping $T : (X, Y, d) \rightrightarrows (X, Y, d)$ is covariant (α, ψ)-contraction.

Definition 3.4. A mapping $T : (X, Y, d) \rightleftarrows (X, Y, d)$ is called a contravariant (α, ψ)-contraction if T is contravariant and there exist some $\alpha : X \times Y \to [0, +\infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y)\, d(Ty, Tx) \leq \psi(d(x, y)), \tag{3.4}$$

for all $(x, y) \in X \times Y$.

Remark 3.1. A mapping $T : (X, Y, d) \rightleftarrows (X, Y, d)$ satisfying the Banach contraction in $\mathfrak{F}$-bip MS (X, Y, d) is convariant (α, ψ)-contraction with

$$\alpha(x, y) = 1$$

for all $(x, y) \in X \times Y$ and $\psi(t) = kt$, for some $k \in [0, 1)$ and for $t \geq 1$. Same remark holds for contravariant mapping $T : (X, Y, d) \rightleftarrows (X, Y, d)$.

Definition 3.5. Let $\alpha : X \times Y \to [0, +\infty)$ be a function. We say that a property (P) holds, if there exists $z \in X \cap Y$ such that $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$ for all $(x, y) \in X \times Y$.

Theorem 3.1. *Let $T : (X, Y, d) \rightrightarrows (X, Y, d)$ be covariant (α, ψ)-contraction. Assume that the following assertions hold*

(i) T is covariant α-admissible,

(ii) there exists $x_0 \in X$, $y_0 \in Y$ such that $\alpha(x_0, y_0) \geq 1$ and $\alpha(x_0, Ty_0) \geq 1$,

(iii) T is continuous or , if (x_i, y_i) is a bisequence in (X, Y, d) such that $\alpha(x_i, y_i) \geq 1$, for all $i \in \mathbb{N}$ with $x_i \to \omega$ and $y_i \to \omega$, as $i \to \infty$ for $\omega \in X \cap Y$, then $\alpha(\omega, y_i) \geq 1$, for all $i \in \mathbb{N}$.

Then T has a fixed point. Furthermore, if the property (P) holds, then fixed point is unique.

Proof. Let $x_0 \in X$ and $y_0 \in Y$ and suppose that $\alpha(x_0, y_0) \geq 1$ and $\alpha(x_0, Ty_0) \geq 1$ by the assumption (ii). Define the bisequence (x_i, y_i) in (X, Y, d) by

$$x_{i+1} = Tx_i \text{ and } y_{i+1} = Ty_i,$$

for all $i \in \mathbb{N}$. As T is covariant α-admissible mapping by the hypothesis (i), so we have

$$\begin{aligned}
\alpha(x_0, y_0) &\geq 1 \text{ implies that } \alpha(x_1, y_1) = \alpha(Tx_0, Ty_0) \geq 1, \\
\alpha(x_0, y_1) &= \alpha(x_0, Ty_0) \geq 1 \text{ implies } \alpha(x_1, y_2) = \alpha(Tx_0, Ty_1) \geq 1 \\
\alpha(x_1, y_1) &= \alpha(Tx_0, Ty_0) \geq 1 \text{ implies } \alpha(x_2, y_2) = \alpha(Tx_1, Ty_1) \geq 1 \\
\alpha(x_1, y_2) &= \alpha(Tx_0, Ty_1) \geq 1 \text{ implies } \alpha(x_2, y_3) = \alpha(Tx_1, Ty_2) \geq 1 \\
\alpha(x_2, y_2) &= \alpha(Tx_1, Ty_1) \geq 1 \text{ implies } \alpha(x_3, y_3) = \alpha(Tx_2, Ty_2) \geq 1.
\end{aligned}$$

Continuing in this way, we have

$$\alpha(x_{i+1}, y_i) \geq 1 \text{ and } \alpha(x_{i+1}, y_{i+1}) \geq 1, \tag{3.5}$$

for all $i \in \mathbb{N}$. Now by (3.3) and (3.5), we have

$$d(x_i, y_{i+1}) = d(Tx_{i-1}, Ty_i) \leq \alpha(x_{i-1}, y_i)\, d(Tx_{i-1}, Ty_i) \leq \psi(d(x_{i-1}, y_i)), \tag{3.6}$$

for all $i \in \mathbb{N}$. And

$$d(x_{i+1}, y_{i+1}) = d(Tx_i, Ty_i) \le \alpha(x_i, y_i)\, d(Tx_i, Ty_i) \le \psi(d(x_i, y_i)), \tag{3.7}$$

for all $i \in \mathbb{N}$. By (3.6) and mathematical induction, we get

$$d(x_i, y_{i+1}) \le \psi(d(x_{i-1}, y_i)) \le \psi(\psi(d(x_{i-2}, y_{i-1}))) \le \ldots \le \psi^i(d(x_0, y_1)). \tag{3.8}$$

Similarly, by (3.7) and mathematical induction, we have

$$d(x_{i+1}, y_{i+1}) \le \psi(d(x_i, y_i)) \le \psi(\psi(d(x_{i-1}, y_{i-1}))) \le \ldots \le \psi^{i+1}(d(x_0, y_0)) \tag{3.9}$$

for all $i \in \mathbb{N}$. Let $(f, \pi) \in \mathcal{F} \times [0, \infty)$ be such that (D_3) is satisfied. Let $\epsilon > 0$ be fixed. By $(\mathfrak{F}_2)$, there exists $\delta > 0$ such that

$$0 < t < \delta \Longrightarrow f(t) < f(\epsilon) - \pi. \tag{3.10}$$

Let there exists $\epsilon > 0$ and $i(\epsilon) \in \mathbb{N}$ such that

$$\sum_{i \ge i(\epsilon)} \psi^i(d(x_0, y_1)) < \frac{\epsilon}{2},$$

and

$$\sum_{i \ge i(\epsilon)} \psi^{i+1}(d(x_0, y_0)) < \frac{\epsilon}{2}.$$

Now for $p > i \ge i(\epsilon)$, by applying (D_3), we have that $d\left(x_i, y_p\right) > 0$ implies

$$\begin{aligned} f\left(d\left(x_i, y_p\right)\right) &\le f\left(\begin{array}{c} d(x_i, y_{i+1}) + d(x_{i+1}, y_{i+1}) + d(x_{i+1}, y_{i+2}) + \\ \ldots + d\left(x_{p-1}, y_{p-1}\right) + d\left(x_{p-1}, y_p\right) \end{array}\right) + \pi \\ &\le f\left(\sum_{j=i}^{p-1} d\left(x_j, y_{j+1}\right) + \sum_{j=i}^{p-2} d\left(x_{j+1}, y_{j+1}\right)\right) + \pi \\ &\le f\left(\sum_{j=i}^{p-1} \psi^j(d(x_0, y_1)) + \sum_{j=i}^{p-2} \psi^{i+1}(d(x_0, y_0))\right) + \pi \\ &\le f\left(\sum_{i \ge i(\epsilon)} \psi^i(d(x_0, y_1)) + \sum_{i \ge i(\epsilon)} \psi^{i+1}(d(x_0, y_0))\right) + \pi \\ &< f(\epsilon). \end{aligned}$$

