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Research article

Symmetry analysis and conservation laws of time fractional Airy type and other KdV type equations

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Abstract: We study the invariance properties of the fractional time version of the nonlinear class of equations $u_t^\alpha - g(u)\ u_x - f(u)\ u_{xxx} = 0$, where $0 < \alpha < 1$ using some recently developed symmetry-based techniques. The equations reduce to ordinary fractional Airy type, Korteweg-de Vries (KdV) and modified KdV equations through the change of variables provided by the symmetries. Furthermore, we utilize the symmetries to construct conservation laws for the fractional partial differential equations.

Keywords: Airy equation; KdV equation; modified KdV equation; transformation groups; Lie symmetries; conservation laws
Mathematics Subject Classification: 05C35, 05C50

1. Introduction

The Airy equation is a well studied differential equation (DE) with applications in optics, acoustics and quantum mechanics, inter alia. In the former, it is used in the study of optical beams, in quantum mechanics the solution of the Airy equation involves the Airy function which arises also in the solution of certain classes of the time independent Schrödinger equation. In this paper, we will first study some general nonlinear time-fractional equations [3] given by

$$u_t^\alpha - g(u)u_x - f(u)u_{xxx} = 0,\ \text{where } 0 < \alpha < 1, \tag{1.1}$$

in which the Airy type equations correspond to $g(u) = 0$ and $f(u) = u$, viz.,

$$u_t^\alpha - uu_{xxx} = 0, \text{ where } 0 < \alpha < 1, \tag{1.2}$$

The Korteweg-de-Vries (KdV) equation [8] has been studied to describe many phenomena of physics such as evolution and interaction of nonlinear waves, and particularly shallow water waves. Furthermore, the KdV equation has applications in ion-acoustic waves, hydro-magnetic waves, plasma physics and the lattice dynamic. The time-fractional order KdV equation has been studied using a variation method by El-Wakil et al. [1], and is given by, as a special case of (1.2), viz.,

$$u_t^\alpha + uu_x + u_{xxx} = 0, \text{ where } 0 < \alpha < 1. \tag{1.3}$$

The KdV de is known to possess infinitely many conservation laws, has a bi-Hamiltonian property, possesses a Lagrangian when differentiated with respect to x, among other properties. It would be interesting to see if these translate to the time-fractional case.

Another special case of (1.2) is the modified KdV equation [10],

$$u_t^\alpha + u^2u_x + u_{xxx} = 0, \quad \text{ where } 0 < \alpha < 1 \tag{1.4}$$

which has many applications in soliton theory, calculation of conservation laws of KdV equation, inverse scattering transform, ultrashort few-optical cycle solitons in nonlinear media, ion acoustic solitons and in-traffic jam studies [11].

We study the invariance properties of the classes of time-fractional equations with a view to obtain the reduction of the equations and the conservation laws that are linked to the symmetries of the equations. The method adopted to construct the conservation laws is derived from the formulae of Noether's theorem [4, 5].

Some of the preliminaries required in the analysis are presented. Here, $u_t^\alpha = D_t^\alpha u$ is a fractional derivative of the function u with respect to t of order $\alpha, 0 < t < T(T \to \infty), x \in \Omega \subset R$. Here, we will take u_t^α to be the Riemann-Liouville left-sided time-fractional derivative ${}_0D_t^\alpha$ [2]

$${}_0D_t^\alpha u = D_t^n({}_0I_t^{n-\alpha}u), \tag{1.5}$$

where, ${}_0I_t^{n-\alpha}u$ is the left-sided time-fractional integral of order $n - \alpha$ defined by

$$({}_0I_t^{n-\alpha}u)(t, x) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{u(\theta, x)}{(t-\theta)^{1-n+\alpha}}d\theta, \tag{1.6}$$

where $\Gamma(\cdot)$ is the Gamma function and D_t^n denotes the total derivative operator with respect to t of order n,

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + \dots. \tag{1.7}$$

Recall that the Erdëlyi-Kober fractional differential operator [9] used in the calculation of fractional integral is given by,

$$(\mathcal{P}_\delta^{\zeta,\alpha})(z) = \prod_{j=0}^{m-1}(\zeta + j - \frac{1}{\delta}z\frac{d}{dz})(\mathcal{K}_\delta^{\zeta+\alpha,m-\alpha}h)(z), z > 0, \delta > 0, m = [\alpha] + 1, \tag{1.8}$$

where

$$(\mathcal{K}_\delta^{\zeta,\alpha}h)(z) = \frac{1}{\Gamma(\alpha)}\int_0^\infty (p-1)^{\alpha-1}(p)^{-(\zeta+\alpha)}h(zp^{\frac{1}{\delta}})dp. \tag{1.9}$$

2. Lie symmetries

In this section, we will analyze the invariance (Lie point symmetries) and conservation laws of various classes of the nonlinear fractional partial differential equation (1.2).

