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Research article

Simpson-type inequalities by means of tempered fractional integrals

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Abstract: The latest iterations of Simpson-type inequalities (STIs) are the topic of this paper. These inequalities were generated via convex functions and tempered fractional integral operators (TFIOs). To get these sorts of inequalities, we employ the well-known Hölder inequality and the inequality of exponent mean. The subsequent STIS are a generalization of several works on this topic that use the fractional integrals of Riemann-Liouville (FIsRL). Moreover, distinctive outcomes can be achieved through unique selections of the parameters.

Keywords: Simpson-type inequalities; convex functions; fractional integrals; Riemann-Liouville fractional integrals; tempered fractional integrals
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1. Introduction

Convex methodology is a fascinating subject that is currently applied in several areas of technology, physics, energy resources, and optimizing theory [1–3]. In these areas, the convexity idea plays a prominent role, particularly in inequalities. Due to their substantial uses in basic, fractional, quantum, interval-valued and fractal calculus, inequalities have an attractive mathematical framework. Numerous scientists have lately devoted their time to exploring inequalities and convexity-related characteristics. There are several different kinds of inequalities that incorporate convex functions in the scientific literature, including those of the Hermite-Hadamard, Simpson and Bullen categories.

There are multiple renowned integral inequalities as a result, but the STIs stand out as being the most prominent.

Fractional calculus (FC) is a renowned title. It is possible to trace the invention of FC back to a correspondence written by Leibniz and L'Hopital. Over the last three centuries, plenty of academics have made significant contributions to the creation of FC notions. As a result, beginning in the earlier century, textbooks on FC began to appear. Examples are Samko, Kilbas, and Marichev (1993), Podlubny (1999) and others. FC may characterize a wide variety of non-classical events that have emerged in the applicable disciplines and technology in recent decades, according to various ideas and studies [4–8].

FC is currently evolving into an effective instrument for illustrating the unusual dynamics that appear in biology, chemistry, physics and other complicated processes due to its strong mathematical properties [5]. Numerous types of fractional derivatives, including the Riemann-Liouville fractional derivative, the Caputo fractional derivative [4, 6], the Hilfer fractional derivative [9, 10] and the Riesz fractional derivative [6] are discussed in practical uses.

We will now provide a number of key concepts in order to construct our primary findings.

Definition 1. The gamma, incomplete gamma and λ-incomplete gamma functions are explained by:

$$\Gamma(\alpha) := \int_0^\infty \theta^{\alpha-1} e^{-\theta} d\theta,$$

$$\curlyvee(\alpha, x) := \int_0^x \theta^{\alpha-1} e^{-\theta} d\theta$$

and

$$\curlyvee_\lambda(\alpha, x) := \int_0^x \theta^{\alpha-1} e^{-\lambda\theta} d\theta,$$

respectively. Here, $0 < \alpha < \infty$ and $\lambda \geq 0$.

Below are some characteristics of the λ-incomplete gamma function:

Remark 1. [11] $\alpha > 0$; $x, \lambda \geq 0$ and $\sigma < \delta$, we have

(i) $\curlyvee_{\lambda(\delta-\sigma)}(\alpha, 1) = \int_0^1 \theta^{\alpha-1} e^{-\lambda(\delta-\sigma)\theta} d\theta = \frac{1}{(\delta-\sigma)^\alpha} \curlyvee_\lambda(\alpha, \delta-\sigma)$,

(ii) $\int_0^1 \curlyvee_{\lambda(\delta-\sigma)}(\alpha, x)\, dx = \frac{\curlyvee_\lambda(\alpha,\delta-\sigma)}{(\delta-\sigma)^\alpha} - \frac{\curlyvee_\lambda(\alpha+1,\delta-\sigma)}{(\delta-\sigma)^{\alpha+1}}$.

The following are the FIRL:

Definition 2. [12] For $\mathcal{F} \in L_1[\sigma, \delta]$, the FIsRL of order $\alpha > 0$ are given by

$$J^\alpha_{\sigma+}\mathcal{F}(x) = \frac{1}{\Gamma(\alpha)} \int_\sigma^x (x-\theta)^{\alpha-1} \mathcal{F}(\theta) d\theta, \quad x > \sigma, \tag{1.1}$$

and

$$J_{\delta-}^{\alpha}\mathcal{F}(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{\delta}(\theta-x)^{\alpha-1}\mathcal{F}(\theta)d\theta,\ \ x<\delta. \tag{1.2}$$

Of course, if $\alpha = 1$, the FIsRL will be equivalent to the classical integrals.

