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Research article

Characterization of ternary derivation of strongly double triangle subspace lattice algebras

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Abstract: Let $\mathcal{D}$ be a strongly double triangle subspace lattice on a nonzero complex reflexive Banach space. In this paper, we characterize the linear maps δ, τ: Alg$\mathcal{D} \to$ Alg$\mathcal{D}$ satisfying $\delta(A)B + A\tau(B) = 0$ for any $A, B \in$ Alg$\mathcal{D}$ with $AB = 0$. This result can be used to characterize linear maps derivable (centralized) at zero point and local centralizers on Alg$\mathcal{D}$, respectively.

Keywords: ternary derivation; derivation; centralizer; strongly double triangle subspace lattice
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1. Introduction and preliminaries

Let $\mathcal{A}$ be an algebra and (γ, δ, τ) be a triple of linear maps of $\mathcal{A}$. Recall that (γ, δ, τ) is a ternary derivation of $\mathcal{A}$ if

$$\gamma(ab) = \delta(a)b + a\tau(b)$$

for all $a, b \in \mathcal{A}$. Clearly, if $\gamma = \delta = \tau$, then γ is a derivation of $\mathcal{A}$; if $\gamma = \delta$ and τ is a derivation of $\mathcal{A}$, then γ is a generalized derivation of $\mathcal{A}$; if $\gamma = \delta$ and $\tau = 0$, then γ is a left centralizer of $\mathcal{A}$. In 2003, Jiménez-Gestal and Pérez-Izquierdo [9] introduced the terminology of ternary derivation and described ternary derivations of the generalized Cayley-Dickson algebras over a field of characteristic not 2 and 3. In [18, 19], Shestakov studied ternary derivations of separable associative and Jordan algebras and Jordan superalgebras, respectively. Recently, ternary derivations of triangular algebras have been precisely described in [2]. We refer the reader to [10, 17] for background information about the definition of ternary derivations.

Let (γ, δ, τ) be a ternary derivation of $\mathcal{A}$. It is easy to verify that δ and τ satisfy

$$ab = 0 \Rightarrow \delta(a)b + a\tau(b) = 0 \tag{1.1}$$

for all $a, b \in \mathcal{A}$. However, the converse of this observation is not necessarily true (see [5, Example 1.1]). It is natural to ask how to characterize linear maps δ and τ satisfying (1.1). Benkovič and Grašič [3] considered linear maps δ and τ satisfying (1.1) on algebras generated by idempotents. In [1], the authors showed that if $\mathcal{A}$ is a unital standard operator algebra on a Banach space X, and $\delta, \tau\colon \mathcal{A} \to B(X)$ are linear maps satisfying (1.1), then there exist $R, S, T \in B(X)$ such that

$$\delta(A) = AS - RA \text{ and } \tau(A) = AT - SA$$

for all $A \in \mathcal{A}$, where $B(X)$ denotes the algebra of all bounded linear operators on X. Recently, Fošner and Ghahramani [5] proved that if linear maps $\delta, \tau\colon \mathrm{Alg}\mathcal{N} \to \mathrm{Alg}\mathcal{N}$ satisfy (1.1), then exists a unique linear map $\gamma\colon \mathrm{Alg}\mathcal{N} \to \mathrm{Alg}\mathcal{N}$ defined by

$$\gamma(A) = RA + AT$$

for some $R, T \in \mathrm{Alg}\mathcal{N}$ such that (γ, δ, τ) is a ternary derivation of $\mathrm{Alg}\mathcal{N}$, where $\mathrm{Alg}\mathcal{N}$ is a nest algebra on a real or complex Banach space with $N \in \mathcal{N}$ complemented whenever $N_- = N$.

In this paper, we will consider linear maps δ and τ satisfying (1.1) on another important kind of reflexive algebra, that is, strongly double triangle subspace lattice algebra. Note that both standard operator algebras on Banach spaces and nest algebras are rich in rank-one operators. This makes it possible to identify certain behavior of the linear maps on some special set of rank-one operators. Actually, the proofs in [1,5] depend heavily on the existence of rank-one operators. However, strongly double triangle subspace lattice algebras contain no rank-one operators [13]. So, the problem is more complicated than the previous one, and we will study this problem by means of operators of even rank.

This paper is organized as follows. In Section 2, all the results of the paper are presented. Section 3 is devoted to the proof of our central result (see Theorem 2.1).