The vector field

$$X = \tau(t, x, u)\frac{\partial}{\partial t} + \xi(t, x, u)\frac{\partial}{\partial x} + \eta(t, x, u)\frac{\partial}{\partial u}, \tag{2.1}$$

is a Lie point symmetry operator of Eq (1.2) if

$$X^{[\alpha,3]}[u_t^\alpha - g(u)u_x - f(u)u_{xxx}] = 0, \tag{2.2}$$

along the solutions of Eq (1.2), where

$$X^{[\alpha,3]} = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u} + \eta^{\alpha,t}\frac{\partial}{\partial u_t^\alpha} + \eta^x\frac{\partial}{\partial u_x} + \eta^{xx}\frac{\partial}{\partial u_{xx}} + \eta^{xxx}\frac{\partial}{\partial u_{xxx}}, \tag{2.3}$$

and

$$\eta^{\alpha,t} = D_t^\alpha\eta + \xi D_t^\alpha(u_x) - D_t^\alpha(\xi u_x) + \tau D_t^\alpha(u_t) - D_t^\alpha(\tau_t), \tag{2.4}$$

$$\eta^x = D_x\eta - u_x D_x\xi - u_t D_x\tau, \tag{2.5}$$

$$\eta^{xx} = D_x\eta^x - u_{xx}D_x\xi - u_{xt}D_x\tau, \tag{2.6}$$

$$\eta^{xxx} = D_x\eta^{xx} - u_{xxx}D_x\xi - u_{xxt}D_x\tau, \tag{2.7}$$

and

$$\begin{aligned} D_t^\alpha\eta &= \frac{\partial^\alpha\eta}{\partial t^\alpha} + (\eta_u\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha\eta_u}{\partial t^\alpha}) + \sum_{n=1}^{\infty}(\alpha n)\frac{\partial^n\eta_u}{\partial t^n}D_t^{\alpha-n}u \\ &+ \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}(\alpha n)(nm)(kr)\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n-\alpha+1)}(-u)^r\frac{\partial^m}{\partial t^m}(u^{k-r})\frac{\partial^{n-m+k}}{\partial t^{n-m+k}}(\eta), \end{aligned}$$

where D_x is the total derivative operator with respect to x,

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + \dots. \tag{2.8}$$

Now using (2.3), the Eq (2.2) becomes

$$\eta^{\alpha,t} - f(u)\eta^{xxx} - g(u)\eta^x - \eta f'(u)u_{xxx} - \eta g'(u)u_x = 0, \tag{2.9}$$

subject to (1.2). Expanding and separating by monomials, we get a system

$$\tau_u = 0, \quad \tau_x = 0, \quad \xi_u = 0, \quad \xi_t = 0, \tag{2.10}$$

$$\eta_{uu} = 0, \quad \eta_{xu} - \xi_{xx} = 0, \quad 2\eta_{tu} - (\alpha-1)\tau_{tt} = 0, \tag{2.11}$$

$$(3\xi_x - \alpha\tau_t)f(u) - \eta f'(u) = 0, \ (\xi_x - \alpha\tau_t)g(u) - \eta g'(u) - (3\eta_{xxu} - \xi_{xxx})f(u) = 0 \tag{2.12}$$

and

$$\partial_t^\alpha\eta - u\partial_t^\alpha\eta_u - g(u)\eta_x - f(u)\eta_{xxx} = 0. \tag{2.13}$$

2.1. Classification of symmetries

(1) In the first case, when $f(u) = u$ and $g(u) = 0$, we get the time-fractional order Airy equation (1.2). After some calculations and the requirement that $\tau|_{t=0} = 0$, we get the generators

$$X_1 = \partial_x, \quad X_2 = x\partial_x + \frac{3}{\alpha}t\partial_t. \tag{2.14}$$

(2) In the case, when $f(u) = -1$ and $g(u) = -u$, we get the KdV equation (1.3). The symmetry generators are

$$X_1 = \partial_x, \quad X_3 = x\partial_x + \frac{1}{\alpha}t\partial_t, \quad X_4 = u\partial_u, \quad X_5 = B(x,t)\partial_u, \tag{2.15}$$

where $B(x,t)$ satisfies the KdV equation (1.3).