for all $j \in \mathbb{N}$. Similarly, for $i > p \ge i(\epsilon)$, by applying (D_3), we have that $d\left(x_i, y_p\right) > 0$ implies

$$\begin{aligned} f\left(d\left(x_i, y_p\right)\right) &\le f\left(\begin{array}{c} d\left(x_p, y_p\right) + d\left(x_p, y_{p+1}\right) + d\left(x_{p+1}, y_{p+1}\right) + \\ \ldots + d(x_i, y_{i+1}) + d(x_i, y_i) \end{array}\right) + \pi \\ &\le f\left(\sum_{j=p}^{i} d\left(x_j, y_j\right) + \sum_{j=i}^{i} d\left(x_j, y_{j+1}\right)\right) + \pi \end{aligned}$$

$$
\begin{aligned}
&\leq f\left(\sum_{j=p}^{i}\psi^{j}\left(d\left(x_0,y_0\right)\right)+\sum_{j=p}^{i}\psi^{i+1}\left(d\left(x_0,y_1\right)\right)\right)+\pi\\
&\leq f\left(\sum_{i\geq i(\epsilon)}\psi^{i}\left(d\left(x_0,y_0\right)\right)+\sum_{i\geq i(\epsilon)}\psi^{i+1}\left(d\left(x_0,y_1\right)\right)\right)+\pi\\
&< f\left(\epsilon\right),
\end{aligned}
$$

for all $j\in\mathbb{N}$. Then by $(\mathfrak{F}_1)$, $d\left(x_i,y_p\right)<\epsilon$, for all $p,i\geq i_0$. Thus (x_i,y_i) is a Cauchy bisequence in $\mathfrak{F}$-bip MS (X,Y,d). Since (X,Y,d) is complete, thus (x_i,y_i) biconverges to a point $\omega\in X\cap Y$. So $(x_i)\to\omega,(y_i)\to\omega$. Also since T is continuous, so

$$
(x_i)\to\omega \text{ implies that } (x_{i+1})=(Tx_i)\to T\omega.
$$

Also since (y_i) has a unique limit ω in $X\cap Y$. Hence $T\omega=\omega$. So T has a fixed point.

Now since a bisequence (x_i,y_i) in (X,Y,d) is such that $\alpha\left(x_i,y_i\right)\geq 1$, for all $i\in\mathbb{N}$ with $x_i\to\omega$ and $y_i\to\omega$, as $i\to\infty$ for $\omega\in X\cap Y$, then by the hypothesis (iii), we have $\alpha\left(\omega,y_i\right)\geq 1$, for all $i\in\mathbb{N}$. Now by (3.4), we have

$$
\begin{aligned}
f\left(d\left(T\omega,\omega\right)\right) &\leq f\left(d\left(T\omega,Ty_i\right)+d\left(Tx_i,Ty_i\right)+d\left(Tx_i,\omega\right)\right)+\pi\\
&\leq f\left(\alpha\left(\omega,y_i\right)d\left(T\omega,Ty_i\right)+\alpha\left(x_i,y_i\right)\left(Tx_i,Ty_i\right)+d\left(x_{i+1},\omega\right)\right)+\pi\\
&\leq f\left(\psi\left(d\left(\omega,y_i\right)\right)+\psi\left(d\left(x_i,y_i\right)\right)+d\left(x_{i+1},\omega\right)\right)+\pi\\
&\leq f\left(\begin{array}{c}\psi\left(d\left(\omega,y_i\right)\right)\\ +\psi\left(d\left(x_i,\omega\right)+d\left(\omega,\omega\right)+d\left(\omega,y_i\right)\right)+d\left(x_{i+1},\omega\right)\end{array}\right)+\pi.
\end{aligned}
$$

Taking the limit as $i\to\infty$ and using the continuity of f and ψ at $t=0$, we have $d\left(T\omega,\omega\right)=0$. Thus $T\omega=\omega$. Hence T has a fixed point.

Now if ϖ is another fixed point of T, then $T\varpi=\varpi$ implies that $\varpi\in X\cap Y$ such that $\omega\neq\varpi$. Then by the property (P), there exists $z\in X\cap Y$ such that

$$
\alpha\left(\omega,z\right)\geq 1 \text{ and } \alpha\left(z,\varpi\right)\geq 1. \tag{3.11}
$$

Since T is covariant α-admissible mapping, so by (3.11), we have

$$
\alpha\left(\omega,T^{i}z\right)\geq 1 \text{ and } \alpha\left(T^{i}z,\varpi\right)\geq 1 \tag{3.12}
$$

for all $i\in\mathbb{N}$. Now by $(\mathfrak{F}_1)$ and (3.3), we have

$$
\begin{aligned}
f\left(d\left(\omega,T^{i}z\right)\right) &\leq f\left(d\left(T\omega,T\left(T^{i-1}z\right)\right)\right)\\
&\leq f\left(\alpha\left(\omega,T^{i-1}z\right)d\left(T\omega,TT^{i-1}z\right)\right)\\
&\leq f\left(\psi\left(d\left(\omega,T^{i-1}z\right)\right)\right)\\
&\leq \ldots\leq f\left(\psi^{i}\left(d\left(\omega,z\right)\right)\right).
\end{aligned} \tag{3.13}
$$

Similarly, we have

$$
f\left(d\left(T^{i}z,\varpi\right)\right) \leq f\left(d\left(T\left(T^{i-1}z\right),T\varpi\right)\right)
$$

$$
\begin{aligned}
&\le f\left(\alpha\left(T^{i-1}z,\varpi\right)d\left(T\left(T^{i-1}z\right),T\varpi\right)\right)\\
&\le f\left(\psi\left(d\left(T^{i-1}z,\varpi\right)\right)\right)\\
&\le \ldots \le f\left(\psi^{i}\left(d\left(T^{i-1}z,\varpi\right)\right)\right). \qquad (3.14)
\end{aligned}
$$

Letting $i \to +\infty$ in (3.13) and (3.14) and using the continuity of f and ψ, we have

$$\lim_{i\to\infty} f\left(d\left(\omega,T^{i}z\right)\right) = -\infty \qquad (3.15)$$

and

$$\lim_{i\to\infty} f\left(d\left(T^{i}z,\varpi\right)\right) = -\infty. \qquad (3.16)$$

Thus from (3.15) and (3.16) by $(\mathfrak{F}_2)$, we have

$$T^{i}z \to \omega \text{ and } T^{i}z \to \varpi$$

that is contradiction because the limit is unique. Hence $\omega = \varpi \in X \cap Y$. □

Example 3.4. Let $X = \mathbb{N} \cup \{0\}$ and $Y = \frac{1}{n} \cup \{0\}$ for $n \in \mathbb{N}$. Define $d : X \times Y \to [0,+\infty)$ as

$$d(x,y) = \begin{cases} 2, & \text{if } (x,y) = (2,1) \\ |x-y|, & \text{otherwise.} \end{cases}$$

Then (X, Y, d) is complete $\mathfrak{F}$-bip MS for $f(t) = \ln(t)$ and $\pi > 2$. Define the covariant mapping $T : X \cup Y \to X \cup Y$ by

$$T(x) = \begin{cases} 0, & \text{if } x \in X - \{0,1\} \\ 1, & \text{if } x \in Y. \end{cases}$$

and $\alpha : X \times Y \to [0,+\infty)$ by

$$\alpha(x,y) = 1$$

for all $(x, y) \in X \times Y$. Then all the conditions of Theorem 3.1 are satisfied with $\psi(t) = \frac{3}{4}t$. Hence, by Theorem 3.1, T must have a unique fixed point, which is $1 \in X \cap Y$.

