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Research article

Some new inequalities for nonnegative matrices involving Schur product

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Abstract: In this study, we focused on the spectral radius of the Schur product. Two new types of the upper bound of $\rho(M \circ N)$, which is the spectral radius of the Schur product of two matrices M, N with nonnegative elements, were established using the Hölder inequality and eigenvalue inclusion theorem. In addition, the obtained new type upper bounds were compared with the classical conclusions. Numerical examples demonstrated that the new type of upper formulas improved the result of Johnson and Horn effectively in some cases, and were sharper than other existing results.

Keywords: nonnegative matrix; Schur product; spectral radius; irreducible
Mathematics Subject Classification: 15A47

1. Introduction

For convenience, we use $C^{n\times n}$ to represent the union of complex matrices of order n, $R^{n\times n}$ denotes the union of all real matrices of order n and R^n denotes the set of vectors of order n.

Given a component-wise nonnegative matrix $M=(m_{ij})\in R^{n\times n}$ and $r>0$, we define $M^{(r)}=(m_{ij}^r)$. If

$$u=(u_1,u_2,\cdots,u_n)\in R^n$$

and $r>0$, we write the following:

$$u^{(r)} = \left(u_1^r, u_2^r, \cdots, u_n^r\right).$$

Let $M \in R^{n\times n}$ ($n \geq 2$), then M is considered a reducible matrix if there exists a permutation matrix P such that

$$PMP^{\mathrm{T}} = \begin{pmatrix} M_{11} & M_{12} \\ O & M_{22} \end{pmatrix},$$

where M_{11} and M_{22} are square matrices of an order of at least one. Otherwise, M is referred to be irreducible.

If the matrix M is nonnegative, i.e., whose elements are nonnegative, then the spectral radius, represented by $\rho(M)$, is a characteristic root of M and any other characteristic root λ is not more than $\rho(M)$ in absolute value. Estimating the spectral radius is widely used in numerical analysis, graph theory, stability theory and other related fields, and it is a relatively active topic in matrix theory research.

The Schur product, often called the Hadamard product, is the consequence of multiplying two matrices element by element to create a new matrix. The Schur product of $M = (m_{ij})$ and $N = (n_{ij})$ is defined to be $M \circ N = (m_{ij} n_{ij})$ (see [1, Definition 5.0.1]). It is worth noting that the two multiplied matrices must have the same number of rows and columns. Schur product plays an essential role in the matrix theory. It can be used in various applications, including replacing matrix multiplication, blind signal separation, feature selection and image processing.

It is obvious that if M, N are two nonnegative matrices, the Schur product $M \circ N$ is nonnegative and the spectral radius $\rho(M \circ N)$ dominates any other characteristic roots of the Schur product $M \circ N$ in absolute value. The study of the Schur product, especially the spectral radius of the Schur product, has attracted the attention of a wide range of scholars. Many studies involving the bound of $\rho(M \circ N)$ can be found in the subsequent works [2,3].

Let M, N be nonnegative matrices. The following classical result can be found in [1],

$$\rho(M \circ N) \leq \rho(M)\rho(N). \tag{1.1}$$

The above-mentioned inequality shows that the spectral radius $\rho(M \circ N)$ is not more than the product of $\rho(M)$ and $\rho(N)$.

The improved result of inequality (1.1) was proposed in [4] as follows:

$$\rho(M \circ N) \leq \sqrt{\rho(M \circ M)\rho(N \circ N)} \leq \rho(M)\rho(N). \tag{1.2}$$

In some cases, the two results mentioned above may be fragile. The following example illustrates this situation. Let $M = I$, the identity matrix of order n and $N = J$, the matrix of all ones with the order n. It is not difficult to observe that

$$\rho(M \circ N) = \rho(I) = 1 \leq \rho(M)\rho(N) = \rho(J) = n$$

and

$$\rho(M \circ N) = \rho(I) = 1 \leq \sqrt{\rho(M \circ M)\rho(N \circ N)} = \sqrt{n} \leq \rho(M)\rho(N) = n.$$

We know that these inequalities can be weak when n is very large. The following result is observed owing to Fang [5]:

$$\rho(M\circ N)\le \max_{1\le i\le n}\left\{2m_{ii}n_{ii}+\rho(M)\rho(N)-m_{ii}\rho(N)-n_{ii}\rho(M)\right\}. \tag{1.3}$$

Liu et al [6] improved inequality (1.3) and derived the following conclusion:

$$\rho(M\circ N)\le \max_{i\ne j}\frac{1}{2}\Big\{m_{ii}n_{ii}+m_{jj}n_{jj}+\Big[\left(m_{ii}n_{ii}-m_{jj}n_{jj}\right)^2 +4\left(\rho(M)-m_{ii}\right)\left(\rho(N)-n_{ii}\right)\left(\rho(M)-m_{jj}\right)\left(\rho(N)-n_{jj}\right)\Big]^{\frac{1}{2}}\Big\}. \tag{1.4}$$

In addition, some significant new boundaries of the spectral radius of the Schur product were introduced in [7–10], which gave better estimations for the spectral radius in some cases. Inspired by the research, we continue to study the upper bound on the spectral radius of the Schur product of two nonnegative matrices. We provided two new types of the upper bound of the spectral radius involving the Schur product using the eigenvalue inclusion theorem and the Hölder inequality. Numerical tests validate that the new type of upper formulas improve the result of Johnson and Horn [1] effectively in some cases and are sharper than other existing results, which approach the real value more efficiently than previous ones.

