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Research article

The fractal-fractional Atangana-Baleanu operator for pneumonia disease: stability, statistical and numerical analyses

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Abstract: The present paper studies pneumonia transmission dynamics by using fractal-fractional operators in the Atangana-Baleanu sense. Our model predicts pneumonia transmission dynamically. Our goal is to generalize five ODEs of the first order under the assumption of five unknowns (susceptible, vaccinated, carriers, infected, and recovered). The Atangana-Baleanu operator is used in addition to analysing existence, uniqueness, and non-negativity of solutions, local and global stability, Hyers-Ulam stability, and sensitivity analysis. As long as the basic reproduction number $\mathscr{R}_0$ is less than one, the free equilibrium point is local, asymptotic, or otherwise global. Our sensitivity statistical analysis shows that $\mathscr{R}_0$ is most sensitive to pneumonia disease density. Further, we compute a numerical solution for the model by using fractal-fractional. Graphs of the results are presented for demonstration of our proposed method. The results of the Atangana-Baleanu fractal-fractional scheme is in excellent agreement with the actual data.

Keywords: fractional derivatives; nonlinear equations; simulation; numerical results; infectious disease; time varying control system
Mathematics Subject Classification: 34C60, 92B05, 92C42, 92D25, 92D30

1. Introduction

Infections of the lungs, such as pneumonia, have a variety of causes. The prevalence of this disease is increasing in all age groups and is a major medical concern. Several researchers are working on mathematical models that describe disease spread and optimal control problems in epidemics because they are highly interesting. As a result, these models play a critical role in predicting the effects of epidemics and diseases on areas and populations, as well as the environment. Researchers have presented models for modeling pneumonia dynamics based on a review of the literature; see, e.g., [1–8]. Based on a mathematical analysis of pneumonia and typhoid characteristics, Tilahun et al. [6] proposed a coinfection model. Tilahun et al. [7] used ordinary differential equations and a few theorems to model pneumonia and meningitis coinfections in 2018.

Since 1970, infectious disease dynamics has emerged as an interdisciplinary field. Epidemiology studies disease spread. Modeling diseases and their effects on humans is described in [5]. Fractional and fractal calculus are combined here. In engineering, physics, biology, and biomedicine, fractal-fractional operators are widely used to model real-world processes. Comparable to classical models, fractional order integrals and fractional derivatives are more precise than classical models. In fractional derivatives, there are three different types of operators: Riemann-Liouville and Caputo, Caputo-Fabrizio and Atangana-Baleanu, which are connected to power laws, exponential decay laws, and extended Mittag-Leffler functions [9–52].

The concept of fractal-fractional order integration and differentiation was developed by Atangana with two orders, i.e., one fractional and the other fractal [53, 54]. Besides, fractal differentiation is equivalent to classical differentiation if the fractal order tends to 1. Fractal behaviors are investigated through the use of these combined operators. Several researchers have shown that fractal-fractional operators better capture real-world mathematics [55–79]. These include but are not limited to, for instance, the HIV/AIDS model [57], Leishmania model [58], tuberculosis model [59], Q fever model [60], hepatitis C virus model [61], AH1N1/09 virus model [62] and tobacco smoking model [63].

This study investigates the formulation of the Tilahun et al. mathematical model of pneumonia transmission dynamics by using fractal-fractional derivatives in the Atangana-Baleanu sense. The model is improved by assuming five unknowns and using this Caputo, and Atangana-Baleanu-type fractional derivatives. The study aims to investigate and compare the solutions to this system, which unique as compared to other studies. The authors construct schemes for this system, using the fractal-fractional Atangana-Baleanu operator to prove the existence, uniqueness, non-negativity and boundedness of solutions. Three levels of stability are established: local, global and Hyers-Ulam. Sensitivity analysis is conducted to assess the impact of parameters on initial disease transmission. The study finds that $\mathscr{R}_0$ is most sensitive to pneumonia disease density. The results show that the scheme's method is effective and suitable for the system defined by Caputo and Atangana-Baleanu fractional derivatives. Simulations using Matlab show that the schemes method is suitable for both types of problems and has approximate solutions that are close to the exact solution. The study also discusses other fractional operators and numerically verifies their mathematical findings for the proposed model's dynamical behavior.

2. Preliminaries

2.1. Fractal-fractional pneumonia model

Definition 1 ([76, 77]). *Consider the fractal to be differentiable on (a, b) of order $0 < \tau_2 \leq 1$ for $\phi \in C((a, b), \mathbb{R})$. The following is a fractal-fractional derivative operator for $\mathtt{t}$ in the Atangana-Baleanu setting:*

$$^{\mathrm{FF-AB}}\mathscr{D}_{0,\mathtt{t}}^{\tau_1,\tau_2}\phi(\mathtt{t}) = \frac{\hbar(\tau_1)}{1-\tau_1}\frac{d}{dt^{\tau_2}}\int_0^{\mathtt{t}} \phi(s)E_{\tau_1}\left[-\frac{\tau_1}{1-\tau_1}(\mathtt{t}-s)^{\tau_1}\right]ds,$$

where, $\hbar(\tau_1) = 1 - \tau_1 + \frac{\tau_1}{\Gamma(\tau_1)}$, *and* $\frac{dh(s)}{ds^{\tau_2}} = \lim_{t\to s}\frac{t(\mathtt{t})-t(s)}{\mathtt{t}^{\tau_2}-\varsigma^{\tau_2}}$.

Definition 2 ([76, 77]). *The fractal-fractional integration operator is given by*

$$^{\mathrm{FF-AB}}I_{0,\mathtt{t}}^{\tau_1,\tau_2}\phi(\mathtt{t}) = \frac{\tau_1\tau_2}{\hbar(\tau_1)\Gamma(\tau_1)}\int_0^{\mathtt{t}} s^{\tau_2-1}\phi(s)(\mathtt{t}-s)^{\tau_1-1}ds + \frac{\tau_2\,(1-\tau_1)\,\vartheta^{\tau_2-1}}{\hbar(\tau_1)}\phi(\mathtt{t}).$$

There are five populations in the pneumonia model: susceptible (x), vaccinated (y), carrier (z), infected (u) and recovered (v). The total human population, denoted by N, can be expressed as $N = \mathrm{x}+\mathrm{y}+\mathrm{z}+\mathrm{u}+\mathrm{v}$. Therefore, our suggested fractal-fractional pneumonia model in the sense of the Atangana-Baleanu derivative looks like this:

$$\begin{aligned}
^{\mathrm{FF-AB}}\mathscr{D}_{0,\mathtt{t}}^{\tau_1,\tau_2}\,\mathrm{x}(\mathtt{t}) &= (1-\mathrm{p})\pi + \phi\,\mathrm{y}(\mathtt{t}) + \delta\,\mathrm{v}(\mathtt{t}) - (\vartheta+\mu+\lambda)\,\mathrm{x}(\mathtt{t}),\\
^{\mathrm{FF-AB}}\mathscr{D}_{0,\mathtt{t}}^{\tau_1,\tau_2}\,\mathrm{y}(\mathtt{t}) &= \mathrm{p}\pi + \vartheta\,\mathrm{x}(\mathtt{t}) - (\phi+\mu+\varepsilon\lambda)\,\mathrm{y}(\mathtt{t}),\\
^{\mathrm{FF-AB}}\mathscr{D}_{0,\mathtt{t}}^{\tau_1,\tau_2}\,\mathrm{z}(\mathtt{t}) &= \varrho\lambda\,\mathrm{x}(\mathtt{t}) + \varrho\varepsilon\lambda\,\mathrm{y}(\mathtt{t}) + \eta(1-\mathrm{q})\,\mathrm{u}(\mathtt{t}) - (\beta+\chi+\mu)\,\mathrm{z}(\mathtt{t}),\\
^{\mathrm{FF-AB}}\mathscr{D}_{0,\mathtt{t}}^{\tau_1,\tau_2}\,\mathrm{u}(\mathtt{t}) &= \lambda(1-\varrho)\,\mathrm{x}(\mathtt{t}) + \varepsilon\lambda(1-\varrho)\,\mathrm{y}(\mathtt{t}) + \chi\,\mathrm{z}(\mathtt{t}) - (\alpha+\eta+\mu)\,\mathrm{u}(\mathtt{t}),\\
^{\mathrm{FF-AB}}\mathscr{D}_{0,\mathtt{t}}^{\tau_1,\tau_2}\,\mathrm{v}(\mathtt{t}) &= \beta\,\mathrm{z}(\mathtt{t}) + \mathrm{q}\eta\,\mathrm{u}(\mathtt{t}) - (\delta+\mu)\,\mathrm{v}(\mathtt{t}),
\end{aligned} \tag{2.1}$$

subject to $\mathrm{x}(0) \geq 0,\ \mathrm{y}(0) \geq 0,\ \mathrm{z}(0) \geq 0,\ \mathrm{u}(0) \geq 0,\ \mathrm{v}(0) \geq 0.$

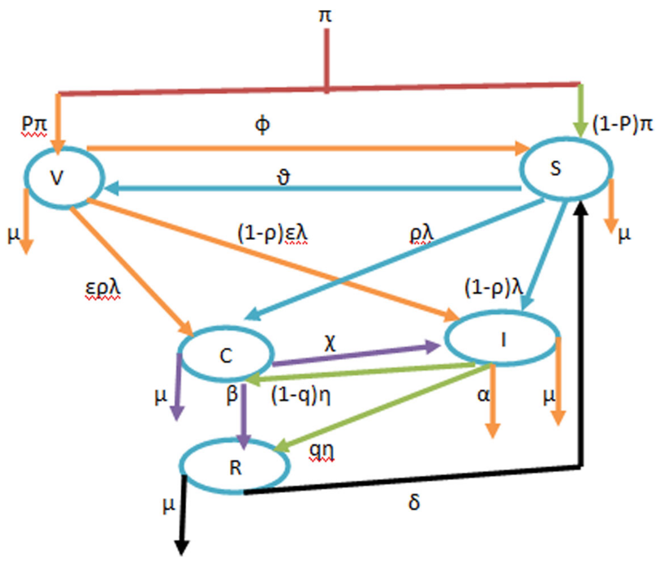


Figure 1. Model flow diagram for (2.1) with $\mathrm{x}(\mathtt{t}) = S(\mathtt{t})$, $\mathrm{y}(\mathtt{t}) = V(\mathtt{t})$, $\mathrm{z}(\mathtt{t}) = C(\mathtt{t})$, $\mathrm{u}(\mathtt{t}) = I(\mathtt{t})$, $\mathrm{v}(\mathtt{t}) = R(\mathtt{t})$ [6].

All of the positive parameters are listed in Table 1. When people get infected, they either join the carrier class z or the infectious class u based on a probability of $1-\varrho$. Let Υ be the transmission coefficient for the carrier. Infection force is defined as $\lambda = \mathrm{a}\,\mathrm{z}+\mathrm{b}\,\mathrm{u}$, where $a = \frac{k\tau\Upsilon}{N}$ represents the carrier compartment transmission and $b = \frac{k\tau}{N}$ represents the infective compartment transmission. The population is $\mathrm{N} = \mathrm{x}(\mathtt{t})+\mathrm{y}(\mathtt{t})+\mathrm{z}(\mathtt{t})+\mathrm{u}(\mathtt{t})+\mathrm{v}(\mathtt{t})$. If $\Upsilon > 1$, carriers are more likely to infect susceptibles than infectious individuals. Both carriers and infectives have the same chance of spreading when $\Upsilon = 1$. Nevertheless, if $\Upsilon < 1$, the infective has a higher chance of contacting the susceptible.

Table 1. The values of the applied parameters.

Parameters symbols	Description	Source	Values
k	The contact rate	Estimated	0.5
ϵ	The transmission coefficient for the carrier	[2]	0.002
τ	The probability that a contact causes infection	[2]	0.89 – 0.99
ϕ	The rate of the susceptible class increased from the vaccinated class	[6]	0.0025
ψ	The proportion of the serotype not covered by the vaccine	Assumed	0.2
δ	The rate at which individuals in the recovery class lose their temporary immunity	[2]	0.1
χ	The rate of infection	[2]	0.001 – 0.01096
p	The rate at which a fraction of the population was vaccinated before the disease outbreak	[2]	0.2
ϑ	The rate of population movement from the susceptible class to the vaccinated class	Assumed	0.008
μ	The natural death rate of the population in all compartments	Estimated	0.01
α	The rate of dying from the disease	Estimated	0.0057
Θ	$\Theta = k\tau$	[6]	0.05
β	Recovery rate after gaining immunity	[6]	0.0115
η	Treatment rate per capita in the infected class moving to the recovered compartment	[6]	0.2
q	Treatment efficacy	[6]	0.5 – 1
Υ	The infection force	Assumed	1.2

2.2. Existence and uniqueness

The matrix form of (1.2) is given by:

$$\begin{aligned} {}^{\mathtt{FF-AB}}\mathscr{D}_{0,\mathtt{t}}^{\tau_1,\tau_2}\Psi(\mathtt{t}) &= \Lambda(\mathtt{t},\Psi(\mathtt{t})) \\ &= (\Upsilon_1(\mathtt{t},\Psi(\mathtt{t})),\Upsilon_2(\mathtt{t},\Psi(\mathtt{t})),\Upsilon_3(\mathtt{t},\Psi(\mathtt{t})),\Upsilon_4(\mathtt{t},\Psi(\mathtt{t})),\Upsilon_5(\mathtt{t},\Psi(\mathtt{t}))), \\ \Psi(\mathtt{t}) &= (\mathrm{x}(\mathtt{t}),\mathrm{y}(\mathtt{t}),\mathrm{z}(\mathtt{t}),\mathrm{u}(\mathtt{t}),\mathrm{v}(\mathtt{t})), \quad \Psi(0) = (\mathrm{x}(0),\mathrm{y}(0),\mathrm{z}(0),\mathrm{u}(0),\mathrm{v}(0)). \end{aligned} \tag{2.2}$$

Define the Banach space $\mathbb{U} = X^5$, where $X = C(I,\mathbb{R})$ is subject to the norm

$$\|\mathscr{H}\| = \max_{\mathtt{t}\in[0,1]} |\mathrm{x}(\mathtt{t})+\mathrm{y}(\mathtt{t})+\mathrm{z}(\mathtt{t})+\mathrm{u}(\mathtt{t})+\mathrm{v}(\mathtt{t})|.$$

$$\mathscr{H}(\Psi)(\mathtt{t}) = \Psi(0) + \frac{\tau_2 \mathtt{t}^{\tau_2-1}(1-\tau_1)}{\mathrm{AB}(\tau_1)}\Lambda(\mathtt{t},\Psi(\mathtt{t})) + \frac{\tau_1\tau_2}{\mathrm{AB}(\tau_1)\Gamma(\tau_1)}\int_0^{\mathtt{t}} \xi^{\tau_2-1}(\mathtt{t}-\xi)^{\tau_2-1}\Lambda(\xi,\Psi(\xi))d\xi. \quad (2.3)$$

Let $\|\mathrm{x}\| \le \eta_1$, $\|\mathrm{y}\| \le \eta_2$, $\|\mathrm{z}\| \le \eta_3$, $\|\mathrm{u}\| \le \eta_4$ and $\|\mathrm{v}\| \le \eta_5$ for some constants $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5 > 0$.