Corollary 3.1. *Let $T : (X, Y, d) \rightrightarrows (X, Y, d)$ be a covariant and continuous mapping. Assume that there exists $\psi \in \Psi$ such that*

$$d(Tx,Ty) \le \psi(d(x,y)),$$

for all $(x, y) \in X \times Y$.

Then T has a unique fixed point.

Proof. Take $\alpha : X \times Y \to [0,+\infty)$ by $\alpha(x,y) = 1$, for $x \in X$ and $y \in Y$ in Theorem 3.1. □

Corollary 3.2. *Let $T : (X, Y, d) \rightrightarrows (X, Y, d)$ be a covariant and continuous mapping. Assume that there exists $0 < k < 1$ such that*

$$d(Tx,Ty) \le kd(x,y),$$

for all $(x, y) \in X \times Y$.

Then T has a unique fixed point.

Proof. Define $\psi : [0, +\infty) \to [0, +\infty)$ by $\psi(t) = kt$, where $0 < k < 1$ in Theorem 3.1. □

Theorem 3.2. *Let $T : (X, Y, d) \rightleftarrows (X, Y, d)$ be contravariant (α, ψ)-contraction. Assume that the following assertions hold*

(i) T is contravariant α-admissible,

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,

(iii) T is continuous or , if (x_i, y_i) is a bisequence in (X, Y, d) such that $\alpha(x_i, y_i) \geq 1$, for all $i \in \mathbb{N}$ with $x_i \to \omega$ and $y_i \to \omega$, as $i \to \infty$ for $\omega \in X \cap Y$, then $\alpha(x_i, \omega) \geq 1$, for all $i \in \mathbb{N}$.

Then T has a fixed point. Furthermore, if the property (P) holds, then fixed point is unique.

Proof. Let $x_0 \in X$ and $y_0 \in Y$ and suppose that $\alpha(x_0, Tx_0) \geq 1$ by the hypothesis (ii). Define the bisequence (x_i, y_i) in (X, Y, d) by

$$y_i = Tx_i \text{ and } x_{i+1} = Ty_i,$$

for all $i \in \mathbb{N}$. As T is contravariant α-admissible mapping by the assumption (i), so we have

$$\begin{aligned}
\alpha(x_0, y_0) &= \alpha(x_0, Tx_0) \geq 1 \text{ implies } \alpha(x_1, y_0) = \alpha(Ty_0, Tx_0) \geq 1 \\
\alpha(x_1, y_0) &\geq 1 \text{ implies } \alpha(x_1, y_1) = \alpha(Ty_0, Tx_1) \geq 1 \\
\alpha(x_1, y_1) &\geq 1 \text{ implies } \alpha(x_2, y_1) = \alpha(Ty_1, Tx_1) \geq 1 \\
\alpha(x_2, y_1) &\geq 1 \text{ implies } \alpha(x_2, y_2) = \alpha(Ty_1, Tx_2) \geq 1.
\end{aligned}$$

Continuing in this way, we have

$$\alpha(x_i, y_i) \geq 1 \text{ and } \alpha(x_{i+1}, y_i) \geq 1 \tag{3.17}$$

for all $i \in \mathbb{N}$. Now by (3.4) and (3.17), we have

$$d(x_i, y_i) = d(Ty_{i-1}, Tx_i) \leq \alpha(x_i, y_{i-1}) d(Ty_{i-1}, Tx_i) \leq \psi(d(x_i, y_{i-1})), \tag{3.18}$$

for all $i \in \mathbb{N}$. And

$$d(x_{i+1}, y_i) = d(Ty_i, Tx_i) \leq \alpha(x_i, y_i) d(Ty_i, Tx_i) \leq \psi(d(x_i, y_i)), \tag{3.19}$$

for all $i \in \mathbb{N}$. By (3.18) and (3.19) and mathematical induction, we get

$$d(x_i, y_i) \leq \psi(d(x_i, y_{i-1})) \leq \psi(\psi(d(x_{i-1}, y_{i-2}))) \leq ... \leq \psi^i(d(x_1, y_0)), \tag{3.20}$$

and

$$d(x_{i+1}, y_i) \leq \psi(d(x_i, y_i)) \leq \psi(\psi(d(x_{i-1}, y_{i-1}))) \leq ... \leq \psi^{i+1}(d(x_0, y_0)), \tag{3.21}$$

for all $i \in \mathbb{N}$. Let $(f, \pi) \in \mathcal{F} \times [0, \infty)$ be such that (D_3) is satisfied. Let $\epsilon > 0$ be fixed. By $(\mathfrak{F}_2)$, there exists $\delta > 0$ such that

$$0 < t < \delta \Longrightarrow f(t) < f(\epsilon) - \pi. \tag{3.22}$$

Let there exists $\epsilon > 0$ and $i(\epsilon) \in \mathbb{N}$ such that

$$\sum_{i \geq i(\epsilon)} \psi^i(d(x_1, y_0)) < \frac{\epsilon}{2},$$

and

$$\sum_{i\geq i(\epsilon)}\psi^{i+1}\left(d\left(x_0,y_0\right)\right)<\frac{\epsilon}{2}.$$

Now for $p>i\geq i(\epsilon)$, by applying (D_3), we have that $d\left(x_i,y_p\right)>0$ implies

$$\begin{aligned}
f\left(d\left(x_i,y_p\right)\right) &\leq f\left(\begin{array}{c} d\left(x_i,y_i\right)+d\left(x_{i+1},y_i\right)+d\left(x_{i+1},y_{i+1}\right)+ \\ \ldots+d\left(x_p,y_{p-1}\right)+d\left(x_p,y_p\right)\end{array}\right)+\pi \\
&\leq f\left(\sum_{j=i}^{p}d\left(x_j,y_j\right)+\sum_{j=i}^{p-1}d\left(x_{j+1},y_j\right)\right)+\pi \\
&\leq f\left(\sum_{j=i}^{p}\psi^{j}\left(d\left(x_1,y_0\right)\right)+\sum_{j=i}^{p-1}\psi^{i+1}\left(d\left(x_0,y_0\right)\right)\right)+\pi \\
&\leq f\left(\sum_{i\geq i(\epsilon)}\psi^{i}\left(d\left(x_1,y_0\right)\right)+\sum_{i\geq i(\epsilon)}\psi^{i+1}\left(d\left(x_0,y_0\right)\right)\right)+\pi \\
&< f\left(\epsilon\right).
\end{aligned}$$

for all $j\in\mathbb{N}$. Similarly, Now for $i>p\geq i(\epsilon)$, by applying (D_3), we have that $d\left(x_i,y_p\right)>0$ implies