2. Main results

The following are some basic lemmas to get our main results.

Lemma 1. [11] Let $M\in R^{n\times n}$ be a nonnegative irreducible matrix, then there is a positive vector x such that $Mx=\rho(M)x$.

Lemma 2. [12] **(**Hölder inequality**)** Let $x=(x_1,x_2,\cdots,x_n)^{\mathrm{T}}\ge 0,\ y=(y_1,y_2,\cdots,y_n)^{\mathrm{T}}\ge 0.$

If $p>1, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then the following is observed:

$$\sum_{i=1}^{n}x_iy_i\le\left(\sum_{i=1}^{n}x_i^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}y_i^q\right)^{\frac{1}{q}}.$$

Lemma 3. [5] Given a nonnegative irreducible matrix $M\in R^{n\times n}$ and a nonnegative nonzero vector $z\in R^n$, if $Mz\le kz$, then $\rho(M)\le k$.

Lemma 4. [13] (Brauer's theorem) Let $M=(m_{ij})\in C^{n\times n}$ $(n\ge 2)$. If λ is the characteristic root of the matrix M, there exists a pair of positive integers (i, j) satisfying the following inequality:

$$\left|\lambda-m_{ii}\right|\left|\lambda-m_{jj}\right|\le\sum_{k\ne i}^{n}\left|m_{ik}\right|\sum_{k\ne j}^{n}\left|m_{jk}\right|,\ i\ne j.$$

Lemma 5. [11] Let M_k be a principal submatrix of the nonnegative matrix M, then

$\rho(M_k) \le \rho(M)$. Specifically, if $M_k \ne M$ and M is irreducible, then $\rho(M_k) < \rho(M)$.

The main findings of this study are presented below.

Theorem 1. If $M = (m_{ij}), N = (n_{ij}) \in R^{n\times n}$ are two nonnegative matrices and $p>1, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then we have the following:

$$\rho(M \circ N) \le \max_{1\le i\le n}\left\{ m_{ii}n_{ii} + \left[\rho\left(M^{(p)}\right) - m_{ii}^{p}\right]^{\frac{1}{p}} \left[\rho\left(N^{(q)}\right) - n_{ii}^{q}\right]^{\frac{1}{q}} \right\}. \tag{2.1}$$

Proof. When $n=1$, inequality (2.1) becomes an equality, and we assume that $n \ge 2$. To demonstrate this problem, we will distinguish two cases.

Case 1. First, we assume that $M \circ N$ is irreducible, then M and N are irreducible, implying that for any $p>1, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, $M^{(p)} = \left(m_{ij}^{p}\right)$ and $N^{(q)} = \left(n_{ij}^{q}\right)$ are nonnegative and irreducible. According to Lemma 1, there exists

$$u = (u_1, u_2, \cdots, u_n)^{\mathrm{T}} > 0$$

and

$$v = (v_1, v_2, \cdots, v_n)^{\mathrm{T}} > 0,$$

such that

$$M^{(p)}u^{(p)} = \rho\left(M^{(p)}\right)u^{(p)} \tag{2.2}$$

and

$$N^{(q)}v^{(q)} = \rho\left(N^{(q)}\right)v^{(q)}, \tag{2.3}$$

where

$$u^{(p)} = \left(u_1^{p}, u_2^{p}, \cdots, u_n^{p}\right)^{\mathrm{T}}$$

and

$$v^{(q)} = \left(v_1^{q}, v_2^{q}, \cdots, v_n^{q}\right)^{\mathrm{T}}.$$

It follows from (2.2) and (2.3) that

$$m_{ii}^{p}u_i^{p} + \sum_{j\ne i}^{n} m_{ij}^{p}u_j^{p} = \rho\left(M^{(p)}\right)u_i^{p},\ \ i = 1, 2, \cdots, n$$

and

$$n_{ii}^{q}v_i^{q} + \sum_{j\ne i}^{n} n_{ij}^{q}v_j^{q} = \rho\left(N^{(q)}\right)v_i^{q},\ i = 1, 2, \cdots, n.$$

Therefore, we get the following:

$$\sum_{j\neq i}^{n} m_{ij}^{p} u_{j}^{p} = \left[\rho\left(M^{(p)} \right) - m_{ii}^{p} \right] u_{i}^{p}, \ \ i = 1, 2, \cdots, n \tag{2.4}$$

and

$$\sum_{j\neq i}^{n} n_{ij}^{q} v_{j}^{q} = \left[\rho\left(N^{(q)} \right) - n_{ii}^{q} \right] v_{i}^{q}, \ \ i = 1, 2, \cdots, n. \tag{2.5}$$