Rewrite (2.1) as follows

$$\begin{aligned}
{}^{\mathrm{FF-AB}}\mathscr{D}_{0,\mathtt{t}}^{\tau_1,\tau_2}\,\mathrm{x}(\mathtt{t}) &= \tau_2 \mathtt{t}^{\tau_2-1}\Upsilon_1(\mathtt{t},\Psi(\mathtt{t})),\\
{}^{\mathrm{FF-AB}}\mathscr{D}_{0,\mathtt{t}}^{\tau_1,\tau_2}\,\mathrm{y}(\mathtt{t}) &= \tau_2 \mathtt{t}^{\tau_2-1}\Upsilon_2(\mathtt{t},\Psi(\mathtt{t})),\\
{}^{\mathrm{FF-AB}}\mathscr{D}_{0,\mathtt{t}}^{\tau_1,\tau_2}\,\mathrm{z}(\mathtt{t}) &= \tau_2 \mathtt{t}^{\tau_2-1}\Upsilon_3(\mathtt{t},\Psi(\mathtt{t})),\\
{}^{\mathrm{FF-AB}}\mathscr{D}_{0,\mathtt{t}}^{\tau_1,\tau_2}\,\mathrm{u}(\mathtt{t}) &= \tau_2 \mathtt{t}^{\tau_2-1}\Upsilon_4(\mathtt{t},\Psi(\mathtt{t})),\\
{}^{\mathrm{FF-AB}}\mathscr{D}_{0,\mathtt{t}}^{\tau_1,\tau_2}\,\mathrm{v}(\mathtt{t}) &= \tau_2 \mathtt{t}^{\tau_2-1}\Upsilon_5(\mathtt{t},\Psi(\mathtt{t})),
\end{aligned}$$

where

$$\begin{aligned}
\Upsilon_1(\mathtt{t},\Psi(\mathtt{t})) &= (1-\mathrm{p})\pi + \phi\,\mathrm{y}(\mathtt{t}) + \delta\,\mathrm{v}(\mathtt{t}) - (\vartheta+\mu+\lambda)\,\mathrm{x}(\mathtt{t}),\\
\Upsilon_2(\mathtt{t},\Psi(\mathtt{t})) &= \mathrm{p}\,\pi + \vartheta\,\mathrm{x}(\mathtt{t}) - (\mu+\lambda\epsilon+\phi)\,\mathrm{y}(\mathtt{t}),\\
\Upsilon_3(\mathtt{t},\Psi(\mathtt{t})) &= \varrho\lambda\,\mathrm{x}(\mathtt{t}) + \varrho\epsilon\lambda\,\mathrm{y}(\mathtt{t}) + \eta(1-\mathrm{q})\,\mathrm{u}(\mathtt{t}) - (\beta+\chi+\mu)\,\mathrm{z}(\mathtt{t}),\\
\Upsilon_4(\mathtt{t},\Psi(\mathtt{t})) &= \lambda(1-\varrho)\,\mathrm{x}(\mathtt{t}) + \varepsilon\lambda(1-\varrho)\,\mathrm{y}(\mathtt{t}) + \chi\,\mathrm{z}(\mathtt{t}) - (\alpha+\eta+\mu)\,\mathrm{u}(\mathtt{t}),\\
\Upsilon_5(\mathtt{t},\Psi(\mathtt{t})) &= \beta\,\mathrm{z}(\mathtt{t}) + \mathtt{q}\eta\,\mathrm{u}(\mathtt{t}) - (\delta+\mu)\,\mathrm{v}(\mathtt{t}).
\end{aligned}$$

Applying fractional integrals, we get

$$\Psi(\mathtt{t}) = \Psi(0) + \frac{\tau_2 \mathtt{t}^{\tau_2-1}(1-\tau_1)}{\hbar(\tau_1)}\Lambda(\mathtt{t},\Psi(\mathtt{t})) + \frac{\tau_1\tau_2}{\hbar(\tau_1)\Gamma(\tau_1)}\int_0^{\mathtt{t}} \xi^{\tau_2-1}(\mathtt{t}-\xi)^{\tau_2-1}\Lambda(\xi,\Psi(\xi))\mathrm{d}\xi.$$

$\Lambda(\mathtt{t},\Psi(\mathtt{t}))$ must satisfy these Lipschitz and growth conditions.

Theorem 1. *For each* $\Psi_1, \Psi_2 \in \mathscr{B}$, $\exists$ *a constant* $\mathrm{A} > 0$ *that satisfies*

$$|\Lambda(\mathtt{t},\Psi_1(\mathtt{t}) - \Lambda(\mathtt{t},\Psi_2(\mathtt{t}))| \leqslant \mathrm{A}\,|\Psi_1(\mathtt{t}) - \Psi_2(\mathtt{t})|, \quad (2.4)$$

where $\mathrm{A} = \max\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$, with $\omega_1 = \mu + 2\lambda + 2\lambda\varrho + 2\vartheta, \omega_2 = \mu + 2\phi + 2\varepsilon\lambda + 2\lambda\varepsilon\varrho, \omega_3 = 2\chi + \mu + 2\beta, \omega_4 = 2\mathtt{q}\eta + \mu + \alpha + 2\eta, \omega_5 = 2\delta + \mu$.

Proof. For each $\Psi_1, \Psi_2 \in \mathscr{B}$, one obtains

$$\begin{aligned}
\|\Lambda(\mathtt{t},\Psi_1(\mathtt{t}) - \Lambda(\mathtt{t},\Psi_2(\mathtt{t}))\| &\le \omega_1|\,\mathrm{x}_1 - \mathrm{x}_2\,| + \omega_2|\,\mathrm{y}_1 - \mathrm{y}_2\,|\\
&\quad + \omega_3|\,\mathrm{z}_1 - \mathrm{z}_2\,| + \omega_4|\,\mathrm{u}_1 - \mathrm{u}_2\,| + \omega_5|\,\mathrm{v}_1 - \mathrm{v}_2\,|\\
&\le \mathrm{A}\,|\Psi_1(\mathtt{t}) - \Psi_2(\mathtt{t})|.
\end{aligned}$$

Therefore, $\Lambda(\mathtt{t},\Psi(\mathtt{t})$ satisfies the Lipschitz condition. □

Theorem 2. *There are constants* $\mathbf{z}_\Psi > 0$ *and* $\mathbb{M}_\Psi$ *satisfies the following, for each* Ψ *in* $\mathscr{B}$,

$$|\Lambda(\mathtt{t},\Psi(\mathtt{t}))| \leqslant \mathbf{z}_\Psi|\Psi(\mathtt{t})| + \mathbb{M}_\Psi.$$

So, there is at least one solution to the suggested model.

Proof. To begin with, we demonstrate that the operator Λ stated in (2.2) is totally continuous. Due to the continuous nature of Ψ, N is also continuous. □

Theorem 3. *Assume that (2.4) is true; then,*

$$\Xi = \Big(\frac{\tau_2 T^{\tau_2-1}(1-\tau_1)}{\hbar(\tau_1)} + \frac{\tau_1\tau_2}{\hbar(\tau_1)\Gamma(\tau_1)} T^{\mu+\tau_2-1}\mathbf{H}(\xi,\tau_2)\Big)\mathrm{A}.$$

So, it has a unique solution.

Proof. For Ψ_1, Ψ_2 in $\mathscr{B}$, we obtain

$$\begin{aligned}
|\mathscr{H}(\Psi_1) - \mathscr{H}(\Psi_2)| &= \max_{t\in[0,T]}\Big|\frac{\tau_2 \mathtt{t}^{\tau_2-1}(1-\tau_1)}{\hbar(\tau_1)}(\Lambda(\mathtt{t},\Psi_1(\mathtt{t})) - \Lambda(\mathtt{t},\Psi_2(\mathtt{t})))\\
&+ \frac{\tau_1\tau_2}{\hbar(\tau_1)\Gamma(\tau_1)}\int_0^{\mathtt{t}} \xi^{\tau_2-1}(t-\xi)^{\tau_2-1}(\Lambda(\xi,\Psi_1(\xi)) - \Lambda(\xi,\Psi_2(\xi)))d\xi\Big|\\
&\leqslant A\left[\frac{\tau_2 \mathtt{t}^{\tau_2-1}(1-\tau_1)}{\hbar(\tau_1)} + \frac{\tau_1\tau_2}{\hbar(\tau_1)\Gamma(\tau_1)} T^{\mu+\tau_2-1}\mathbf{H}(\xi,\tau_2)\right]\|\Psi_1-\Psi_2\|\\
&\leqslant \Xi\|\Psi_1-\Psi_2\|.
\end{aligned}$$

Due to this, $\mathscr{H}$ is a contraction and there is only one solution to the model. □

The non-negativity and boundedness of the solutions of the system (2.1) in the fractional case have been proved in [8].

3. Model analysis

An infection-free equilibrium is $E^0 = (\mathrm{x}_0, \mathrm{y}_0, 0, 0, 0) = \left(\frac{\pi\aleph_1}{\mu}, \frac{\pi\aleph_2}{\mu}, 0, 0, 0\right)$, with $\aleph_1 = \frac{\mu+\phi-p\mu}{\mu+\phi+\vartheta}$ and $\aleph_2 = \frac{\vartheta+p\mu}{\mu+\phi+\vartheta}$ for z = u = v = 0. The endemic equilibrium, given as $E^* = (\mathrm{x}^*, \mathrm{y}^*, \mathrm{z}^*, \mathrm{u}^*, \mathrm{v}^*)$ for $\mathrm{u} > 0, \mathrm{z} > 0$ was acquired by applying the following:

$$\begin{cases}
{}^{\mathtt{FF-AB}}\mathscr{D}_{0,\mathtt{t}}^{\tau_1,\tau_2}\,\mathrm{x}(\mathtt{t}) = 0,\\
{}^{\mathtt{FF-AB}}\mathscr{D}_{0,\mathtt{t}}^{\tau_1,\tau_2}\,\mathrm{y}(\mathtt{t}) = 0,\\
{}^{\mathtt{FF-AB}}\mathscr{D}_{0,\mathtt{t}}^{\tau_1,\tau_2}\,\mathrm{z}(\mathtt{t}) = 0,\\
{}^{\mathtt{FF-AB}}\mathscr{D}_{0,\mathtt{t}}^{\tau_1,\tau_2}\,\mathrm{u}(\mathtt{t}) = 0,\\
{}^{\mathtt{FF-AB}}\mathscr{D}_{0,\mathtt{t}}^{\tau_1,\tau_2}\,\mathrm{v}(\mathtt{t}) = 0.
\end{cases}$$

Thus, $E^* = (\mathrm{x}^*, \mathrm{y}^*, \mathrm{z}^*, \mathrm{u}^*, \mathrm{v}^*) = (\mathrm{E}_5 + \mathrm{E}_6\,\mathrm{u}^*, \mathrm{E}_3 + \mathrm{E}_4\,\mathrm{E}_5 + \mathrm{E}_4\,\mathrm{E}_6\,\mathrm{u}^*, \mathrm{E}_1\,\mathrm{u}^*, \mathrm{u}^*, \mathrm{E}_2\,\mathrm{u}^*)$, with

$$\begin{aligned}
\mathrm{E}_1 &= \frac{(1-\varrho)\eta(1-\mathtt{q}) + \varrho\Phi}{(1-\varrho)(\beta+\chi+\mu) + \varrho\chi}, \quad \mathrm{E}_2 = \frac{\beta\,\mathrm{E}_1 + \mathtt{q}\eta}{\delta+\mu}, \quad \mathrm{E}_3 = \frac{\mathtt{p}\pi}{\phi+\mu+\varepsilon\lambda},\\
\mathrm{E}_4 &= \frac{\theta}{\phi+\mu+\varepsilon\lambda}, \quad \mathrm{E}_5 = \frac{(1-\mathtt{p})\pi + \phi\,\mathrm{E}_3}{\mu+\lambda+\theta-\phi\,\mathrm{E}_4}, \quad \mathrm{E}_6 = \frac{\delta\,\mathrm{E}_2}{\mu+\lambda+\theta-\phi\,\mathrm{E}_4},\\
\mathrm{u}^* &= \frac{\mathtt{p}\pi + \theta\,\mathrm{E}_5 - (\phi+\mu+\varepsilon\lambda)(\mathrm{E}_3 + \mathrm{E}_4\,\mathrm{E}_5)}{-\Psi\,\mathrm{E}_6 + (\mu+\varepsilon\lambda+(\alpha+\eta+\mu))\,\mathrm{E}_4\,\mathrm{E}_6}.
\end{aligned}$$