$$\begin{aligned}
f\left(d\left(x_i,y_p\right)\right) &\leq f\left(\begin{array}{c} d\left(x_i,y_{i-1}\right)+d\left(x_{i-1},y_{i-1}\right)+d\left(x_{i-1},y_{i-2}\right)+ \\ \ldots+d\left(x_p,y_{p-1}\right)+d\left(x_p,y_p\right)\end{array}\right)+\pi \\
&\leq f\left(\sum_{j=p}^{i-1}d\left(x_j,y_j\right)+\sum_{j=i}^{i}d\left(x_j,y_{j-1}\right)\right)+\pi \\
&\leq f\left(\begin{array}{c} \sum_{j=p}^{i-1}\psi^{j}\left(d\left(x_1,y_0\right)\right) \\ +\sum_{j=p}^{i}\psi^{i+1}\left(d\left(x_0,y_0\right)\right)\end{array}\right)+\pi \\
&\leq f\left(\begin{array}{c} \sum_{i\geq i(\epsilon)}\psi^{j+1}\left(d\left(x_0,y_0\right)\right) \\ +\sum_{i\geq i(\epsilon)}\psi^{i}\left(d\left(x_1,y_0\right)\right)\end{array}\right)+\pi \\
&< f\left(\epsilon\right),
\end{aligned}$$

for all $j\in\mathbb{N}$. Then by $(\mathfrak{F}_1)$, $d\left(x_i,y_p\right)<\epsilon$, for all $p,i\geq i_0$. Thus (x_i,y_i) is a Cauchy bisequence in (X,Y,d). Since (X,Y,d) is complete, thus (x_i,y_i) biconverges to a point $\omega\in X\cap Y$. So $(x_i)\to\omega,(y_i)\to\omega$. Also since T is continuous, so

$$(x_i)\to\omega \text{ implies that } (y_i)=(Tx_i)\to T\omega.$$

Also since (y_i) has a unique limit ω in $X\cap Y$. Hence $T\omega=\omega$. So T has a fixed point. Now since a bisequence (x_i,y_i) in (X,Y,d) is such that $\alpha\left(x_i,y_i\right)\geq 1$, for all $i\in\mathbb{N}$ with $x_i\to\omega$ and $y_i\to\omega$, as $i\to\infty$ for $\omega\in X\cap Y$, then by the hypothesis (iii), we have $\alpha\left(x_i,\omega\right)\geq 1$, for all $i\in\mathbb{N}$. Now by (3.4), we have

$$f\left(d\left(T\omega,\omega\right)\right) \leq f\left(d\left(T\omega,Tx_i\right)+d\left(Ty_i,Tx_i\right)+d\left(Ty_i,\omega\right)\right)+\pi$$

$$\begin{aligned} &\le f\left(\alpha\left(x_i,\omega\right)d\left(T\omega,Tx_i\right)+\alpha\left(x_i,y_i\right)d\left(Ty_i,Tx_i\right)+d\left(x_{i+1},\omega\right)\right)+\pi \\ &\le f\left(\psi\left(d\left(x_i,\omega\right)\right)+\psi\left(d\left(x_i,y_i\right)\right)+d\left(x_{i+1},\omega\right)\right)+\pi \\ &\le f\left(\begin{array}{c}\psi\left(d\left(x_i,\omega\right)\right)\\ +\psi\left(d\left(x_i,\omega\right)+d\left(\omega,\omega\right)+d\left(\omega,y_i\right)\right)+d\left(x_{i+1},\omega\right)\end{array}\right)+\pi. \end{aligned}$$

Taking the limit as $i \to \infty$ and using the continuity of f and ψ at $t = 0$, we have $d(T\omega, \omega) = 0$. Thus $T\omega = \omega$. Hence T has a fixed point. Uniqueness of the fixed point is same as given in Theorem 3.1. □

Example 3.5. Let $X=\{9, 10, 18, 20\}$ and $Y=\{3, 5, 11, 18\}$. Define the usual metric $d : X \times Y \to [0,\infty)$ by

$$d(x, y) = 2^{|x-y|}.$$

Then (X, Y, d) is complete $\mathfrak{F}$-bip MS. Define the contravariant mapping $T : X \cup Y \to X \cup Y$ by

$$T(x) = \begin{cases} 18, & \text{if } x \in X \cup \{11\} \\ 9, & \text{otherwise.} \end{cases}$$

Then all the conditions of Theorem 3.2 are satisfied with $\psi(\mathrm{t}) = \frac{3}{4}\mathrm{t}$. Hence, by Theorem 3.2, T must have a unique fixed point, which is $1 \in X \cap Y$.

Corollary 3.3. *Let $T : (X, Y, d) \rightleftarrows (X, Y, d)$ be a contravariant and continuous mapping. Assume that there exists $\psi \in \Psi$ such that*

$$d(Ty, Tx) \le \psi(d(x, y)),$$

for all $(x, y) \in X \times Y$.

Then T has a unique fixed point.

Proof. Take $\alpha : X \times Y \to [0, +\infty)$ by $\alpha(x, y) = 1$, for $x \in X$ and $y \in Y$ in Theorem 3.2. □

Corollary 3.4. *Let $T : (X, Y, d) \rightleftarrows (X, Y, d)$ be a contravariant and continuous mapping. Assume that there exists $0 < k < 1$ such that*

$$d(Ty, Tx) \le kd(x, y),$$

for all $(x, y) \in X \times Y$.

Then T has a unique fixed point.

Proof. Define $\psi : [0, +\infty) \to [0, +\infty)$ by $\psi(t) = kt$, where $0 < k < 1$ in Theorem 3.2. □

Now, we derive coupled fixed point theorems from the our obtained results.

Definition 3.6. Let $\mathcal{R} : (X{\times}Y, Y{\times}X) \rightrightarrows (X, Y)$ be a covariant mapping. A point $(a, b) \in X{\times}Y$ is said to be a coupled fixed point of $\mathcal{R}$ if

$$\mathcal{R}(a, b) = a \text{ and } \mathcal{R}(b, a) = b.$$

Lemma 3.1. *Let $\mathcal{R} : (X{\times}Y, Y{\times}X) \rightrightarrows (X, Y)$ be a covariant mapping. If we define a covariant mapping $\aleph : (X{\times}Y, Y{\times}X) \rightrightarrows (X{\times}Y, Y{\times}X)$ by*

$$\aleph(x, y) = (\mathcal{R}(x, y), \mathcal{R}(y, x)),$$

for all $(x, y) \in X{\times}Y$, then (x, y) is a coupled fixed point of $\mathcal{R}$ if only if (x, y) is a fixed point of $\aleph$.

We state a property ($P^/$) which is required in our result.

($P^/$) there exists $(z_1, z_2) \in (X\times Y) \cap (Y\times X)$ such that

$$\alpha((x,y),(z_1,z_2)) \geq 1,\ \alpha((z_2,z_1),(y,x)) \geq 1,$$

and

$$\alpha((u,v),(z_1,z_2)) \geq 1,\ \alpha((z_2,z_1),(u,v)) \geq 1,$$

for all $(x,y) \in X\times Y$ and $(u,v) \in Y\times X$.