Let

$$z = \left(z_1, z_2, \cdots, z_n \right) \in R^n,$$

where $z_i = u_i v_i$. We define $A = M \circ N$. By Lemma 2, Eqs (2.4) and (2.5), for any $p > 1, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, and we obtain the following:

$$\begin{aligned}
\left(Az \right)_i &= m_{ii} n_{ii} z_i + \sum_{j\neq i}^{n} m_{ij} n_{ij} z_j \\
&= m_{ii} n_{ii} z_i + \sum_{j\neq i}^{n} \left(m_{ij} u_j \right) \left(n_{ij} v_j \right) \\
&\leq m_{ii} n_{ii} z_i + \left(\sum_{j\neq i}^{n} m_{ij}^{p} u_{j}^{p} \right)^{\frac{1}{p}} \left(\sum_{j\neq i}^{n} n_{ij}^{q} v_{j}^{q} \right)^{\frac{1}{q}} \\
&= m_{ii} n_{ii} z_i + \left[\rho\left(M^{(p)} \right) - m_{ii}^{p} \right]^{\frac{1}{p}} u_i \left[\rho\left(N^{(q)} \right) - n_{ii}^{q} \right]^{\frac{1}{q}} v_i \\
&= m_{ii} n_{ii} z_i + \left[\rho\left(M^{(p)} \right) - m_{ii}^{p} \right]^{\frac{1}{p}} \left[\rho\left(N^{(q)} \right) - n_{ii}^{q} \right]^{\frac{1}{q}} z_i \\
&= \left\{ m_{ii} n_{ii} + \left[\rho\left(M^{(p)} \right) - m_{ii}^{p} \right]^{\frac{1}{p}} \left[\rho\left(N^{(q)} \right) - n_{ii}^{q} \right]^{\frac{1}{q}} \right\} z_i.
\end{aligned}$$

Based on Lemma 3, there exists some i such that

$$\begin{aligned}
\rho\left(M \circ N \right) &\leq m_{ii} n_{ii} + \left[\rho\left(M^{(p)} \right) - m_{ii}^{p} \right]^{\frac{1}{p}} \left[\rho\left(N^{(q)} \right) - n_{ii}^{q} \right]^{\frac{1}{q}} \\
&\leq \max_{1\leq i\leq n} \left\{ m_{ii} n_{ii} + \left[\rho\left(M^{(p)} \right) - m_{ii}^{p} \right]^{\frac{1}{p}} \left[\rho\left(N^{(q)} \right) - n_{ii}^{q} \right]^{\frac{1}{q}} \right\}.
\end{aligned}$$

Case 2. Now, we consider the matrix $M \circ N$ to be reducible. At this point, there is a permutation matrix $P = \left(p_{ij} \right)$ of size n with the following:

$$p_{12} = p_{23} = \cdots = p_{n-1,n} = p_{n1} = 1.$$

The remaining elements are zero, where $M + \varepsilon P$, $N + \varepsilon P$ are nonnegative and irreducible for any sufficiently small $\varepsilon > 0$. Using continuity theory and combining it with Case 1, we can achieve our desired result. Thus, the proof of Theorem 1 is done.

In Theorem 1, when $p=q=2$, we will obtain the following result.

Theorem 2. If $M=(m_{ij}), N=(n_{ij})\in R^{n\times n}$ are two nonnegative matrices, then we have the following:

$$\rho(M\circ N)\le \max_{1\le i\le n}\left\{m_{ii}n_{ii}+\sqrt{\left[\rho\left(M^{(2)}\right)-m_{ii}^2\right]\left[\rho\left(N^{(2)}\right)-n_{ii}^2\right]}\right\}. \tag{2.6}$$

Remark 1. For any $i=1,2,\cdots,n$, we have the following:

$$\begin{aligned}&\left[\rho\left(M^{(2)}\right)-m_{ii}^2\right]\left[\rho\left(N^{(2)}\right)-n_{ii}^2\right]\\&=\rho\left(M^{(2)}\right)\rho\left(N^{(2)}\right)+m_{ii}^2n_{ii}^2-n_{ii}^2\rho\left(M^{(2)}\right)-m_{ii}^2\rho\left(N^{(2)}\right)\\&\le\rho\left(M^{(2)}\right)\rho\left(N^{(2)}\right)+m_{ii}^2n_{ii}^2-2m_{ii}n_{ii}\sqrt{\rho\left(M^{(2)}\right)\rho\left(N^{(2)}\right)}\\&=\left(\sqrt{\rho\left(M^{(2)}\right)\rho\left(N^{(2)}\right)}-m_{ii}n_{ii}\right)^2.\end{aligned}$$

