Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(12): 29577–29603.
DOI: 10.3934/math.20231515
Received: 13 September 2023
Revised: 26 October 2023
Accepted: 26 October 2023
Published: 01 November 2023

Research article

Long-time behavior for a nonlinear Timoshenko system: Thermal damping versus weak damping of variable-exponents type

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Abstract: In this work, we consider a nonlinear thermoelastic Timoshenko system with a time-dependent coefficient where the heat conduction is given by Coleman-Gurtin [1]. Consequently, the Fourier and Gurtin-Pipkin laws are special cases. We prove that the system is exponentially and polynomially stable. The equality of the wave speeds is not imposed unless the system is not fully damped by the thermoelasticity effect. In other words, the thermoelasticity is only coupled to the first equation in the system. By constructing a suitable Lyapunov functional, we establish exponential and polynomial decay rates for the system. We noticed that the decay sometimes depends on the behavior of the thermal kernel, the variable exponent, and the time-dependent coefficient. Our results extend and improve some earlier results in the literature especially the recent results by Fareh [2], Mustafa [3] and Al-Mahdi and Al-Gharabli [4].

Keywords: thermoelastic Timoshenko system; variable exponents; embedding theory; Coleman-Gurtin's law; general decay; energy method
Mathematics Subject Classification: 35L20, 45K05, 74F05, 93D23

1. Introduction

In this work, we investigate the asymptotic behavior of solutions of the following one-dimensional thermoelastic Timoshenko system:

$$\begin{cases} \rho_1\phi_{tt} - \kappa(\phi_x + \psi)_x + \gamma\theta_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa(\phi_x + \psi) + \beta(t)|\psi_t|^{\nu(x)-2}\psi_t = 0, \\ \rho_3\theta_t + \tau q_x + \gamma\phi_{xt} = 0, \end{cases} \tag{1.1}$$

with the following Dirichlet boundary conditions:

$$\begin{cases} \phi(0,t) = \phi(L,t) = 0, \ t \geq 0, \\ \psi(0,t) = \psi(L,t) = 0, \ t \geq 0, \\ \theta(0,t) = \theta(L,t) = 0, \ t \geq 0, \end{cases} \tag{1.2}$$

and initial data

$$\begin{cases} \phi(x,0) = \phi_0(x), \ \psi(x,0) = \psi_0(x), \ \theta(x,0) = \theta_0(x), \ x \in [0,L], \\ \phi_t(x,0) = \phi_1(x), \ \psi_t(x,0) = \psi_1(x), \ \ x \in [0,L], \end{cases} \tag{1.3}$$

where $\rho_1, \rho_2, \rho_3, \kappa, b, \gamma > 0$ are positive physical parameters from thermoelasticity theory and $\beta(t)$ is the time-dependent coefficient of the damping term. The unknown variables $(\phi, \psi, \theta) : [0, L] \times \mathbb{R}_+$ are functions of (x, t) and represent the transverse displacement, rotational angle of the filament of the beam and the temperature, respectively. The initial conditions $\phi_0, \phi_1, \psi_0, \psi_1$, and the history function θ_0 are fixed data.

The heat flux q in (1.1) is given by the Coleman-Gurtin's law [1]:

$$\tau q(t) + (1-\alpha)\theta_x + \alpha \int_0^\infty \mu(s)\theta_x(x, t-s)ds = 0, \qquad \alpha \in (0,1), \tag{1.4}$$

where the Fourier (1.8) law and the Gurtin-Pipkin (1.11) law are special cases. In (1.4), the function μ is the heat conductivity relaxation kernel given by $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ which satisfies some conditions that will be mentioned later.

Substituting Eq (1.4) into (1.1), we get

$$\begin{cases} \rho_1\phi_{tt} - \kappa(\phi_x + \psi)_x + \gamma\theta_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa(\phi_x + \psi) + \beta(t)|\psi_t|^{\nu(x)-2}\psi_t = 0, \\ \rho_3\theta_t - \widehat{\alpha}\theta_{xx} - \alpha\int_0^\infty \mu(s)\theta_{xx}(x, t-s)ds + \gamma\phi_{xt} = 0, \\ \phi(x,0) = \phi_0(x), \ \phi_t(x,0) = \phi_1(x), \ \psi(x,0) = \psi_0(x), \\ \psi_t(x,0) = \psi_1(x), \ \theta(x,-t) = \theta_0(x,t), \\ \phi(0,t) = \phi(L,t) = \psi(0,t) = \psi(L,t) = \theta(0,t) = \theta(1,t) = 0. \end{cases} \tag{1.5}$$

where $\widehat{\alpha} = (1-\alpha)$ with $\alpha \in (0,1)$.

This work aims to address the stability problem of the system (1.5) with a focus on the interaction between the thermoelastic dissipation and another weak damping with variable exponent nonlinearity. Unlike the previous stability results in the literature, we prove the exponential and polynomial stability of the solutions without imposing the equality of the wave speeds. This is important because the case of equal speeds is purely mathematical and physically never satisfied [5]. Therefore, the stability result obtained without any restriction on the coefficients is more realistic than that obtained with a stability condition.

The system (1.5) was originally introduced by Timoshenko [6] as follows:

$$\begin{cases} \rho_1\phi_{tt} - \kappa(\phi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa(\phi_x + \psi) = 0, \end{cases} \tag{1.6}$$

as a model of the motion of a thick beam where ϕ is the transverse displacement of the beam, and $-\psi$ is the rotational angle of the filament of the beam. ρ_1, ρ_2, b and κ are fixed positive physical constants.

The issue of stability of Timoshenko systems (1.6) has attracted a great deal of attention in the last decades. Various types of damping mechanisms have been used to stabilize these systems such as boundary and/or internal feedback, heat or thermoelasticity, memory, and Kelvin-Voigt damping, have been used. See, for example, [7–20].

The above studies have shown that the exponential stability of system (1.6) is achieved in the presence of linear dampings in both equations of (1.6) without imposing any condition on the speeds of wave propagation. However, if the damping effect is acting on only one equation, the system is exponentially stable if and only if it has equal speeds of wave propagation; that is

$$\frac{\kappa}{\rho_1} = \frac{b}{\rho_2}. \tag{1.7}$$

In this paper, we focus on the thermoelastic dissipations which refer to the mechanisms by which the energy is dissipated in thermoelastic materials. Thermoelastic damping is a source of intrinsic material damping that occurs due to the coupling between the elastic field in a structure caused by deformation and the temperature field. When a vibrating structure undergoes strain, it causes a change in internal energy, resulting in regions becoming hotter or cooler. Thermoelastic damping occurs when there is a lack of thermal equilibrium between different parts of the vibrating structure, leading to irreversible heat flow driven by temperature gradients and energy dissipation. For more information on the heat conduction equations, we refer to [21]. One of the thermoelastic dissipations is the heat conduction via a heat flux. The heat flux comes in different types. One of these types is the one defined by Fourier's law, which is

$$\tau q = -\theta_x, \tag{1.8}$$

where q is the heat flux and the coefficient τ is a positive constant called the coefficient of thermal conductivity of the material. However, if the model is subjected to the Fourier's law (1.8), this leads to a parabolic equation. Consequently, the heat propagates at an infinite speed, that is, any thermal disturbance produced at some point in the body has an instantaneous effect elsewhere in the body. To overcome this physical paradox, many theories were developed. The first theory was proposed by Green and Naghdi [22–24] who expanded three new theories based on entropy equality rather than the entropy inequality. They called them thermoelasticity of type I, type II and type III, respectively. Lord and Shulman [25] proposed the second theory and suggested replacing Fourier's law (1.8) by the following Cattaneo's law:

$$\tau_0 q_t + q + \tau\theta_x = 0, \tag{1.9}$$

where the positive constant τ_0, also known as the thermal relaxation time, stands for the delay in the heat flux reaction to the temperature gradient. According to this theory, the system becomes fully hyperbolic. This means the heat propagates with a finite speed and is viewed as a wave-like propagation rather than a diffusion phenomenon. A wave-like thermal disturbance is referred to as a second sound (where the first sound is the usual sound). A nonclassical theory predicting the occurrence of such disturbances is known as thermoelasticity with finite wave speeds or second sound thermoelasticity. It is worth to mention that the constitutive equation for the heat flux in the type III theory is given by

$$q = -c_1\alpha_x - c_2\theta_x, \tag{1.10}$$

where

$$\alpha = \alpha_0(x) + \int_0^t \theta(x,s)ds.$$

It is also worth mentioning that the type III thermoelasticity and the second sound thermoelasticity cannot describe the memory effect that reigns in some materials, particularly at a low temperature. This fact leads to the search for a more general constitutive assumption relating the heat flux to the thermal memory. Due to this feature, a more general constitutive assumption connecting the heat flux to the thermal memory must be sought. Gurtin and Pipkin [26] assumed that the heat flux depends on the integrated history of the temperature gradient and established a general nonlinear theory for which thermal disturbances propagate with a finite speed. According to this theory, Gurtin and Pipkin [26] proposed the following linearized constitutive equation for q; that is

$$\tau q = -\int_0^\infty \mu(s)\,\theta_x(t-s)\,ds, \tag{1.11}$$

where q represents the heat flux depending on the history of the temperature gradient due to the kernel μ. The function $\mu = \mu(s)$ is the relaxation kernel of the thermal conductivity, which is a bounded convex function on $\mathbb{R}_+$ with total mass of 1; that is

$$\int_0^\infty \mu(s)ds = 1,$$

and satisfies some other conditions mentioned in the Assumption (A_1) (below). The presence of the convolution term (1.11) renders the Timoshenko system coupled with the heat equation into a fully hyperbolic system, which allows the heat to propagate with a finite speed and describes the memory effect of heat conduction.

It is observed that various selections of $\mu(s)$ lead to different distinct constitutive models. In particular, if we choose $\mu(s) = \kappa\delta(s)$ and $\mu(s) = \frac{\kappa}{\tau_0}e^{-\frac{s}{\tau_0}}$, $\tau_0 > 0$, we get Fourier's law (1.8) and Cattaneo's law, respectively, where $\delta(s)$ is the Dirac mass weighted at 0. In other words, (1.11) is a generalized form of Fourier's and Cattaneo's laws. For more information on this topic, we direct the reader to [27–32].

As a summary, condition (1.7) is never physically satisfied, due to the physical meaning of the constants appearing in the Timoshenko systems (see [33] for more details). Therefore, results that are obtained with this condition compulsorily holding are definitely not in tandem with physical reality. Thus, any result concerning Timoshenko systems that is obtained without condition (1.7) is of great interest, and much desired. In this paper, we aim to stabilize the Timoshenko systems (1.1)–(1.3) without imposing the equal speed of wave propagation (1.7).

Before starting the analysis and proof of our results, we compare our model with others that have been investigated in the literature.