The basic reproduction number is given by [7, 8],

$$\mathscr{R}_0 = \frac{[\varrho\, \mathrm{b}\chi + \mathrm{a}\,\varrho(\alpha+\eta+\mu) + \mathrm{a}\eta(1-\varrho)(1-\mathrm{q}) + \mathrm{b}(\beta+\chi+\mu)(1-\varrho)]}{(\beta+\chi+\mu)(\alpha+\eta+\mu) - \eta(1-\mathrm{q})\chi}(\mathrm{x}_0 + \epsilon\,\mathrm{y}_0).$$

3.1. Stability of an infection-free equilibrium E_0

Lemma 1. *If $\mathscr{R}_0 = 1$, E_0 is locally asymptotically stable for model (2.1), and unstable if $\mathscr{R}_0 > 1$. Moreover, model (2.1) has a globally asymptotically stable E_0.*

Proof. The first part follows as in [8]. We present a positive definite Lyapunov function:

$$L_1 = \left(\mathrm{x} - \mathrm{x}_0 - \mathrm{x}_0 \ln \frac{\mathrm{x}}{\mathrm{x}_0}\right) + \left(\mathrm{y} - \mathrm{y}_0 - \mathrm{y}_0 \ln \frac{\mathrm{y}}{\mathrm{y}_0}\right).$$

One obtains

$$\begin{aligned}
{}^{\mathrm{FF-AB}}\mathscr{D}_{0,\mathrm{t}}^{\tau_1,\tau_2} L_1 &\le \left(\frac{\mathrm{x}-\mathrm{x}_0}{\mathrm{x}}\right){}^{\mathrm{FF-AB}}\mathscr{D}_{0,\mathrm{t}}^{\tau_1,\tau_2}\,\mathrm{x} + \left(\frac{\mathrm{y}-\mathrm{y}_0}{\mathrm{y}}\right){}^{\mathrm{FF-AB}}\mathscr{D}_{0,\mathrm{t}}^{\tau_1,\tau_2}\,\mathrm{y} \\
&= \left(\frac{\mathrm{x}-\mathrm{x}_0}{\mathrm{x}}\right)\Big((1-\mathrm{p})\pi + \phi\,\mathrm{y}(\mathrm{t}) + \delta\,\mathrm{v}(\mathrm{t}) - (\vartheta+\mu+\lambda)\,\mathrm{x}(\mathrm{t})\Big) \\
&+ \left(\frac{\mathrm{y}-\mathrm{y}_0}{\mathrm{y}}\right)\Big(\mathrm{p}\pi + \vartheta\,\mathrm{x}(\mathrm{t}) - (\phi+\mu+\varepsilon\lambda)\,\mathrm{y}(\mathrm{t})\Big).
\end{aligned}$$

At E_0, one obtains

$$\begin{aligned}
{}^{\mathrm{FF-AB}}\mathscr{D}_{0,\mathrm{t}}^{\tau_1,\tau_2} L_1 &\le \left(\frac{\mathrm{x}-\mathrm{x}_0}{\mathrm{x}}\right){}^{\mathrm{FF-AB}}\mathscr{D}_{0,\mathrm{t}}^{\tau_1,\tau_2}\,\mathrm{x} + \left(\frac{\mathrm{y}-\mathrm{y}_0}{\mathrm{y}}\right){}^{\mathrm{FF-AB}}\mathscr{D}_{0,\mathrm{t}}^{\tau_1,\tau_2}\,\mathrm{y} \\
&= \left(\mathrm{x}-\mathrm{x}_0\right)\left(\frac{(1-\mathrm{p})\pi}{\mathrm{x}} + \frac{\phi\,\mathrm{y}}{\mathrm{x}} + \frac{\delta\,\mathrm{v}}{\mathrm{x}} - (\vartheta+\mu+\lambda)\right) \\
&+ \left(\mathrm{y}-\mathrm{y}_0\right)\left(\frac{\mathrm{p}\pi}{\mathrm{y}} + \frac{\vartheta\,\mathrm{x}}{\mathrm{y}} - (\phi+\mu+\varepsilon\lambda)\right) \\
&= -\frac{(1-\mathrm{p})\pi}{\mathrm{x}\,\mathrm{x}_0}\left(\mathrm{x}-\mathrm{x}_0\right)^2 - \frac{(\alpha+\eta+\mu)V}{\mathrm{x}\,\mathrm{x}_0}\left(\mathrm{x}-\mathrm{x}_0\right)^2 \\
&- \frac{\delta\,\mathrm{v}}{\mathrm{x}\,\mathrm{x}_0}\left(\mathrm{x}-\mathrm{x}_0\right)^2 - \frac{\mathrm{p}\pi}{\mathrm{y}\,\mathrm{y}_0}\left(\mathrm{y}-\mathrm{y}_0\right)^2 - \frac{\vartheta\,\mathrm{x}}{\mathrm{y}\,\mathrm{y}_0}\left(\mathrm{y}-\mathrm{y}_0\right)^2.
\end{aligned}$$

Thus, ${}^{\mathrm{FF-AB}}\mathscr{D}_{0,\mathrm{t}}^{\tau_1,\tau_2} L_1 < 0$ for all $(\mathrm{x}, \mathrm{y}, \mathrm{z}, \mathrm{u}, \mathrm{v}) \in \Lambda$. Moreover, ${}^{\mathrm{FF-AB}}\mathscr{D}_{0,\mathrm{t}}^{\tau_1,\tau_2} L_1 = 0$ implies that $\mathrm{x} = \mathrm{x}_0$, $\mathrm{y} = \mathrm{y}_0$, $\mathrm{z} = \mathrm{z}_0$, $\mathrm{u} = \mathrm{u}_0$ and $\mathrm{v} = \mathrm{v}_0$. So, $\{E_0\}$ is the only set satisfying that ${}^{\mathrm{FF-AB}}\mathscr{D}_{0,\mathrm{t}}^{\tau_1,\tau_2} L_1 = 0$. □

3.2. Stability of an endemic equilibrium E^*

Lemma 2. *E^* exists when $\mathscr{R}_0 > 1$; otherwise, there is no endemic equilibrium.*

Proof. The following characteristics are required for a disease to be endemic: ${}^{\mathrm{FF-AB}}\mathscr{D}_{0,\mathrm{t}}^{\tau_1,\tau_2}\,\mathrm{z}(\mathrm{t}) > 0$ and ${}^{\mathrm{FF-AB}}\mathscr{D}_{0,\mathrm{t}}^{\tau_1,\tau_2}\,\mathrm{u}(\mathrm{t}) > 0$, that is,

$$\begin{aligned}
{}^{\mathrm{FF-AB}}\mathscr{D}_{0,\mathrm{t}}^{\tau_1,\tau_2}\,\mathrm{z}(\mathrm{t}) &= \varrho\lambda\,\mathrm{x}(\mathrm{t}) + \varrho\varepsilon\lambda\,\mathrm{y}(\mathrm{t}) + \eta(1-\mathrm{q})\,\mathrm{u}(\mathrm{t}) - (\beta+\chi+\mu)\,\mathrm{z}(\mathrm{t}) > 0, \\
{}^{\mathrm{FF-AB}}\mathscr{D}_{0,\mathrm{t}}^{\tau_1,\tau_2}\,\mathrm{u}(\mathrm{t}) &= \lambda(1-\varrho)\,\mathrm{x}(\mathrm{t}) + \varepsilon\lambda(1-\varrho)\,\mathrm{y}(\mathrm{t}) + \chi\,\mathrm{z}(\mathrm{t}) - (\alpha+\eta+\mu)\,\mathrm{u}(\mathrm{t}) > 0.
\end{aligned} \tag{3.1}$$

Given (3.1), based on the first inequality,

$$(\beta+\chi+\mu)\,\mathrm{z}(\mathtt{t}) < \varrho\lambda\,\mathrm{x}(\mathtt{t}) + \varrho\varepsilon\lambda\,\mathrm{y}(\mathtt{t}) + \eta(1-\mathtt{q})\,\mathrm{u}(\mathtt{t}).$$

Then,

$$\mathrm{z}(\mathtt{t}) < \frac{\varrho\alpha\Big(\frac{\mathrm{u}(\mathtt{t})+\Upsilon\,\mathrm{y}(\mathtt{t})}{N}\Big)(\mathrm{x}(\mathtt{t})+\varepsilon\,\mathrm{y}(\mathtt{t})) + \eta(1-\mathtt{q})\,\mathrm{u}(\mathtt{t})}{(\beta+\chi+\mu)}.$$

Because $\frac{(\mathrm{x}(\mathtt{t})+\varepsilon\,\mathrm{y}(\mathtt{t}))}{N} \leqslant 1$, one obtains

$$\mathrm{z}(\mathtt{t}) < \frac{\varrho\alpha\,\mathrm{u}(\mathtt{t}) + \eta(1-\mathtt{q})\,\mathrm{u}(\mathtt{t})}{(\beta+\chi+\mu)-\varrho\alpha\Upsilon}. \tag{3.2}$$

As a result of the second inequality of (3.1),

$$(\alpha+\eta+\mu)\,\mathrm{u}(\mathtt{t}) < \lambda(1-\varrho)\,\mathrm{x}(\mathtt{t}) + \varepsilon\lambda(1-\varrho)\,\mathrm{y}(\mathtt{t}) + \chi\,\mathrm{y}(\mathtt{t}).$$

Then,

$$\mathrm{u}(\mathtt{t}) < \frac{(1-\varrho)\alpha\Big(\frac{\mathrm{u}(\mathtt{t})+\Upsilon\,\mathrm{y}(\mathtt{t})}{N}\Big)(\mathrm{x}(\mathtt{t})+\varepsilon\,\mathrm{y}(\mathtt{t})) + \chi\,\mathrm{y}(\mathtt{t})}{(\alpha+\eta+\mu)}.$$

Using the fact that $\frac{(\mathrm{x}(\mathtt{t})+\varepsilon\,\mathrm{y}(\mathtt{t}))}{N} \leqslant 1$, one obtains

$$\mathrm{u}(\mathtt{t}) < \frac{(1-\varrho)\alpha\,\mathrm{u}(\mathtt{t}) + (1-\varrho)\alpha\Upsilon\,\mathrm{y}(\mathtt{t}) + \chi\,\mathrm{y}(\mathtt{t})}{(\alpha+\eta+\mu)}. \tag{3.3}$$

Substituting (3.2) into (3.3), one obtains

$$\mathrm{u}(\mathtt{t}) < \frac{(1-\varrho)\alpha\,\mathrm{u}((\beta+\chi+\mu)-\varrho\alpha\Upsilon) + (1-\varrho)\alpha\Upsilon(\varrho\alpha\,\mathrm{u} + \eta(1-\mathtt{q})\,\mathrm{u}) + \chi(\varrho\alpha\,\mathrm{u} + \eta(1-\mathtt{q})\,\mathrm{u})}{(\beta+\chi+\mu-\varrho\alpha\Upsilon)(\alpha+\eta+\mu)}.$$

After rearranging and canceling $\mathrm{u}(\mathtt{t})$, one gets

$$\begin{aligned}1 &< \alpha\Big[\frac{(1-\varrho)(\Upsilon\eta(1-\mathtt{q})+(\beta+\chi+\mu))}{(\alpha+\eta+\mu)(\beta+\chi+\mu)-\chi\eta(1-\mathtt{q})} + \frac{\varrho(\Upsilon(\alpha+\eta+\mu)+\chi)}{(\alpha+\eta+\mu)(\beta+\chi+\mu)-\chi\eta(1-\mathtt{q})}\Big]\\ &\leqslant \alpha\Big[\frac{(1-\varrho)(\Upsilon\eta(1-\mathtt{q})+(\beta+\chi+\mu))}{(\beta+\chi+\mu)(\alpha+\eta+\mu)-\eta(1-\mathtt{q})\chi} + \frac{\varrho(\Upsilon(\alpha+\eta+\mu)+\chi)}{(\beta+\chi+\mu)(\alpha+\eta+\mu)-\eta(1-\mathtt{q})\chi}\Big]\Big(\frac{\pi\aleph_1}{\mu} + \frac{\pi\aleph_2}{\mu}\Big)\\ &= \mathscr{R}_0.\end{aligned}$$

Thus, $\mathscr{R}_0 > 1$ creates a unique endemic equilibrium. □

Lemma 3 ([8]). *E^* is locally asymptotically stable for $\mathscr{R}_0 > 1$. Moreover, E^* is globally asymptotically stable.*

Proof. Define

$$\begin{aligned}L_2 &= \Big(\mathrm{x}-\mathrm{x}^*-\mathrm{x}^*\ln\frac{\mathrm{x}}{\mathrm{x}^*}\Big) + \Big(\mathrm{y}-\mathrm{y}^*-\mathrm{y}^*\ln\frac{\mathrm{y}}{\mathrm{y}^*}\Big) + \Big(\mathrm{z}-\mathrm{z}^*-\mathrm{z}^*\ln\frac{\mathrm{z}}{\mathrm{z}^*}\Big)\\ &\quad + \Big(\mathrm{u}-\mathrm{u}^*-\mathrm{u}^*\ln\frac{\mathrm{u}}{\mathrm{u}^*}\Big) + \Big(\mathrm{v}-\mathrm{v}^*-\mathrm{v}^*\ln\frac{\mathrm{v}}{\mathrm{v}^*}\Big).\end{aligned}$$