Therefore, we obtain the following:

$$\sqrt{\left[\rho\left(M^{(2)}\right)-m_{ii}^2\right]\left[\rho\left(N^{(2)}\right)-n_{ii}^2\right]}\le\sqrt{\rho\left(M^{(2)}\right)\rho\left(N^{(2)}\right)}-m_{ii}n_{ii}.$$

This implies that

$$m_{ii}n_{ii}+\sqrt{\left[\rho\left(M^{(2)}\right)-m_{ii}^2\right]\left[\rho\left(N^{(2)}\right)-n_{ii}^2\right]}\le\sqrt{\rho\left(M^{(2)}\right)\rho\left(N^{(2)}\right)}. \tag{2.7}$$

In terms of Theorem 2, inequalities (1.2) and (2.7) and noticing that $M^{(2)}=M\circ M, N^{(2)}=N\circ N$, we get the following:

$$\begin{aligned}\rho(M\circ N)&\le\max_{1\le i\le n}\left\{m_{ii}n_{ii}+\sqrt{\left[\rho\left(M^{(2)}\right)-m_{ii}^2\right]\left[\rho\left(N^{(2)}\right)-n_{ii}^2\right]}\right\}\\&\le\sqrt{\rho\left(M^{(2)}\right)\rho\left(N^{(2)}\right)}\\&=\sqrt{\rho(M\circ M)\rho(N\circ N)}\\&\le\rho(M)\rho(N).\end{aligned}$$

Therefore, the bound in (2.6) is sharper than $\rho(M)\rho(N)$ known in [1].

Now, we give an example to illustrate our conclusion. We consider again the numerical example in the introduction. If M is an identity matrix of order n and N denotes the matrix of all ones with the order $n,$ then we obtain the following:

$$\max_{1\le i\le n}\left\{m_{ii}n_{ii}+\sqrt{\left[\rho\left(M^{(2)}\right)-m_{ii}^2\right]\left[\rho\left(N^{(2)}\right)-n_{ii}^2\right]}\right\}=1+\sqrt{\left(1-1^2\right)\left(n-1^2\right)}=1.$$

It is surprising to see that our bound is the actual value of the spectral radius.

Next, we establish the second inequality for $\rho(M\circ N)$.

Theorem 3. If $M=(m_{ij}), N=(n_{ij})\in R^{n\times n}$ are two nonnegative matrices and $p>1, q>1$ with

$\frac{1}{p}+\frac{1}{q}=1$, then we have the following:

$$\rho(M\circ N)\le \max_{i\neq j}\frac{1}{2}\Big\{m_{ii}n_{ii}+m_{jj}n_{jj}+\Big[\left(m_{ii}n_{ii}-m_{jj}n_{jj}\right)^2 +4\left(\rho\left(M^{(p)}\right)-m_{ii}^p\right)^{\frac{1}{p}}\left(\rho\left(N^{(q)}\right)-n_{ii}^q\right)^{\frac{1}{q}}\left(\rho\left(M^{(p)}\right)-m_{jj}^p\right)^{\frac{1}{p}}\left(\rho\left(N^{(q)}\right)-n_{jj}^q\right)^{\frac{1}{q}}\Big]^{\frac{1}{2}}\Big\}. \tag{2.8}$$

Proof. The conclusion holds with equality when $n=1$. Next, we assume that $n\ge 2$. To demonstrate this problem, we will discuss two cases.

Case 1. First, we suppose that $M\circ N$ is irreducible, then M and N are irreducible. Clearly, for any $p>1, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, and $M^{(p)}=\left(m_{ij}^p\right)$ and $N^{(q)}=\left(n_{ij}^q\right)$ are nonnegative and irreducible. According to Lemma 1, there exists

$$u=\left(u_1,u_2,\cdots,u_n\right)^{\mathrm{T}}>0$$

and

$$v=\left(v_1,v_2,\cdots,v_n\right)^{\mathrm{T}}>0,$$

such that

$$M^{(p)}u^{(p)}=\rho\left(M^{(p)}\right)u^{(p)}$$

and

$$N^{(q)}v^{(q)}=\rho\left(N^{(q)}\right)v^{(q)},$$

where

$$u^{(p)}=\left(u_1^p,u_2^p,\cdots,u_n^p\right)^{\mathrm{T}}$$

and

$$v^{(q)}=\left(v_1^q,v_2^q,\cdots,v_n^q\right)^{\mathrm{T}}.$$

Therefore,

$$\sum_{k\neq i}^{n}\frac{m_{ik}^p u_k^p}{u_i^p}=\rho\left(M^{(p)}\right)-m_{ii}^p,\quad \sum_{k\neq j}^{n}\frac{m_{jk}^p u_k^p}{u_j^p}=\rho\left(M^{(p)}\right)-m_{jj}^p,\quad i,j=1,2,\cdots,n, \tag{2.9}$$

and

$$\sum_{k\neq i}^{n}\frac{n_{ik}^q v_k^q}{v_i^q}=\rho\left(N^{(q)}\right)-n_{ii}^q,\quad \sum_{k\neq j}^{n}\frac{n_{jk}^q v_k^q}{v_j^q}=\rho\left(N^{(q)}\right)-n_{jj}^q,\quad i,j=1,2,\cdots,n. \tag{2.10}$$