Let us introduce some notations used in this paper. Throughout, X will be a nonzero reflexive complex Banach space with topological dual X^*. For $A \in B(X)$, by $\ker(A)$, $\mathrm{ran}(A)$ and $\mathrm{rank}(A)$, we denote the kernel, the range and the rank of A, respectively. For nonzero vectors $e^* \in X^*$ and $f \in X$, we define the rank-one operator $e^* \otimes f$ by $x \mapsto e^*(x)f$ for $x \in X$. For any nonempty subset $L \subseteq X$, by $L^\perp$ we denote the annihilator of L, that is,

$$L^\perp = \{f^* \in X^* : f^*(x) = 0 \text{ for all } x \in L\}.$$

Dually, for any nonempty subset $M \subseteq X^*$, ${}^\perp M$ denotes its pre-annihilator, that is,

$${}^\perp M = \{x \in X : f^*(x) = 0 \text{ for all } f^* \in M\}.$$

A family $\mathcal{L}$ of closed subspaces of X is called a subspace lattice on X if it contains (0) and X, and is closed under the operations closed linear span $\vee$ and intersection $\cap$ in the sense that $\vee_{\gamma\in\Gamma} L_\gamma \in \mathcal{L}$ and $\cap_{\gamma\in\Gamma} L_\gamma \in \mathcal{L}$ for every family $\{L_\gamma\colon \gamma \in \Gamma\}$ of elements in $\mathcal{L}$. Given a subspace lattice $\mathcal{L}$ on X, the associated subspace lattice algebra $\mathrm{Alg}\mathcal{L}$ is the set of operators on X leaving every subspace in $\mathcal{L}$ invariant, that is,

$$\mathrm{Alg}\mathcal{L} = \{A \in B(X) : Ax \in L \quad \text{for every} \quad x \in L \quad \text{and for every} \quad L \in \mathcal{L}\}.$$

A double triangle subspace lattice on X is a set

$$\mathcal{D} = \{(0), K, L, M, X\}$$

of subspaces of X satisfying

$$K \cap L = L \cap M = M \cap K = (0)$$

and

$$K \vee L = L \vee M = M \vee K = X.$$

We say that $\mathcal{D}$ is a strongly double triangle subspace lattice if one of the three sums $K + L$, $L + M$ and $M + K$ is closed. Observe that

$$\mathcal{D}^{\perp} = \{(0), K^{\perp}, L^{\perp}, M^{\perp}, X^{*}\}$$

is a double triangle subspace lattice on the reflexive Banach space X^{*}. Put

$$K_0 = K \cap (L + M),\ L_0 = L \cap (M + K),\ M_0 = M \cap (K + L)$$

and

$$K_p = K^{\perp} \cap (L^{\perp} + M^{\perp}),\ L_p = L^{\perp} \cap (M^{\perp} + K^{\perp}),\ M_p = M^{\perp} \cap (K^{\perp} + L^{\perp}).$$

It follows from [11, Lemma 2.2] that

$$\dim K_0 = \dim L_0 = \dim M_0$$

and

$$\dim K_p = \dim L_p = \dim M_p.$$

We close this section by summarizing some lemmas on strongly double triangle subspace lattice algebras, which will be used to prove our main results.

Lemma 1.1. *([11]) Let*

$$\mathcal{D} = \{(0), K, L, M, X\}$$

be a double triangle subspace lattice on a nonzero complex reflexive Banach space X. Then the following statements hold.

(1) $K_0 \subseteq K \subseteq {}^{\perp}K_p$, $L_0 \subseteq L \subseteq {}^{\perp}L_p$ *and* $M_0 \subseteq M \subseteq {}^{\perp}M_p$.
(2) $K_0 \cap L_0 = L_0 \cap M_0 = M_0 \cap K_0 = (0)$.
(3) $K_p \cap L_p = L_p \cap M_p = M_p \cap K_p = (0)$.
(4) $K_0 + L_0 = L_0 + M_0 = M_0 + K_0 = K_0 + L_0 + M_0$.
(5) $K_p + L_p = L_p + M_p = M_p + K_p = K_p + L_p + M_p$.

Lemma 1.2. *([11]) Let*

$$\mathcal{D} = \{(0), K, L, M, X\}$$

be a double triangle subspace lattice on a nonzero complex reflexive Banach space X. Then the following statements hold.