Thus, one obtains

$$
\begin{aligned}
{}^{\mathtt{FF-AB}}\mathscr{D}_{0,\mathrm{t}}^{\tau_1,\tau_2} L_2 &\leq \Big(\frac{\mathrm{x}-\mathrm{x}^*}{\mathrm{x}}\Big){}^{\mathtt{FF-AB}}\mathscr{D}_{0,\mathrm{t}}^{\tau_1,\tau_2}\,\mathrm{x} + \Big(\frac{\mathrm{y}-\mathrm{y}^*}{\mathrm{y}}\Big){}^{\mathtt{FF-AB}}\mathscr{D}_{0,\mathrm{t}}^{\tau_1,\tau_2}\,\mathrm{y} + \Big(\frac{\mathrm{z}-\mathrm{z}^*}{\mathrm{z}}\Big){}^{\mathtt{FF-AB}}\mathscr{D}_{0,\mathrm{t}}^{\tau_1,\tau_2}\,\mathrm{z} \\
&+ \Big(\frac{\mathrm{u}-\mathrm{u}^*}{\mathrm{u}}\Big){}^{\mathtt{FF-AB}}\mathscr{D}_{0,\mathrm{t}}^{\tau_1,\tau_2}\,\mathrm{u} + \Big(\frac{\mathrm{v}-\mathrm{v}^*}{\mathrm{v}}\Big){}^{\mathtt{FF-AB}}\mathscr{D}_{0,\mathrm{t}}^{\tau_1,\tau_2}\,\mathrm{v} \\
&= -\Big(\mathrm{x}-\mathrm{x}^*\Big)^2\frac{(1-\mathrm{p})\pi}{\mathrm{x}\,\mathrm{x}^*} - \Big(\mathrm{x}-\mathrm{x}^*\Big)^2\frac{(\alpha+\eta+\mu)\,\mathrm{y}}{\mathrm{x}\,\mathrm{x}^*} - \Big(\mathrm{x}-\mathrm{x}^*\Big)^2\frac{\delta\,\mathrm{v}}{\mathrm{x}\,\mathrm{x}^*} - \Big(\mathrm{y}-\mathrm{y}^*\Big)^2\frac{\mathrm{p}\pi}{\mathrm{y}\,\mathrm{y}^*} \\
&- \Big(\mathrm{y}-\mathrm{y}^*\Big)^2\frac{\vartheta\,\mathrm{x}}{\mathrm{y}\,\mathrm{y}^*} - \Big(\mathrm{z}-\mathrm{z}^*\Big)^2\frac{\varrho\lambda\,\mathrm{x}}{\mathrm{z}\,\mathrm{z}^*} - \Big(\mathrm{z}-\mathrm{z}^*\Big)^2\frac{\varrho\varepsilon\lambda\,\mathrm{y}}{\mathrm{z}\,\mathrm{z}^*} - \Big(\mathrm{z}-\mathrm{z}^*\Big)^2\frac{\eta(1-\mathrm{q})\,\mathrm{u}}{\mathrm{z}\,\mathrm{z}^*} - \Big(\mathrm{u}-\mathrm{u}^*\Big)^2 \\
&\times \frac{\lambda(1-\varrho)\,\mathrm{x}}{\mathrm{u}\,\mathrm{u}^*} - \Big(\mathrm{u}-\mathrm{u}^*\Big)^2\frac{\varepsilon\lambda(1-\varrho)\,\mathrm{y}}{\mathrm{u}\,\mathrm{u}^*} - \Big(\mathrm{u}-\mathrm{u}^*\Big)^2\frac{\chi\,\mathrm{z}}{\mathrm{u}\,\mathrm{u}^*} - \Big(\mathrm{v}-\mathrm{v}^*\Big)^2\frac{\beta\,\mathrm{z}}{\mathrm{v}\,\mathrm{v}^*} - \Big(\mathrm{v}-\mathrm{v}^*\Big)^2\frac{\mathrm{q}\eta\,\mathrm{u}}{\mathrm{v}\,\mathrm{v}^*}.
\end{aligned}
$$

Thus, ${}^{\mathtt{FF-AB}}\mathscr{D}_{0,\mathrm{t}}^{\tau_1,\tau_2} L_2 < 0$ for all $(\mathrm{x},\mathrm{y},\mathrm{z},\mathrm{u},\mathrm{v}) \in \Lambda$. Furthermore, ${}^{\mathtt{FF-AB}}\mathscr{D}_{0,\mathrm{t}}^{\tau_1,\tau_2} L_2 = 0$ implies that $\mathrm{x} = \mathrm{x}^*$, $\mathrm{y} = \mathrm{y}^*$, $\mathrm{z} = \mathrm{z}^*$, $\mathrm{u} = \mathrm{u}^*$, and $\mathrm{v} = \mathrm{v}^*$. Therefore, according to Theorem 5, E^* is globally asymptotically stable. □

3.3. Hyers-Ulam stability

The Hyers-Ulam stability has been motivated by the work done in [80, 81].

Definition 3. *The constants $\zeta_i > 0$, for $i \in \mathbb{N}_1^5$ must meet the following conditions for every $\zeta_i > 0$, $i \in \mathbb{N}_1^5$, for model (2.1) to have Hyers-Ulam stability:*

$$
\begin{aligned}
&\Big|\mathrm{x}(\mathtt{t}) - \frac{\tau_2(1-\tau_1)\mathtt{t}^{\tau_2-1}}{\hbar(\tau_1)}\Upsilon_1(\mathtt{t},\Psi(\mathtt{t})) - \frac{\tau_1\tau_2}{\hbar(\tau_1)\Gamma(\tau_1)}\int_0^{\mathtt{t}}\xi^{\tau_2-1}(\mathtt{t}-\xi)^{\tau_1-1}\Upsilon_1(\xi,\Psi(\xi))d\xi\Big| \leq \zeta_1, \\
&\Big|\mathrm{y}(\mathtt{t}) - \frac{\tau_2(1-\tau_1)\mathtt{t}^{\tau_2-1}}{\hbar(\tau_1)}\Upsilon_2(\mathtt{t},\Psi(\mathtt{t})) - \frac{\tau_1\tau_2}{\hbar(\tau_1)\Gamma(\tau_1)}\int_0^{\mathtt{t}}\xi^{\tau_2-1}(\mathtt{t}-\xi)^{\tau_1-1}\Upsilon_2(\xi,\Psi(\xi))d\xi\Big| \leq \zeta_2, \\
&\Big|\mathrm{z}(\mathtt{t}) - \frac{\tau_2(1-\tau_1)\mathtt{t}^{\tau_2-1}}{\hbar(\tau_1)}\Upsilon_3(\mathtt{t},\Psi(\mathtt{t})) - \frac{\tau_1\tau_2}{\hbar(\tau_1)\Gamma(\tau_1)}\int_0^{\mathtt{t}}\xi^{\tau_2-1}(\mathtt{t}-\xi)^{\tau_1-1}\Upsilon_3(\xi,\Psi(\xi))d\xi\Big| \leq \zeta_3, \\
&\Big|\mathrm{u}(\mathtt{t}) - \frac{\tau_2(1-\tau_1)\mathtt{t}^{\tau_2-1}}{\hbar(\tau_1)}\Upsilon_4(\mathtt{t},\Psi(\mathtt{t})) - \frac{\tau_1\tau_2}{\hbar(\tau_1)\Gamma(\tau_1)}\int_0^{\mathtt{t}}\xi^{\tau_2-1}(\mathtt{t}-\xi)^{\tau_1-1}\Upsilon_4(\xi,\Psi(\xi))d\xi\Big| \leq \zeta_4, \\
&\Big|\mathrm{v}(\mathtt{t}) - \frac{\tau_2(1-\tau_1)\mathtt{t}^{\tau_2-1}}{\hbar(\tau_1)}\Upsilon_5(\mathtt{t},\Psi(\mathtt{t})) - \frac{\tau_1\tau_2}{\hbar(\tau_1)\Gamma(\tau_1)}\int_0^{\mathtt{t}}\xi^{\tau_2-1}(\mathtt{t}-\xi)^{\tau_1-1}\Upsilon_5(\xi,\Psi(\xi))d\xi\Big| \leq \zeta_5.
\end{aligned}
$$

In the model (2.1), an approximation is $\Big(\mathrm{x}_1(\mathtt{t}),\mathrm{y}_1(\mathtt{t}),\mathrm{z}_1(\mathtt{t}),\mathrm{u}_1(\mathtt{t}),\mathrm{v}_1(\mathtt{t})\Big)$, which satisfies the following:

$$
\begin{aligned}
\mathrm{x}_1(\mathtt{t}) &= \frac{\tau_2(1-\tau_1)\mathtt{t}^{\tau_2-1}}{\hbar(\tau_1)}\Upsilon_1(\mathtt{t},\mathrm{x}_1(\mathtt{t})) + \frac{\tau_1\tau_2}{\hbar(\tau_1)\Gamma(\tau_1)}\int_0^{\mathtt{t}}\xi^{\tau_2-1}(\mathtt{t}-\xi)^{\tau_1-1}\Upsilon_1(\xi,\mathrm{x}_1(\xi))d\xi, \\
\mathrm{y}_1(\mathtt{t}) &= \frac{\tau_2(1-\tau_1)\mathtt{t}^{\tau_2-1}}{\hbar(\tau_1)}\Upsilon_2(\mathtt{t},\mathrm{y}_1(\mathtt{t})) + \frac{\tau_1\tau_2}{\hbar(\tau_1)\Gamma(\tau_1)}\int_0^{\mathtt{t}}\xi^{\tau_2-1}(\mathtt{t}-\xi)^{\tau_1-1}\Upsilon_2(\xi,\mathrm{y}_1(\xi))d\xi, \\
\mathrm{z}_1(\mathtt{t}) &= \frac{\tau_2(1-\tau_1)\mathtt{t}^{\tau_2-1}}{\hbar(\tau_1)}\Upsilon_3(\mathtt{t},\mathrm{z}_1(\mathtt{t})) + \frac{\tau_1\tau_2}{\hbar(\tau_1)\Gamma(\tau_1)}\int_0^{\mathtt{t}}\xi^{\tau_2-1}(\mathtt{t}-\xi)^{\tau_1-1}\Upsilon_3(\xi,\mathrm{z}_1(\xi))d\xi,
\end{aligned}
$$

$$\mathrm{u}_1(\mathtt{t}) = \frac{\tau_2(1-\tau_1)\mathtt{t}^{\tau_2-1}}{\hbar(\tau_1)}\Upsilon_4(\mathtt{t},\mathrm{u}_1(\mathtt{t})) + \frac{\tau_1\tau_2}{\hbar(\tau_1)\Gamma(\tau_1)}\int_0^{\mathtt{t}} \xi^{\tau_2-1}(\mathtt{t}-\xi)^{\tau_1-1}\Upsilon_4(\xi,\mathrm{u}_1(\xi))d\xi,$$

$$\mathrm{v}_1(\mathtt{t}) = \frac{\tau_2(1-\tau_1)\mathtt{t}^{\tau_2-1}}{\hbar(\tau_1)}\Upsilon_5(\mathtt{t},\mathrm{v}_1(\mathtt{t})) + \frac{\tau_1\tau_2}{\hbar(\tau_1)\Gamma(\tau_1)}\int_0^{\mathtt{t}} \xi^{\tau_2-1}(\mathtt{t}-\xi)^{\tau_1-1}\Upsilon_5(\xi,\mathrm{v}_1(\xi))d\xi,$$

so that

$$\begin{aligned}
\left|\mathrm{x}-\mathrm{x}_1\right| &\leq \mathrm{v}_1\,\omega_1,\\
\left|\mathrm{y}-\mathrm{y}_1\right| &\leq \mathrm{v}_2\,\omega_2,\\
\left|\mathrm{z}-\mathrm{z}_1\right| &\leq \mathrm{v}_3\,\omega_3,\\
\left|\mathrm{u}-\mathrm{u}_1\right| &\leq \mathrm{v}_4\,\omega_4,\\
\left|\mathrm{v}-\mathrm{v}_1\right| &\leq \mathrm{v}_5\,\omega_5.
\end{aligned} \tag{3.4}$$

Theorem 4. *If (3.1) is true, then model (2.1) has Hyers-Ulam stability.*

Proof.

$$\begin{aligned}
\left|\mathrm{x}-\mathrm{x}_1\right| &= \Big|\frac{\tau_2(1-\tau_1)\mathtt{t}^{\tau_2-1}}{\hbar(\tau_1)}\Big(\Upsilon_1(\mathtt{t},\mathrm{x}(\mathtt{t}))-\Upsilon_1(\mathtt{t},\mathrm{x}_1(\mathtt{t}))\Big)\\
&\quad+\frac{\tau_1\tau_2}{\hbar(\tau_1)\Gamma(\tau_1)}\int_0^{\mathtt{t}}\xi^{\tau_2-1}(\mathtt{t}-\xi)^{\tau_1-1}\Big(\Upsilon_1(\xi,\mathrm{x}(\xi))-\Upsilon_1(\xi,\mathrm{x}_1(\xi))\Big)d\xi\Big|\\
&\leq \frac{\tau_2(1-\tau_1)\mathtt{t}^{\tau_2-1}}{\hbar(\tau_1)}\omega_1\left\|\mathrm{x}-\mathrm{x}_1\right\| + \frac{\tau_1\tau_2}{\hbar(\tau_1)\Gamma(\tau_1)}\int_0^{\mathtt{t}}\xi^{\tau_2-1}(\mathtt{t}-\xi)^{\tau_1-1}\omega_1\left\|\mathrm{x}-\mathrm{x}_1\right\|d\xi\\
&\leq \Big(\frac{\tau_2(1-\tau_1)}{\hbar(\tau_1)}+\frac{\tau_1\tau_2\Gamma(\tau_2)}{\hbar(\tau_1)\Gamma(\tau_1+\tau_2)}\Big)\omega_1\left\|\mathrm{x}-\mathrm{x}_1\right\|.
\end{aligned}$$

Then,

$$\left|\mathrm{x}-\mathrm{x}_1\right| \leq \mathrm{v}_1\,\omega_1,\ \text{ with }\ \mathrm{v}_1 = \Big(\frac{\tau_2(1-\tau_1)}{\hbar(\tau_1)}+\frac{\tau_1\tau_2\Gamma(\tau_2)}{\hbar(\tau_1)\Gamma(\tau_1+\tau_2)}\Big)\left\|\mathrm{x}-\mathrm{x}_1\right\|.$$