Many studies have thoroughly examined fractional STIs. Scientists use fractional calculus to expand the classical notions of derivative and integral to non-integer orders. Researchers have found this subject attractive in recent years [13, 14]. STIs were examined by Riemann-Liouville fractional integrals and other varieties of fractional integrals. In addition to modeling several kinds of biology-related mathematical problems, fractional derivatives are additionaly utilized for modeling reactions in chemicals, physics, and problems in engineering [15, 16].

The fundamental descriptions and updated forms of the TFIOs are now reviewed.

Definition 3. [17, 18] The TFIOs $\mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathcal{F}$ and $\mathcal{J}_{\delta-}^{(\sigma,\lambda)}\mathcal{F}$ of order $\alpha>0$ and $\lambda\geq 0$ are given by

$$\mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathcal{F}(x)=\frac{1}{\Gamma(\alpha)}\int_{\sigma}^{x}(x-\theta)^{\alpha-1}e^{-\lambda(x-\theta)}\mathcal{F}(\theta)d\theta,\ \ x\in[\sigma,\delta], \tag{1.3}$$

and

$$\mathcal{J}_{\delta-}^{(\alpha,\lambda)}\mathcal{F}(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{\delta}(\theta-x)^{\alpha-1}e^{-\lambda(\theta-x)}\mathcal{F}(\theta)d\theta,\ \ \ x\in[\sigma,\delta], \tag{1.4}$$

respectively for $\mathcal{F}\in L_1[\sigma,\delta]$.

Obviously, if we assume $\lambda = 0$, then the TFIO (1.3) equals to the FIRL in (1.1). Moreover, the TFIO in (1.4) becomes to the FIRL in (1.2) when $\lambda = 0$.

An extension of fractional calculus is known as fractional tempered calculus. As far as we are aware, Buschman's prior work [19] was the first to disclose the terms of fractional integration with weakly singular and exponential kernels (see [6, 20, 21] for further details regarding the various definitions of the TFIOs). For the case of convex functions, Mohammed et al. [11] showed numerous Hermite-Hadamard-type connected with the tempered fractional integrals that encompass previously reported results such as Riemann integrals and Riemann-Liouville fractional integrals. To be more explicit, the authors established a number of Hermite-Hadamard-type inequalities involving tempered fractional integrals using the method developed by Sarikaya et al. [22] and Sarikaya and Yildirim [23].

We focus on the most recent incarnations of Simpson-type inequalities. Convex functions and tempered fractional integral operators were used to construct these inequalities. We use the well-known Hölder inequality and the exponent mean inequality to obtain various types of inequalities. The obtained STIs are a generalization of various works on this issue that employ Riemann-Liouville fractional integrals. Furthermore, it is evident that by selecting particular parameter values, the obtained results are novel and can be simplified to the conclusions of [24, 25].

Following the introduction are three sections that comprise the full study format. In order to construct our major conclusions, the basic terms of the TFIOs and FIsRL are described here. The renowned gamma, incomplete gamma function and λ-incomplete gamma function are additionally defined. With the use of TFIOs and convex mappings, we demonstrate a novel form of STIs in Section 2. More specifically, some of the established inequalities will employ Hölder and exponent-mean inequalities, both of which are popular in the literature. We will also make several observations and corollaries. The final section, Section 2, will present concepts that will guide mathematicians. It will be made known to researchers that fresh iterations of the inequalities we have collected can be produced via various fractional integrals.

2. Formulation of the issue

In this part, we employ the TFIOs to build STIs using differentiable convex functions. To begin, let us create a new identity in order to obtain STIs.

Lemma 1. *Let $\mathcal{F} : [\sigma, \delta] \to \mathbb{R}$ be a differentiable mapping on (σ, δ) such that $\mathcal{F}' \in L_1[\sigma, \delta]$. Then, the following equality holds:*

$$\frac{\Gamma(\alpha)}{2 \curlyvee_\lambda \left(\alpha, \frac{\delta-\sigma}{2}\right)} \left[\mathcal{J}_{\sigma+}^{(\alpha,\lambda)} \mathcal{F}\left(\frac{\sigma+\delta}{2}\right) + \mathcal{J}_{\delta-}^{(\alpha,\lambda)} \mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right] - \frac{1}{6}\left[\mathcal{F}(\sigma) + 4\mathcal{F}\left(\frac{\sigma+\delta}{2}\right) + \mathcal{F}(\delta)\right] \tag{2.1}$$

$$= \left(\frac{\delta-\sigma}{2}\right)^{\alpha+1} \frac{1}{\curlyvee_\lambda \left(\alpha, \frac{\delta-\sigma}{2}\right)} \int_0^1 A_\sigma^{(\alpha,\lambda)}(\theta) d\theta.$$

where,

$$\begin{aligned} A_\sigma^{(\alpha,\lambda)}(\theta) &= \left\{\frac{1}{3} \curlyvee_{\lambda\left(\frac{\delta-\sigma}{2}\right)} (\alpha, 1) - \frac{1}{2} \curlyvee_{\lambda\left(\frac{\delta-\sigma}{2}\right)} (\alpha, \theta)\right\} \\ &\times \left[\mathcal{F}'\left(\frac{1-\theta}{2}\sigma + \frac{1+\theta}{2}\delta\right) - \mathcal{F}'\left(\frac{1+\theta}{2}\sigma + \frac{1-\theta}{2}\delta\right)\right]. \end{aligned}$$