- In the case of $\beta = 0$, $\alpha = 0$, with the heat flux defined by Fourier's law (1.8), this model was investigated by Munõz Rivera and Racke [34], Almeida Júnior et al. [10], Alves et al. [35] and Alves et al. [36] who proved that the solution of the system is exponentially stable if and only if the wave speeds are equal $\left(\dfrac{k}{\rho_1} = \dfrac{b}{\rho_2}\right)$.

- In the case of $\beta = 0$, $\alpha = 0$, with the heat flux q defined by the Cattaneo's law (1.9), Fernandez Sare and Racke [37] showed that the solution of the Timoshenko system coupled with the heat equation described by the Cattaneo law loses the exponential stability even if the wave speeds are equal. Santos et al. [30] then produced a stability number and proved that the solution of the system is exponentially stable if and only if that stability number is zero.
- In the case of $\beta = 0$, $\alpha = 1$, with the heat flux q defined by the Gurtin-Pipkin law,

$$\tau q = -\int_{-\infty}^{t} \mu(t-s)\,\theta_x(x,s)\,ds. \tag{1.12}$$

  This model was investigated by Dell'Oro and Pata [38] and Hanni et al. [39], who proved that the solution of the system is exponentially stable under some conditions with stability numbers. However, Fareh [2] demonstrated that the stability number has no effect on the exponential stability of system when it is fully damped, meaning when all equations are damped by the heat conduction.
- In the case of $\beta \neq 0$ and without the thermoelasticity dissipation (the third equation in the system (1.5)), Mustafa [3] proved that the system is exponentially and polynomially stable when the wave speeds are equal $\left(\frac{k}{\rho_1} = \frac{b}{\rho_2}\right)$ and the variable exponents under some conditions. Al-Mahdi and Al-Gharabli [4] obtained similar results, but with the addition of another damping of variable exponent type in the first equation of the system.
- In the case of $\beta \neq 0$, $\alpha \neq 0$ and $\alpha \neq 1$ such as in the system (1.5), the problem had not been investigated. This is the first study to investigate the interaction between the thermoelasticity dissipation and weak damping with variable exponent. We prove that the system is exponentially and polynomially stable without imposing the equality of the wave speeds, where the system is not full damped by the heat conduction, unlike the one in Fareh [2].

For the existence, uniqueness and stability analysis of some other classes of differential equations, we refer to [40–42].

2. Assumptions and transformations

This section is devoted to the hypotheses and certain transformations that our problem requires. Initially, we consider the convolution kernel $\mu : \mathbb{R}_+ \to \mathbb{R}_+$, which is a $C^2(\mathbb{R}_+)$ convex non-increasing function satisfying:

(A_1) $\mu(0) > 0$, $\lim_{s\to\infty} \mu(s) = 0$, and $\int_0^\infty \mu(s)ds = 1$. Furthermore, there exists a positive nonincreasing differentiable function ξ such that

$$-\mu''(s) \leq \xi(s)\mu'(s), \quad \text{for almost every } s > 0, \tag{2.1}$$

and $\int_0^{+\infty} \xi(s)\mathrm{d}s = +\infty$.

We introduce the memory kernel $g = -\mu'$; that is $\mu(s) = \int_s^{+\infty} g(r)dr$, and it satisfies the following conditions:

(A_2) $g:\mathbb{R}_+\to\mathbb{R}_+,\ g\in C^1(\mathbb{R}_+),\ g(0)>0,\ g_0=\int_0^\infty g(s)ds=\mu(0)>0$, and $\int_0^\infty sg(s)ds=1$. Condition (2.1) implies that

$$g'(s)\le -\xi(s)g(s),\quad \text{for almost every } s>0. \tag{2.2}$$

(A_3) We assume that there exists a positive constant Δ such that

$$\|\Psi_{0x}(s)\|^2\le\Delta,\quad \forall s>0, \tag{2.3}$$

where $\Psi(x)$ and $\Psi_0(x)$ are defined (below) in (2.5) and (2.6), respectively.
We define $L_g=\left\{\Psi:\mathbb{R}_+\to H_0^1(0,L):\ \|\Psi\|_{L_g}^2:=\int_0^\infty g(s)\|\Psi_x(s)\|^2ds<\infty\right\}$, which is a Hilbert space.

(A_4) The time-dependent coefficient $\beta:[0,\infty)\to(0,\infty)$ is a nonincreasing C^1 function satisfying $\int_0^\infty\beta(s)ds=\infty$.

(A_5) The variable exponent $v(x)$:
$v:[0,L]\to[1,\infty)$ is a continuous function such that

$$v_1:=\operatorname{essinf}_{x\in[0,L]}v(x),\qquad v_2:=\operatorname{esssup}_{x\in[0,L]}v(x),$$

and $1<v_1\le v(x)\le v_2<\infty$. Moreover, the variable function v satisfies the log-Hölder continuity condition; that is, for any δ with $0<\delta<1$, there exists a constant $A>0$ such that,

$$|v(x)-v(y)|\le -\frac{A}{\log|x-y|},\ \text{for all } x,y\in\Omega,\ \text{with } |x-y|<\delta. \tag{2.4}$$

Remark 2.1. (1) Condition (2.2) was introduced for the first time in (2008) by Messaoudi [43]. For more results for the history of the relaxation functions, we refer to [44–52].
(2) The class of the function μ that satisfies the condition (A_1) is not empty. For example, the following functions satisfies the condition (A_1):

$$\mu(s)=\frac{1}{(1+s)^2},\ \mu(s)=e^{-s},\ \mu(s)=2a\left(\sqrt{s}+1\right)e^{-\sqrt{s}},$$

where a is a positive real number chosen so that the condition (A_1) holds.

Using the Dafermos method [44], we introduce the following variable $\Psi:(0,1)\times\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ by

$$\Psi(x,t,s)=\int_{t-s}^t\theta(x,r)dr, \tag{2.5}$$

with $\Psi(x,0,s)=\int_0^s\theta_0(x,r)dr$. Simple calculations yield the following

$$\begin{aligned}&\Psi_t(x,t,s)+\Psi_s(x,t,s)=\theta(x,t), && (0,L)\times\mathbb{R}_+\times\mathbb{R}_+,\\ &\Psi(0,t,s)=\Psi(L,t,s)=\Psi(x,t,0)=0, && (0,L)\times\mathbb{R}_+\times\mathbb{R}_+,\end{aligned}$$

where the subscripts t and s means the partial derivatives with respect to t and s, respectively. Direct integrations, using the properties of Ψ, show that

$$\int_0^\infty \mu(s)\theta_{xx}(x,t-s)ds = \underbrace{\lim_{y\to\infty}\mu(y)\int_{t-y}^{t}\theta_{xx}(x,r)dr\Big|_{y=0}^{y=s}}_{\to 0} - \int_0^\infty \mu'(s)\int_{t-s}^{t}\theta_{xx}(x,r)drds$$

$$= \int_0^\infty g(s)\Psi_{xx}(x,t,s)ds.$$

Gathering all the transformations, system (1.5) becomes

$$\begin{cases}\rho_1\phi_{tt} - \kappa(\phi_x+\psi)_x + \gamma\theta_x = 0,\\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa(\phi_x+\psi) + \beta(t)|\psi_t|^{\nu(x)-2}\psi_t = 0,\\ \rho_3\theta_t - \widehat{\alpha}\theta_{xx} - \alpha\int_0^\infty g(s)\Psi_{xx}(t-s)ds + \gamma\phi_{xt} = 0,\\ \phi(x,0)=\phi_0(x),\ \phi_t(x,0)=\phi_1(x),\ \psi(x,0)=\psi_0(x),\ \psi_t(x,0)=\psi_1(x),\\ \Psi_t(x,t,s)+\Psi_s(x,t,s)=\theta(x,t),\\ \theta(x,-t)=\theta_0(x,t),\ \Psi_0(x,s)=\int_0^s\theta_0(x,r)dr,\\ \phi(0,t)=\phi(L,t)=\psi(0,t)=\psi(L,t)=\theta(0,t)=\theta(L,t)=0,\\ \Psi(0,t,s)=\Psi(L,t,s)=\Psi(x,t,0)=0.\end{cases} \tag{2.6}$$

where $\widehat{\alpha} = 1-\alpha$ and $\alpha \in (0,1)$.

The utility of the presence of nonstandard damping with variable exponent can be seen in modeling physical phenomena such as flows of electro-rheological fluids or fluids with temperature-dependent viscosity, nonlinear viscoelasticity, filtering processes through a porous media, and image processing. These models comprise hyperbolic, parabolic or elliptic equations with varying exponents of nonlinearity and nonlinear gradients of the unknown solution. Further information on these issues can be found in [53–55].

Now, we state without proof, the following existence result:

Proposition 2.1. *For any* $(\phi_0,\phi_1,\psi_0,\psi_1,\theta_0) \in \mathcal{H}$, *and assuming that the hypotheses* $(A_1 - A_5)$ *holds, then, the system* (2.6) *has a unique global (weak) solution*

$$\phi,\phi_t,\psi,\psi_t,\theta \in C(\mathbb{R}_+;\mathcal{H}). \tag{2.7}$$

Moreover, if $(\phi_0,\psi_0,\theta_0) \in \mathcal{V}$, *then the system* (2.6) *has a strong solution*

$$\phi,\psi,\theta \in C^1(\mathbb{R}_+;\mathcal{H}) \cap C(\mathbb{R}_+;\mathcal{V}), \tag{2.8}$$

where

$$\mathcal{H} := H_0^1(0,L)\times L^2(0,L)\times H_0^1(0,L)\times L^2(0,L)\times L_g, \tag{2.9}$$

and

$$\mathcal{V} := \left(H^2(0,L)\cap H_0^1(0,L)\right)\times\left(H^2(0,L)\cap H_0^1(0,L)\right)\times H_0^1(0,L). \tag{2.10}$$

The above existence result can be proved by using the Faedo-Galerkin method and repeating the steps in [56–60].

3. The stability results

In this section, we state the outcomes of our study. We present three different theorems based on the varying range of the variable exponents. In addition, we provide some examples and remarks.