Theorem 3.3. *Let $\mathcal{R} : (X\times Y, Y\times X) \rightrightarrows (X,Y)$ be a covariant mapping. Assume that there exist $\alpha : (X\times Y)\times(X\times Y) \to [0,+\infty)$ and $\psi \in \Psi$ such that*

$$\alpha((x,y),(u,v))\, d(\mathcal{R}(x,y), \mathcal{R}(u,v)) \leq \psi\left(\frac{d(x,u)+d(v,y)}{2}\right), \tag{3.23}$$

for all $(x,y),(u,v) \in X\times Y$, and the following hypotheses also hold
(i) $\alpha((x,y),(u,v)) \geq 1$ implies

$$\alpha((\mathcal{R}(x,y), \mathcal{R}(y,x)),(\mathcal{R}(u,v), \mathcal{R}(v,u))) \geq 1,$$

(ii) there exists $(x_0,y_0) \in X\times Y$ such that

$$\alpha((x_0,y_0),(\mathcal{R}(y_0,x_0), \mathcal{R}(x_0,y_0))) \geq 1,$$

and

$$\alpha((\mathcal{R}(x_0,y_0), \mathcal{R}(y_0,x_0)),(x_0,y_0)) \geq 1,$$

(iii) $\mathcal{R}$ is continuous or, if (x_i,y_i) is a bisequence in (X,Y,d) such that

$$\alpha((x_i,y_i),(y_{i+1},x_{i+1})) \geq 1,$$

and

$$\alpha((y_{i+1},x_{i+1}),(x_i,y_i)) \geq 1,$$

for all $i \in \mathbb{N}$ with $x_i \to x$ and $y_i \to y$, as $i \to \infty$ for $(x,y) \in X\cap Y$, then

$$\alpha((x_i,y_i),(x,y)) \geq 1 \text{ and } \alpha((x,y),(x_i,y_i)) \geq 1,$$

for all $i \in \mathbb{N}$.

Then $\mathcal{R}$ has a coupled fixed point. Furthermore, if the property ($P^/$) holds, then the coupled fixed point is unique.

Proof. Let $L = X\times Y$ and $H = Y\times X$ and

$$\delta((x,y),(u,v)) = d(x,u) + d(v,y),$$

for all $(x,y) \in L$ and $(u,v) \in H$. Then (L,H,δ) is a complete $\mathfrak{F}$-bipolar metric space. By (3.23), we have

$$\alpha((x,y),(u,v))\, d(\mathcal{R}(x,y), \mathcal{R}(u,v)) \leq \psi\left(\frac{d(x,u)+d(v,y)}{2}\right), \tag{3.24}$$

and

$$\alpha\left((x,y),(u,v)\right)d\left(\mathcal{R}(x,y),\mathcal{R}(u,v)\right)\leq\psi\left(\frac{d(x,u)+d(v,y)}{2}\right).\tag{3.25}$$

Combining (3.24) and (3.25), we obtain

$$\beta(\varkappa,\varrho)\,d(\mathcal{R}\varkappa,\mathcal{R}\varrho)\leq\psi(\delta(\varkappa,\varrho)),$$

for all $\varkappa=(\varkappa_1,\varkappa_2)\in L$ and $\varrho=(\varrho_1,\varrho_2)\in H$. And the function $\beta: L\times H\to[0,+\infty)$ is defined as

$$\beta(\varkappa,\varrho)=\min\{\alpha((\varkappa_1,\varkappa_2),(\varrho_1,\varrho_2)),\alpha((\varrho_2,\varrho_1),(\varkappa_2,\varkappa_1))\},$$

and $\aleph:(L,H)\rightrightarrows(L,H)$ is defined by

$$\aleph(x,y)=(\mathcal{R}(x,y),\mathcal{R}(y,x)).$$

Then $\aleph$ is continuous and covariant (β,ψ)-contraction. Now we suppose that $\beta(\varkappa,\varrho)\geq1$. Then by (i), we have $\beta(\aleph\varkappa,\aleph\varrho)\geq1$. By condition (ii), there exists $(x_0,y_0)\in L$ (or $(y_0,x_0)\in H$) such that

$$\beta((x_0,y_0),\aleph(x_0,y_0))\geq1,$$

(or $\beta(\aleph(x_0,y_0),(y_0,x_0,))\geq1$). Since $\aleph$ is continuous, so $\aleph$ has a fixed point. Now if (x_i,y_i) is a bisequence in $L=X\times Y$ and (y_i,x_i) be a bisequence in $H=Y\times X$ such that $\alpha((x_i,y_i),(y_{i+1},x_{i+1}))\geq1$ and $(x_i,y_i)\to(x,y)$ as $n\to\infty$. Then by (iii), we have $\alpha((x_i,y_i),(y,x))\geq1$. Thus all the conditions of Theorem 3.1 are satisfied and $\aleph$ has a fixed point. Hence by Lemma (3.1), $\mathcal{R}$ has a coupled fixed point. Now since the property ($P^{/}$) holds, thus $\mathcal{R}$ has a unique coupled fixed point. □

Example 3.6. Let $L_n(\mathbb{R})$ and $U_n(\mathbb{R})$ be the sets of all $n\times n$ lower and upper triangular matrices on set of real numbers $\mathbb{R}$, respectively. Define $d: U_n(\mathbb{R})\times L_n(\mathbb{R})\to\mathbb{R}^+$ by

$$d(E,F)=\sum_{i,j=1}^{n}\left|\ell_{ij}-\hbar_{ij}\right|$$

for all $E=\left(\ell_{ij}\right)_{n\times n}\in U_n(\mathbb{R})$ and $F=\left(\hbar_{ij}\right)_{n\times n}\in L_n(\mathbb{R})$. Then $(U_n(\mathbb{R}),L_n(\mathbb{R}),d)$ is complete $\mathfrak{F}$-bip MS. Now we define

$$\mathcal{R}:(U_n(\mathbb{R})\times L_n(\mathbb{R}),L_n(\mathbb{R})\times U_n(\mathbb{R}))\rightrightarrows(U_n(\mathbb{R}),L_n(\mathbb{R}))$$

by

$$\mathcal{R}(E,F)=\left(\frac{\ell_{ij}+\hbar_{ij}}{5}\right)_{n\times n}$$

where $\left(E=\left(\ell_{ij}\right)_{n\times n},F=\left(\hbar_{ij}\right)_{n\times n}\right)\in U_n(\mathbb{R})^2\cup L_n(\mathbb{R})^2$. Then $\mathcal{R}$ is a covariant mapping which is continuous. Now we define

$$\alpha:(U_n(\mathbb{R})\times L_n(\mathbb{R}))\times(L_n(\mathbb{R})\times U_n(\mathbb{R}))\to[0,+\infty)$$

by

$$\alpha\left((E,F),(E^{/},F^{/})\right)=\begin{cases}1, & \text{if } \ell_{ij}\geq\hbar_{ij},\kappa_{ij}\geq\varsigma_{ij}\\ 0, & \text{otherwise.}\end{cases}$$