Proof. Employing the integration by parts, we obtain

$$I_1 = \int_0^1 \left\{\frac{1}{3} \curlyvee_{\lambda\left(\frac{\delta-\sigma}{2}\right)} (\alpha, 1) - \frac{1}{2} \curlyvee_{\lambda\left(\frac{\delta-\sigma}{2}\right)} (\alpha, \theta)\right\} \mathcal{F}'\left(\frac{1-\theta}{2}\sigma + \frac{1+\theta}{2}\delta\right) d\theta \tag{2.2}$$

$$= \frac{2}{\delta-\sigma} \left\{\frac{1}{3} \curlyvee_{\lambda\left(\frac{\delta-\sigma}{2}\right)} (\alpha, 1) - \frac{1}{2} \curlyvee_{\lambda\left(\frac{\delta-\sigma}{2}\right)} (\alpha, \theta)\right\} \mathcal{F}\left(\frac{1-\theta}{2}\sigma + \frac{1+\theta}{2}\delta\right)\Bigg|_0^1$$

$$+ \frac{1}{\delta-\sigma} \int_0^1 \theta^{\alpha-1} e^{-\lambda\left(\frac{\delta-\sigma}{2}\right)\theta} \mathcal{F}\left(\frac{1-\theta}{2}\sigma + \frac{1+\theta}{2}\delta\right) d\theta$$

$$= \frac{2^\alpha \Gamma(\alpha)}{(\delta-\sigma)^{\alpha+1}} \mathcal{J}_{\delta-}^{(\alpha,\lambda)} \mathcal{F}\left(\frac{\sigma+\delta}{2}\right) - \frac{2^\alpha}{3(\delta-\sigma)^{\alpha+1}} \curlyvee_\lambda \left(\alpha, \frac{\delta-\sigma}{2}\right)\left[2f\left(\frac{\sigma+\delta}{2}\right) + \mathcal{F}(\delta)\right].$$

Similar to the foregoing process, applying the integration by parts, we have

$$I_2 = \int_0^1 \left\{ \frac{1}{3} \curlyvee_{\lambda\left(\frac{\delta-\sigma}{2}\right)} (\alpha, 1) - \frac{1}{2} \curlyvee_{\lambda\left(\frac{\delta-\sigma}{2}\right)} (\alpha, \theta) \right\} \mathcal{F}' \left(\frac{1+\theta}{2}\sigma + \frac{1-\theta}{2}\delta \right) d\theta \tag{2.3}$$

$$= \frac{2^\alpha}{3(\delta-\sigma)^{\alpha+1}} \curlyvee_\lambda \left(\alpha, \frac{\delta-\sigma}{2}\right) \left[2f\left(\frac{\sigma+\delta}{2}\right) + \mathcal{F}(\sigma) \right] - \frac{2^\alpha \Gamma(\alpha)}{(\delta-\sigma)^{\alpha+1}} \mathcal{J}_{\sigma+}^{(\alpha,\lambda)} \mathcal{F}\left(\frac{\sigma+\delta}{2}\right).$$

From (2.2) and (2.3), if we examine the following calculation

$$\left(\frac{\delta-\sigma}{2}\right)^{\alpha+1} \frac{1}{\curlyvee_\lambda \left(\alpha, \frac{\delta-\sigma}{2}\right)} [I_1 - I_2]$$

$$= \frac{\Gamma(\alpha)}{2 \curlyvee_\lambda \left(\alpha, \frac{\delta-\sigma}{2}\right)} \left[\mathcal{J}_{\sigma+}^{(\alpha,\lambda)} \mathcal{F}\left(\frac{\sigma+\delta}{2}\right) + \mathcal{J}_{\delta-}^{(\alpha,\lambda)} \mathcal{F}\left(\frac{\sigma+\delta}{2}\right) \right] - \frac{1}{6} \left[\mathcal{F}(\sigma) + 4\mathcal{F}\left(\frac{\sigma+\delta}{2}\right) + \mathcal{F}(\delta) \right].$$