(1) *Every finite-rank operator of* Alg$\mathcal{D}$ *has even rank (possibly zero).*
(2) *If $e, f \in X$ and $e^{*}, f^{*} \in X^{*}$ are nonzero vectors satisfying $e \in K_0$, $f \in L_0$, $e + f \in M_0$ and $e^{*} \in K_p$, $f^{*} \in L_p$, $e^{*} + f^{*} \in M_p$, then $R = e^{*} \otimes f - f^{*} \otimes e$ is a rank-two operator in* Alg$\mathcal{D}$*. Moreover, every rank-two operator in* Alg$\mathcal{D}$ *has this form for such vectors $e, f \in X$ and $e^{*}, f^{*} \in X^{*}$.*
(3) Alg$\mathcal{D}$ *contains a nonzero finite-rank operator if and only if* $\dim K_0 \neq 0$ *and* $\dim K_p \neq 0$.

Lemma 1.3. *([11]) Let*

$$\mathcal{D} = \{(0), K, L, M, X\}$$

be a strongly double triangle subspace lattice on a nonzero complex reflexive Banach space X. If $K + L$ is closed, then the following statements hold.

(1) *K_0 is dense in K, L_0 is dense in L.*
(2) *$K_0 + L_0 + M_0$ is dense in X.*
(3) *$K_p + L_p + M_p$ is dense in X^*.*

Lemma 1.4. *([11]) Let*

$$\mathcal{D} = \{(0), K, L, M, X\}$$

be a strongly double triangle subspace lattice on a complex reflexive Banach space X. If $\mathrm{Alg}\mathcal{D}$ contains a rank-two operator, then the following statements hold.

(1) $\mathrm{span}\{\mathrm{ran}(R) : R \in \mathrm{Alg}\mathcal{D},\ \mathrm{rank}(R) = 2\} = K_0 + L_0 + M_0$.
(2) $\cap\{\ker(R) : R \in \mathrm{Alg}\mathcal{D},\ \mathrm{rank}(R) = 2\} = {}^{\perp}(K_p + L_p + M_p)$.

The above two lemmas show that strongly double triangle subspace lattice algebras are rich in rank-two operators. We refer readers [11, 13, 15] to more properties of strongly double triangle subspace lattice algebras.

2. The main results

In this section, we present the results of this paper. The central result is as follows.

Theorem 2.1. *Let*

$$\mathcal{D} = \{(0), K, L, M, X\}$$

be a strongly double triangle subspace lattice on a nonzero complex reflexive Banach space X. Suppose δ, τ: $\mathrm{Alg}\mathcal{D} \to \mathrm{Alg}\mathcal{D}$ are linear maps satisfying the relation (1.1). *Then exists a linear map γ: $\mathrm{Alg}\mathcal{D} \to \mathrm{Alg}\mathcal{D}$ such that (γ, δ, τ) is a ternary derivation. Moreover, there exist densely defined, closed linear operators $R, T, S : \mathcal{M} \to X$ such that*

$$\tau(A)x = (TA + AS)x,$$

$$\delta(A)x = (RA - AT)x$$

and

$$\gamma(A)x = (RA + AS)x,$$

for all $A \in \mathrm{Alg}\mathcal{D}$ and all $x \in \mathcal{M}$. Here $\mathcal{M}$ is invariant under every element of $\mathrm{Alg}\mathcal{D}$.

The proof of Theorem 2.1 will be given in Section 3. We now give some applications. In the following corollaries, $\mathcal{D}$ will always denote a strongly double triangle subspace lattice on a nonzero complex reflexive Banach space.

First of all, as an immediate consequence, we get the following corollary.

Corollary 2.2. *Let (γ, δ, τ) be a ternary derivation of $\mathrm{Alg}\mathcal{D}$. Then (γ, δ, τ) is of the form in Theorem 2.1.*

Recall that a linear map δ: Alg$\mathcal{D} \to$ Alg$\mathcal{D}$ is derivable at zero point if

$$\delta(A)B + A\delta(B) = 0$$

for any $A, B \in$ Alg$\mathcal{D}$ with $AB = 0$. In [7,8,15,21], the authors studied the linear maps derivable at zero point for some algebras. Recently, Fallahi and Ghahramani [4] considered another type of derivable maps at zero point, and they showed that if $\mathcal{A}$ is a standard operator algebra on a Banach space X, and δ is a linear map from $\mathcal{A}$ into itself, then δ satisfies

$$b\delta(a) + \delta(b)a = 0$$

for any $a, b \in \mathcal{A}$ with $ab = 0$ if and only if $\delta = 0$. Here, applying Theorem 2.1, we can get the main result in [15].

Corollary 2.3. *([15, Theorem 2.3]) Suppose δ: Alg$\mathcal{D} \to$ Alg$\mathcal{D}$ is a linear map derivable at zero point. Then there exist a complex scalar λ and a derivation d such that*

$$\delta(A) = d(A) + \lambda A$$

for all $A \in$ Alg$\mathcal{D}$.