Next, we define two nonsingular positive diagonal matrices as follows:

$$U = \mathrm{diag}\left(u_1, u_2, \cdots, u_n\right)$$

and

$$V = \mathrm{diag}\left(v_1, v_2, \cdots, v_n\right).$$

Let

$$\tilde{M} = U^{-1}MU = \begin{pmatrix} m_{11} & \frac{m_{12}u_2}{u_1} & \cdots & \frac{m_{1n}u_n}{u_1} \\ \frac{m_{21}u_1}{u_2} & m_{22} & \cdots & \frac{m_{2n}u_n}{u_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{m_{n1}u_1}{u_n} & \frac{m_{n2}u_2}{u_n} & \cdots & m_{nn} \end{pmatrix}$$

and

$$\tilde{N} = V^{-1}NV = \begin{pmatrix} n_{11} & \frac{n_{12}v_2}{v_1} & \cdots & \frac{n_{1n}v_n}{v_1} \\ \frac{n_{21}v_1}{v_2} & n_{22} & \cdots & \frac{n_{2n}v_n}{v_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{n_{n1}v_1}{v_n} & \frac{n_{n2}v_2}{v_n} & \cdots & n_{nn} \end{pmatrix}.$$

It is simple to see that

$$\begin{aligned} \tilde{M} \circ \tilde{N} &= \begin{pmatrix} m_{11}n_{11} & \frac{m_{12}n_{12}u_2v_2}{u_1v_1} & \cdots & \frac{m_{1n}n_{1n}u_nv_n}{u_1v_1} \\ \frac{m_{21}n_{21}u_1v_1}{u_2v_2} & m_{22}n_{22} & \cdots & \frac{m_{2n}n_{2n}u_nv_n}{u_2v_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{m_{n1}n_{n1}u_1v_1}{u_nv_n} & \frac{m_{n2}n_{n2}u_2v_2}{u_nv_n} & \cdots & m_{nn}n_{nn} \end{pmatrix} \\ &= \left(UV\right)^{-1}\left(M \circ N\right)\left(UV\right). \end{aligned}$$

Therefore,

$$\rho\left(\tilde{M} \circ \tilde{N}\right) = \rho\left(M \circ N\right).$$

From Lemmas 2 and 4, as well as Eqs (2.9) and (2.10), for any $p > 1, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, there exists a pair $\left(i,\ j\right)$ such that

$$
\begin{aligned}
&\left|\rho(M\circ N)-m_{ii}n_{ii}\right|\left|\rho(M\circ N)-m_{jj}n_{jj}\right| \\
&\le \sum_{k\ne i}^{n}\frac{m_{ik}n_{ik}u_k v_k}{u_i v_i}\sum_{k\ne j}^{n}\frac{m_{jk}n_{jk}u_k v_k}{u_j v_j} \\
&\le \left(\sum_{k\ne i}^{n}\frac{m_{ik}^{p}u_k^{p}}{u_i^{p}}\right)^{\frac{1}{p}}\left(\sum_{k\ne i}^{n}\frac{n_{ik}^{q}v_k^{q}}{v_i^{q}}\right)^{\frac{1}{q}}\left(\sum_{k\ne j}^{n}\frac{m_{jk}^{p}u_k^{p}}{u_j^{p}}\right)^{\frac{1}{p}}\left(\sum_{k\ne j}^{n}\frac{n_{jk}^{q}v_k^{q}}{v_j^{q}}\right)^{\frac{1}{q}} \\
&=\left[\rho\left(M^{(p)}\right)-m_{ii}^{p}\right]^{\frac{1}{p}}\left[\rho\left(N^{(q)}\right)-n_{ii}^{q}\right]^{\frac{1}{q}}\left[\rho\left(M^{(p)}\right)-m_{jj}^{p}\right]^{\frac{1}{p}}\left[\rho\left(N^{(q)}\right)-n_{jj}^{q}\right]^{\frac{1}{q}}.
\end{aligned}
\tag{2.11}
$$