This ends the proof of Lemma 1. □

Theorem 1. *Assume that* $\mathcal{F} : [\sigma, \delta] \to \mathbb{R}$ *is a differentiable function on* (σ, δ) *and* $|\mathcal{F}'|$ *is convex on* $[\sigma, \delta]$. *Under these conditions, the following inequality is derived:*

$$\left| \frac{\Gamma(\alpha)}{2 \curlyvee_\lambda \left(\alpha, \frac{\delta-\sigma}{2}\right)} \left[\mathcal{J}_{\sigma+}^{(\alpha,\lambda)} \mathcal{F}\left(\frac{\sigma+\delta}{2}\right) + \mathcal{J}_{\delta-}^{(\alpha,\lambda)} \mathcal{F}\left(\frac{\sigma+\delta}{2}\right) \right] - \frac{1}{6} \left[\mathcal{F}(\sigma) + 4\mathcal{F}\left(\frac{\sigma+\delta}{2}\right) + \mathcal{F}(\delta) \right] \right|$$

$$\leq \left(\frac{\delta-\sigma}{2}\right)^{\alpha+1} \frac{\varphi_1(\alpha, \lambda)}{\curlyvee_\lambda \left(\alpha, \frac{\delta-\sigma}{2}\right)} \left\{ |\mathcal{F}'(\sigma)| + |\mathcal{F}'(\delta)| \right\}.$$

Here, λ-incomplete gamma function are denoted as $\curlyvee_\lambda (\alpha, \theta)$ and

$$\varphi_1(\alpha, \lambda) = \int_0^1 \left| \frac{1}{3} \curlyvee_{\lambda\left(\frac{\delta-\sigma}{2}\right)} (\alpha, 1) - \frac{1}{2} \curlyvee_{\lambda\left(\frac{\delta-\sigma}{2}\right)} (\alpha, \theta) \right| d\theta. \tag{2.4}$$

Proof. First, let us take the absolute value of both sides of (2.1). Then, we obtain

$$\left| \frac{\Gamma(\alpha)}{2 \curlyvee_\lambda \left(\alpha, \frac{\delta-\sigma}{2}\right)} \left[\mathcal{J}_{\sigma+}^{(\alpha,\lambda)} \mathcal{F}\left(\frac{\sigma+\delta}{2}\right) + \mathcal{J}_{\delta-}^{(\alpha,\lambda)} \mathcal{F}\left(\frac{\sigma+\delta}{2}\right) \right] - \frac{1}{6} \left[\mathcal{F}(\sigma) + 4\mathcal{F}\left(\frac{\sigma+\delta}{2}\right) + \mathcal{F}(\delta) \right] \right| \tag{2.5}$$

$$\leq \left(\frac{\delta-\sigma}{2}\right)^{\alpha+1} \frac{1}{\curlyvee_\lambda \left(\alpha, \frac{\delta-\sigma}{2}\right)} \int_0^1 \left| \frac{1}{3} \curlyvee_{\lambda\left(\frac{\delta-\sigma}{2}\right)} (\alpha, 1) - \frac{1}{2} \curlyvee_{\lambda\left(\frac{\delta-\sigma}{2}\right)} (\alpha, \theta) \right| \left| \mathcal{F}' \left(\frac{1-\theta}{2}\sigma + \frac{1+\theta}{2}\delta \right) \right| d\theta$$

$$+\left(\frac{\delta-\sigma}{2}\right)^{\alpha+1}\frac{1}{2\curlyvee_{\lambda}\left(\alpha,\frac{\delta-\sigma}{2}\right)}\int_{0}^{1}\left|\frac{1}{3}\curlyvee_{\lambda\left(\frac{\delta-\sigma}{2}\right)}(\alpha,1)-\frac{1}{2}\curlyvee_{\lambda\left(\frac{\delta-\sigma}{2}\right)}(\alpha,\theta)\right|\left|\mathcal{F}'\left(\frac{1+\theta}{2}\sigma+\frac{1-\theta}{2}\delta\right)\right|d\theta.$$

From the fact that $|\mathcal{F}'|$ is convex on $[\sigma,\delta]$. Then, it yields

$$\left|\frac{\Gamma(\alpha)}{2\curlyvee_{\lambda}\left(\alpha,\frac{\delta-\sigma}{2}\right)}\left[\mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\mathcal{J}_{\delta-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]-\frac{1}{6}\left[\mathcal{F}(\sigma)+4\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\mathcal{F}(\delta)\right]\right|$$

$$\leq\left(\frac{\delta-\sigma}{2}\right)^{\alpha+1}\frac{1}{\curlyvee_{\lambda}\left(\alpha,\frac{\delta-\sigma}{2}\right)}\int_{0}^{1}\left|\frac{1}{3}\curlyvee_{\lambda\left(\frac{\delta-\sigma}{2}\right)}(\alpha,1)-\frac{1}{2}\curlyvee_{\lambda\left(\frac{\delta-\sigma}{2}\right)}(\alpha,\theta)\right|d\theta$$