Similarly, one obtains

$$\begin{aligned}
\left|\mathrm{y}-\mathrm{y}_1\right| &\leq \mathrm{v}_2\,\omega_2,\ \text{ with }\ \mathrm{v}_2 = \Big(\frac{\tau_2(1-\tau_1)}{\hbar(\tau_1)}+\frac{\tau_1\tau_2\Gamma(\tau_2)}{\hbar(\tau_1)\Gamma(\tau_1+\tau_2)}\Big)\left\|\mathrm{y}-\mathrm{y}_1\right\|,\\
\left|\mathrm{z}-\mathrm{z}_1\right| &\leq \mathrm{v}_3\,\omega_3,\ \text{ with }\ \mathrm{v}_3 = \Big(\frac{\tau_2(1-\tau_1)}{\hbar(\tau_1)}+\frac{\tau_1\tau_2\Gamma(\tau_2)}{\hbar(\tau_1)\Gamma(\tau_1+\tau_2)}\Big)\left\|\mathrm{z}-\mathrm{z}_1\right\|,\\
\left|\mathrm{u}-\mathrm{u}_1\right| &\leq \mathrm{v}_4\,\omega_4,\ \text{ with }\ \mathrm{v}_4 = \Big(\frac{\tau_2(1-\tau_1)}{\hbar(\tau_1)}+\frac{\tau_1\tau_2\Gamma(\tau_2)}{\hbar(\tau_1)\Gamma(\tau_1+\tau_2)}\Big)\left\|\mathrm{u}-\mathrm{u}_1\right\|,\\
\left|\mathrm{v}-\mathrm{v}_1\right| &\leq \mathrm{v}_5\,\omega_5,\ \text{ with }\ \mathrm{v}_5 = \Big(\frac{\tau_2(1-\tau_1)}{\hbar(\tau_1)}+\frac{\tau_1\tau_2\Gamma(\tau_2)}{\hbar(\tau_1)\Gamma(\tau_1+\tau_2)}\Big)\left\|\mathrm{v}-\mathrm{v}_1\right\|.
\end{aligned}$$

Hence, the results follows. □

3.4. Sensitivity analysis

According to the parameters of our model, the following equation yields the sensitivity index of $\mathscr{R}_0$:

$$\Gamma_{\omega}^{\mathscr{R}_0} = \frac{\partial \mathscr{R}_0}{\partial \omega} \times \frac{\omega}{\mathscr{R}_0}, \tag{3.5}$$

where ω is a value from Table 1. Table 1 lists the sensitivity indices of $\mathscr{R}_0$. It is easy to verify that

$$\frac{\partial \mathscr{R}_0}{\partial \varrho} = \frac{[\mathrm{b}\chi + \mathrm{a}(\alpha+\eta+\mu) - (\mathrm{a}\eta(1-\mathrm{q}) + \mathrm{b}(\beta+\chi+\mu))]}{(\beta+\chi+\mu)(\alpha+\eta+\mu) - \eta(1-\mathrm{q})\chi}(\mathrm{x}_0 + \varepsilon\,\mathrm{y}_0) = -0.0025 < 0,$$

$$\begin{aligned}\frac{\partial \mathscr{R}_0}{\partial \mu} &= \frac{(a\varrho + \mathrm{b}(1-\varrho))\Big((\beta+\chi+\mu)(\alpha+\eta+\mu) - \eta(1-\mathrm{q})\chi\Big)}{\Big((\beta+\chi+\mu)(\alpha+\eta+\mu) - \eta(1-\mathrm{q})\chi\Big)^2}(\mathrm{x}_0 + \varepsilon\,\mathrm{y}_0)\\ &- \frac{[\varrho(\mathrm{b}\chi + \mathrm{a}(\alpha+\eta+\mu)) + (1-\varrho)(\mathrm{a}\eta(1-\mathrm{q}) + \mathrm{b}(\beta+\chi+\mu))]\Big(2\alpha+\eta+\mu+\beta+\chi\Big)}{\Big((\beta+\chi+\mu)(\alpha+\eta+\mu) - \eta(1-\mathrm{q})\chi\Big)^2}(\mathrm{x}_0 + \varepsilon\,\mathrm{y}_0)\\ &= 1.1058 > 0,\end{aligned}$$

$$\begin{aligned}\frac{\partial \mathscr{R}_0}{\partial \alpha} &= \frac{\Big(a\varrho(\beta+\chi+\mu)(\alpha+\eta+\mu) - a\varrho\eta(1-\mathrm{q})\chi\Big)}{\Big((\beta+\chi+\mu)(\alpha+\eta+\mu) - \eta(1-\mathrm{q})\chi\Big)^2}(\mathrm{x}_0 + \varepsilon\,\mathrm{y}_0)\\ &- \frac{[\varrho(\mathrm{b}\chi + \mathrm{a}(\alpha+\eta+\mu)) + (1-\varrho)(\mathrm{a}\eta(1-\mathrm{q}) + \mathrm{b}(\beta+\chi+\mu))]\Big(\beta+\chi+\mu\Big)}{\Big((\beta+\chi+\mu)(\alpha+\eta+\mu) - \eta(1-\mathrm{q})\chi\Big)^2}(\mathrm{x}_0 + \varepsilon\,\mathrm{y}_0)\\ &= -0.1418 < 0,\end{aligned}$$

$$\begin{aligned}\frac{\partial \mathscr{R}_0}{\partial \eta} &= \frac{\mathrm{a}(\varrho + (1-\varrho)(1-\mathrm{q}))\Big((\beta+\chi+\mu)(\alpha+\eta+\mu) - \eta(1-\mathrm{q})\chi\Big)}{\Big((\beta+\chi+\mu)(\alpha+\eta+\mu) - \eta(1-\mathrm{q})\chi\Big)^2}(\mathrm{x}_0 + \varepsilon\,\mathrm{y}_0)\\ &- \frac{[\varrho(\mathrm{b}\chi + \mathrm{a}(\alpha+\eta+\mu)) + (1-\varrho)(\mathrm{a}\eta(1-\mathrm{q}) + \mathrm{b}(\beta+\chi+\mu))]\Big(\beta+\chi+\mu-(1-\mathrm{q})\chi\Big)}{\Big((\beta+\chi+\mu)(\alpha+\eta+\mu) - \eta(1-\mathrm{q})\chi\Big)^2}(\mathrm{x}_0 + \varepsilon\,\mathrm{y}_0)\\ &= -0.0662 < 0,\end{aligned}$$

$$\begin{aligned}\frac{\partial \mathscr{R}_0}{\partial \beta} &= \frac{\mathrm{b}(1-\varrho)\Big((\beta+\chi+\mu)(\alpha+\eta+\mu) - \eta(1-\mathrm{q})\chi\Big)}{\Big((\beta+\chi+\mu)(\alpha+\eta+\mu) - \eta(1-\mathrm{q})\chi\Big)^2}(\mathrm{x}_0 + \varepsilon\,\mathrm{y}_0)\\ &- \frac{[\varrho(\mathrm{b}\chi + \mathrm{a}(\alpha+\eta+\mu)) + (1-\varrho)(\mathrm{a}\eta(1-\mathrm{q}) + \mathrm{b}(\beta+\chi+\mu))]\Big(\alpha+\eta+\mu\Big)}{\Big((\beta+\chi+\mu)(\alpha+\eta+\mu) - \eta(1-\mathrm{q})\chi\Big)^2}(\mathrm{x}_0 + \varepsilon\,\mathrm{y}_0)\\ &= -0.1938 < 0,\end{aligned}$$

$$\frac{\partial \mathscr{R}_0}{\partial \chi} = \frac{b\Big((\beta+\chi+\mu)(\alpha+\eta+\mu)-\eta(1-\mathrm{q})\chi\Big)}{\Big((\beta+\chi+\mu)(\alpha+\eta+\mu)-\eta(1-\mathrm{q})\chi\Big)^2}(\mathrm{x}_0+\varepsilon\,\mathrm{y}_0)$$
$$-\frac{[\varrho(\mathrm{b}\chi+\mathrm{a}(\alpha+\eta+\mu))+(1-\varrho)(\mathrm{a}\eta(1-\mathrm{q})+\mathrm{b}(\beta+\chi+\mu))]\Big(\alpha+\eta+\mu-\eta(1-\mathrm{q})\Big)}{\Big((\beta+\chi+\mu)(\alpha+\eta+\mu)-\eta(1-\mathrm{q})\chi\Big)^2}(\mathrm{x}_0+\varepsilon\,\mathrm{y}_0)$$
$$= 0.0835 > 0,$$

$$\frac{\partial \mathscr{R}_0}{\partial q} = \frac{-(\mathrm{a}(1-\varrho)\eta)\Big((\beta+\chi+\mu)(\alpha+\eta+\mu)-\eta(1-\mathrm{q})\chi\Big)}{\Big((\beta+\chi+\mu)(\alpha+\eta+\mu)-\eta(1-\mathrm{q})\chi\Big)^2}(\mathrm{x}_0+\varepsilon\,\mathrm{y}_0)$$
$$-\frac{[\varrho(\mathrm{b}\chi+\mathrm{a}(\alpha+\eta+\mu))+(1-\varrho)(\mathrm{a}\eta(1-\mathrm{q})+\mathrm{b}(\beta+\chi+\mu))]\Big(\eta\chi\Big)}{\Big((\beta+\chi+\mu)(\alpha+\eta+\mu)-\eta(1-\mathrm{q})\chi\Big)^2}(\mathrm{x}_0+\varepsilon\,\mathrm{y}_0)$$
$$= -0.0052 < 0,$$

$$\frac{\partial \mathscr{R}_0}{\partial a} = \frac{\Big(\varrho(\alpha+\eta+\mu)+(1-\varrho)\eta(1-\mathrm{q})\Big)}{\Big((\beta+\chi+\mu)(\alpha+\eta+\mu)-\eta(1-\mathrm{q})\chi\Big)}(\mathrm{x}_0+\varepsilon\,\mathrm{y}_0) = 2.2216 > 0,$$
$$\frac{\partial \mathscr{R}_0}{\partial b} = \frac{\Big(\varrho\chi+(1-\varrho)(\beta+\chi+\mu)\Big)}{\Big((\beta+\chi+\mu)(\alpha+\eta+\mu)-\eta(1-\mathrm{q})\chi\Big)}(\mathrm{x}_0+\varepsilon\,\mathrm{y}_0) = 3.9229 > 0,$$

$$\frac{\partial \mathscr{R}_0}{\partial \varepsilon} = \frac{\mathrm{y}_0[\varrho(\mathrm{b}\chi+\mathrm{a}(\alpha+\eta+\mu))+(1-\varrho)(\mathrm{a}\eta(1-\mathrm{q})+\mathrm{b}(\beta+\chi+\mu))]}{(\beta+\chi+\mu)(\alpha+\eta+\mu)-\eta(1-\mathrm{q})\chi} = 0.0025 > 0.$$

The sensitivity index of each parameter in the model obtained as in Table 1 by applying (3.5). The sensitivity indexes of Table 1 indicate that $\mathscr{R}_0$ increases as the parameters χ, μ, a, b and ϵ are increased. In contrast, the values of other parameters are fixed. Based on these indices, it appears that disease endemicity has increased. In contrast, when the parameters β, η, q, α and ϱ are decreased while the rest of the parameters are maintained, $\mathscr{R}_0$ decreases.

4. Numerical procedures

Atangana-Baleanu fractal-fractional operators are implemented via Lagrangian piecewise interpolation for the proposed model.

4.1. Adams-Bashforth-Moulton method application

As in [82, 83], consider system (2.2) in the following case:

$$^{z}\mathscr{D}_{0,\mathtt{t}}^{\tau_1}\Psi(\mathtt{t}) = \Lambda(\mathtt{t},\Psi(\mathtt{t})),$$

subject to ceil function $n = [\tau_1]$ and for $t \in [0, T]$, $0 < \tau_1 \leq 1$ with ${}^{z}\mathscr{D}_{0,t}^{\tau_1}\Psi(0) = \Psi_0^{(\kappa)}$, $\kappa = 0, 1, 2, ..., n-1$. Volterra's integral equation of system (2.3) is given by

$$\Psi = \sum_{\kappa=0}^{n-1} \frac{t^{\kappa}}{\kappa!}\ \Psi_0^{(\kappa)} + \frac{1}{\Gamma(\tau_1)} \int_0^{t} (t-\xi)^{\tau_1 - 1} \Lambda(\xi, \Psi(\xi))\, d\xi. \tag{4.1}$$

It is easy to reconstruct Eq (4.1) by using the product rule for rectangles,

$$\int_0^{t_{n+1}} (t_{n+1} - \xi)^{\tau_1 - 1} \Lambda(\xi, \Psi(\xi))\, d\xi \simeq \sum_{\kappa=0}^{n}\ \Psi_{\kappa,n+1} \Lambda(t_\kappa,\ g_h(t_\kappa)),$$

where $\mathbb{A}_{\kappa,n+1}$ is given by

$$\mathbb{A}_{\kappa,n+1} = \begin{cases} n^{\tau_1+1} - (n-\tau_1)(n+1)^{\tau_1} & \text{if} \quad \kappa = 0, \\ (n-\kappa+2)^{\tau_1+1} + (n-\kappa)^{\tau_1+1} - 2(n-\kappa+1)^{\tau_1+1} & \text{if} \quad 1 \leq \kappa \leq n, \\ 1 & \text{if} \quad \kappa = n+1. \end{cases}$$

Let $\{t_n = nh : n = -k, -k+1, ..., -1, 0, 1, ..., N\}$, with $h = T/N$. Then, (4.1) can be discretized as follows:

$$\Psi_h(t_{n+1}) = \sum_{\kappa=0}^{n-1} \frac{t_{n+1}^{\kappa}}{\kappa!} \Psi_0^{(\kappa)} + \frac{h^{\tau_1}}{\Gamma(\tau_1+2)} \Lambda(t_{n+1}, \Psi(t_{n+1})) + \frac{h^{\tau_1}}{\Gamma(\tau_1+2)} \sum_{\kappa=0}^{n}\ \mathbb{A}_{\kappa,n+1} \Lambda(t_n, \Psi(t_n)). \tag{4.2}$$