Theorem 3.1. *Assume that the hypotheses* $(A_1 - A_5)$ *hold and* $v_1 = v_2 = 2$. *Then, there exist constants* $\gamma_0 \in (0,1)$ *and* $\delta_1 > 0$ *such that, for all* $t \in \mathbb{R}_+$ *and for all* $\delta_0 \in (0, \gamma_0]$, *the energy functional* (4.1) *satisfies*

$$E(t) \leq \delta_1 \left(1 + \int_0^t (g(s))^{1-\delta_0} ds\right) e^{-\delta_0 \int_0^t (\beta\xi)(s) ds} + \frac{\alpha \Delta \widehat{c}}{\delta_0} \int_t^{+\infty} g(s) ds, \tag{3.1}$$

where $\delta_1 = \max\{E(0), \frac{\alpha\Delta\widehat{c}}{\delta_0}(g(0))^{\delta_0}\}$, *the constant* Δ *is introduced in* (A_3), *and* $\widehat{c}$ *is a positive constant depends on the coefficients of the system and the Poincaré's constant.*

Examples 3.1. This example illustrates the result in Theorem (3.1). Let $\mu(s) = e^{-s}$ and $\beta \equiv 1$, then it is easy to check that $\mu(s)$ satisfies condition (A_1). Therefore, the energy decay (3.1) becomes for $\delta_0 = \frac{1}{2}$,

$$E(t) \leq ce^{-t}. \tag{3.2}$$

This is an exponential decay.

Let $\mu(s) = \frac{1}{(1+s)^2}$ and $\beta \equiv 1$, then it is easy to check that $\mu(s)$ satisfies condition (A_1). Therefore, the energy decay (3.1) becomes for $\delta_0 = \frac{1}{2}$, and a positive constant C,

$$E(t) \leq \frac{C}{(1+t)^2}. \tag{3.3}$$

This is a polynomial decay. Then, we note that $\lim_{t\to\infty} E(t) = 0$.

Theorem 3.2. *Assume that* $(A_1 - A_5)$ *hold,* $1 < v_1 < 2$ *and* $v_2 \neq 2$. *Then, the energy functional* (4.1) *satisfies for a positive constants* C,

$$E(t) \leq C(1+t)^{\frac{-1}{\kappa}} (\xi\beta)^{-\frac{\kappa+1}{\kappa}} \left[1 + \int_0^t (\xi\beta)^{\frac{\kappa+1}{\kappa}}(s) h^{\kappa+1}(s)(1+s)^{\frac{1}{\kappa}} ds\right], \tag{3.4}$$

where $\kappa = \max\left\{\frac{2-v_1}{2v_1-2}, \frac{v_2}{2} - 1\right\}$.

Examples 3.2. This example illustrates the result in Theorem (3.2). Let $\mu(s) = 2a\left(\sqrt{s}+1\right)e^{-\sqrt{s}}$, then, $\xi(s) = \frac{1}{2\sqrt{s}}$ and $g(s) = ae^{-\sqrt{s}}$. Therefore, $\int_t^\infty g(s) = 2a\left(\sqrt{s}+1\right)e^{-\sqrt{s}}$. Then, $h(t) = \xi(t)\int_t^\infty g(s) = a\left(\frac{\sqrt{t}+1}{\sqrt{t}}e^{-\sqrt{t}}\right)$. Then in case if $\kappa = \frac{2-v_1}{2v_1-2} > 0$, the energy decay (3.4) becomes for $\beta(t) = (1+t)^{-\lambda}$, and $0 \leq \lambda \leq 1$,

$$E(t) \leq C(1+t)^{-\frac{2v_1-2}{2-v_1}}(1+t)^{\frac{v_1}{4-2v_1}-\lambda}\left(1 + \int_0^t (1+s)^{\frac{5v_1-4}{2(2-v_1)}-\lambda}(\xi\beta)^{\frac{v_1}{2-v_1}}(s) h^{\frac{v_1}{2-v_1}} ds\right). \tag{3.5}$$

Then, we obtain polynomial stability of the form

$$E(t) \leq C(1+t)^{\frac{4-3v_1}{2(2-v_1)}-\lambda}. \tag{3.6}$$

It is clear that for $\lambda = 0$ and $v_1 > \frac{4}{3}$, $\lim_{t\to\infty} E(t) = 0$. If $\lambda = 1$, then we have for any $1 < v_1 < 2$, $\lim_{t\to\infty} E(t) = 0$.

In the second case, if $\kappa = \frac{v_2}{2} - 1 > 0$, then we obtain for the same function polynomial stability of the form

$$E(t) \leq C(1+t)^{\frac{v_2-4}{2(v_2-2)}-\lambda}. \tag{3.7}$$

Then, we note that for $\lambda = 0$ and $v_1 < v_2 < 4$, we have $\lim_{t\to\infty} E(t) = 0$. Also, if $\lambda = 1$, then we have for any $1 < v_1 \leq v_2$, $\lim_{t\to\infty} E(t) = 0$.

Theorem 3.3. *Assume that* $(A_1 - A_5)$ *hold,* $v_1 \geq 2$ *and* $v_2 > 2$*. Then, the energy functional* (4.1) *satisfies for a positive constants C,*

$$E(t) \leq C(1+t)^{-\frac{2}{v_2-2}}(\xi\beta)^{-\frac{v_2}{v_2-2}}\left(1 + \int_0^t (1+s)^{\frac{2}{v_2-2}}(\xi\beta)^{\frac{v_2}{v_2-2}}(s)h^{\frac{v_2}{2}}ds\right). \tag{3.8}$$

4. Technical lemmas

In this section, we state and establish several lemmas needed for the proofs of our main results in Section 3. Through this paper, the constant c denotes to the generic positive constant and $\widehat{c}$ is a positive constant depends on the coefficients of the system and the Poincaré's constant.

Lemma 4.1. *The energy of system (2.6) is defined by*

$$E(t) = \frac{1}{2}\left[\rho_1\|\phi_t\|^2 + \rho_2\|\psi_t\|^2 + b\|\psi_x\|^2 + \rho_3\|\theta\|^2 + \kappa\|(\phi_x+\psi)\|^2 + \alpha\int_0^\infty g(s)|\Psi_x|^2 ds\right]dx, \tag{4.1}$$

and satisfies

$$E'(t) = -\widehat{\alpha}\int_0^L \theta_x^2 dx + \frac{\alpha}{2}\int_0^L\int_0^\infty g'(s)|\Psi_x|^2 ds dx - \beta(t)\int_0^L |\psi_t|^{v(x)}dx \leq 0, \ \forall\, t \geq 0. \tag{4.2}$$

Proof. By multiplying $(2.6)_1$ by ϕ_t, $(2.6)_2$ by ψ_t and $(2.6)_3$ by θ, using integration by parts and adding the results, it is easy to arrive at the proof of (4.2).

Lemma 4.2. *The second-order energy of the system (2.6) is defined by*

$$\begin{aligned} E_1(t) = \frac{1}{2}\Big[&\rho_1\|\phi_{tt}\|^2 + \rho_2\|\psi_{tt}\|^2 + b\|\psi_{xt}\|^2 + \rho_3\|\theta_t\|^2 + \kappa\|(\phi_x+\psi)_t\|^2 \\ &+ \alpha\int_0^\infty g(s)|\Psi_{xt}|^2 ds\Big]dx, \end{aligned} \tag{4.3}$$

and satisfies the following uniform bound

$$E_1(t) \leq C, \ \ \forall\, t \geq 0. \tag{4.4}$$

Proof. By taking the derivative of all the equations of the system (2.6) with respect to t, then multiplying $(2.6)_1$ by ϕ_{tt}, $(2.6)_2$ by ψ_{tt} and $(2.6)_3$ by θ_t, and integrating over $(0, L)$, we get

$$\begin{aligned} E_1'(t) = &-\widehat{\alpha}\int_0^L \theta_{tx}^2 dx + \frac{\alpha}{2}\int_0^L\int_0^\infty g'(s)|\Psi_{xt}|^2 ds dx \\ &- \beta(t)\int_0^L (v(x)-1)\,|\psi_t|^{v(x)-2}\psi_{tt}^2 dx - \beta'(t)\int_0^L |\psi_t|^{v(x)-2}\psi_t\psi_{tt}dx. \end{aligned} \tag{4.5}$$

Using the fact $g' \leq 0$, and applying Young's inequality, we have

$$E_1'(t) \leq -\beta'(t) \int_0^L \left[\frac{|\psi_t|^{v(x)-2}\psi_t}{\sqrt{\beta(t)\,(v(x)-1)\,|\psi_t|^{v(x)-2}}} \right] \left[\psi_{tt}\,\sqrt{\beta(t)\,(v(x)-1)\,|\psi_t|^{v(x)-2}} \right] dx$$

$$-\beta(t) \int_0^L (v(x)-1)\,|\psi_t|^{v(x)-2}\psi_{tt}^2 dx$$

$$\leq \int_0^L \left[\frac{-\beta'(t)|\psi_t|^{v(x)-2}\psi_t}{2\sqrt{\beta(t)\,(v(x)-1)\,|\psi_t|^{v(x)-2}}} \right]^2 + \left[\psi_{tt}\,\sqrt{\beta(t)\,(v(x)-1)\,|\psi_t|^{v(x)-2}} \right]^2 dx \tag{4.6}$$

$$-\beta(t) \int_0^L (v(x)-1)\,|\psi_t|^{v(x)-2}\psi_{tt}^2 dx$$

$$\leq \frac{1}{4} \int_0^L \left(\frac{-\beta'(t)}{\beta(t)} \right)^2 \frac{\beta(t)|\psi_t|^{v(x)}}{(v(x)-1)} dx.$$

As in the argument in [3], since $\int_0^\infty \beta(t)dt = \infty$, one can show that $\lim\limits_{t\to\infty} \frac{-\beta'(t)}{\beta(t)} \neq \infty$. In fact, if $\lim\limits_{t\to\infty} \frac{-\beta'(t)}{\beta(t)} = \infty$, then for a given $M > 0$, there exists $\varepsilon > 0$ such that $\frac{-\beta'(t)}{\beta(t)} \geq M$ for any $t > 0$. Integrating this inequality, we find $M \int_\varepsilon^t \beta(s)ds \leq -\int_\varepsilon^t \beta'(s)ds \leq \beta(\varepsilon)$. This is a contradiction with $\int_M^\infty \beta(t)dt = \infty$. Hence, we conclude that $\frac{-\beta'(t)}{\beta(t)}$ is bounded; that is, for some positive constant m_0, we have $\frac{-\beta'(t)}{\beta(t)} \leq m_0$. Thus,

$$E_1'(t) \leq \frac{m_0^2}{4(v_1-1)}\beta(t) \int_0^L |\psi_t|^{v(x} dx = \frac{m_0^2}{4(v_1-1)} \left(-E'(t)\right). \tag{4.7}$$

Integrating this inequality over $(0,t)$, we get

$$[E_1(t) - E_1(0)] \leq \left(\frac{m_0^2 \beta}{4(v_1-1)} \right) [E(0) - E(t)]. \tag{4.8}$$

Since $-E(t) \leq 0$, we get

$$E_1(t) \leq \left(\frac{m_0^2 \beta}{4(v_1-1)} \right) E(0) + E_1(0) = C, \tag{4.9}$$

where C is a positive constant independent of t. This is the desired result.