$$\times\left(\frac{1-\theta}{2}|\mathcal{F}'(\delta)|+\frac{1+\theta}{2}|\mathcal{F}'(\sigma)|+\frac{1-\theta}{2}|\mathcal{F}'(\sigma)|+\frac{1+\theta}{2}|\mathcal{F}'(\delta)|\right)d\theta$$

$$=\left(\frac{\delta-\sigma}{2}\right)^{\alpha+1}\frac{1}{\curlyvee_{\lambda}\left(\alpha,\frac{\delta-\sigma}{2}\right)}\int_{0}^{1}\left|\frac{1}{3}\curlyvee_{\lambda\left(\frac{\delta-\sigma}{2}\right)}(\alpha,1)-\frac{1}{2}\curlyvee_{\lambda\left(\frac{\delta-\sigma}{2}\right)}(\alpha,\theta)\right|\left[|\mathcal{F}'(\sigma)|+|\mathcal{F}'(\delta)|\right]d\theta.$$

The proof of Theorem 1 is completed. □

Remark 2. If we set $\lambda=0$ in Theorem 1, then we have

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha}}\left[J_{\sigma+}^{\alpha}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+J_{\delta-}^{\alpha}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]-\frac{1}{6}\left[\mathcal{F}(\sigma)+4\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\mathcal{F}(\delta)\right]\right|$$

$$\leq\left(\frac{\delta-\sigma}{12}\right)\left\{\frac{4\alpha}{\alpha+1}\left(\frac{2}{3}\right)^{\frac{1}{\alpha}}+\frac{1-2\alpha}{\alpha+1}\right\}\left[|\mathcal{F}'(\delta)|+|\mathcal{F}'(\sigma)|\right],$$

which is given by Chen and Huang in [24, Corollary 2.4].

Remark 3. If we choose $\alpha=1$ and $\lambda=0$ in Theorem 2, then the following inequality holds:

$$\left|\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathcal{F}(x)\,dx-\frac{1}{6}\left[\mathcal{F}(\sigma)+4\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\mathcal{F}(\delta)\right]\right|\leq\frac{5(\delta-\sigma)}{72}\left[|\mathcal{F}'(\delta)|+|\mathcal{F}'(\sigma)|\right],$$

which is given by Sarikaya et al. in [25, Corollary 1].

Theorem 2. *If $\mathcal{F}:[\sigma,\delta]\to\mathbb{R}$ is a differentiable mapping on (σ,δ) and $|\mathcal{F}'|^{q}$ is convex on $[\sigma,\delta]$ with $q>1$, then, the following inequalities can be written*

$$\left|\frac{\Gamma(\alpha)}{2 \curlyvee_{\lambda}\left(\alpha, \frac{\delta-\sigma}{2}\right)}\left[\mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\mathcal{J}_{\delta-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]-\frac{1}{6}\left[\mathcal{F}(\sigma)+4\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\mathcal{F}(\delta)\right]\right|$$

$$\leq\left(\frac{\delta-\sigma}{2}\right)^{\alpha+1}\frac{\left(\psi_1^p(\alpha,\lambda)\right)^{\frac{1}{p}}}{\curlyvee_{\lambda}\left(\alpha, \frac{\delta-\sigma}{2}\right)}\left[\left(\frac{|\mathcal{F}'(\sigma)|^q+3|\mathcal{F}'(\delta)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\mathcal{F}'(\sigma)|^q+|\mathcal{F}'(\delta)|^q}{4}\right)^{\frac{1}{q}}\right].$$

Here, $\frac{1}{p}+\frac{1}{q}=1$, and

$$\psi_1^p(\alpha,\lambda)=\int_0^1\left|\frac{1}{3}\curlyvee_{\lambda\left(\frac{\delta-\sigma}{2}\right)}(\alpha,1)-\frac{1}{2}\curlyvee_{\lambda\left(\frac{\delta-\sigma}{2}\right)}(\alpha,\theta)\right|^p d\theta.$$