Furthermore, Lemma 5 denotes that

$$\rho(M\circ N)>m_{ii}n_{ii}$$

for $i=1,2,\cdots,n$. From inequality (2.11), we obtain the following:

$$
\begin{aligned}
\rho(M\circ N)\le \frac{1}{2}\Bigg\{ & m_{ii}n_{ii}+m_{jj}n_{jj}+\Bigg[\left(m_{ii}n_{ii}-m_{jj}n_{jj}\right)^{2} \\
&+4\left(\rho\left(M^{(p)}\right)-m_{ii}^{p}\right)^{\frac{1}{p}}\left(\rho\left(N^{(q)}\right)-n_{ii}^{q}\right)^{\frac{1}{q}}\left(\rho\left(M^{(p)}\right)-m_{jj}^{p}\right)^{\frac{1}{p}}\left(\rho\left(N^{(q)}\right)-n_{jj}^{q}\right)^{\frac{1}{q}}\Bigg]^{\frac{1}{2}}\Bigg\} \\
\le \max_{i\ne j}\frac{1}{2}\Bigg\{ & m_{ii}n_{ii}+m_{jj}n_{jj}+\Bigg[\left(m_{ii}n_{ii}-m_{jj}n_{jj}\right)^{2} \\
&+4\left(\rho\left(M^{(p)}\right)-m_{ii}^{p}\right)^{\frac{1}{p}}\left(\rho\left(N^{(q)}\right)-n_{ii}^{q}\right)^{\frac{1}{q}}\left(\rho\left(M^{(p)}\right)-m_{jj}^{p}\right)^{\frac{1}{p}}\left(\rho\left(N^{(q)}\right)-n_{jj}^{q}\right)^{\frac{1}{q}}\Bigg]^{\frac{1}{2}}\Bigg\}.
\end{aligned}
$$

Case 2. If the matrix $M\circ N$ is reducible, we can use the method of Theorem 1 to prove similarly.

Remark 2. Following that, we compare the bound in (2.1) of Theorem 1 to the bound in (2.8) of Theorem 3. We assume, without losing generality, the following:

$$m_{ii}n_{ii}+\left[\rho\left(M^{(p)}\right)-m_{ii}^{p}\right]^{\frac{1}{p}}\left[\rho\left(N^{(q)}\right)-n_{ii}^{q}\right]^{\frac{1}{q}}\ge m_{jj}n_{jj}+\left[\rho\left(M^{(p)}\right)-m_{jj}^{p}\right]^{\frac{1}{p}}\left[\rho\left(N^{(q)}\right)-n_{jj}^{q}\right]^{\frac{1}{q}}.$$

As a result, we can express the inequality above as follows:

$$\left[\rho\left(M^{(p)}\right)-m_{jj}^{p}\right]^{\frac{1}{p}}\left[\rho\left(N^{(q)}\right)-n_{jj}^{q}\right]^{\frac{1}{q}}\le\left[\rho\left(M^{(p)}\right)-m_{ii}^{p}\right]^{\frac{1}{p}}\left[\rho\left(N^{(q)}\right)-n_{ii}^{q}\right]^{\frac{1}{q}}+m_{ii}n_{ii}-m_{jj}n_{jj}.$$

Therefore,

$$
\begin{aligned}
&\left(m_{ii}n_{ii}-m_{jj}n_{jj}\right)^2+4\left[\rho\left(M^{(p)}\right)-m_{ii}^p\right]^{\frac{1}{p}}\left[\rho\left(N^{(q)}\right)-n_{ii}^q\right]^{\frac{1}{q}}\left[\rho\left(M^{(p)}\right)-m_{jj}^p\right]^{\frac{1}{p}}\left[\rho\left(N^{(q)}\right)-n_{jj}^q\right]^{\frac{1}{q}}\\
&\qquad\le\left(m_{ii}n_{ii}-m_{jj}n_{jj}\right)^2+4\left[\rho\left(M^{(p)}\right)-m_{ii}^p\right]^{\frac{1}{p}}\left[\rho\left(N^{(q)}\right)-n_{ii}^q\right]^{\frac{1}{q}}\\
&\qquad\times\left\{\left[\rho\left(M^{(p)}\right)-m_{ii}^p\right]^{\frac{1}{p}}\left[\rho\left(N^{(q)}\right)-n_{ii}^q\right]^{\frac{1}{q}}+m_{ii}n_{ii}-m_{jj}n_{jj}\right\}\\
&\qquad=\left(m_{ii}n_{ii}-m_{jj}n_{jj}\right)^2+4\left[\rho\left(M^{(p)}\right)-m_{ii}^p\right]^{\frac{2}{p}}\left[\rho\left(N^{(q)}\right)-n_{ii}^q\right]^{\frac{2}{q}}\\
&\qquad+4\left[\rho\left(M^{(p)}\right)-m_{ii}^p\right]^{\frac{1}{p}}\left[\rho\left(N^{(q)}\right)-n_{ii}^q\right]^{\frac{1}{q}}\left(m_{ii}n_{ii}-m_{jj}n_{jj}\right)\\
&\qquad=\left[\left(m_{ii}n_{ii}-m_{jj}n_{jj}\right)+2\left(\rho\left(M^{(p)}\right)-m_{ii}^p\right)^{\frac{1}{p}}\left(\rho\left(N^{(q)}\right)-n_{ii}^q\right)^{\frac{1}{q}}\right]^2.
\end{aligned}
\tag{2.12}
$$