(3) In the case, $g(u) = u^n, n \neq 1$ and $f(u) = 1$, we get the modified KdV equation (1.4) for $n = 2$ and it has symmetry generators

$$X_1 = \partial_x, \quad X_6 = x\partial_x + \frac{3}{\alpha}t\partial_t - \frac{2}{n}u\partial_u. \tag{2.16}$$

2.2. A brief discussion on reductions by scaling symmetries

Notice that, the Eq (1.2) is not invariant in time t, and consequently the Riemann-Liouvile fractional equation does not admit a traveling wave or steady state solutions. We discuss the reduction based on the scaling symmetry $X_2 = x\partial_x + \frac{3}{\alpha}t\partial_t$, whose new invariants are found by the system of first order ordinary differential equation

$$\frac{dx}{x} = \frac{dt}{\frac{3}{\alpha}t} = \frac{du}{0} \tag{2.17}$$

viz.,

$$y = xt^{-\frac{\alpha}{3}}, \quad u(x,t) = w(y). \tag{2.18}$$

By use of operator (1.8), one can analyze further and after some cumbersome calculations, the Eq (1.2) reduces to

$$(\mathcal{P}^{1-\alpha,\alpha}_{\frac{3}{\alpha}}w)(y) - ww''' = 0. \tag{2.19}$$

Similarly, the symmetry generator $x\partial_x + \frac{3}{\alpha}t\partial_t - u\partial_u$ of the Eq (1.4) reduces the mKdV to an ordinary fractional equation as

$$(\mathcal{P}^{1-\alpha,\alpha}_{\frac{3}{\alpha}}w)(y) + w^2w' + w''' = 0. \tag{2.20}$$

3. Conservation laws

A vector (Φ^t, Φ^x) is a conserved vector or a conservation law associated with conservation law of (1.2), if

$$D_t\Phi^t + D_x\Phi^x = 0, \tag{3.1}$$

subjected to the solutions of Eq (1.2). In case, $D_t\Phi^t + D_x\Phi^x$ vanishes identically and is not subjected to (1.2), then we obtain the trivial conservation law. In physics, a conservation law states that a particular measurable property of an isolated physical system does not change as the system evolves over time. Exact conservation laws include conservation of mass-energy, conservation of linear momentum, conservation of angular momentum, and conservation of electric charge. There are also many approximate conservation laws, which apply to such quantities as mass, parity, lepton number, baryon number, strangeness, hypercharge, and so on. These quantities are conserved in certain classes of physics processes, but not in all.

A local conservation law is usually expressed mathematically as a continuity equation, which is a partial differential equation that gives a relation between the amount of the quantity and the "transport" of that quantity. It states that the amount of the conserved quantity at a point or within a volume can only change by the amount of the quantity that flows in or out of the volume.

One particularly important result concerning conservation laws is Noether's theorem, which states that there is a one-to-one correspondence between each one of them and a differentiable symmetry of nature. For example, the conservation of energy follows from the time-invariance of physical systems, and the conservation of angular momentum arises from the fact that physical systems behave the same regardless of how they are oriented in space.

To construct the conservation laws, we will follow Ibragimov's formal Lagrangian method [6], which relies on an adjoint equation.

The formal Lagrangian is given by

$$\mathcal{L} = m(x,t)(u_t^\alpha - g(u)u_x - f(u)u_{xxx}), \tag{3.2}$$

for which the action integral is

$$\int_0^T \int_\Omega \mathcal{L}(t, x, , u, u_t^\alpha, u_x, u_{xx}, u_{xxx})dxdt \tag{3.3}$$

and the Euler operator is

$$\frac{\delta \mathcal{L}}{\delta u} = \frac{\partial \mathcal{L}}{\partial u} + (D_t^\alpha)^* \frac{\partial \mathcal{L}}{\partial u_t^\alpha} - D_x \frac{\partial \mathcal{L}}{\partial u_x} + D_x D_x \frac{\partial \mathcal{L}}{\partial u_{xx}} + D_x D_x D_x \frac{\partial \mathcal{L}}{\partial u_{xxx}} \tag{3.4}$$

where $(D_t^\alpha)^*$ is the adjoint, in the Frechet sense, of D_t^α, so that

$$\frac{\delta \mathcal{L}}{\delta u} = up_{xxx} + 3u_x p_{xx} + 3u_{xx} p_x + (D_t^\alpha)^* p \tag{3.5}$$

with $\frac{\delta \mathcal{L}}{\delta p} = u_t^\alpha - uu_{xxx}$. The function $p(x,t)$ is the solution of the adjoint equation of (1.2).