Proof. If we use Hölder's inequality in (2.5), then we obtain

$$\left|\frac{\Gamma(\alpha)}{2 \curlyvee_{\lambda}\left(\alpha, \frac{\delta-\sigma}{2}\right)}\left[\mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\mathcal{J}_{\delta-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]-\frac{1}{6}\left[\mathcal{F}(\sigma)+4\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\mathcal{F}(\delta)\right]\right|$$

$$\leq\frac{\left(\frac{\delta-\sigma}{2}\right)^{\alpha+1}}{\curlyvee_{\lambda}\left(\alpha, \frac{\delta-\sigma}{2}\right)}\left(\int_0^1\left|\frac{1}{3}\curlyvee_{\lambda\left(\frac{\delta-\sigma}{2}\right)}(\alpha,1)-\frac{1}{2}\curlyvee_{\lambda\left(\frac{\delta-\sigma}{2}\right)}(\alpha,\theta)\right|^p d\theta\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathcal{F}'\left(\frac{1-\theta}{2}\sigma+\frac{1+\theta}{2}\delta\right)\right|^q d\theta\right)^{\frac{1}{q}}$$

$$+\frac{\left(\frac{\delta-\sigma}{2}\right)^{\alpha+1}}{\curlyvee_{\lambda}\left(\alpha, \frac{\delta-\sigma}{2}\right)}\left(\int_0^1\left|\frac{1}{3}\curlyvee_{\lambda\left(\frac{\delta-\sigma}{2}\right)}(\alpha,1)-\frac{1}{2}\curlyvee_{\lambda\left(\frac{\delta-\sigma}{2}\right)}(\alpha,\theta)\right|^p d\theta\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathcal{F}'\left(\frac{1+\theta}{2}\sigma+\frac{1-\theta}{2}\delta\right)\right|^q d\theta\right)^{\frac{1}{q}}.$$

If we apply the convexity of $|\mathcal{F}'|^q$ on $[\sigma,\delta]$, then we have

$$\left|\frac{\Gamma(\alpha)}{2 \curlyvee_{\lambda}\left(\alpha, \frac{\delta-\sigma}{2}\right)}\left[\mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\mathcal{J}_{\delta-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]-\frac{1}{6}\left[\mathcal{F}(\sigma)+4\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\mathcal{F}(\delta)\right]\right|$$

$$\leq\left(\frac{\delta-\sigma}{2}\right)^{\alpha+1}\frac{1}{\curlyvee_{\lambda}\left(\alpha, \frac{\delta-\sigma}{2}\right)}\left\{\left(\int_0^1\left|\frac{1}{3}\curlyvee_{\lambda\left(\frac{\delta-\sigma}{2}\right)}(\alpha,1)-\frac{1}{2}\curlyvee_{\lambda\left(\frac{\delta-\sigma}{2}\right)}(\alpha,\theta)\right|^p d\theta\right)^{\frac{1}{p}}\right.$$

$$\left.\times\left[\left(\frac{|\mathcal{F}'(\sigma)|^q+3|\mathcal{F}'(\delta)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\mathcal{F}'(\sigma)|^q+|\mathcal{F}'(\delta)|^q}{4}\right)^{\frac{1}{q}}\right]\right\}$$

$$= \left(\frac{\delta-\sigma}{2}\right)^{\alpha+1} \frac{\left(\psi_1^{\alpha}(\lambda,p)\right)^{\frac{1}{p}}}{\curlyvee_\lambda\left(\alpha,\frac{\delta-\sigma}{2}\right)} \left[\left(\frac{|\mathcal{F}'(\sigma)|^q+3|\mathcal{F}'(\delta)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\mathcal{F}'(\sigma)|^q+|\mathcal{F}'(\delta)|^q}{4}\right)^{\frac{1}{q}}\right].$$

Hence, the proof of Theorem 2 is completed. □

Remark 4. If we consider $\lambda = 0$ in Theorem 2, then, the following result is obtained

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha}}\left[J_{\delta-}^{\alpha}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+J_{\sigma+}^{\alpha}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]-\frac{1}{6}\left[\mathcal{F}(\sigma)+\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\mathcal{F}(\delta)\right]\right|$$

$$\leq \frac{\delta-\sigma}{12}\left(\int_0^1 |2-3\theta^{\alpha}|^p\,d\theta\right)^{\frac{1}{p}}\left[\left(\frac{|\mathcal{F}'(\sigma)|^q+3|\mathcal{F}'(\delta)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\mathcal{F}'(\sigma)|^q+|\mathcal{F}'(\delta)|^q}{4}\right)^{\frac{1}{q}}\right],$$

which is established in [24, Corollary 2.10].

Remark 5. If we choose $\alpha = 1$ and $\lambda = 0$ in Theorem 2, then, the following double inequality holds:

$$\left|\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathcal{F}(x)\,dx-\frac{1}{6}\left[\mathcal{F}(\sigma)+4\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\mathcal{F}(\delta)\right]\right|$$

$$\leq \frac{\delta-\sigma}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{|\mathcal{F}'(\sigma)|^q+3|\mathcal{F}'(\delta)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\mathcal{F}'(\sigma)|^q+|\mathcal{F}'(\delta)|^q}{4}\right)^{\frac{1}{q}}\right]$$

which is given in [25, Theorem 4].