From inequalities (2.8) and (2.12), we obtain the following:

$$
\begin{aligned}
\rho\left(M\circ N\right)&\le\max_{i\ne j}\frac{1}{2}\left\{m_{ii}n_{ii}+m_{jj}n_{jj}+\left[\left(m_{ii}n_{ii}-m_{jj}n_{jj}\right)^2\right.\right.\\
&\qquad\left.\left.+4\left(\rho\left(M^{(p)}\right)-m_{ii}^p\right)^{\frac{1}{p}}\left(\rho\left(N^{(q)}\right)-n_{ii}^q\right)^{\frac{1}{q}}\left(\rho\left(M^{(p)}\right)-m_{jj}^p\right)^{\frac{1}{p}}\left(\rho\left(N^{(q)}\right)-n_{jj}^q\right)^{\frac{1}{q}}\right]^{\frac{1}{2}}\right\}\\
&\le\max_{i\ne j}\frac{1}{2}\left\{m_{ii}n_{ii}+m_{jj}n_{jj}+\left(m_{ii}n_{ii}-m_{jj}n_{jj}\right)+2\left[\rho\left(M^{(p)}\right)-m_{ii}^p\right]^{\frac{1}{p}}\left[\rho\left(N^{(q)}\right)-n_{ii}^q\right]^{\frac{1}{q}}\right\}\\
&=\max_{1\le i\le n}\left\{m_{ii}n_{ii}+\left[\rho\left(M^{(p)}\right)-m_{ii}^p\right]^{\frac{1}{p}}\left[\rho\left(N^{(q)}\right)-n_{ii}^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}
$$

As a result, the bound in (2.8) of Theorem 3 is more precise than that in (2.1) of Theorem 1.

When $p=q=2$ in Theorem 3, we get the following conclusion.

Theorem 4. If

$$M=\left(m_{ij}\right),N=\left(n_{ij}\right)\in R^{n\times n}$$

are two nonnegative matrices, then we have the following:

$$
\begin{aligned}
\rho\left(M\circ N\right)&\le\max_{i\ne j}\frac{1}{2}\left\{m_{ii}n_{ii}+m_{jj}n_{jj}+\left[\left(m_{ii}n_{ii}-m_{jj}n_{jj}\right)^2\right.\right.\\
&\qquad\left.\left.+4\sqrt{\left(\rho\left(M^{(2)}\right)-m_{ii}^2\right)\left(\rho\left(N^{(2)}\right)-n_{ii}^2\right)\left(\rho\left(M^{(2)}\right)-m_{jj}^2\right)\left(\rho\left(N^{(2)}\right)-n_{jj}^2\right)}\right]^{\frac{1}{2}}\right\}.
\end{aligned}
$$

Remark 3. According to the previous proofs, we have the following conclusions:

$$\rho(M\circ N)\le \max_{i\ne j}\frac{1}{2}\Bigg\{m_{ii}n_{ii}+m_{jj}n_{jj}+\Bigg[\left(m_{ii}n_{ii}-m_{jj}n_{jj}\right)^2$$
$$+4\sqrt{\left(\rho\left(M^{(2)}\right)-m_{ii}^2\right)\left(\rho\left(N^{(2)}\right)-n_{ii}^2\right)\left(\rho\left(M^{(2)}\right)-m_{jj}^2\right)\left(\rho\left(N^{(2)}\right)-n_{jj}^2\right)}\Bigg]^{\frac{1}{2}}\Bigg\}$$
$$\le \max_{1\le i\le n}\left\{m_{ii}n_{ii}+\sqrt{\left[\rho\left(M^{(2)}\right)-m_{ii}^2\right]\left[\rho\left(N^{(2)}\right)-n_{ii}^2\right]}\right\}$$
$$\le\sqrt{\rho(M\circ M)\rho(N\circ N)}$$
$$\le\rho(M)\rho(N).$$

The above inequalities show that our bounds are improvements of the result of Johnson and Horn [1].

3. Numerical examples

Here, we discuss certain concrete examples to show that our new upper bounds are more precise than earlier results.

Example 1. First, we will employ two 4×4 matrices as in [6]:

$$M=\begin{pmatrix}4 & 1 & 0 & 2\\ 1 & 0.05 & 1 & 1\\ 0 & 1 & 4 & 0.5\\ 1 & 0.5 & 0 & 4\end{pmatrix},\ N=\begin{pmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{pmatrix}.$$

With the Schur product $M\circ N=M$, we see that

$$\rho(M\circ N)=\rho(M)=5.7339.$$

From inequality (1.1), we have

$$\rho(M\circ N)\le\rho(M)\rho(N)=22.9336.$$

From inequality (1.3) in [5], we get

$$\rho(M\circ N)\le\max_{1\le i\le n}\left\{2m_{ii}n_{ii}+\rho(M)\rho(N)-m_{ii}\rho(N)-n_{ii}\rho(M)\right\}=17.1017.$$