Then, we have

$$\begin{aligned} d\left(\mathcal{R}(E,F),\mathcal{R}(E^{/},F^{/})\right) &= d\left(\left(\frac{\ell_{ij}+\hbar_{ij}}{5}\right)_{n\times n},\left(\frac{\kappa_{ij}+\varsigma_{ij}}{5}\right)_{n\times n}\right) \\ &= \sum_{i,j=1}^{n}\left|\frac{\ell_{ij}+\hbar_{ij}-\kappa_{ij}-\varsigma_{ij}}{5}\right| \\ &\leq \sum_{i,j=1}^{n}\left|\frac{\ell_{ij}-\kappa_{ij}}{5}\right|+\left|\frac{\hbar_{ij}-\varsigma_{ij}}{5}\right| \\ &= \frac{1}{5}\left(d\left(E,E^{/}\right)+d\left(F,F^{/}\right)\right) \\ &= \psi\left(\frac{d\left(E,E^{/}\right)+d\left(F,F^{/}\right)}{2}\right) \end{aligned}$$

for all $E = \left(\ell_{ij}\right)_{n\times n}, F = \left(\hbar_{ij}\right)_{n\times n} \in U_n(\mathbb{R})$ and $E^{/} = \left(\kappa_{ij}\right)_{n\times n}, F^{/} = \left(\varsigma_{ij}\right)_{n\times n} \in L_n(\mathbb{R})$. Thus the inequality (3.23) is satisfied for $\psi(t) = \frac{2}{5}t$, for $t > 0$. Furthermore, the assumptions (i) and (ii) of Theorem are also satisfied for $(x_0, y_0) = (I_n, I_n)$. Hence $\mathcal{R}$ has a coupled fixed point, that is, $(0_{n\times n}, 0_{n\times n}) \in U_n(\mathbb{R}) \cap L_n(\mathbb{R})$, where $(0_{n\times n}, 0_{n\times n})$ is a null matrix.

4. Discussion

In this section, we show that our established results are generalization of some previous results of literature.

Remark 4.1. If we define $\alpha : X \times Y \to [0,+\infty)$ by $\alpha(x,y) = 1$ and $\psi(t) = kt$, where $0 < k < 1$ in Theorem 3.1, then we deduce the principal result of Rawat et al. [20].

Remark 4.2. Taking $f(t) = \ln(t)$, for $t > 0$ and $\pi = 0$ in Definition 2.4, then $\mathfrak{F}$-bip MS is reduced to bip MS. Thus the main result of Gürdal et al. [13] is direct consequence of above result.

Remark 4.3. If we take $X = Y$ in Definition 2.4, then $\mathfrak{F}$-bip MS is reduced to $\mathfrak{F}$-MS and we derive the leading result of Hussain et al. [7] from above Corollary.

Remark 4.4. Taking $\alpha((x,y),(u,v)) = 1$ and $\psi(t) = kt$, where $0 < k < 1$ in Theorem 3.3, we can get the leading result of Mutlu et al. [9].

5. Application

5.1. Integral equations

The metric fixed point theory is a distinguished and influential mechanism used to examine differential and integral equations. Keeping in mind that most of real-life problems can be transformed into the problem of differential and integral equations, we can conclude the significance of the metric fixed point theory in qualitative science and technology. Specifically, differential and integral equations appear in various scientific concerns to include the different developments in engineering, economy, game theory, optimal control, and so on. In the present section, we discuss the existence and uniqueness of solution of an integral equation.

Theorem 5.1. *Considering an integral equation*

$$\varphi(x) = g(x) + \int_{X\cup Y} K(x, y, \varphi(x))dy, \tag{5.1}$$

where $X \cup Y$ is considered as Lebesgue measurable set. Suppose that

(i) $K : \left(X^2 \cup Y^2\right) \times [0, \infty) \to [0, \infty)$ and $f \in \mathcal{L}^{\infty}(X) \cup \mathcal{L}^{\infty}(Y)$,

(ii) there is a continuous function $\Upsilon : X^2 \cup Y^2 \to [0, \infty)$ such that

$$|K(x, y, \varphi(y)) - K(x, y, \phi(y))| \leq \frac{1}{2}\Upsilon(x, y)\,|\phi(y) - \varphi(y)|,$$

for all $x, y \in \left(X^2 \cup Y^2\right)$,

(iii) $\left\|\int_{X\cup Y} \Upsilon(x, y)\, dy\right\| \leq 1$, that is. $\sup_{x\in X\cup Y} \int_{X\cup Y} |\Upsilon(x, y)|\, dy \leq 1$.

Then the integral equation (5.1) possesses a unique solution in $\mathcal{L}^{\infty}(X) \cup \mathcal{L}^{\infty}(Y)$.

Proof. Let $\Xi = \mathcal{L}^{\infty}(X)$ and $\Theta = \mathcal{L}^{\infty}(Y)$ be normed linear spaces, where X and Y are Lebesgue measureable sets and $m(X \cup Y) < \infty$. Consider $d : \Xi \times \Theta \to [0, \infty)$ to be defined by

$$d(\xi, \zeta) = \|\xi - \zeta\|_{\infty}$$

for all $\xi, \zeta \in \Xi \times \Theta$. Then (Ξ, Θ, d) is a complete $\mathfrak{F}$-bip MS. Define the mapping $I : \Xi \cup \Theta \to \Xi \cup \Theta$ by

$$I(\varphi(x)) = g(x) + \int_{X\cup Y} K(x, y, \varphi(x))dy$$

for $x \in X \cup Y$. Now we have

$$\begin{aligned}
d(I(\varphi(x)), I(\phi(x))) &= \|I(\varphi(x)) - I(\phi(x))\| \\
&= \left|\int_{X\cup Y} K(x, y, \varphi(x))dy - \int_{X\cup Y} K(x, y, \phi(x))dy\right| \\
&\leq \int_{X\cup Y} |K(x, y, \varphi(x)) - K(x, y, \phi(x))|\, dy \\
&\leq \int_{X\cup Y} \frac{1}{2}\Upsilon(x, y)\,|\phi(y) - \varphi(y)|\, dy \\
&\leq \frac{1}{2}\|\phi(y) - \varphi(y)\| \int_{X\cup Y} |\Upsilon(x, y)|\, dy \\
&\leq \frac{1}{2}\|\phi - \varphi\| \sup_{x\in X\cup Y} \int_{X\cup Y} |\Upsilon(x, y)|\, dy \\
&\leq \frac{1}{2}\|\phi - \varphi\| \\
&= \psi(d(\phi, \varphi)).
\end{aligned}$$

Define $\psi : [0, +\infty) \to [0, +\infty)$ by $\psi(t) = \frac{1}{2}t$, for $t > 0$. Hence by using Theorem 3.1, I possesses a unique fixed point in $\Xi \cup \Theta$. □

5.2. Homotopy theory

Homotopy theory is an elementary and primitive section of algebraic topology where topological objects are considered to be homotopy equivalence. In the past five decades, some solid connections have arises between this theory and various other fields of mathematics. For example, this trend plays an outstanding aspect in making more powerful ties between the homotopy theory and fixed point theory, which have received substantial consideration in the last few years, (see [23–25]).