Proof. By Theorem 2.1, there exists a linear map γ: Alg$\mathcal{D} \to$ Alg$\mathcal{D}$ such that (γ, δ, δ) is a ternary derivation. That is,

$$\gamma(AB) = \delta(A)B + A\delta(B)$$

for every $A, B \in$ Alg$\mathcal{D}$. Let P be any idempotent in Alg$\mathcal{D}$. Then,

$$P(I - P) = (I - P)P = 0.$$

Since δ is derivable at zero point, we can get

$$0 = \delta(P)(I - P) + P\delta(I - P) = \delta(I - P)P + (I - P)\delta(P),$$

which further implies that $\delta(I)P = P\delta(I)$ for every idempotent $P \in$ Alg$\mathcal{D}$. Then by the proof of [15, Theorem 2.3], there exists a constant λ such that $\delta(I) = \lambda I$. Define a linear map d: Alg$\mathcal{D} \to$ Alg$\mathcal{D}$ by

$$d(A) = \gamma(A) - 2\lambda A$$

for every $A \in$ Alg$\mathcal{D}$. By the definition of γ, we have

$$\gamma(A) - 2\lambda A = \delta(A) - \lambda A.$$

Then

$$\begin{aligned} d(AB) &= \gamma(AB) - 2\lambda AB \\ &= \delta(A)B + A\delta(B) - 2\lambda AB \\ &= (\delta(A) - \lambda A)B + A(\delta(B) - \lambda B) \\ &= (\gamma(A) - 2\lambda A)B + A(\gamma(B) - 2\lambda B) \\ &= d(A)B + Ad(B). \end{aligned}$$

The proof is complete. □

Recall that a linear map ϕ: $\mathrm{Alg}\mathcal{D} \to \mathrm{Alg}\mathcal{D}$ is a left (right) centralizer if

$$\phi(AB) = \phi(A)B(\phi(AB) = A\phi(B))$$

for all $A, B \in \mathrm{Alg}\mathcal{D}$ and ϕ is a centralizer if it is both left and right centralizer. There have been a number of papers on the study of conditions under which centralizers on algebras can be determined by the action on some sets of operators (see [6, 12, 16] and the references therein). Among others, in [6], the authors studied the structure of linear maps ϕ on certain operator algebras $\mathcal{A}$ satisfying

$$\phi(ab - ba) = \phi(a)b - b\phi(a)$$

or

$$\phi(ab - ba) = a\phi(b) - \phi(b)a$$

for any $a, b \in \mathcal{A}$ with $ab = 0$. A linear map ϕ: $\mathrm{Alg}\mathcal{D} \to \mathrm{Alg}\mathcal{D}$ is said to be left (right) centralized at zero point if

$$\phi(A)B = 0(A\phi(B) = 0)$$

for any $A, B \in \mathrm{Alg}\mathcal{D}$ with $AB = 0$ and ϕ is said to be centralized at zero point if it is both left centralized and right centralized at zero point. The following corollary characterizes linear maps on $\mathrm{Alg}\mathcal{D}$ which are centralized at zero point.

Corollary 2.4. *Suppose δ, τ, ϕ: $\mathrm{Alg}\mathcal{D} \to \mathrm{Alg}\mathcal{D}$ are linear maps. The following statements hold.*

(1) *If δ is left centralized at zero point, then there exists $T \in \mathrm{Alg}\mathcal{D}$ such that $\delta(A) = TA$ for all $A \in \mathrm{Alg}\mathcal{D}$.*
(2) *If τ is right centralized at zero point, then there exists $T \in \mathrm{Alg}\mathcal{D}$ such that $\tau(A) = AT$ for all $A \in \mathrm{Alg}\mathcal{D}$.*
(3) *Suppose ϕ is centralized at zero point, then there exists $\lambda \in \mathbb{C}$ such that $\phi(A) = \lambda A$ for all $A \in \mathrm{Alg}\mathcal{D}$.*

Proof. (1) By Theorem 2.1, there exists a linear map γ: $\mathrm{Alg}\mathcal{D} \to \mathrm{Alg}\mathcal{D}$ such that

$$\gamma(AB) = \delta(A)B$$

for every $A, B \in \mathrm{Alg}\mathcal{D}$. By the definition of γ, we have $\gamma(A) = \delta(A)$ for every $A \in \mathrm{Alg}\mathcal{D}$. So, $\delta(BA) = \delta(B)A$ for all $A, B \in \mathrm{Alg}\mathcal{D}$. Taking $B = I$ in the above equation, we arrive at $\delta(A) = TA$ for all $A \in \mathrm{Alg}\mathcal{D}$, where $T = \delta(I)$.