Lemma 4.3. *(Gagliardo-Nirenberg interpolation inequality). For some $c > 0$ and any $v > 2$, we have*

$$\|\psi\|_v \leq \|\psi_x\|_2^{\frac{1}{2}-\frac{1}{v}} \|\psi\|_2^{\frac{1}{2}+\frac{1}{v}}, \quad \forall\ \psi \in \mathcal{W}^{1,2}(0,L). \tag{4.10}$$

As a consequence of the above interpolation inequality (4.10), we have for $1 < v_1 < 2$,

$$
\begin{aligned}
\|\psi_t\|_{\frac{v_1}{v_1-1}} &\leq \|\psi_{xt}\|_2^{\frac{2-v_1}{2v_1}} \|\psi_t\|_2^{\frac{3v_1-2}{2v_1}} \leq (E_1'(t))^{\frac{2-v_1}{2v_1}} \|\psi_t\|_2^{\frac{3v_1-2}{2v_1}} \\
&\leq (E_1'(t))^{\frac{2-v_1}{2v_1}} \|\psi_t\|_2^{\frac{3v_1-2}{2v_1}} \leq (C)^{\frac{2-v_1}{4v_1}} \|\psi_t\|_2^{2\frac{3v_1-2}{4v_1}} \\
&\leq c\,(E(t))^{\frac{3v_1-2}{4v_1}}.
\end{aligned} \tag{4.11}
$$

Lemma 4.4. *Under the assumptions* (A_4) *and* (A_5), *we have the following estimates:*

$$
\begin{aligned}
&c\beta(t)\int_0^L \psi_t^2 dx \leq -cE'(t), \quad \text{if } v_1 = v_2 = 2, \\
&c\beta(t)\int_0^L \psi_t^2 dx \leq c\varepsilon\beta E(t) - C_\varepsilon E^{-\widehat{\kappa}}\left(E'(t)\right), \quad \text{if } v_1 \geq 2,\ v_2 > 2, \\
&c\beta(t)\int_0^L \psi_t^2 dx \leq c\varepsilon_1\beta E + c\varepsilon_2\beta E - C_{\varepsilon_1}\left(E'(t)\right)E^{-\widehat{\kappa}} - C_{\varepsilon_2}\left(E'(t)\right)E^{-\widehat{\kappa}}, \quad \text{if } 1 < v_1 < 2, \quad v_2 \neq 2,
\end{aligned} \tag{4.12}
$$

where $\widehat{\kappa} = \frac{v_2}{2} - 1 > 0$.

Proof. The proof of $(4.12)_1$ can be achieved directly by imposing $v(x) = 2$ and combining with (4.2). To prove $(4.12)_2$, we set the following partitions:

$$
\Omega_1 = \{x \in [0, L] : |\psi_t| \geq 1\} \quad \text{and} \quad \Omega_2 = \{x \in [0, L] : |\psi_t| < 1\}. \tag{4.13}
$$

Using of Hölder and Young inequalities and (4.1), we obtain for Ω_1,

$$
c\beta(t)\int_{\Omega_1} \psi_t^2 dx \leq c\beta(t)\int_0^L |\psi_t|^{v(x)}\,dx \leq -cE'(t), \tag{4.14}
$$

and for Ω_2, we get

$$
\begin{aligned}
c\beta(t)\int_{\Omega_2} \psi_t^2 \mathrm{d}x &\leq c\beta(t)\left(\int_{\Omega_2} |\psi_t|^{v_2}\,\mathrm{d}x\right)^{\frac{2}{v_2}} \\
&\leq c\beta(t)\left(\int_{\Omega_2} |\psi_t|^{v(x)}\,\mathrm{d}x\right)^{\frac{2}{v_2}} \\
&\leq \beta^{1-\frac{2}{v_2}}\left(\beta\int_0^L |\psi_t|^{v(x)}\,\mathrm{d}x\right)^{\frac{2}{v_2}} \\
&\leq c\beta^{1-\frac{2}{v_2}}\left(-E'(t)\right)^{\frac{2}{v_2}}.
\end{aligned} \tag{4.15}
$$

Note that

$$
\beta^{1-\frac{2}{v_2}}\left(-E'(t)\right)^{\frac{2}{v_2}} = \frac{\beta^{1-\frac{2}{v_2}} E^{\widehat{\kappa}} \left(-E'(t)\right)^{\frac{2}{v_2}}}{E^{\widehat{\kappa}}},
$$

where $\widehat{\kappa} = \frac{v_2}{2} - 1 > 0$. Using Young's inequality for $\zeta = \frac{v_2}{2}$ and $\zeta^* = \frac{v_2}{v_2-2}$ gives for a positive constant ε,

$$
\frac{\beta^{1-\frac{2}{v_2}} E^{\widehat{\kappa}} \left(-E'(t)\right)^{\frac{2}{v_2}}}{E^{\widehat{\kappa}}} \leq \frac{\varepsilon\beta E^{\widehat{\kappa}+1} + C_\varepsilon\left(-E'(t)\right)}{E^{\widehat{\kappa}}}. \tag{4.16}
$$

Combining (4.15) and (4.16), we get

$$\begin{aligned} c\beta(t)\int_{\Omega_2}\psi_t^2\mathrm{d}x &\leq \frac{\varepsilon\beta E^{\widehat{\kappa}+1}+C_\varepsilon\left(-E'(t)\right)}{E^{\widehat{\kappa}}} \\ &\leq c\varepsilon\beta E^{\widehat{\kappa}}+C_\varepsilon E^{\widehat{-\kappa}}\left(-E'(t)\right). \end{aligned} \tag{4.17}$$

Combining (4.14) and (4.17), the proof of $(4.12)_2$ is completed. To prove $(4.12)_3$, we consider two cases:

Case 1. If $v_2 \leq 2$, then on Ω_2, we have

$$c\beta(t)\int_{\Omega_2}\psi_t^2\mathrm{d}x \leq c\beta(t)\int_0^L|\psi_t|^{v(x)}\,dx \leq -cE'(t). \tag{4.18}$$

However, on Ω_1, using the estimate in (4.11), we have

$$\begin{aligned} &c\beta(t)\int_{\Omega_1}\psi_t^2dx = c\beta(t)\int_{\Omega_1}\psi_t\psi_t dx \leq c\beta\left(\int_{\Omega_1}|\psi_t|^{v_1}\,\mathrm{d}x\right)^{\frac{1}{v_1}}\left(\int_{\Omega_1}|\psi_t|^{\frac{v_1}{v_1-1}}\,\mathrm{d}x\right)^{\frac{v_1-1}{v_1}} \\ &\leq c\beta^{1-\frac{1}{v_1}}\left(\beta\int_0^L|\psi_t|^{v(x)}\,\mathrm{d}x\right)^{\frac{1}{v_1}}\|\psi_t\|_{\frac{v_1}{v_1-1}} \\ &\leq c\beta\left(-E'(t)\right)^{\frac{1}{v_1}}\left(E(t)\right)^{\frac{3v_1-2}{4v_1}}. \end{aligned} \tag{4.19}$$

Using Young's inequality for $\gamma = v_1$ and $\gamma^* = \frac{v_1}{v_1-1}$, we have for a positive constant ε,

$$c\beta(t)\int_{\Omega_1}\psi_t^2dx \leq \varepsilon\beta\left[E(t)\right]^{\frac{3v_1-2}{4(v_1-1)}} + C_\varepsilon\left(-E'(t)\right). \tag{4.20}$$

Combining (4.18) and (4.18), we obtain

$$c\beta(t)\int_0^L\psi_t^2dx \leq -cE'(t) + c\varepsilon\beta\left[E(t)\right]^{\frac{3v_1-2}{4(v_1-1)}} + C_\varepsilon\left(-E'(t)\right). \tag{4.21}$$

Since $\frac{3v_1-2}{4(v_1-1)} > 1$, we have

$$\int_0^L\psi_t^2dx \leq -E'(t) + c\varepsilon\beta E(t) + C_\varepsilon\left(-E'(t)\right). \tag{4.22}$$

Case 2. If $v_2 > 2$, on Ω_2, we have

$$\begin{aligned} c\beta(t)\int_{\Omega_2}\psi_t^2\mathrm{d}x &\leq c\beta(t)\left(\int_{\Omega_2}|\psi_t|^{v_2}\,\mathrm{d}x\right)^{\frac{2}{v_2}} \\ &\leq c\beta\left(\int_{\Omega_2}|\psi_t|^{v(x)}\,\mathrm{d}x\right)^{\frac{2}{v_2}} \leq c\beta^{1-\frac{2}{v_1}}\left(\beta\int_0^L|\psi_t|^{v(x)}\,\mathrm{d}x\right)^{\frac{2}{v_2}} \\ &\leq c\beta\left(-E'(t)\right)^{\frac{2}{v_2}}, \end{aligned} \tag{4.23}$$

and on Ω_1, we have

$$c\beta(t)\int_{\Omega_1}\psi_t^2dx \leq c\beta\left(-E'(t)\right)^{\frac{1}{v_1}}\left(E(t)\right)^{\frac{3v_1-2}{4v_1}}. \tag{4.24}$$

Now, a combination of (4.23) and (4.24), we find

$$c\beta(t)\int_0^L \psi_t^2 \mathrm{d}x \quad \leq \quad c\beta\left(-E'(t)\right)^{\frac{2}{\nu_2}} + c\beta\left(-E'(t)\right)^{\frac{1}{\nu_1}}\left(E(t)\right)^{\frac{3\nu_1-2}{4\nu_1}}. \tag{4.25}$$

Multiply the last inequality by $E^{\widehat{\kappa}}$ where $\widehat{\kappa} = \frac{\nu_2}{2} - 1 > 0$, we get

$$cE^{\widehat{\kappa}}\beta(t)\int_0^L \psi_t^2 \mathrm{d}x \quad \leq \quad c\beta E^{\widehat{\kappa}}\left(-E'(t)\right)^{\frac{2}{\nu_2}} + c\beta E^{\widehat{\kappa}}\left(-E'(t)\right)^{\frac{1}{\nu_1}}\left(E(t)\right)^{\frac{3\nu_1-2}{4\nu_1}}. \tag{4.26}$$

Using Young's inequality two times, to get for a positive constant $\varepsilon_1, \varepsilon_2 > 0$,

$$\begin{aligned} cE^{\widehat{\kappa}}\beta(t)\int_0^L \psi_t^2 \mathrm{d}x &\leq c\varepsilon_1\beta E^{\widehat{\kappa}+1} + C_{\varepsilon_1}\left(-E'(t)\right) + cC_{\varepsilon_2}\left(-E'(t)\right) + c\beta\varepsilon_2 E^{\frac{\nu_1}{\nu_1-1}\left(\frac{3\nu_1-2}{4\nu_1}+\widehat{\kappa}\right)} \\ &\leq c\beta\varepsilon_1 E^{\widehat{\kappa}+1} + C_{\varepsilon_1}\left(-E'(t)\right) + cC_{\varepsilon_2}\left(-E'(t)\right) + c\beta\varepsilon_2 E^{\frac{\nu_2}{2}}. \end{aligned} \tag{4.27}$$

The above estimate is obtained by using the facts that $\frac{\nu_1}{\nu_1-1}\left(\frac{3\nu_1-2}{4\nu_1}+\widehat{\kappa}\right) \geq \frac{\nu_2}{2}$ and $E(t)$ is a nonincreasing function. Therefore, we have

$$c\beta(t)\int_0^L \psi_t^2 \mathrm{d}x \quad \leq \quad c\varepsilon_1\beta E + c\varepsilon_2\beta E - C_{\varepsilon_1}\left(E'(t)\right)E^{-\widehat{\kappa}} - C_{\varepsilon_2}\left(E'(t)\right)E^{-\widehat{\kappa}}. \tag{4.28}$$

where $\widehat{\kappa} = \frac{\nu_2}{2} - 1 > 0$. This is the end of the proof.