From inequality (1.4) in [6], we obtain

$$\rho(M\circ N)\le\max_{i\ne j}\frac{1}{2}\Bigg\{m_{ii}n_{ii}+m_{jj}n_{jj}+\Bigg[\left(m_{ii}n_{ii}-m_{jj}n_{jj}\right)^2$$
$$+4\left(\rho(M)-m_{ii}\right)\left(\rho(N)-n_{ii}\right)\left(\rho(M)-m_{jj}\right)\left(\rho(N)-n_{jj}\right)\Bigg]^{\frac{1}{2}}\Bigg\}=11.6478.$$

According to Theorem 2 in this study, we obtain

$$\rho(M\circ N)\le\max_{1\le i\le n}\left\{m_{ii}n_{ii}+\sqrt{\left[\rho\left(M^{(2)}\right)-m_{ii}^2\right]\left[\rho\left(N^{(2)}\right)-n_{ii}^2\right]}\right\}=7.3573.$$

If we apply Theorem 4 in this study, we will get

$$\rho(M\circ N)\le\max_{i\neq j}\frac{1}{2}\Bigg\{m_{ii}n_{ii}+m_{jj}n_{jj}+\Bigg[\left(m_{ii}n_{ii}-m_{jj}n_{jj}\right)^2$$
$$+4\sqrt{\left(\rho\left(M^{(2)}\right)-m_{ii}^2\right)\left(\rho\left(N^{(2)}\right)-n_{ii}^2\right)\left(\rho\left(M^{(2)}\right)-m_{jj}^2\right)\left(\rho\left(N^{(2)}\right)-n_{jj}^2\right)}\Bigg]^{\frac{1}{2}}\Bigg\}=6.7142.$$

Example 2. Now, we present the second example and look at the following two 4×4 nonnegative matrices:

$$M=\begin{pmatrix}4&1&1&1\\2&5&1&1\\0&2&4&1\\1&1&1&4\end{pmatrix},\quad N=\begin{pmatrix}1&1&0&0\\1&3&2&0\\0&1&4&3\\0&0&1&5\end{pmatrix}.$$

Then, the Schur product will be as follows:

$$M\circ N=\begin{pmatrix}4&1&0&0\\2&15&2&0\\0&2&16&3\\0&0&1&20\end{pmatrix}.$$

We also observe that

$$\rho(M\circ N)=20.7439,\ \rho(M\circ M)=26.1755,$$

and

$$\rho(N\circ N)=25.9286.$$

From inequality (1.1), we acquire

$$\rho(M\circ N)\le\rho(M)\rho(N)=50.1274.$$

According to inequality (1.3) in [5], we get

$$\rho(M\circ N)\le\max_{1\le i\le n}\left\{2m_{ii}n_{ii}+\rho(M)\rho(N)-m_{ii}\rho(N)-n_{ii}\rho(M)\right\}=25.6463.$$

From inequality (1.4) in [6], we obtain

$$\rho(M\circ N)\le\max_{i\neq j}\frac{1}{2}\Bigg\{m_{ii}n_{ii}+m_{jj}n_{jj}+\Bigg[\left(m_{ii}n_{ii}-m_{jj}n_{jj}\right)^2$$
$$+4\left(\rho(M)-m_{ii}\right)\left(\rho(N)-n_{ii}\right)\left(\rho(M)-m_{jj}\right)\left(\rho(N)-n_{jj}\right)\Bigg]^{\frac{1}{2}}\Bigg\}=25.5209.$$

According to Theorem 2, we obtain

$$\rho(M\circ N)\le\max_{1\le i\le n}\left\{m_{ii}n_{ii}+\sqrt{\left[\rho\left(M^{(2)}\right)-m_{ii}^2\right]\left[\rho\left(N^{(2)}\right)-n_{ii}^2\right]}\right\}=26.0512.$$

However, if we apply Theorem 4, we will get

$$\rho\left(M\circ N\right)\le\max_{i\ne j}\frac{1}{2}\left\{m_{ii}n_{ii}+m_{jj}n_{jj}+\left[\left(m_{ii}n_{ii}-m_{jj}n_{jj}\right)^{2}\right.\right.$$
$$\left.\left.+4\sqrt{\left(\rho\left(M^{(2)}\right)-m_{ii}^{2}\right)\left(\rho\left(N^{(2)}\right)-n_{ii}^{2}\right)\left(\rho\left(M^{(2)}\right)-m_{jj}^{2}\right)\left(\rho\left(N^{(2)}\right)-n_{jj}^{2}\right)}\right]^{\frac{1}{2}}\right\}=24.0030.$$

The data calculated above shows that our results are more precise than previous results in some cases.

4. Conclusions

In this study, we focused on the spectral radius of the Schur product for two nonnegative matrices. We presented two new types of upper bounds of the spectral radius by utilizing Brauer's theorem and the Hölder inequality. The obtained upper bounds improved the classical conclusion of Johnson and Horn [1].

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that no conflict of interest.

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