In order to construct the conservation laws, we require the Noether operator [6] in its role in constructing conserved vectors. In its fractional setup, the operation of the Noether operator on the formal Lagrangian $\mathcal{L}$ leads to the conservation laws [7]. Given a Lie point symmetry and vector field of a time-fractional system of PDEs ($u = u(t,x)$)

$$X = \tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial u} + \eta \frac{\partial}{\partial u} \tag{3.6}$$

the respective Noether operators for the Riemann-Liouville case are given by [6]

$$N^t = \tau \mathcal{I} + \sum_{k=0}^{n-1} (-1)^k {}_0D_t^{\alpha-1-k}(W) D_t^k \frac{\partial}{\partial({}_0D_t^\alpha u)} - (-1)^n J(W, D_t^n \frac{\partial}{\partial({}_0D_t^\alpha u)}),$$

$$N^x = \xi\mathcal{I} + W(\frac{\partial}{\partial u_x} - D_x\frac{\partial}{\partial u_{xx}} + D_xD_x\frac{\partial}{\partial u_{xxx}}) + D_x(W)(\frac{\partial}{\partial u_{xx}} - D_x\frac{\partial}{\partial u_{xxx}}) + D_xD_x(W)\frac{\partial}{\partial u_{xxx}},$$

where $W = \eta - u_t\tau - u_x\xi$ is the characteristic of the vector field, $\mathcal{I}$ is the identity operator,

$$J(g,h) = \frac{1}{\Gamma(n-\alpha)}\int_0^t\int_t^T \frac{g(\theta,x)h(\nu,x)}{(\nu-\theta)^{\alpha+1-n}}d\nu d\theta \tag{3.7}$$

and $n = [\alpha] + 1$. Then, the components of conservation laws are

$$T^t = N^t\mathcal{L}, \quad T^x = N^x\mathcal{L}. \tag{3.8}$$

Since

$$\frac{\partial}{\partial u} = -pu_{xxx}, \quad \frac{\partial}{\partial u_x} = 0, \quad \frac{\partial}{\partial u_{xx}} = 0, \quad \frac{\partial}{\partial u_{xxx}} = -pu, \quad \frac{\partial}{\partial u_t^\alpha} = -pu_{xxx}, \tag{3.9}$$

we have,

$$\begin{aligned}
\Phi^t &= \tau\mathcal{L} + D_t^{\alpha-1}(W)p + J(W,p_t),\\
\Phi^x &= \xi\mathcal{L} + W(\frac{\partial\mathcal{L}}{\partial u_x} - D_x\frac{\partial\mathcal{L}}{\partial u_{xx}} + D_xD_x\frac{\partial\mathcal{L}}{\partial u_{xxx}}) + D_x(W)(\frac{\partial\mathcal{L}}{\partial u_{xx}} - D_x\frac{\partial\mathcal{L}}{\partial u_{xxx}}) + D_xD_x(W)\frac{\partial\mathcal{L}}{\partial u_{xxx}}.
\end{aligned}$$

In the case of the Airy type equation (1.2), we have the following non-trivial conserved vectors (Φ^t, Φ^x) associated with the vectors field V.