Lemma 4.5. *Assuming that* (A_4) *and* (A_5) *hold, then for any* $\lambda > 0$*, we have*

$$\begin{aligned} -\beta(t)\int_0^L \psi|\psi_t|^{\nu(x)-2}\psi_t dx \leq (c_1 + c_e)\,\lambda \int_0^L |\psi_x|^2 dx + \beta(t)\int_{\Omega*} C_\lambda(x)|\psi_t|^{2\nu(x)-2}dx \\ + \beta(t)\int_{\Omega**} C_\lambda(x)|\psi_t|^{\nu(x)}dx. \end{aligned} \tag{4.29}$$

Proof. We consider the following partition:

$$\Omega_* = \{x \in [0, L] : \nu(\nu) < 2\}, \quad \Omega_{**} = \{x \in [0, L] : \nu(x) \geq 2\}.$$

Therefore,

$$-\beta(t)\int_0^L \psi|\psi_t|^{\nu(x)-2}\psi_t dx = -\beta(t)\int_{\Omega*} \psi|\psi_t|^{\nu(x)-2}\psi_t dx - \beta(t)\int_{\Omega**} \psi|\psi_t|^{\nu(x)-2}\psi_t dx. \tag{4.30}$$

Since β is a nonincreasing function, we find

$$-\beta(t)\int_{\Omega*} \psi|\psi_t|^{\nu(x)-2}\psi_t dx \leq \lambda c_e \int_{\Omega*} \psi_x^2 dx + \beta(t)\int_{\Omega*} C_\lambda(x)|\psi_t|^{2\nu(x)-2}dx, \tag{4.31}$$

where c_e is the embedding constant. On the other hand,

$$-\beta(t)\int_{\Omega_{**}}\psi|\psi_t|^{\nu(x)-2}\psi_t dx \le \lambda\beta(t)\int_{\Omega_{**}}|\psi_t|^{\nu(x)}dx+\beta(t)\int_{\Omega_{**}}C_\lambda(x)|\psi_t|^{\nu(x)}dx. \tag{4.32}$$

We estimate the first integral in (4.32) as follows:

$$\begin{aligned}
&\lambda\beta(t)\int_{\Omega_{**}}|\psi|^{\nu(x)}dx \le \lambda\beta(t)\int_0^L|\psi|^{\nu(x)}dx = \lambda\beta(t)\int_{\Omega_+}|\psi|^{\nu(x)}dx+\lambda\beta(t)\int_{\Omega_-}|\psi|^{\nu(x)}dx\\
&\le \lambda\beta(t)\int_{\Omega_+}|\psi|^{\nu_2}dx+\lambda\beta(t)\int_{\Omega_-}|\psi|^{\nu_1}dx \le \lambda\beta(t)\int_0^L|\psi|^{\nu_2}dx+\lambda\beta(t)\int_0^L|\psi|^{\nu_1}dx\\
&\le \lambda c_e^{\nu_1}\|\psi_x\|_2^{\nu_1}+\lambda c_e^{\nu_2}\|\psi_x\|_2^{\nu_2} \le \lambda\left(c_e^{\nu_1}\|\psi_x\|_2^{\nu_1-2}+c_e^{\nu_2}\|\psi_x\|_2^{\nu_2-2}\right)\|\psi_x\|_2^2\\
&\le \lambda\left(c_e^{\nu_1}\left(\frac{2}{b}E(0)\right)^{\nu_1-2}+c_e^{\nu_2}\left(\frac{2}{b}E(0)\right)^{\nu_2-2}\right)\|\psi_x\|_2^2 \le c_1\lambda\|\psi_x\|_2^2,
\end{aligned} \tag{4.33}$$

where

$$\Omega_+=\{x\in[0,L]:|\psi(x,t)|\ge 1\},\quad \Omega_-=\{x\in[0,L]:|\psi(x,t)|<1,\}$$

and

$$c_1=\left(c_e^{\nu_1}\left(\frac{2}{b}E(0)\right)^{\nu_1-2}+c_e^{m_2}\left(\frac{2}{b}E(0)\right)^{\nu_2-2}\right). \tag{4.34}$$

Therefore, (4.32) becomes

$$-\beta(t)\int_{\Omega_{**}}\psi|\psi_t|^{\nu(x)-2}\psi_t dx \le c_1\lambda\int_0^L|\psi_x|^2dx+\beta(t)\int_{\Omega_{**}}C_\lambda(x)|\psi_t|^{\nu(x)}dx. \tag{4.35}$$

Combining (4.31) and (4.35), the proof is finished.

Lemma 4.6. The functional

$$F_1(t):=\rho_1\rho_3\int_0^L\phi_t\int_0^x\theta(y)dydx,$$

satisfies

$$F_1'(t)\le -\frac{\rho_1\gamma_1}{2}\int_0^L\phi_t^2dx+\varepsilon_1\int_0^L(\phi_x+\psi)^2\,dx+\widehat{c}\int_0^L\theta_x^2dx+\alpha\widehat{c}\int_0^L\int_0^\infty g(s)|\Psi_x|^2dsdx. \tag{4.36}$$

Proof. The derivative of $F_1(t)$ with imposing $(2.6)_3$, is given by

$$F_1'(t)=\rho_3\int_0^L\rho_1\phi_{tt}\int_0^x\theta(y)dydx+\rho_1\int_0^L\phi_t\left[\widehat{\alpha}\theta_x+\alpha\int_0^\infty g(s)\Psi_x(x,t,s)ds-\gamma\phi_t\right]dx.$$

Using $(2.6)_1$ and integrating by parts, we arrive at

$$\begin{aligned}
F_1'(t)=&-\rho_3\kappa\int_0^L\theta\,(\phi_x+\psi)\,dx+\rho_3\gamma\int_0^L\theta^2dx+\rho_1\widehat{\alpha}\int_0^L\phi_t\theta_xdx\\
&-\rho_1\gamma\int_0^L\phi_t^2dx+\rho_1\alpha\int_0^L\phi_t\int_0^\infty g(s)\Psi_x(x,t,s)dsdx.
\end{aligned} \tag{4.37}$$

Apply Young's and Cauchy-Schwarz inequalities, we get for positive constant $\varepsilon, \varepsilon_1, \varepsilon_2$

$$\begin{aligned} &\rho_1\alpha \int_0^L \phi_t \int_0^\infty g(s)\eta_x(x,t,s)dsdx \\ &\le \varepsilon_2 \int_0^L \phi_t^2 dx + \frac{\alpha^2\rho_1^2}{4\varepsilon_2}\int_0^1 \left(\int_0^\infty \sqrt{g(s)}\sqrt{g(s)}\,\Psi_x ds\right)^2 dx \\ &\le \varepsilon_2 \int_0^L \phi_t^2 dx + \frac{\alpha^2\rho_1^2}{4\varepsilon}\left(\int_0^\infty g(s)ds\right)\int_0^L \int_0^\infty g(s)|\Psi_x|^2 dsdx \\ &\le \varepsilon_2 \int_0^L \phi_t^2 dx + \frac{g_0\alpha^2\rho_1^2}{4\varepsilon}\int_0^L \int_0^\infty g(s)|\Psi_x|^2 dsdx, \end{aligned} \tag{4.38}$$

and

$$\begin{aligned} -\rho_3\kappa \int_0^L \theta\,(\phi_x+\psi)\,dx &\le \varepsilon_1 \int_0^L (\phi_x+\psi)^2\,dx + \frac{\rho_3^2\kappa^2}{4\varepsilon_1}\int_0^L \theta^2 dx, \\ \rho_1\widehat{\alpha}\int_0^L \phi_t\theta_x dx &\le \varepsilon_2 \int_0^L \phi_t^2 dx + \frac{\widehat{\alpha}^2\rho_1^2}{4\varepsilon_2}\int_0^L \theta_x^2 dx. \end{aligned} \tag{4.39}$$

Choosing $\varepsilon_2 = \frac{\rho_1\gamma}{4}$ and using Poincaré's inequality, the estimate (4.36) is established.

Lemma 4.7. The functional

$$F_2(t) := \rho_2 \int_0^L \psi_t\psi dx,$$

satisfies the estimate

$$\begin{aligned} F_2'(t) \le -\frac{b}{4}\int_0^L \psi_x^2 dx + \rho_2 \int_0^L \psi_t^2 + \widehat{c}\int_0^L (\phi_x+\psi)^2 dx \\ + c\beta(t)\int_{\Omega_*} |\psi_t|^{2\nu(x)-2}dx + c\beta(t)\int_{\Omega_{**}} |\psi_t|^{\nu(x)}dx. \end{aligned} \tag{4.40}$$

Proof. We have

$$F_2'(t) = \rho_2 \int_0^L \psi_t^2 dx + \rho_2 \int_0^L \psi\psi_{tt}dx. \tag{4.41}$$

From Eq $(2.6)_2$, we find

$$\begin{aligned} \rho_2 \int_0^L \psi\psi_{tt}dx &= \int_0^L \psi\left[b\psi_{xx} - \kappa(\phi_x+\psi) - \beta(t)|\psi_t|^{\nu(x)-2}\psi_t\right]dx \\ &= -b\int_0^L \psi_x^2 dx - \kappa\int_0^L \psi(\phi_x+\psi)dx - \beta(t)\int_0^L \psi|\psi_t|^{\nu(x)-2}\psi_t dx. \end{aligned} \tag{4.42}$$

Imposing this in (4.41), we get

$$F_2'(t) = \rho_2 \int_0^L \psi_t^2 dx - b\int_0^L \psi_x^2 dx - \kappa\int_0^L \psi(\phi_x+\psi)dx - \beta(t)\int_0^L \psi|\psi_t|^{\nu(x)-2}\psi_t dx. \tag{4.43}$$

Young's inequality, we get for ε

$$-\kappa \int_0^L \psi(\phi_x + \psi)dx \leq \varepsilon \int_0^L \psi^2 dx + \frac{\kappa^2}{4\varepsilon} \int_0^L (\phi_x + \psi)^2 dx. \tag{4.44}$$

Now, using the estimate in (4.29), taking $\lambda = \frac{b}{2(c_e+c_1)}$ and choosing $\varepsilon = \frac{b}{4c_e}$, the proof is completed.