The predicted value $\Psi_h^p(t_{n+1})$ is determined as follows:

$$\Psi_h^p(t_{n+1}) = \sum_{\kappa=0}^{\ell-1} \frac{t_{n+1}^{\kappa}}{\kappa!}\ \Psi_0^{(\kappa)} + \frac{1}{\Gamma(\tau_1)} \sum_{\kappa=0}^{n}\ \mathbb{B}_{\kappa,n+1} \Lambda(t_\kappa, \Psi(t_\kappa)),$$

where

$$\mathbb{B}_{\kappa,n+1} = \frac{h^{\tau_1}}{\tau_1} \left((n-\kappa+1)^{\tau_1} - (n-\kappa)^{\tau_1} \right), \quad \text{if} \quad 1 \leq \kappa \leq n.$$

According to (4.2), (2.1) is as follows:

$$\begin{aligned}
\mathrm{x}_{n+1} &= \mathrm{x}_0 + \frac{h^{\tau_1}}{\Gamma(\tau_1+2)}\left[(1-\mathrm{p})\pi + \phi\,\mathrm{y}^p_{n+1} + \delta\,\mathrm{v}^p_{n+1} - (\vartheta+\mu+\lambda)\,\mathrm{x}^p_{n+1}\right]\\
&\quad + \frac{h^{\tau_1}}{\Gamma(\tau_1+2)}\sum_{\kappa=0}^{n}\mathbb{A}_{\kappa,n+1}[(1-\mathrm{p})\pi + \phi\,\mathrm{y}_\kappa + \delta w_\kappa - (\vartheta+\mu+\lambda)\,\mathrm{x}_\kappa],\\
\mathrm{y}_{n+1} &= \mathrm{y}_0 + \frac{h^{\tau_1}}{\Gamma(\tau_1+2)}\left[\mathrm{p}\pi + \vartheta\,\mathrm{x}^p_{n+1} - (\phi+\mu+\varepsilon\lambda)\,\mathrm{y}^p_{n+1}\right]\\
&\quad + \frac{h^{\tau_1}}{\Gamma(\tau_1+2)}\sum_{\kappa=0}^{n}\mathbb{A}_{\kappa,n+1}[\mathrm{p}\pi + \vartheta\,\mathrm{x}_\kappa - (\phi+\mu+\varepsilon\lambda)\,\mathrm{y}_\kappa],\\
\mathrm{z}_{n+1} &= \mathrm{z}_0 + \frac{h^{\tau_1}}{\Gamma(\tau_1+2)}\left[\varrho\lambda\,\mathrm{x}^p_{n+1} + \varrho\varepsilon\lambda\,\mathrm{y}^p_{n+1} + \eta(1-\mathrm{q})\,\mathrm{u}^p_{n+1} - (\beta+\chi+\mu)\,\mathrm{z}^p_{n+1}\right]\\
&\quad + \frac{h^{\tau_1}}{\Gamma(\tau_1+2)}\sum_{\kappa=0}^{n}\mathbb{A}_{\kappa,n+1}[\varrho\lambda\,\mathrm{x}_\kappa + \varrho\varepsilon\lambda\,\mathrm{y}_\kappa + \eta(1-\mathrm{q})u_\kappa - (\beta+\chi+\mu)\,\mathrm{z}_\kappa],\\
\mathrm{u}_{n+1} &= \mathrm{u}_0 + \frac{h^{\tau_1}}{\Gamma(\tau_1+2)}\left[\lambda(1-\varrho)\,\mathrm{x}^p_{n+1} + \varepsilon\lambda(1-\varrho)\,\mathrm{y}^p_{n+1} + \chi\,\mathrm{z}^p_{n+1} - (\alpha+\eta+\mu)\,\mathrm{u}^p_{n+1}\right]\\
&\quad + \frac{h^{\tau_1}}{\Gamma(\tau_1+2)}\sum_{\kappa=0}^{n}\mathbb{A}_{\kappa,n+1}[\lambda(1-\varrho)\,\mathrm{x}_\kappa + \varepsilon\lambda(1-\varrho)\,\mathrm{y}_\kappa + \chi\,\mathrm{z}_\kappa - (\alpha+\eta+\mu)u_\kappa],\\
\mathrm{v}_{n+1} &= \mathrm{v}_0 + \frac{h^{\tau_1}}{\Gamma(\tau_1+2)}\left[\beta\,\mathrm{z}^p_{n+1} + \mathrm{q}\eta\,\mathrm{u}^p_{n+1} - (\delta+\mu)\,\mathrm{v}^p_{n+1}\right]\\
&\quad + \frac{h^{\tau_1}}{\Gamma(\tau_1+2)}\sum_{\kappa=0}^{n}\mathbb{A}_{\kappa,n+1}[\beta\,\mathrm{z}_\kappa + \mathrm{q}\eta u_\kappa - (\delta+\mu)w_\kappa],
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{x}^p_{n+1} &= \mathrm{x}_0 + \frac{h^{\tau_1}}{\Gamma(\tau_1+2)}\sum_{\kappa=0}^{n}\mathbb{B}_{\kappa,n+1}[(1-\mathrm{p})\pi + \phi\,\mathrm{y}_\kappa + \delta w_\kappa - (\vartheta+\mu+\lambda)\,\mathrm{x}_\kappa],\\
\mathrm{y}^p_{n+1} &= \mathrm{y}_0 + \frac{h^{\tau_1}}{\Gamma(\tau_1+2)}\sum_{\kappa=0}^{n}\mathbb{B}_{\kappa,n+1}\left[\mathrm{p}\pi + \vartheta\,\mathrm{x}_\kappa - (\phi+\mu+\varepsilon\lambda)\,\mathrm{y}_\kappa\right],
\end{aligned}$$

$$\begin{aligned}
\mathrm{z}^p_{n+1} &= \mathrm{z}_0 + \frac{h^{\tau_1}}{\Gamma(\tau_1+2)}\sum_{\kappa=0}^{n}\mathbb{B}_{\kappa,n+1}\left[\varrho\lambda\,\mathrm{x}_\kappa + \varrho\varepsilon\lambda\,\mathrm{y}_\kappa + \eta(1-\mathrm{q})u_\kappa - (\beta+\chi+\mu)\,\mathrm{z}_\kappa\right],\\
\mathrm{u}^p_{n+1} &= \mathrm{u}_0 + \frac{h^{\tau_1}}{\Gamma(\tau_1+2)}\sum_{\kappa=0}^{n}\mathbb{B}_{\kappa,n+1}\left[\lambda(1-\varrho)\,\mathrm{x}_\kappa + \varepsilon\lambda(1-\varrho)\,\mathrm{y}_\kappa + \chi\,\mathrm{z}_\kappa - (\alpha+\eta+\mu)\,\mathrm{u}_\kappa\right],\\
\mathrm{v}^p_{n+1} &= \mathrm{v}_0 + \frac{h^{\tau_1}}{\Gamma(\tau_1+2)}\sum_{\kappa=0}^{n}\mathbb{B}_{\kappa,n+1}\left[\beta\,\mathrm{z}_\kappa + \mathrm{q}\eta\,\mathrm{u}_\kappa - (\delta+\mu)\,\mathrm{v}_\kappa\right].
\end{aligned}$$

4.2. Application of the fractal-fractional Atangana-Baleanu scheme

$$^{\mathtt{FF-AB}}\mathscr{D}^{\tau_1,\tau_2}_{0,\mathrm{t}}\Psi(\mathrm{t}) = \Lambda(\mathrm{t},\Psi(\mathrm{t})).$$

The Antangana-Baleanu integral gives us

$$\vartheta(\mathtt{t}) = \Psi(0) + \frac{1-\tau_1}{\hbar(\tau_1)}\Lambda(\mathtt{t},\Psi(\mathtt{t})) + \frac{\tau_1}{\hbar(\tau_1)\Gamma(\tau_1)}\int_0^{\mathtt{t}} (\mathtt{t}-\xi)^{\tau_1-1}\xi^{\tau_2-1}\Lambda(\xi,\Psi(\xi))d\xi.$$

Replacing $\mathtt{t}$ with $\mathtt{t}_{n+1}$ we have

$$\Psi^{n+1} = \Psi(0) + \frac{1-\tau_1}{\hbar(\tau_1)}\Lambda\left(\mathtt{t}_{n+1},\Psi(\mathtt{t})\right) + \frac{\tau_1}{\hbar(\tau_1)\Gamma(\tau_1)}\int_0^{\mathtt{t}_{n+1}} (\mathtt{t}_{n+1}-\xi)^{\tau_1-1}\xi^{\tau_2-1}\Lambda(\xi,\Psi(\xi))d\xi.$$

Application of the two-step Lagrange polynomial yields

$$\begin{aligned}\Lambda(\mathtt{t},(y,\Psi(\mathtt{t})) &= \frac{\left(y-\mathtt{t}_{\xi-1}\right)\Lambda(\mathtt{t},\left(\mathtt{t}_{\xi},\Psi\left(\mathtt{t}_{\xi}\right)\right)}{\mathtt{t}_{\xi}-\mathtt{t}_{\xi-1}} - \frac{\left(y-\mathtt{t}_{\xi}\right)\Lambda\left(\mathtt{t}_{\xi-1},\Psi\left(\mathtt{t}_{\xi-1}\right)\right.}{\mathtt{t}_{\xi}-\mathtt{t}_{\xi-1}} \\ &= \frac{\Lambda(\mathtt{t},\left(\mathtt{t}_{\xi},\Psi\left(\mathtt{t}_{\xi}\right)\right)\left(x-\mathtt{t}_{\xi-1}\right)}{\mathtt{t}_{\xi}-\mathtt{t}_{\xi-1}} - \frac{\Lambda\left(\mathtt{t}_{\xi-1},\Psi\left(\mathtt{t}_{\xi-1}\right)\left(y-\mathtt{t}_{\xi}\right)\right.}{\mathtt{t}_{\xi}-\mathtt{t}_{\xi-1}} \\ &= \frac{\Lambda(\mathtt{t},\left(\mathtt{t}_{\xi},\Psi\left(\mathtt{t}_{\xi}\right)\right)\left(y-\mathtt{t}_{\xi-1}\right)}{h} - \frac{\Lambda\left(\mathtt{t}_{\xi-1},\Psi\left(\mathtt{t}_{\xi-1}\right)\left(y-\mathtt{t}_{\xi}\right)\right.}{h}.\end{aligned}$$

By using the Lagrange polynomial to solve the given problem, we obtain

$$\begin{aligned}\Psi^{n+1} &= \Psi(0) + \frac{1-\tau_1}{\hbar(\tau_1)}\Lambda(\mathtt{t},(\mathtt{t}_n,\Psi(\mathtt{t}_n)) \\ &+ \frac{\tau_1}{\hbar(\tau_1)\Gamma(\tau_1)}\sum_{\xi=1}^{n}\left(\frac{\Lambda(\mathtt{t},\left(\mathtt{t}_{\xi},\Psi\left(\mathtt{t}_{\xi}\right)\right)}{h}\int_{\mathtt{t}_{\xi}}^{\mathtt{t}_{\xi}+1}\left(\xi-\mathtt{t}_{\xi}-1\right)(\mathtt{t}_{n+1}-\xi)^{\tau_1-1}d\xi\right. \\ &\left.-\frac{\Lambda(\mathtt{t},\left(\mathtt{t}_{\xi-1},\Psi\left(\mathtt{t}_{\xi-1}\right)\right)}{h}\int_{\mathtt{t}_{\xi}}^{\mathtt{t}_{n+1}}\left(\xi-\mathtt{t}_{\xi}\right)(\mathtt{t}_{n+1}-\xi)^{\tau_1-1}d\xi\right).\end{aligned}$$

Now, solving the integral we get

$$\begin{aligned}\Psi^{n+1} &= \Psi(0) + \frac{1-\tau_1}{\hbar(\tau_1)}\Lambda(\mathtt{t},(\mathtt{t}_n,\Psi(\mathtt{t}_n)) + \frac{\tau_1 h^{\tau_1}}{\Gamma(\tau_1+2)} \\ &\times\sum_{\xi=1}^{n}\left[\Lambda(\mathtt{t},\left(\mathtt{t}_{\xi},\Psi\left(\mathtt{t}_{\xi}\right)\right)((n-\xi+1)^{\tau_1}(n-\xi+2+\tau_1)\;-(n-\xi)^{\tau_1}(n-\xi+2+2\tau_1))\right. \\ &\left.-\Lambda(\mathtt{t},\left(\mathtt{t}_{\xi-1},\Psi_{\xi-1}\right)\left((n-\xi+1)^{\tau_1+1}-(n-\xi+1+\tau_1)(n-\xi)^{\tau_1}\right)\right].\end{aligned}$$

Now, replacing the value of $\Lambda(y,\Psi(\mathtt{t}))$, we get

$$\begin{aligned}\Psi^{n+1} &= \Psi(0) + \tau_2\mathtt{t}^{\tau_2-1}\frac{1-\tau_1}{\hbar(\tau_1)}\Lambda\left(\mathtt{t}_{\xi},\Psi\left(\mathtt{t}_{\xi}\right)\right) + \tau_2\mathtt{t}^{\tau_2-1}\frac{\tau_1 h^{\tau_1}}{\Gamma(\tau_1+2)} \\ &\times\sum_{\xi=1}^{n}\left[\Lambda\left(\mathtt{t}_{\xi},\Psi\left(\mathtt{t}_{\xi}\right)\right)((n+1-\xi)^{\tau_1}(n-\xi+2+\tau_1)\;-(n-\xi)^{\tau_1}(n-\xi+2+2\tau_1))\right. \\ &\left.-\Lambda\left(\mathtt{t}_{\xi-1},\Psi_{\xi-1}\right)\left((n-\xi+1)^{\tau_1+1}-(n-\xi+1+\tau_1)(n-\xi)^{\tau_1}\right)\right].\end{aligned}$$