(i) $X = \partial_x$ $(W = -u_x)$**-linear momentum**

$$\begin{aligned}
\Phi^t &= pu_{xt}^{\alpha-1} + J(-u_x, p_t),\\
\Phi^x &= p(u_t^\alpha - 2uu_{xx}) - p_x(2{u_x}^2 + uu_{xx}) - p_{xx}uu_{xx}.
\end{aligned}$$

(ii) $X = x\partial_x + \frac{3}{\alpha}t\partial_t$ $(W = -xu_x - \frac{3}{\alpha}tu_t)$

$$\begin{aligned}
\Phi^t &= \frac{3}{\alpha}ptu_t^\alpha - \frac{3}{\alpha}ptuu_{xxx} - pxu_{xt}^{\alpha-1} - \frac{3}{\alpha}pD_t^{\alpha-1}(tu_t) + J(-xu_x - \frac{3}{\alpha}tu_t, p_t),\\
\Phi^x &= pxu_t^\alpha - p{u_x}^2 + 2puu_{xx} + \frac{3}{\alpha}ptuu_{xxt} + \frac{3}{\alpha}ptu_tu_{xx} - \frac{3}{\alpha}tpu_xu_{xt} + 2xp_x{u_x}^2\\
&+ \frac{6}{\alpha}tp_xu_xu_t - p_xuu_x - xup_xu_{xx} - \frac{3}{\alpha}tup_xu_{xt} + xup_{xx} + \frac{3}{\alpha}tup_{xx}u_t.
\end{aligned}$$

Similarly, we have the following cases of non-trivial conserved vectors (Φ^t, Φ^x) associated with the vectors field V of the KdV equation (1.3).

(i) $X = \partial_x$ $(W = -u_x)$**-linear momentum**

$$\begin{aligned}
\Phi^t &= pu_{xt}^{\alpha-1} + J(-u_x, p_t),\\
\Phi^x &= pu_t^\alpha + p_xu_{xx} - p_{xx}u_x.
\end{aligned}$$

(ii) $X = u\partial_u$ $(W = u)$

$$\begin{aligned}
\Phi^t &= pu_t^{\alpha-1} + J(u, p_t),\\
\Phi^x &= pu^2 + pu_{xx} - p_xu_x + p_{xx}u.
\end{aligned}$$

(iii) $X = x\partial_x + \frac{1}{\alpha}t\partial_t \ (W = -xu_x - \frac{1}{\alpha}tu_t)$

$$\begin{aligned}
\Phi^t &= \frac{1}{\alpha}pt(u_t^\alpha + uu_x + u_{xxx}) - pxu_{xt}^{\alpha-1} - \frac{1}{\alpha}pD_t^{\alpha-1}(tu_t) + J(-xu_x - \frac{1}{\alpha}tu_t, p_t),\\
\Phi^x &= pxu_t^\alpha - pu_{xx} - \frac{1}{\alpha}ptuu_t - \frac{1}{\alpha}ptu_{xxt} + p_xu_x + p_xu_{xx} + \frac{1}{\alpha}tp_xu_{xt} - xp_{xx}u_x - \frac{1}{\alpha}tp_{xx}u_t.
\end{aligned}$$

The following non-trivial conserved vectors (Φ^t, Φ^x) associated with the vectors field V of the modified KdV equation (1.4) are

(i) $X = \partial_x \ (W = -u_x)$**-linear momentum**

$$\begin{aligned}
\Phi^t &= pu_{xt}^{\alpha-1} + J(-u_x, p_t),\\
\Phi^x &= pu_t^\alpha + p_xu_{xx} - p_{xx}u_x.
\end{aligned}$$

(ii) $X = x\partial_x + \frac{3}{\alpha}t\partial_t - u\partial_u \ (W = -u - xu_x - \frac{3}{\alpha}tu_t)$

$$\begin{aligned}
\Phi^t &= \frac{3}{\alpha}pt(u_t^\alpha + u^2u_x + u_{xxx}) - pu_t^{\alpha-1} - pxu_{xt}^{\alpha-1} - \frac{3}{\alpha}pD_t^{\alpha-1}(tu_t)\\
&+ J(-u - xu_x - \frac{3}{\alpha}tu_t, p_t),\\
\Phi^x &= xpu_t^\alpha - pu^3 - 3pu_{xx} - \frac{3}{\alpha}tpu^2u_t - \frac{3}{\alpha}tpu_{xxt} + 2p_xu_x + xp_xu_{xx} + \frac{3}{\alpha}tp_xu_{xt} - up_{xx}\\
&- xp_{xx}u_x - \frac{3}{\alpha}tu_tp_{xx}.
\end{aligned}$$

4. Conclusions

We have studied, the fractional-time version of the nonlinear class of equations using the symmetry approach. In specific cases, it was shown that the equations reduced to ordinary fractional Airy type, KdV and modified KdV equations via the change of variables provided by the symmetries. We also utilized the symmetries to construct conservation laws for the fractional partial differential equations.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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