Lemma 4.8. The functional

$$F_3(t) := -\rho_1 \int_0^L (\phi_x + \psi) \int_0^x \phi_t(y) dy dx,$$

satisfies

$$F_1'(t) \leq -\frac{\kappa}{2} \int_0^L (\phi_x + \psi)^2 \, dx + \rho_1(1 + \varepsilon_3) \int_0^L \phi_t^2 dx + \frac{c}{\varepsilon_3} \int_0^L \psi_t^2 dx + \widehat{c} \int_0^L \theta_x^2 dx. \tag{4.45}$$

Proof. Differentiating $F_3(t)$, and imposing $(2.6)_1$ gives

$$F_3'(t) = -\rho_1 \int_0^L (\phi_x + \psi)_t \int_0^x \phi_t(y) dy dx - \int_0^L (\phi_x + \psi) \left[\kappa (\phi_x + \psi) - \gamma\theta\right] dx.$$

Simplifying, this equation becomes

$$F_3'(t) = -\rho_1 \int_0^L \phi_{xt} \int_0^x \phi_t(y) dy dx - \rho_1 \int_0^L \psi_t \int_0^x \phi_t(y) dy dx - \kappa \int_0^L (\phi_x + \psi)^2 \, dx$$
$$+ \gamma \int_0^L (\phi_x + \psi) \, \theta dx. \tag{4.46}$$

Integration by parts leads to

$$F_3'(t) = \rho_1 \int_0^L \phi_t^2 dx - \rho_1 \int_0^L \psi_t \int_0^x \phi_t(y) dy dx - \kappa \int_0^L (\phi_x + \psi)^2 \, dx + \gamma \int_0^L (\phi_x + \psi) \, \theta dx. \tag{4.47}$$

Using Young's and Cauchy-Schwarz inequalities, we get for positive constant $\varepsilon_2, \varepsilon_3$

$$\begin{aligned} -\rho_1 \int_0^L \psi_t \int_0^x \phi_t(y) dy dx &\leq \varepsilon_3 \int_0^L \phi_t^2 dx + \frac{c}{\varepsilon_3} \int_0^L \psi_t^2 dx, \\ \gamma \int_0^L (\phi_x + \psi) \, \theta dx &\leq \varepsilon_2 \int_0^L (\phi_x + \psi)^2 \, dx + \frac{c}{\varepsilon_2} \int_0^L \theta^2 dx. \end{aligned} \tag{4.48}$$

Combining (4.47) and (4.48), choosing $\varepsilon_2 = \frac{\kappa}{2}$, and using Poincaré's inequality, we get the estimate (4.45).

Lemma 4.9. *Assume that $(A_1 - A_5)$ hold. Then there exist strictly positive constants μ, μ_1 such that the functional*

$$\mathcal{H}(t) = \mu E(t) + \mu_1 F_1(t) + F_2(t) + \mu_3 F_3(t), \tag{4.49}$$

satisfies, for all $t \in \mathbb{R}_+$,

$$\mathcal{H}(t) \sim E(t), \tag{4.50}$$

and

$$\mathcal{H}'(t) \leq -cE(t) + c\int_0^L \psi_t^2 dx + \alpha\widehat{c}\int_0^L\int_0^\infty g(s)|\Psi_x|^2 ds dx + c\beta(t)\int_{\Omega*}|\psi_t|^{2v(x)-2}dx. \tag{4.51}$$

Proof. Using the previous functionals and setting $\varepsilon_1\mu_1 = 1$ and $\varepsilon_3\mu_3 = 1$, we end up with

$$\begin{aligned}\mathcal{H}'(t) \leq& -(\mu - \widehat{c}\mu_1 - \widehat{c}\mu_3)\widehat{\alpha}\int_0^L \theta_x^2 dx - \frac{b}{4}\int_0^L \psi_x^2 dx \\ &- \left[\frac{\gamma}{2}\mu_1 - \widehat{c}\right]\rho_1\int_0^L \phi_t^2 dx - \left[\frac{\kappa}{2}\mu_3 - \widehat{c} - 1\right]\int_0^L (\phi_x+\psi)^2 dx \\ &+ \rho_2\widehat{c}\mu_3^2\int_0^L \psi_t^2 dx + \alpha\mu_1\widehat{c}\int_0^L\int_0^\infty g(s)|\Psi_x|^2 ds dx \\ &- [\mu - c\mu_1]\beta(t)\int_0^L C_\lambda(x)|\psi_t|^{v(x)}dx + c\mu_1\beta(t)\int_{\Omega*}|\psi_t|^{2v(x)-2}dx.\end{aligned} \tag{4.52}$$

Choosing μ_1 so that $\frac{\gamma}{2}\mu_1 - \widehat{c} > 0$, and μ_3 so that $\frac{\kappa}{2}\mu_3 - \widehat{c} > 1$ then we find that for some positive constant m,

$$\begin{aligned}\mathcal{H}'(t) \leq& -(\mu - \widehat{c}\mu_1 - \widehat{c})\widehat{\alpha}\int_0^L \theta_x^2 dx - m\int_0^L \left(\phi_t^2 + \psi_x^2 + (\phi_x+\psi)^2\right)dx \\ &+ c\int_0^L \psi_t^2 dx + c\beta(t)\int_{\Omega*}|\psi_t|^{2v(x)-2}dx + \alpha\widehat{c}\int_0^L\int_0^\infty g(s)|\Psi_x|^2 ds dx.\end{aligned} \tag{4.53}$$

On the other hand, exploiting Young's and Poincaré inequalities, keeping in mind the energy functional (4.1) and the fact that $\phi_x^2 \leq 2(\phi_x+\psi)^2 + 2\psi^2$, we obtain

$$|\mathcal{H}(t) - \mu E(t)| \leq cE(t). \tag{4.54}$$

Taking μ sufficiently large so that $\mu - \widehat{c}\mu_1 - \widehat{c} > 0$ and $\mu - c > 0$ yields the equivalence (4.50) and the estimate (4.51).

Lemma 4.10. *If the assumptions* $(A_1 - A_5)$ *hold, then we have the following estimates:*

$$c\beta(t)\int_{\Omega_*}|\psi_t|^{2v(x)-2}dx = \begin{cases} 0, & v_1 \geq 2; \\ c\varepsilon E(t) + c\beta(t)E^{\frac{2v_1-2}{2-v_1}}\int_{\Omega_*} C_\varepsilon|\psi_t|^{v(x)}, & 1 < v_1 < 2. \end{cases} \tag{4.55}$$

Proof. The first estimate is clear because if $v_1 \geq 2$, then $meas(\Omega_*) = 0$. However, if $1 < v_1 < 2$, then by using Young's inequality, we have

$$E^{\frac{2-v_1}{2v_1-2}}|\psi_t|^{2v(x)-2} \leq \varepsilon E^{\frac{v(x)}{2-v(x)}\frac{2-v_1}{2v_1-2}} + C_\varepsilon|\psi_t|^{v(x)}. \tag{4.56}$$

Since $\frac{v(x)}{2-v(x)}\frac{2-v_1}{2v_1-2} \geq \frac{2-v_1}{2v_1-2} + 1$, and E is nonincreasing, we obtain

$$E^{\frac{2-v_1}{2v_1-2}}|\psi_t|^{2v(x)-2} \leq c\varepsilon E^{\frac{2-v_1}{2v_1-2}+1} + C_\varepsilon|\psi_t|^{v(x)}. \tag{4.57}$$

Multiplying by $\frac{\beta(t)}{E^{\frac{2-\nu_1}{2\nu_1-2}}}$ and integrating with respect to x,

$$c\beta(t)\int_{\Omega_*}|\psi_t|^{2\nu(x)-2}dx \quad \leq \quad c\varepsilon E(t)+c\beta(t)E^{\frac{2\nu_1-2}{2-\nu_1}}\int_{\Omega_*}C_\varepsilon|\psi_t|^{\nu(x)}. \tag{4.58}$$

This finishes the proof.

5. The proofs of the stability theorems

In this section, we prove our main decay theorems stated in Section 3.

5.1. The proof of Theorem 3.1

Proof. To prove the energy decay in (3.1), we multiply (4.51) by $\beta(t)$ to get

$$\begin{aligned}\beta\mathcal{H}'(t) \leq -c\beta E(t)+c\beta\int_0^L\psi_t^2dx+\alpha\beta\widehat{c}\int_0^L\int_0^\infty g(s)|\Psi_x|^2dsdx\\ +c\beta(t)\int_{\Omega*}|\psi_t|^{2\nu(x)-2}dx.\end{aligned} \tag{5.1}$$

Combining (4.2) with (5.1), and using the estimate of $c\beta(t)\int_0^L\psi_t^2dx$ in $(4.12)_1$, then (5.1) becomes

$$\mathcal{L}'(t)\leq -c\beta E(t)+\widehat{c}\alpha\beta\int_0^t g(s)|\Psi_x|^2ds+\widehat{c}\alpha\beta\int_t^\infty g(s)|\Psi_x|^2ds, \tag{5.2}$$

where $\mathcal{L}=\beta\mathcal{H}+c\beta E\sim E$. Using (4.1) and (4.2) and the fact that ξ and g are nonincreasing, we find that

$$\begin{aligned}\widehat{c}\alpha\beta\xi(t)\int_0^t g(s)|\Psi_x|^2ds &\leq -\widehat{c}\alpha\beta\int_0^t g'(s)|\Psi_x|^2ds\\ &\leq -c\alpha\beta E'(t), \quad \forall t\in\mathbb{R}_+.\end{aligned} \tag{5.3}$$

Multiplying (5.2) by $\xi(t)$ and combining with (5.3) and the bound in the hypothesis (A_3), we get

$$\mathcal{F}'(t)\leq -c\xi(t)\beta E(t)+\widehat{c}\Delta\alpha\xi(t)\int_t^\infty g(s)ds. \tag{5.4}$$

where $\mathcal{F}=\xi\mathcal{L}+c\alpha\beta\xi\sim E$. Let $h(t)=\widehat{c}(\xi\beta)(t)\Delta\int_t^{+\infty}g(s)ds$. Then, (5.4) becomes

$$\mathcal{F}'(t)\leq -\gamma_0(\beta\xi)\mathcal{F}(t)+\alpha h(t), \tag{5.5}$$

for some $\gamma_0>0$. This last inequality remains true for any $\delta_0\in(0,\gamma_0]$; that is

$$\mathcal{F}'(t)\leq -\delta_0(\beta\xi)(t)\mathcal{F}(t)+\alpha h(t), \quad \forall t\in\mathbb{R}_+.$$