Theorem 5.2. *Let* (X, Y, d) *be a complete* $\mathfrak{F}$*-bip MS. Suppose that* Ξ *be an open subset of* X *and* Θ *be an open subset of* Y *and* $\left(\overline{\Xi}, \overline{\Theta}\right)$ *be a closed subset of* (X, Y) *and* $(\Xi, \Theta) \subseteq \left(\overline{\Xi}, \overline{\Theta}\right)$. *Suppose* $\mathcal{S} : \left(\overline{\Xi} \cup \overline{\Theta}\right) \times [0, 1] \to X \cup Y$ *satisfies*

(hom1) $x \neq \mathcal{S}(x, q)$ *for each* $x \in \partial\Xi \cup \partial\Theta$ *and* $q \in [0, 1]$,

(hom2) for all $x \in \overline{\Xi}$, $y \in \overline{\Theta}$ *and* $q \in [0, 1]$

$$d(\mathcal{S}(y, q), \mathcal{S}(x, q)) \leq \psi(d(x, y)),$$

where $\psi \in \Psi$,

(hom3) there exists $M \geq 0$ *such that*

$$d(\mathcal{S}(x, r), \mathcal{S}(y, o)) \leq M|r - o|,$$

for all $x \in \overline{\Xi}$, $y \in \overline{\Theta}$ *and* $r, o \in [0, 1]$.

Then $\mathcal{S}(\cdot, 0)$ *has a fixed point iff* $\mathcal{S}(\cdot, 1)$ *has a fixed point.*

Proof. Let $G_1 = \{\tau \in [0, 1] : x = \mathcal{S}(x, \tau), x \in \Xi\}$ and $G_1 = \{o \in [0, 1] : y = \mathcal{S}(y, o), y \in \Theta\}$. Since $\mathcal{S}(\cdot, 0)$ has a fixed point in $\Xi \cup \Theta$, then we get $0 \in G_1 \cap G_2$. Thus $G_1 \cap G_2 \neq \emptyset$. Now, we shall prove that $G_1 \cap G_2$ is both open and closed in $[0, 1]$ and so, by connectedness, $G_1 = G_2 = [0, 1]$. Let $(\{\tau_i\}_{i=1}^{\infty}), (\{o_i\}_{i=1}^{\infty}) \subseteq (G_1, G_2)$ with $(\tau_i, o_i) \to (\mu, \mu) \in [0, 1]$ as $i \to \infty$. We also claim that $\mu \in G_1 \cap G_2$. Since $(\tau_i, o_i) \in G_1 \cap G_2$, for $i \in \mathbb{N} \cup \{0\}$. Hence there exists a bisequence $(x_i, y_i) \in (\Xi, \Theta)$ such that $y_i = \mathcal{S}(x_i, \tau_i)$ and $x_{i+1} = \mathcal{S}(y_i, o_i)$. Also, we get

$$\begin{aligned} d(x_{i+1}, y_i) &= d(\mathcal{S}(y_i, o_i), \mathcal{S}(x_i, \tau_i)) \\ &\leq \psi(d(x_i, y_i)). \end{aligned}$$

And,

$$\begin{aligned} d(x_i, y_i) &= d(\mathcal{S}(y_{i-1}, o_{i-1}), \mathcal{S}(x_i, \tau_i)) \\ &\leq \psi(d(x_i, y_{i-1})). \end{aligned}$$

Doing the similar process as we did in Theorem 3.1, one can easily show that (x_i, y_i) is a Cauchy bisequence in (Ξ, Θ). Since (Ξ, Θ) is complete, so there exists $\mu_1 \in \Xi \cap \Theta$ such that $\lim_{i\to\infty}(x_i) = \lim_{i\to\infty}(y_i) = \mu_1$. Now, we have

$$\begin{aligned} f(d(\mathcal{S}(\mu_1, o), y_i)) &= f(d(\mathcal{S}(\mu_1, o), \mathcal{S}(x_i, \tau_i))) \\ &\leq f(\psi(d(x_i, \mu_1))) = -\infty, \end{aligned}$$

whenever $i \to \infty$. Hence by $(\mathfrak{F}_2)$, we get $d(\mathcal{S}(\mu_1, o), \mu_1) = 0$, which implies that $\mathcal{S}(\mu_1, o) = \mu_1$. Similarly, $\mathcal{S}(\mu_1, \tau) = \nu_1$. Thus $\tau = o \in G_1 \cap G_2$, and evidently $G_1 \cap G_2$ is closed set in $[0, 1]$.

Next, we have to prove that $G_1 \cap G_2$ is open in $[0,1]$. Suppose $(\tau_0, o_0) \in (G_1, G_2)$, then there is a bisequence (x_0, y_0) so that $x_0 = \mathcal{S}(x_0, \tau_0)$, $y_0 = \mathcal{S}(y_0, o_0)$. Since $\Xi \cup \Theta$ is open, so there exists $r > 0$ so that $B_d(x_0, r) \subseteq \Xi \cup \Theta$ and $B_d(r, y_0) \subseteq \Xi \cup \Theta$. Choose $\tau \in (o_0 - \epsilon, o_0 + \epsilon)$ and $o \in (\tau_0 - \epsilon, \tau_0 + \epsilon)$ such that

$$|\tau - o_0| \leq \frac{1}{M^i} < \frac{\epsilon}{2}$$
$$|o - \tau_0| \leq \frac{1}{M^i} < \frac{\epsilon}{2},$$

and

$$|\tau_0 - o_0| \leq \frac{1}{M^i} < \frac{\epsilon}{2}.$$

Hence, we have

$$y \in \overline{B_{G_1 \cup G_2}(x_0, r)} = \{y : y_0 \in \Theta : d(x_0, y) \leq r + d(x_0, y_0)\},$$

and

$$x \in \overline{B_{G_1 \cup G_2}(r, y_0)} = \{x : x_0 \in \Xi : d(x, y_0) \leq r + d(x_0, y_0)\}.$$

Moreover, we have

$$\begin{aligned} d(\mathcal{S}(x,\tau), y_0) &= d(\mathcal{S}(x,\tau), \mathcal{S}(y_0, o_0)) \\ &\leq d(\mathcal{S}(x,\tau), \mathcal{S}(y, o_0)) \\ &\quad + d(\mathcal{S}(x_0,\tau), \mathcal{S}(y, o_0)) \\ &\quad + d(\mathcal{S}(x_0,\tau), \mathcal{S}(y_0, o_0)) \\ &\leq 2M|\tau - o_0| + d(\mathcal{S}(x_0,\tau), \mathcal{S}(y, o_0)) \\ &\leq \frac{2}{M^i - 1} + \psi(d(x_0, y)) \\ &\leq \frac{2}{M^i - 1} + d(x_0, y). \end{aligned}$$

Letting $i \to \infty$, we get

$$d(\mathcal{S}(x,\tau), y_0) \leq d(x_0, y) \leq r + d(x_0, y_0).$$

By similar way, we get

$$d(x_0, \mathcal{S}(y, o)) \leq d(x, y_0) \leq r + d(x_0, y_0).$$

But

$$\begin{aligned} d(x_0, y_0) &= d(\mathcal{S}(x_0, \tau_0), \mathcal{S}(y_0, o_0)) \\ &\leq M|\tau_0 - o_0| \leq \frac{1}{M^{i-1}} \to 0 \end{aligned}$$

as $i \to \infty$, which implies that $x_0 = y_0$. Therefore, for each fixed o, $o = \tau \in (o_0 - \epsilon, o_0 + \epsilon)$ and

$$\mathcal{S}(\cdot, \tau) : \overline{B_{G_1 \cup G_2}(x_0, r)} \to \overline{B_{G_1 \cup G_2}(x_0, r)}.$$