Theorem 3. *Consider the existence of a differentiable function such that $\mathcal{F} : [\sigma,\delta] \to \mathbb{R}$ on (σ,δ). Let us also consider that the function $|\mathcal{F}'|^q$ is convex on $[\sigma,\delta]$ with $q \geq 1$. Then, the following inequality holds:*

$$\left|\frac{\Gamma(\alpha)}{2\curlyvee_\lambda\left(\alpha,\frac{\delta-\sigma}{2}\right)}\left[\mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\mathcal{J}_{\delta-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]-\frac{1}{6}\left[\mathcal{F}(\sigma)+4\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\mathcal{F}(\delta)\right]\right|$$

$$\leq \left(\frac{\delta-\sigma}{2}\right)^{\alpha+1}\frac{(\varphi_1(\alpha,\lambda))^{1-\frac{1}{q}}}{\curlyvee_\lambda\left(\alpha,\frac{\delta-\sigma}{2}\right)}$$

$$\times\left\{\left(\frac{(\varphi_1(\alpha,\lambda)+\varphi_2(\alpha,\lambda))}{2}|\mathcal{F}'(\delta)|^q+\frac{(\varphi_1(\alpha,\lambda)-\varphi_2(\alpha,\lambda))}{2}|\mathcal{F}'(\sigma)|^q\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{(\varphi_1(\alpha,\lambda)-\varphi_2(\alpha,\lambda))}{2}\left|\mathcal{F}'(\delta)\right|^q+\frac{(\varphi_1(\alpha,\lambda)+\varphi_2(\alpha,\lambda))}{2}\left|\mathcal{F}'(\sigma)\right|^q\right)^{\frac{1}{q}}\Bigg\}.$$

Here, $\varphi_1(\alpha,\lambda)$ is described as in (2.4) and

$$\varphi_2(\alpha,\lambda)=\int_0^1\theta\left|\frac{1}{3}\curlyvee_{\lambda\left(\frac{\delta-\sigma}{2}\right)}(\alpha,1)-\frac{1}{2}\curlyvee_{\lambda\left(\frac{\delta-\sigma}{2}\right)}(\alpha,\theta)\right|d\theta.$$

Proof. With the help of the power-mean inequality, we get

$$\left|\frac{\Gamma(\alpha)}{2\curlyvee_\lambda\left(\alpha,\frac{\delta-\sigma}{2}\right)}\left[\mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\mathcal{J}_{\delta-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]-\frac{1}{6}\left[\mathcal{F}(\sigma)+4\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\mathcal{F}(\delta)\right]\right|$$

$$\leq\left(\frac{\delta-\sigma}{2}\right)^{\alpha+1}\frac{1}{\curlyvee_\lambda\left(\alpha,\frac{\delta-\sigma}{2}\right)}\left\{\left(\int_0^1\left|\frac{1}{3}\curlyvee_{\lambda\left(\frac{\delta-\sigma}{2}\right)}(\alpha,1)-\frac{1}{2}\curlyvee_{\lambda\left(\frac{\delta-\sigma}{2}\right)}(\alpha,\theta)\right|d\theta\right)^{1-\frac{1}{q}}\right.$$

$$\times\left[\left(\int_0^1\left|\frac{1}{3}\curlyvee_{\lambda\left(\frac{\delta-\sigma}{2}\right)}(\alpha,1)-\frac{1}{2}\curlyvee_{\lambda\left(\frac{\delta-\sigma}{2}\right)}(\alpha,\theta)\right|\left|\mathcal{F}'\left(\frac{1-\theta}{2}\sigma+\frac{1+\theta}{2}\delta\right)\right|^q d\theta\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\int_0^1\left|\frac{1}{3}\curlyvee_{\lambda\left(\frac{\delta-\sigma}{2}\right)}(\alpha,1)-\frac{1}{2}\curlyvee_{\lambda\left(\frac{\delta-\sigma}{2}\right)}(\alpha,\theta)\right|\left|\mathcal{F}'\left(\frac{1+\theta}{2}\sigma+\frac{1-\theta}{2}\delta\right)\right|^q d\theta\right)^{\frac{1}{q}}\right]\right\}.$$