Therefore, direct integration leads to

$$\mathcal{F}(T)\leq e^{-\delta_0\int_0^T(\beta\xi)(s)ds}\left(\mathcal{F}(0)+\alpha\int_0^T e^{\delta_0\int_0^t(\beta\xi)(s)ds}h(t)dt\right),$$

and the fact that $\mathcal{F} \sim E$ gives

$$E(T) \leq \gamma_1 e^{-\delta_0 \int_0^T (\beta\xi)(s)ds} \left(\mathcal{F}(0) + \alpha \int_0^T e^{\delta_0 \int_0^t (\beta\xi)(s)ds} h(t)dt \right). \tag{5.6}$$

We note that

$$e^{\delta_0 \int_0^t (\beta\xi)(s)ds} h(t) = \frac{\Delta\widehat{c}}{\delta_0} \left(e^{\delta_0 \int_0^t (\beta\xi)(s)ds} \right)' \int_t^{+\infty} g(s)ds, \quad \forall t \in \mathbb{R}_+.$$

Then, integration by parts gives

$$\begin{aligned}
&\int_0^T e^{\delta_0 \int_0^t (\beta\xi)(s)ds} h(t)dt \\
&= \frac{\Delta\widehat{c}}{\delta_0} \left(e^{\delta_0 \int_0^T (\beta\xi)(s)ds} \int_T^{+\infty} g(s)ds - \int_0^{+\infty} g(s)ds + \int_0^T e^{\delta_0 \int_0^t (\beta\xi)(s)ds} g(t)dt \right).
\end{aligned}$$

Combining with (5.6), we have

$$E(T) \leq \gamma_1 \left(\mathcal{F}(0) + \frac{\alpha\Delta\widehat{c}}{\delta_0} \int_0^T e^{\delta_0 \int_0^t (\beta\xi)(s)ds} g(t)dt \right) e^{-\delta_0 \int_0^T \xi(s)ds} + \frac{\alpha\Delta\widehat{c}}{\delta_0} \int_T^{+\infty} g(s)ds. \tag{5.7}$$

We note that

$$\left(e^{\int_0^t \xi(s)ds} g(t) \right)' \leq 0, \quad \forall t \in \mathbb{R}_+.$$

We have $e^{\int_0^t (\beta\xi)(s)ds} g(t) \leq g(0)$ and

$$\int_0^T e^{\delta_0 \int_0^t (\beta\xi)(s)ds} g(t)dt \leq (g(0))^{\delta_0} \int_0^T (g(t))^{1-\delta_0} dt. \tag{5.8}$$

Finally, combining (5.7) and (5.8), we obtain

$$E(t) \leq \delta_1 \left(1 + \int_0^t (g(s))^{1-\delta_0} ds \right) e^{-\delta_0 \int_0^t (\beta\xi)(s)ds} + \frac{\alpha\Delta\widehat{c}}{\delta_0} \int_t^{+\infty} g(s)ds, \tag{5.9}$$

where $\delta_1 = \max\{\gamma_1 \mathcal{F}(0), \frac{\alpha\Delta\widehat{c}}{\delta_0} (g(0))^{\delta_0}\}$. Thus, the proof of (3.1) is completed.

5.2. The proof of Theorem 3.2

Proof. To prove (3.4), we first multiply (4.51) by $\beta(t)$, to get

$$\begin{aligned}
\beta\mathcal{H}'(t) \leq &-c\beta E(t) + c\beta \int_0^L \psi_t^2 dx + \alpha\beta\widehat{c} \int_0^L \int_0^\infty g(s)|\Psi_x|^2 dsdx \\
&+ c\beta(t) \int_{\Omega_*} |\psi_t|^{2\nu(x)-2} dx.
\end{aligned} \tag{5.10}$$

Combining (5.10) and (4.55), and choosing ε small enough, we arrive at

$$\beta\mathcal{H}'(t) \leq -c\beta E(t) + c\beta \int_0^L \psi_t^2 dx + \alpha\beta\widehat{c} \int_0^L \int_0^\infty g(s)|\Psi_x|^2 dsdx - c\beta E'(t) E^{\frac{2\nu_1-2}{2-\nu_1}}. \tag{5.11}$$

Using the estimate of $c\beta \int_0^L \psi_t^2 dx$ in $(4.12)_3$, we have

$$\begin{aligned} \beta\mathcal{H}'(t) \le& -c\beta E(t)) + c\varepsilon_1\beta E + c\varepsilon_2\beta E - C_{\varepsilon_1}E'(t)E^{-\kappa} - C_{\varepsilon_2}E'(t)E^{-\kappa} - c\beta E'(t)E^{\frac{2\nu_1-2}{2-\nu_1}} \\ &+ \widehat{c}\alpha\beta \int_0^t g(s)ds + \Delta\alpha\beta\widehat{c}\int_t^\infty g(s)ds. \end{aligned} \tag{5.12}$$

Multiplying (5.12) by $E^{\widetilde{\kappa}}(t)$, where $\widetilde{\kappa} = \max\left\{\frac{2-\nu_1}{2\nu_1-2}, \frac{\nu_2}{2} - 1\right\}$, we get

$$\begin{aligned} E^{\widetilde{\kappa}}(t)\beta\mathcal{H}'(t) \le& -c\beta E^{\widetilde{\kappa}+1}(t) + c\varepsilon_1 E^{\widetilde{\kappa}+1}(t) + \varepsilon_2\beta E^{\widetilde{\kappa}+1}(t) - C_{\varepsilon_1}E'(t) - C_{\varepsilon_2}E'(t) \\ &+ \widehat{c}\alpha\beta E^{\widetilde{\kappa}}(t)\int_0^t g(s)ds + \Delta\alpha\beta\widehat{c}E^{\widetilde{\kappa}}(t)\int_t^\infty g(s)ds, \end{aligned}$$

and choosing $\varepsilon_1, \varepsilon_2$ small enough, then we get

$$\mathcal{Y}'(t) \le -c\beta E^{\widetilde{\kappa}+1}(t) + c\beta E^{\widetilde{\kappa}}(t)\int_0^t g(s)ds + c\beta E^{\widetilde{\kappa}}(t)\int_t^\infty g(s)ds, \tag{5.13}$$

where $\mathcal{Y} = E^{\widetilde{\kappa}}\beta\mathcal{H} + c\beta E \sim E$. Multiplying (5.35) by $\beta^{\widetilde{\kappa}}\xi^{\widetilde{\kappa}+1}(t)$, using (4.2), and using that ξE is nonincreasing, we get

$$\mathcal{F}'(t) \le -c(\xi\beta)^{\widetilde{\kappa}+1}(t)\mathcal{F}^{\widetilde{\kappa}+1}(t) + (\xi\beta E)^{\widetilde{\kappa}}(t)h(t), \tag{5.14}$$

where $h(t) = \Delta\alpha\widehat{c}(\xi\beta)(t)\int_t^\infty g(s)ds$ and $\mathcal{F} = \xi\mathcal{Y} + c\beta E \sim E$. Use of Young's inequality, with $q = \widetilde{\kappa} + 1$ and $q^* = \frac{\widetilde{\kappa}+1}{\widetilde{\kappa}}$, gives for some positive constant c_1 and c_2,

$$\mathcal{F}'(t) \le -c_1(\xi\beta)^{\widetilde{\kappa}+1}(t)\mathcal{F}^{\widetilde{\kappa}+1}(t) + c_2 h^{\widetilde{\kappa}+1}(t). \tag{5.15}$$

Multiply both sides of (5.35) by $(\xi\beta)^\eta$, $\eta > 1$, thus, we get

$$\xi^\eta\mathcal{F}'(t) \le -c_1(\xi\beta)^{\widetilde{\kappa}+1+\eta}(t)\mathcal{F}^{\widetilde{\kappa}+1}(t) + c_2(\xi\beta)^\eta h^{\widetilde{\kappa}+1}(t). \tag{5.16}$$

Let $\chi := \xi\beta > 0$, which is nonincreasing, we find that

$$(\chi^\eta\mathcal{F}(t))' \le -c_1\chi^{\widetilde{\kappa}+1+\eta}(t)\mathcal{F}^{\widetilde{\kappa}+1}(t) + c_2\chi^\eta h^{\widetilde{\kappa}+1}(t). \tag{5.17}$$

Setting $\varphi = \chi^\eta\mathcal{F}$ and noting $\eta = \frac{\widetilde{\kappa}+1}{\widetilde{\kappa}}$, one finds that

$$\varphi'(t) \le -c_1\varphi^{\widetilde{\kappa}+1}(t) + c_2\chi^\eta(t)h^{\widetilde{\kappa}+1}(t). \tag{5.18}$$

Let

$$H(t) := \varphi(t) - \Lambda(t); \quad \text{where} \quad \Lambda(t) = c_2(1+t)^{\frac{-1}{\widetilde{\kappa}}}\int_0^t \chi^\eta(s)h^{\widetilde{\kappa}+1}(s)(1+s)^{\frac{1}{\widetilde{\kappa}}}ds. \tag{5.19}$$

From the definition of Λ, we have

$$c_2\chi^\eta(t)h^{\widetilde{\kappa}+1}(t) = \Lambda'(t) + \frac{c_2}{\widetilde{\kappa}}(1+t)^{\frac{-1}{\widetilde{\kappa}}-1}\int_0^t \chi^\eta(s)h^{\widetilde{\kappa}+1}(s)(1+s)^{\frac{1}{\widetilde{\kappa}}}ds, \tag{5.20}$$

since $\chi^\eta(s)h^{\widetilde{\kappa}+1}(1+s)^{\frac{1}{\widetilde{\kappa}}} > 0$, then we have for all $t \ge t_0 > 0$

$$\nu := \int_0^{t_0} \chi^\eta(s)h^{\widetilde{\kappa}+1}(s)(1+s)^{\frac{1}{\widetilde{\kappa}}}ds \le \int_0^t \chi^\eta(s)h^{\widetilde{\kappa}+1}(s)(1+s)^{\frac{1}{\widetilde{\kappa}}}ds,$$

and then

$$\frac{\int_0^t \chi^{\eta}(s)h^{\widetilde{\kappa}+1}(s)(1+s)^{\frac{1}{\kappa}}ds}{\nu} \geq 1, \quad \forall t \geq t_0.$$