As a result, the numerical scheme above rewritten as follows:

$$\begin{aligned} \mathrm{x}^{n+1} &= \mathrm{x}(0) + \tau_2 \mathtt{t}^{\tau_2 - 1} \frac{1-\tau_1}{\hbar(\tau_1)} \Upsilon_1\left(\mathtt{t}_\xi, \mathrm{x}\left(\mathtt{t}_\xi\right)\right) + \tau_2 \mathtt{t}^{\tau_2 - 1} \frac{\tau_1 h^{\tau_1}}{\Gamma(\tau_1+2)} \\ &\times \sum_{\xi=1}^{n} \Big[\Upsilon_1\left(\mathtt{t}_\xi, \mathrm{x}\left(\mathtt{t}_\xi\right)\right)\left((n-\xi+1)^{\tau_1}(n-\xi+2+\tau_1) \; -(n-\xi)^{\tau_1}(n-\xi+2+2\tau_1)\right) \\ &-\Upsilon_1\left(\mathtt{t}_{\xi-1}, \mathrm{x}_{\xi-1}\right)\left((n-\xi+1)^{\tau_1+1} - (n-\xi+1+\tau_1)(n-\xi)^{\tau_1}\right)\Big], \end{aligned}$$

$$\begin{aligned} \mathrm{y}^{n+1} &= \mathrm{y}(0) + \tau_2 \mathtt{t}^{\tau_2 - 1} \frac{1-\tau_1}{\hbar(\tau_1)} \Upsilon_2\left(\mathtt{t}_\xi, \mathrm{y}\left(\mathtt{t}_\xi\right)\right) + \tau_2 \mathtt{t}^{\tau_2 - 1} \frac{\tau_1 h^{\tau_1}}{\Gamma(\tau_1+2)} \\ &\times \sum_{\xi=1}^{n} \Big[\Upsilon_2\left(\mathtt{t}_\xi, \mathrm{y}\left(\mathtt{t}_\xi\right)\right)\left((n-\xi+1)^{\tau_1}(n-\xi+2+\tau_1) \; -(n-\xi)^{\tau_1}(n-\xi+2+2\tau_1)\right) \\ &-\Upsilon_2\left(\mathtt{t}_{\xi-1}, \mathrm{y}_{\xi-1}\right)\left((n-\xi+1)^{\tau_1+1} - (n-\xi+1+\tau_1)(n-\xi)^{\tau_1}\right)\Big], \end{aligned}$$

$$\begin{aligned} \mathrm{z}^{n+1} &= \mathrm{z}(0) + \tau_2 \mathtt{t}^{\tau_2 - 1} \frac{1-\tau_1}{\hbar(\tau_1)} \Upsilon_3\left(\mathtt{t}_\xi, \mathrm{z}\left(\mathtt{t}_\xi\right)\right) + \tau_2 \mathtt{t}^{\tau_2 - 1} \frac{\tau_1 h^{\tau_1}}{\Gamma(\tau_1+2)} \\ &\times \sum_{\xi=1}^{n} \Big[\Upsilon_3\left(\mathtt{t}_\xi, \mathrm{z}\left(\mathtt{t}_\xi\right)\right)\left((n-\xi+1)^{\tau_1}(n-\xi+2+\tau_1) \; -(n-\xi)^{\tau_1}(n-\xi+2+2\tau_1)\right) \\ &-\Upsilon_3\left(\mathtt{t}_{\xi-1}, \mathrm{z}_{\xi-1}\right)\left((n-\xi+1)^{\tau_1+1} - (n-\xi+1+\tau_1)(n-\xi)^{\tau_1}\right)\Big], \end{aligned}$$

$$\begin{aligned} \mathrm{u}^{n+1} &= \mathrm{u}(0) + \tau_2 \mathtt{t}^{\tau_2 - 1} \frac{1-\tau_1}{\hbar(\tau_1)} \Upsilon_4\left(\mathtt{t}_\xi, \mathrm{u}\left(\mathtt{t}_\xi\right)\right) + \tau_2 \mathtt{t}^{\tau_2 - 1} \frac{\tau_1 h^{\tau_1}}{\Gamma(\tau_1+2)} \\ &\times \sum_{\xi=1}^{n} \Big[\Upsilon_4\left(\mathtt{t}_\xi, \mathrm{u}\left(\mathtt{t}_\xi\right)\right)\left((n-\xi+1)^{\tau_1}(n-\xi+2+\tau_1) \; -(n-\xi)^{\tau_1}(n-\xi+2+2\tau_1)\right) \\ &-\Upsilon_4\left(\mathtt{t}_{\xi-1}, \mathrm{u}_{\xi-1}\right)\left((n-\xi+1)^{\tau_1+1} - (n-\xi+1+\tau_1)(n-\xi)^{\tau_1}\right)\Big], \end{aligned}$$

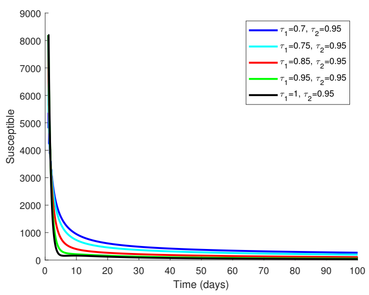
$$\begin{aligned} \mathrm{v}^{n+1} &= \mathrm{v}(0) + \tau_2 \mathtt{t}^{\tau_2 - 1} \frac{1-\tau_1}{\hbar(\tau_1)} \Upsilon_5\left(\mathtt{t}_\xi, \mathrm{v}\left(\mathtt{t}_\xi\right)\right) + \tau_2 \mathtt{t}^{\tau_2 - 1} \frac{\tau_1 h^{\tau_1}}{\Gamma(\tau_1+2)} \\ &\times \sum_{\xi=1}^{n} \Big[\Upsilon_5\left(\mathtt{t}_\xi, \mathrm{v}\left(\mathtt{t}_\xi\right)\right)\left((n-\xi+1)^{\tau_1}(n-\xi+2+\tau_1) \; -(n-\xi)^{\tau_1}(n-\xi+2+2\tau_1)\right) \\ &-\Upsilon_5\left(\mathtt{t}_{\xi-1}, \mathrm{v}_{\xi-1}\right)\left((n-\xi+1)^{\tau_1+1} - (n-\xi+1+\tau_1)(n-\xi)^{\tau_1}\right)\Big]. \end{aligned}$$

Table 2. Sensitivity index of the applied parameters.

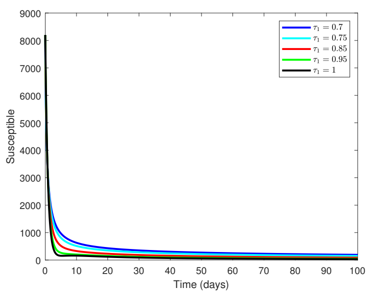
Parameters	Sensitivity index
β	-0.4669
η	-0.3987
q	-0.1264
α	-0.0973
ϱ	-0.0035
χ	0.0101
μ	0.1332
a	0.2676
b	0.9450
ϵ	$5.9220 \times e^{-04}$

4.3. Numerical simulations and discussion

The above analyses are displayed in Figures 2–16, which display the time series of the model (2.1) under the following initial conditions: $\mathrm{x}(0) = 8200, \mathrm{y}(0) = 2800, \mathrm{z}(0) = 200, \mathrm{u}(0) = 210, \mathrm{v}(0) = 200$. Figures 2–6 show that the time series of the model (2.1) with different trajectories of infected states tends to zero whenever $\mathrm{R}_0 = 0.0083 < 1$. The proposed model was simulated for approximately 100 days for different fractal fractional-order values τ_1 and τ_2. According to these parameters, $E_0 = (0.1192, 0.2961, 0, 0, 0)$ is asymptoticaly stable. As predicted, the solutions of (2.1) converge to the unique disease-free equilibrium E_0. The biological implication is that we need to bring R_0 to below 1 to ensure a reduction of the disease in the country.

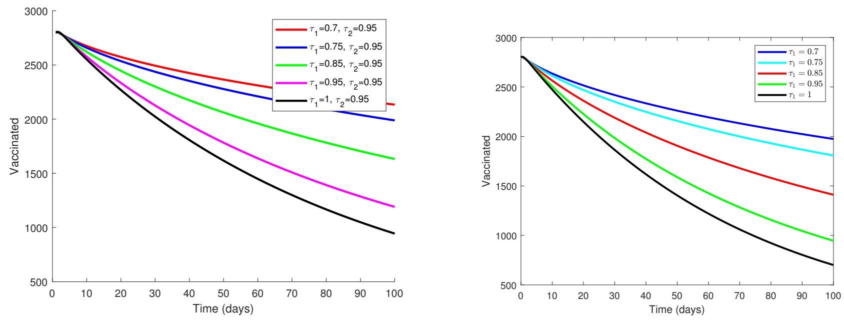


(a) Under the fractal-fractional Atangana-Baleanu operator



(b) Under the Caputo operator

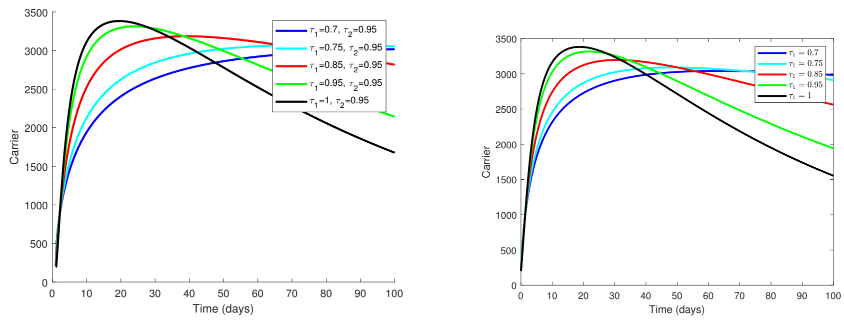
Figure 2. For $\mathscr{R}_0 = 0.0083 < 1$ with different fractional-order τ_1 values with a fixed $\tau_2 = 0.95$, a time series plot of the susceptible (x) is shown.



(a) Under the fractal-fractional Atangana-Baleanu operator

(b) Under the Caputo operator

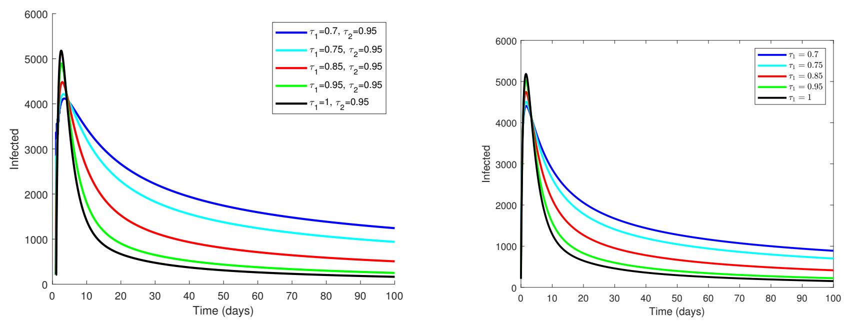
Figure 3. For $\mathscr{R}_0 = 0.0083 < 1$ with different fractional-order τ_1 values with a fixed $\tau_2 = 0.95$, a time series plot of the vaccinated (y) is shown.



(a) Under the fractal-fractional Atangana-Baleanu operator

(b) Under the Caputo operator

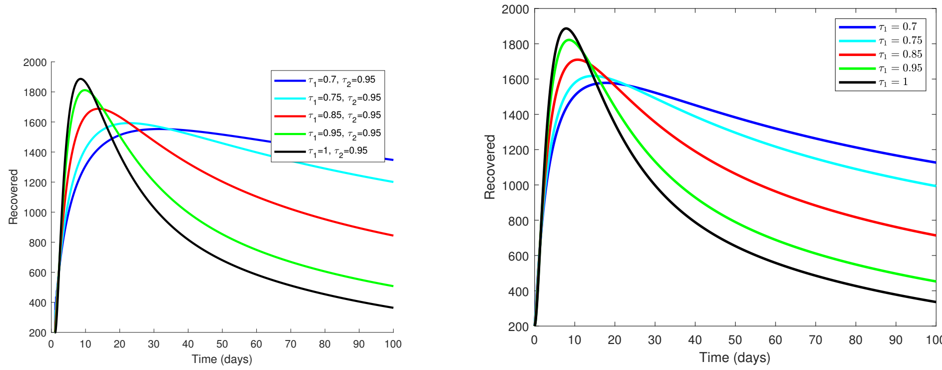
Figure 4. For $\mathscr{R}_0 = 0.0083 < 1$ with different fractional-order τ_1 values with a fixed $\tau_2 = 0.95$, a time series plot of the carrier (z) is shown.



(a) Under the fractal-fractional Atangana-Baleanu operator

(b) Under the Caputo operator

Figure 5. For $\mathscr{R}_0 = 0.0083 < 1$ with different fractional-order τ_1 values with a fixed $\tau_2 = 0.95$, a time series plot of the infected (u) is shown.



(a) Under the fractal-fractional Atangana-Baleanu operator

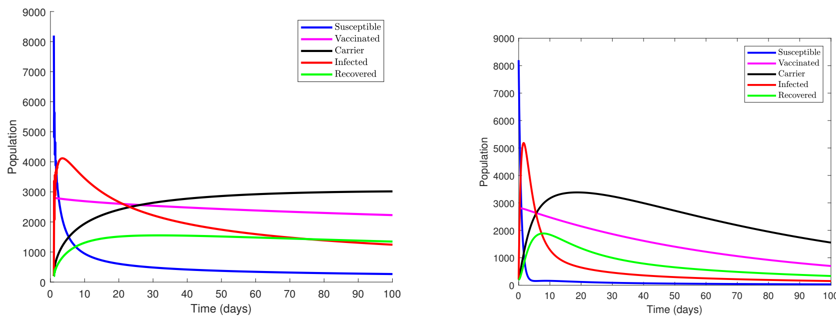
(b) Under the Caputo operator

Figure 6. For $\mathscr{R}_0 = 0.0083 < 1$ with different fractional-order τ_1 values with a fixed $\tau_2 = 0.95$, a time series plot of the recovered (v) is shown.