Since all the hypothesis of Theorem 3.1 hold, $\mathcal{S}(\cdot, \tau)$ has a fixed point in $\overline{\Xi} \cap \overline{\Theta}$, which must be in $\Xi \cap \Theta$. Then $\tau = o \in G_1 \cap G_2$ for each $o \in (o_0 - \epsilon, o_0 + \epsilon)$. Hence $(o_0 - \epsilon, o_0 + \epsilon) \in G_1 \cap G_2$ which gives $G_1 \cap G_2$ is open in $[0, 1]$. The proof of the converse can be established by doing the similar procedure. □

6. Conclusions

In the present research article, we have defined the notion of (α, ψ)-contraction in the setting of $\mathfrak{F}$-bipolar metric and established fixed point results. In this way, the major results of Gürdal et al. [13], Rawat et al. [20], Hussain et al. [7] and Mutlu et al. [8] are derived. Some non-trivial examples are also provided to show the validity of the established results. The solution of an integral equation and homotopy problem is also investigated.

For future work, the obtained theorems in this article can be expanded to fuzzy mappings and multivalued mappings in the setting of $\mathfrak{F}$-bip MS. Furthermore, one can prove common fixed point results for (α, ψ)-contractions. As applications of these results in the setting of $\mathfrak{F}$-bip MS, some differential and integral inclusions can be investigated.

Use of AI tools declaration

The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflicts of interests.

References

1. S. Banach, Sur les operations dans les ensembles abstracts ET leur applications aux equations integrals, *Fund. Math.*, **3** (1922), 133–181. https://doi.org/10.4064/fm-3-1-133-181
2. M. Frechet, Sur quelques points du calcul fonctionnel, *Rend. Circ. Matem. Palermo*, **22** (1906), 1–72. https://doi.org/10.1007/BF03018603
3. I. A. Bakhtin, The contraction mapping principle in almost metric spaces, *Funct. Anal.*, **30** (1989), 26–37.
4. S. Czerwik, Contraction mappings in b-metric spaces, *Acta Math. Inform. Univ. Ostra.*, **1** (1993), 5–11.
5. A. Branciari, A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces, *Publ. Math. Debrecen*, **57** (2000), 31–37.
6. M. Jleli, B. Samet, On a new generalization of metric spaces, *J. Fixed Point Theory Appl.*, **20** (2018), 128. https://doi.org/10.1007/s11784-018-0606-6
7. A. Hussain, T. Kanwal, Existence and uniqueness for a neutral differential problem with unbounded delay via fixed point results, *T. A Razmadze Math. In.*, **172** (2018), 481–490. https://doi.org/10.1016/j.trmi.2018.08.006
8. A. Mutlu, U. Gürdal, Bipolar metric spaces and some fixed point theorems, *J. Nonlinear Sci. Appl.*, **9** (2016), 5362–5373.
9. A. Mutlu, K. Ozkan, U. Gürdal, Coupled fixed point theorems on bipolar metric spaces, *Eur. J. Pure Appl. Math.*, **10** (2017), 655–667.

10. G. N. V. Kishore, D. R. Prasad, B. S. Rao, V. S. Baghavan, Some applications via common coupled fixed point theorems in bipolar metric spaces, *J. Crit. Rev.*, **7** (2019), 601–607.
11. G. N. V. Kishore, K. P. R. Rao, H. Isik, B. S. Rao, A. Sombabu, Covariant mappings and coupled fixed point results in bipolar metric spaces, *Int. J. Nonlinear Anal. Appl.*, **12** (2021), 1–15. https://doi.org/10.22075/IJNAA.2021.4650
12. B. S. Rao, G. N. V. Kishore, G. K. Kumar, Geraghty typecontraction and common coupled fixed point theorems in bipolar metric spaces with applications to homotopy, *Int. J. Math. Trends Technol.*, **63** (2018), 25–34.
13. U. Gürdal, A. Mutlu, K. Ozkan, Fixed point results for α-ψ-contractive mappings in bipolar metric spaces, *J. Inequal. Spec. Funct.*, **11** (2020), 64–75.
14. G. N. V. Kishore, K. P. R. Rao, A. Sombabu, R. V. N. S. Rao, Related results to hybrid pair of mappings and applications in bipolar metric spaces, *J. Math.*, **2019** (2019), 8485412. https://doi.org/10.1155/2019/8485412
15. G. N. V. Kishore, R. P. Agarwal, B. S. Rao, R. V. N. S. Rao, Caristi type cyclic contraction and common fixed point theorems in bipolar metric spaces with applications, *Fixed Point Theory Appl.*, **2018** (2018), 21. https://doi.org/10.1186/s13663-018-0646-z
16. G. N. V. Kishore, H. Isik, H. Aydi, B. S. Rao, D. R. Prasad, On new types of contraction mappings in bipolar metric spaces and applications, *J. Linear Topol. Algebra*, **9** (2020), 253–266.
17. C. G. Moorthy, G. Siva, Bipolar multiplicative metric spaces and fixed point theorems of covariant and contravariant mappings, *Math. Anal. Convex Optim.*, **2** (2021), 39–49. https://doi.org/10.52547/maco.2.1.4
18. A. Mutlu, K. Ozkan, U. Gürdal, Some common fixed point theorems in bipolar metric spaces, *Turk. J. Math. Comput. Sci.*, **14** (2022), 346–354. https://doi.org/10.47000/tjmcs.1099118
19. K. Roy, M. Saha, Generalized contractions and fixed point theorems over bipolar cone tvs b-metric spaces with an application to Homotopy theory, *Mat. Vesnik*, **72** (2020), 281–296.
20. S. Rawat, R. C. Dimri, A. Bartwal, $\mathfrak{F}$-Bipolar metric spaces and fixed point theorems with applications, *J. Math. Comput. Sci.*, **26** (2022), 184–195.
21. B. Alamri, Fixed point results in $\mathfrak{F}$-bipolar metric spaces with applications, *Mathematics*, **11** (2023), 2399. https://doi.org/10.3390/math11102399
22. B. Samet, C. Vetro, P. Vetro, Fixed point theorem for $\alpha - \psi$ contractive type mappings, *Nonlinear Anal. Theor.*, **75** (2012), 2154–2165.
23. A. Butt, I. Beg, A. Iftikhar, Fixed points on ordered metric spaces with application in homotopy theory, *J. Fixed Point Theory Appl.*, **20** (2018), 21. https://doi.org/10.1007/s11784-018-0496-7
24. M. E. Ege, C. Alaca, Fixed point results and an application to homotopy in modular metric spaces, *J. Nonlinear Sci. Appl.*, **8** (2015), 900–908.
25. Humaira, M. Sarwar, P. Kumam, Common fixed point results for fuzzy mappings on complex-valued metric spaces with Homotopy results, *Symmetry*, **11** (2019), 61. https://doi.org/10.3390/sym11010061

AIMS Press