Thus, (5.20) yields, $\forall t \geq t_0$,

$$c_2\chi^{\eta}(t)h^{\widetilde{\kappa}+1}(t) \leq \Lambda'(t) + \frac{1}{\widetilde{\kappa}}c_2^{\widetilde{\kappa}}\nu^{\widetilde{\kappa}}c_2^{\widetilde{\kappa}+1}\left[(1+t)^{\frac{-1}{\kappa}}\right]^{\widetilde{\kappa}+1}\left[\int_0^t \chi^{\eta}(s)h^{\widetilde{\kappa}+1}(s)(1+s)^{\frac{1}{\kappa}}ds\right]^{\widetilde{\kappa}+1}, \tag{5.21}$$

we can choose c_2 large enough so that $\frac{1}{\kappa}c_2^{\widetilde{\kappa}}\nu^{\widetilde{\kappa}} \leq c_1$, and then we get

$$c_2\chi^{\eta}(t)h^{\widetilde{\kappa}+1}(t) \leq \Lambda'(t) + c_1\Lambda^{\widetilde{\kappa}+1}, \quad \forall t \geq t_0. \tag{5.22}$$

Now using (5.22) and the definition of H, we get, $\forall t \geq t_0$,

$$\begin{aligned} H'(t) = \varphi'(t) - \Lambda'(t) &\leq -c_1\varphi^{\widetilde{\kappa}+1}(t) + c_2\chi^{\eta}(t)h^{\widetilde{\kappa}+1}(t) - \Lambda'(t) \\ &\leq -c_1\left[(H+\Lambda)^{\widetilde{\kappa}+1}(t)\right] + c_2\chi^{\eta}(t)h^{\widetilde{\kappa}+1}(t) - \Lambda'(t). \end{aligned} \tag{5.23}$$

Since $H(0) > 0$. Then there exists $t_1 > 0$ such that $H(t) > 0, \forall t \in [0, t_1)$. Hence,

$$\begin{aligned} H'(t) &\leq -c_1\left[H^{\widetilde{\kappa}+1}(t) + \Lambda^{\widetilde{\kappa}+1}(t)\right] + c_2\chi^{\eta}(t)h^{\widetilde{\kappa}+1}(t) - \Lambda'(t), \quad \forall\, t \in [t_0, t_1). \\ &\leq -c_1\left[H^{\widetilde{\kappa}+1}(t) + \Lambda^{\widetilde{\kappa}+1}(t) - \frac{c_2}{c_1}\chi^{\eta}(t)h^{\widetilde{\kappa}+1}(t) + \frac{1}{c_1}\Lambda'(t)\right]. \end{aligned} \tag{5.24}$$

Thus,

$$H'(t) \leq -c_1H^{\widetilde{\kappa}+1}(t), \quad \forall\, t \in [t_0, t_1). \tag{5.25}$$

Integrate over (t_0, t), we have

$$H(t) \leq \frac{c}{(t-t_0)^{\frac{1}{\kappa}}}, \quad \forall\, t \in [t_0, t_1). \tag{5.26}$$

If $t_1 = +\infty$, then using the definitions of H and Λ, we have, for t large enough,

$$\varphi(t) \leq C(1+t)^{\frac{-1}{\kappa}}\left[1 + \int_0^t \chi^{\eta}(s)h^{\widetilde{\kappa}+1}(s)(1+s)^{\frac{1}{\kappa}}ds\right]. \tag{5.27}$$

If $t_1 < +\infty$, then there exists $t_2 > t_1$ such that $H(t) \leq 0, \forall\, t_1 \leq t < t_2$. Hence, (5.19) yields $\varphi(t) \leq \Lambda(t), \forall\, t_1 \leq t < t_2$. Therefore, we get (5.27). If $t_2 = +\infty$, we are done. Otherwise, there exists $t_3 > t_2$ such that $H(t_2) = 0$ and $H(t) > 0, \forall\, t_2 < t < t_3$. Consequently, we obtain (5.27) by repeating the steps (5.24)–(5.26) on $[t_2, t_3)$. Therefore, (5.27) remains valid for all $t \geq t_0$. Multiplying (5.27) by $\chi^{-\eta}$ and recalling the definition of φ then we have, for $\eta = \frac{\widetilde{\kappa}+1}{\widetilde{\kappa}}$, the following

$$\mathcal{F}(t) \leq C(1+t)^{\frac{-1}{\kappa}}\chi^{-\frac{\widetilde{\kappa}+1}{\widetilde{\kappa}}}\left[1 + \int_0^t \chi^{\frac{\widetilde{\kappa}+1}{\widetilde{\kappa}}}(s)h^{\widetilde{\kappa}+1}(s)(1+s)^{\frac{1}{\kappa}}ds\right]. \tag{5.28}$$

Using the fact $\mathcal{F} \sim E$, we have two cases:

If $\widetilde{\kappa} = \frac{2-\nu_1}{2\nu_1-2} > 0$, then $\widetilde{\kappa} + 1 = \frac{\nu_1}{2\nu_1-2}$ and $\frac{\widetilde{\kappa}+1}{\widetilde{\kappa}} = \frac{\nu_1}{2-\nu_1}$, we get

$$E(t) \leq C(1+t)^{-\frac{2\nu_1-2}{2-\nu_1}}\chi^{-\frac{\nu_1}{2-\nu_1}}\left(1 + \int_0^t (1+s)^{\frac{2\nu_1-2}{2-\nu_1}}\chi^{\frac{\nu_1}{2-\nu_1}}(s)h^{\frac{1}{\nu_1-1}}ds\right). \tag{5.29}$$

If $\widetilde{\kappa} = \frac{\nu_2}{2} - 1 > 0$, we have

$$E(t) \leq C(1+t)^{-\frac{2}{\nu_2-2}}\chi^{-\frac{\nu_2}{\nu_2-2}}\left(1 + \int_0^t (1+s)^{\frac{2}{\nu_2-2}}\chi^{\frac{\nu_2}{\nu_2-2}}(s)h^{\frac{\nu_2}{2}}ds\right). \tag{5.30}$$

This establishes (3.4).

5.3. The proof of Theorem (3.3)

Proof. Multiplying (4.51) by β, recalling the estimate of $c\beta\int_0^L \psi_t^2 dx$ in $(4.12)_2$, we get

$$\beta\mathcal{H}'(t) \leq -c\beta E(t) + c\varepsilon\beta E(t) - C_\varepsilon\left(E'(t)\right)E^{-\widehat{\kappa}} + \widehat{c}\alpha\beta\int_0^t g(s)|\Psi_x|_2^2 ds + \widehat{c}\alpha\beta\int_t^\infty g(s)|\Psi_x|^2 dx. \tag{5.31}$$

Multiplying (5.31) by $E^{\widehat{\kappa}}$ where $\widehat{\kappa} = \frac{\nu_2}{2} - 1 > 0$, we get

$$\beta E^{\widehat{\kappa}}\mathcal{H}'(t) \leq -c\beta E^{\widehat{\kappa}+1}(t) + c\varepsilon E^{\widehat{\kappa}+1} - C_\varepsilon\left(-E'(t)\right) + \widehat{c}\beta\alpha E^{\widehat{\kappa}}\int_0^t g(s)|\Psi_x|_2^2 ds + \widehat{c}\beta\alpha E^{\widehat{\kappa}}\int_t^\infty g(s)|\Psi_x|^2 dx.$$

Choosing ε sufficiently small, we get

$$\mathcal{L}(t) \leq -c\beta E^{\widehat{\kappa}+1}(t) + \widehat{c}\alpha\beta E^{\widehat{\kappa}}\int_0^t g(s)|\Psi_x|_2^2 ds + \widehat{c}\alpha\beta E^{\widehat{\kappa}}\int_t^\infty g(s)|\Psi_x|^2 dx., \qquad \forall t \geq 0, \tag{5.32}$$

where $\mathcal{L} = \beta E^{\widehat{\kappa}}\mathcal{H} + c\beta E \sim E$. Multiplying (5.32) by $(\xi\beta)^{\widehat{\kappa}}\xi$, combining with (4.2) and (5.3), and imposing Assumption (A_3), we get

$$\mathcal{Z}'(t) \leq -c(\xi\beta)^{\widehat{\kappa}+1}E^{\widehat{\kappa}+1}(t) + \Delta\widehat{c}\alpha\beta(\xi\beta E)^{\widehat{\kappa}}\xi\int_t^\infty g(s)ds. \tag{5.33}$$

where $\mathcal{Z} = \xi\beta\mathcal{L} + 2\alpha\beta\widehat{c}E \sim E$. Setting $h(t) = \Delta\alpha\widehat{c}(\xi\beta)(t)\int_t^{+\infty} g(s)ds$.

Then, (5.33) becomes

$$\mathcal{Z}'(t) \leq -c(\xi\beta)^{\widehat{\kappa}+1}(t)E^{\widehat{\kappa}+1}(t) + c(\xi\beta E)^{\widehat{\kappa}}(t)h(t). \tag{5.34}$$

Put $\chi := \xi\beta$, which is a positive nonincreasing, then we get

$$\mathcal{Z}'(t) \leq -c\chi^{\widehat{\kappa}+1}(t)\mathcal{Z}^{\widehat{\kappa}+1}(t) + c\,(\chi E)^{\widehat{\kappa}}\,h(t). \tag{5.35}$$

Using Young's inequality, with $q = \widehat{\kappa} + 1$ and $q^* = \frac{\widehat{\kappa}+1}{\widehat{\kappa}}$, we get for some positive constant c_1 and c_2,

$$\mathcal{Z}'(t) \leq -c_1\chi^{\widehat{\kappa}+1}(t)\mathcal{Z}^{\widehat{\kappa}+1}(t) + c_2 h^{\widehat{\kappa}+1}(t). \tag{5.36}$$

Repeating the same steps in the last part of the proof of Theorem (3.2), replacing $\widetilde{\kappa}$ by $\widehat{\kappa} = \frac{\nu_2-2}{2}$, we complete the proof of the decay (3.8).

6. Conclusions

In this work, we considered a nonlinear Timoshenko system with Coleman-Gurtin's heat flux. We proved that the system is exponentially and polynomially stable even without imposing the equality of the wave speeds unless, the system is not fully damped by the thermoelasticity effect as in [2]. By constructing a suitable Lyapunov functional, we studied the interaction between the effective of the nonlinear feedback with variable exponent and the thermal heat. We noticed that the decay depends on the behavior of the thermal kernel, the range of the variable exponent, and the time-dependent coefficient. Our results extend and improve many earlier results. To our knowledge, this is the first study of thermoelastic systems where the thermal kernel satisfies (A_1) in the current form (ξ is a function not a constant). Moreover, this is the first study of the competition between variable exponents and infinite memory.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The author would like to acknowledge the support provided by King Fahd University of Petroleum & Minerals (KFUPM), Saudi Arabia. The support provided by the Interdisciplinary Research Center for Construction & Building Materials (IRC-CBM) at King Fahd University of Petroleum & Minerals (KFUPM), Saudi Arabia, for funding this work through Project No. INCB2315, is also greatly acknowledged.

Competing interests

The author declares no competing interests.

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