Figures 2–6 show the results of the fractal-fractional Atangana-Baleanu and the Adams-Bashforth-Moulton methods for pneumonia transmission, with (a) $\tau_1 = 0.7$, $\tau_2 = 0.95$, (b) $\tau_1 = 0.75$, $\tau_2 = 0.95$, (c) $\tau_1 = 0.85$, $\tau_2 = 0.95$ (d) $\tau_1 = 0.95$, $\tau_2 = 0.95$ and (e) $\tau_1 = 1$, $\tau_2 = 0.95$. Comparisons between the ordinary differential system, ABC fractal, and fractional derivative, can also be seen in Figures 2–6.

Figures 2–6 show the influence of varying τ_1 between 0.7 and 1 with a fixed $\tau_2 = 0.95$ on model dynamics. The black curve in each of these figures represents the numerical results of model (2.1) when the fractional order is equal to 1. From the results of Figures 2–6, it follows that the variation of the fractional parameter has a great impact on the quantitative dynamics of the model. Indeed, in Figure 5, the classes of infected humans peak after 10 years and decrease according to the decrease of the fractional parameter τ_1.

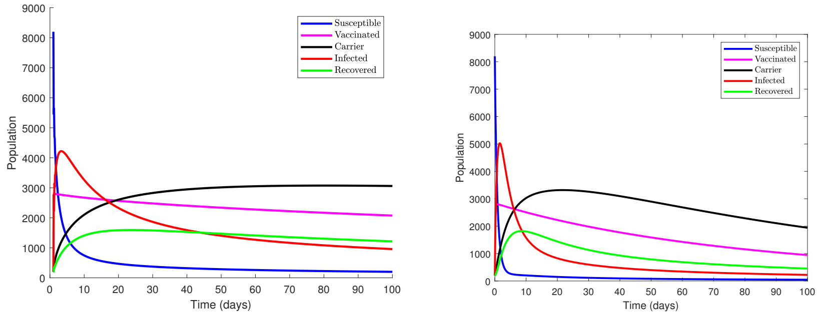
Lemmas 2 and 3 are validated numerically in Figures 2–6. It is clear that varying the fractional order parameter τ_1 does not influence the model dynamics whenever $\mathscr{R}_0 = 0.0083 < 1$. Indeed, whatever the value of τ_1, the infected compartments tend to zero asymptotically whenever $\mathscr{R}_0 = 0.0083 < 1$. This validates the fact that the pneumonia-free equilibrium of the fractional model is globally asymptotically stable whenever $\mathscr{R}_0 = 0.0083 < 1$.



(a) Under the fractal-fractional Atangana-Baleanu operator

(b) Under the Caputo operator

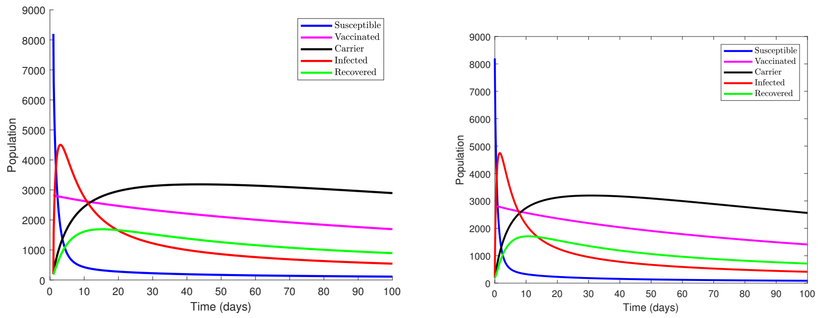
Figure 7. Dynamics of system (2.1) for (a) $\tau_1 = 0.7$, $\tau_2 = 0.95$, (b) $\tau_1 = 0.7$.



(a) Under the fractal-fractional Atangana-Baleanu operator

(b) Under the Caputo operator

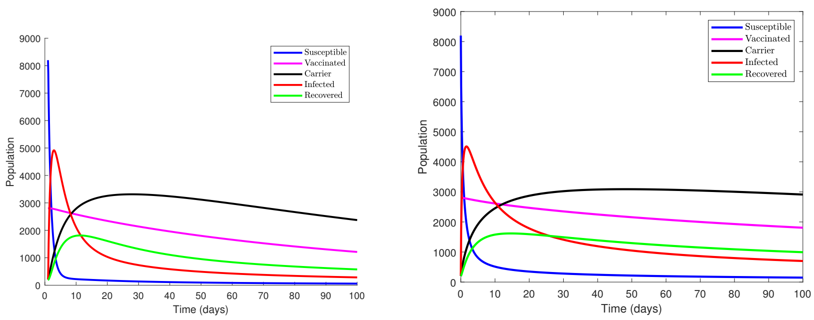
Figure 8. Dynamics of system (2.1) for (a) $\tau_1 = 0.75$, $\tau_2 = 0.95$, (b) $\tau_1 = 0.75$.



(a) Under the fractal-fractional Atangana-Baleanu operator

(b) Under the Caputo operator

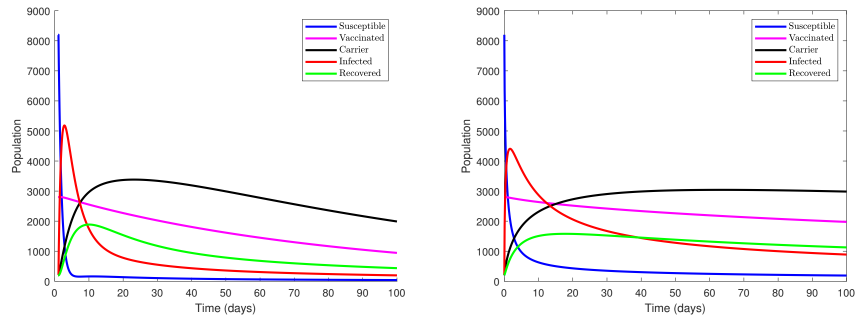
Figure 9. Dynamics of system (2.1) for (a) $\tau_1 = 0.85$, $\tau_2 = 0.95$, (b) $\tau_1 = 0.85$.



(a) Under the fractal-fractional Atangana-Baleanu operator

(b) Under the Caputo operator

Figure 10. Dynamics of system (2.1) for (a) $\tau_1 = 0.95$, $\tau_2 = 0.95$, (b) $\tau_1 = 0.95$.

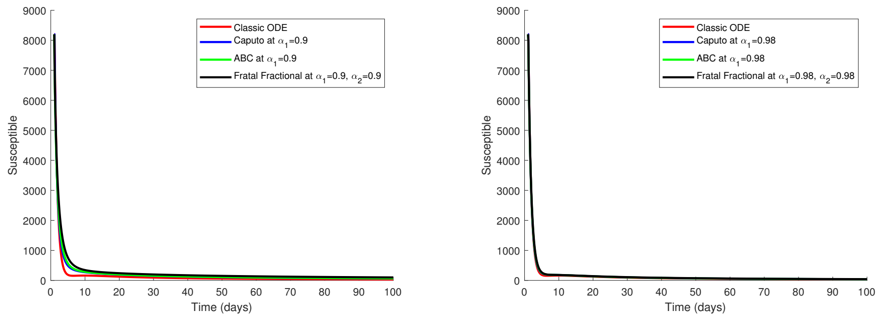


(a) Under the fractal-fractional Atangana-Baleanu operator

(b) Under the Caputo operator

Figure 11. Dynamics of all five compartments for (a) $\tau_1 = 1, \tau_2 = 0.95$, (b) $\tau_1 = 1$.

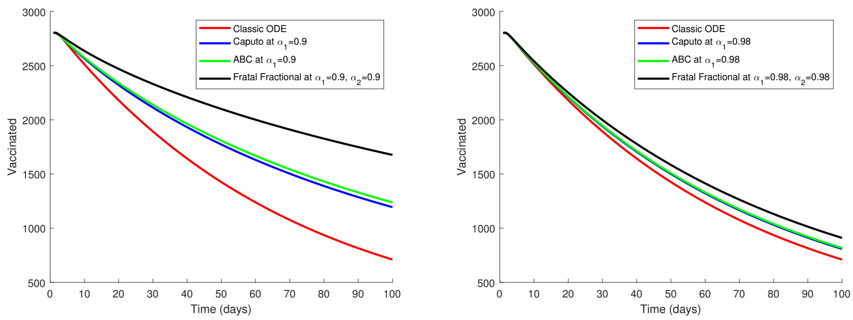
Figures 7–11 show the phase plots (x − y − z − u − v) for different values, i.e., (a) $\tau_1 = 0.7, \tau_2 = 0.95$, (b) $\tau_1 = 0.75, \tau_2 = 0.95$, (c) $\tau_1 = 0.85, \tau_2 = 0.95$ (d) $\tau_1 = 0.95, \tau_2 = 0.95$ and (e) $\tau_1 = 1, \tau_2 = 0.95$.



(a)

(b)

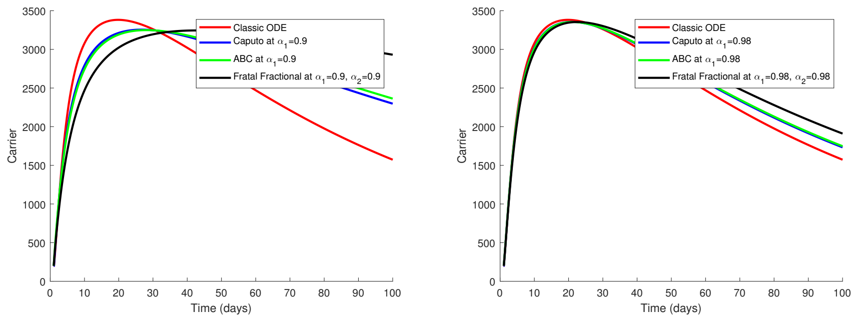
Figure 12. Comparison between the three numerical schemes: ODE, Caputo and fractal-fractional in two cases (a) and (b).



(a)

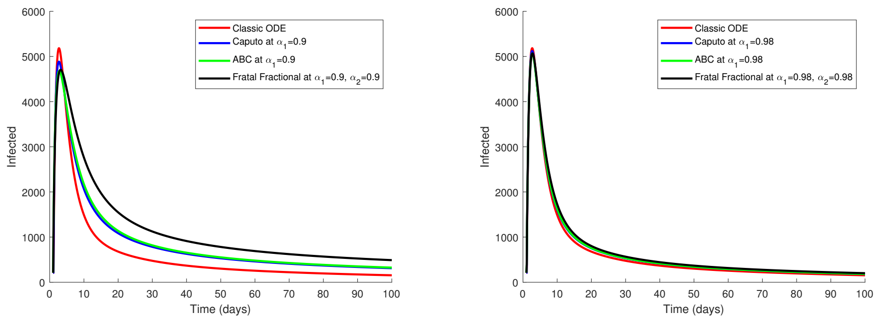
(b)

Figure 13. Comparison between the three numerical schemes: ODE, Caputo and fractal-fractional in two cases (a) and (b).



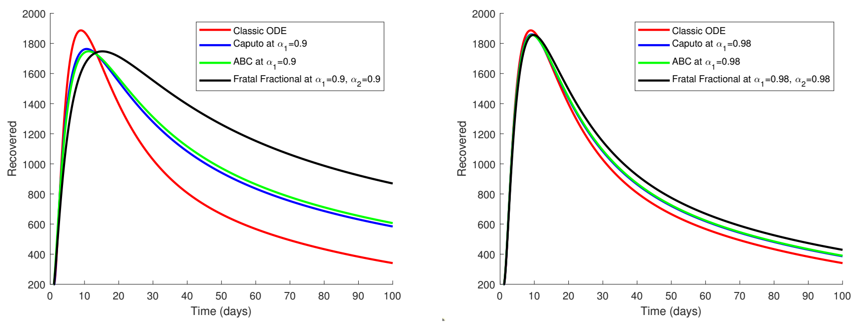
(a) (b)

Figure 14. Comparison between the three numerical schemes: ODE, Caputo and fractal-fractional in two cases (a) and (b).



(a) (b)

Figure 15. Comparison between the three numerical schemes: ODE, Caputo and fractal-fractional in two cases (a) and (b).



(a) (b)

Figure 16. Comparison between the three numerical schemes: ODE, Caputo and fractal-fractional in two cases (a) and (b).

Comparison between ordinary differential system, ABC, ABC fractal fractional derivative schemes can be seen in Figures 12–16.

5. Conclusions

A fractal fractional-order mathematical model based on the Atangana-Baleanu operator was constructed to describe pneumonia transmission in a population. The Caputo operator was used to analyze the dynamics of the virus, and a fractal fractional derivative was used to maximize the number of recovered populations. For models of infectious diseases, the vaccination rate coefficient is considered as a control to reduce the disease burden. It is important to prove the existence of optimal control, characterize the optimal control, prove the uniqueness of optimal control and compute the optimal control numerically. The model was subjected to dynamic analysis, and the results show that the rate at which susceptible individuals contract an infectious disease is the most significant parameter. In this simulation, the value of $\mathscr{R}_0 = 0.0083 < 1$, which is smaller than 1. As you can see, disease spread is controlled, and the number of infected people is reduced to zero. We also see that each function tends to its equilibrium point, and that the equilibrium point becomes stable as the system approaches its equilibrium point. We also note that the total number of susceptible humans decreases rapidly according to the increase of the fractional parameter (Figure 2).

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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