Since $|\mathcal{F}'|^q$ is convex on $[\sigma,\delta]$, we obtain

$$\left|\frac{\Gamma(\alpha)}{2\curlyvee_\lambda\left(\alpha,\frac{\delta-\sigma}{2}\right)}\left[\mathcal{J}_{\sigma+}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\mathcal{J}_{\delta-}^{(\alpha,\lambda)}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]-\frac{1}{6}\left[\mathcal{F}(\sigma)+4\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\mathcal{F}(\delta)\right]\right|$$

$$\leq\left(\frac{\delta-\sigma}{2}\right)^{\alpha+1}\frac{1}{\curlyvee_\lambda\left(\alpha,\frac{\delta-\sigma}{2}\right)}\left\{\left(\int_0^1\left|\frac{1}{3}\curlyvee_{\lambda\left(\frac{\delta-\sigma}{2}\right)}(\alpha,1)-\frac{1}{2}\curlyvee_{\lambda\left(\frac{\delta-\sigma}{2}\right)}(\alpha,\theta)\right|d\theta\right)^{1-\frac{1}{q}}\right.$$

$$\times\left[\left(\int_0^1\left|\frac{1}{3}\curlyvee_{\lambda\left(\frac{\delta-\sigma}{2}\right)}(\alpha,1)-\frac{1}{2}\curlyvee_{\lambda\left(\frac{\delta-\sigma}{2}\right)}(\alpha,\theta)\right|\left(\frac{1+\theta}{2}\left|\mathcal{F}'(\delta)\right|^q+\frac{1-\theta}{2}\left|\mathcal{F}'(\sigma)\right|^q\right)d\theta\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^1\left|\frac{1}{3}\curlyvee_{\lambda\left(\frac{\delta-\sigma}{2}\right)}(\alpha,1)-\frac{1}{2}\curlyvee_{\lambda\left(\frac{\delta-\sigma}{2}\right)}(\alpha,\theta)\right|\left(\frac{1+\theta}{2}\left|\mathcal{F}'(\sigma)\right|^q+\frac{1-\theta}{2}\left|\mathcal{F}'(\delta)\right|^q\right)d\theta\right)^{\frac{1}{q}}\Bigg]\Bigg]\Bigg\}.$$

□

Remark 6. If $\lambda = 0$ in Theorem 3, then we have

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha}}\left[J_{\sigma+}^{\alpha}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+J_{\delta-}^{\alpha}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)\right]-\frac{1}{6}\left[\mathcal{F}(\sigma)+4\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\mathcal{F}(\delta)\right]\right|$$

$$\leq\left(\frac{\delta-\sigma}{2}\right)\alpha\left(\varphi_1(\alpha,0)\right)^{1-\frac{1}{q}}$$

$$\times\left\{\left(\frac{(\varphi_1(\alpha,0)+\varphi_2(\alpha,0))}{2}\left|\mathcal{F}'(\delta)\right|^q+\frac{(\varphi_1(\alpha,0)-\varphi_2(\alpha,0))}{2}\left|\mathcal{F}'(\sigma)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{(\varphi_1(\alpha,0)-\varphi_2(\alpha,0))}{2}\left|\mathcal{F}'(\delta)\right|^q+\frac{(\varphi_1(\alpha,0)+\varphi_2(\alpha,0))}{2}\left|\mathcal{F}'(\sigma)\right|^q\right)^{\frac{1}{q}}\right\},$$

where

$$\begin{cases}\varphi_1(\alpha,0)=\frac{1}{6\alpha}\left[\frac{4\alpha}{\alpha+1}\left(\frac{2}{3}\right)^{\frac{1}{\alpha}}+\frac{1-2\alpha}{\alpha+1}\right],\\ \varphi_2(\alpha,0)=\frac{1}{6\alpha}\left[\frac{2\alpha}{\alpha+2}\left(\frac{2}{3}\right)^{\frac{2}{\alpha}}+\frac{1-\alpha}{\alpha+2}\right].\end{cases}$$

This coincides with [24, Corollary 2.13].

Remark 7. Let us consider $\alpha = 1$ and $\lambda = 0$ in Theorem 3. Then, the following inequality holds:

$$\left|\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathcal{F}(x)\,dx-\frac{1}{6}\left[\mathcal{F}(\sigma)+4\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\mathcal{F}(\delta)\right]\right|$$

$$\leq\frac{5(\delta-\sigma)}{72}\left[\left(\frac{61\left|\mathcal{F}'(\delta)\right|^q+29\left|\mathcal{F}'(\sigma)\right|^q}{90}\right)^{\frac{1}{q}}+\left(\frac{29\left|\mathcal{F}'(\delta)\right|^q+61\left|\mathcal{F}'(\sigma)\right|^q}{90}\right)^{\frac{1}{q}}\right],$$

which is given by Sarikaya et al. in [25, Theorem 10 (for $s = 1$)].

3. Concluding remarks

In this study, we use the TFIOs to prove STIs. In these inequalities, convexity of the function, Hölder and exponant-mean inequalities are applied. Furthermore, specific variable choices in the theorems, generalizations of several articles, and new findings were discovered. The researchers may deduce novel inequalities of other fractional kinds connected to these STIs in the future. Different types of convexities can also be used to demonstrate novel STIs. These kinds of inequalities will spark fresh research in a variety of mathematical domains. Furthermore, various results can be obtained from our outcomes by making individual parameter selections.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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