(2) The proof is similar to the proof of (1).

(3) By (1) and (2), there exists $T \in \mathrm{Alg}\mathcal{D}$ such that

$$\delta(A) = TA = AT$$

for all $A \in \mathrm{Alg}\mathcal{D}$. By the proof of Corollary 2.3, there exists $\lambda \in \mathbb{C}$ such that $\delta(A) = \lambda A$ for all $A \in \mathrm{Alg}\mathcal{D}$. □

The other direction of the study of centralizers by the local actions is the well-known local maps problem. See, for example [12, 20]. Recently, in [14], Molnár introduced a new type of local maps, and calculated the operational reflexive closures of some important classes of transformations. We say that a linear map ϕ: Alg$\mathcal{D} \to$ Alg$\mathcal{D}$ is a local left (right) centralizer of Alg$\mathcal{D}$, if for every $A \in$ Alg$\mathcal{D}$, there exists a left (right) centralizer ϕ_A, depending on A, such that $\phi(A) = \phi_A(A)$. Similarly, we can define a local centralizer. Now, we study the local centralizers on strongly double triangle subspace lattice algebras.

Corollary 2.5. *The following statements hold.*

(1) *Every local left centralizer on* Alg$\mathcal{D}$ *is a left centralizer.*
(2) *Every local right centralizer on* Alg$\mathcal{D}$ *is a right centralizer.*
(3) *Every local centralizer on* Alg$\mathcal{D}$ *is a centralizer.*

Proof. We only show (1). By a similar argument, we can get (2) and (3). Take $A, B \in$ Alg$\mathcal{D}$ such that $AB = 0$. Now we show that $\phi(A)B = 0$. For $A \in$ Alg$\mathcal{D}$, there exists a left centralizer ϕ_A, depending on A, such that $\phi(A) = \phi_A(A)$. From this, we get

$$\phi(A)B = \phi_A(A)B = \phi_A(I)AB = 0.$$

Applying Corollary 2.4, we see that ϕ is a left centralizer. □

3. Proof of Theorem 2.1

Let

$$\mathcal{D} = \{(0), K, L, M, X\}$$

be a strongly double triangle subspace lattice on a nonzero complex reflexive Banach space X. Note that

$$\dim K_0 = \dim L_0 = \dim M_0$$

and

$$\dim K_p = \dim L_p = \dim M_p.$$

It is easy to prove that $\dim K_0 \neq 0$ and $\dim K_p \neq 0$. It follows from Lemma 1.2 (3) that Alg$\mathcal{D}$ contains nonzero finite-rank operators. In this section, we will complete the proof of Theorem 2.1.

First, we need the following lemma, which comes from [1].

Lemma 3.1. *Let $\mathcal{A}$ be an algebra with unit I and $A \in \mathcal{A}$. Suppose δ, τ: $\mathcal{A} \to \mathcal{A}$ are linear maps satisfying the relation* (1.1). *Then the following statements hold.*

(1) $\delta(AP) + AP\tau(I) = \delta(A)P + A\tau(P)$ *for every idempotent $P \in \mathcal{A}$.*
(2) $\delta(P)A + P\tau(A) = \delta(I)PA + \tau(PA)$ *for every idempotent $P \in \mathcal{A}$.*

Now we are at a position to give the proof of our main result.

Proof of Theorem 2.1. Without loss of generality, we may assume that $K + L$ is closed. We will prove the theorem by checking several claims.

Claim 1. *Let $A \in$ Alg$\mathcal{D}$. Then the following statements hold.*

(1) $\delta(AR) + AR\tau(I) = \delta(A)R + A\tau(R)$ *for every rank-two operator $R \in$ Alg$\mathcal{D}$.*

(2) $\delta(R)A + R\tau(A) = \delta(I)RA + \tau(RA)$ *for every rank-two operator* $R \in \mathrm{Alg}\mathcal{D}$.

We only show (1). With the same argument, one can get (2). By Lemma 1.2, we can write

$$R = e^* \otimes f - f^* \otimes e$$

for some nonzero vectors $e \in L_0$, $f \in M_0$, $e + f \in K_0$ and $e^* \in L_p$, $f^* \in M_p$, $e^* + f^* \in K_p$. Now we consider two cases:
Case 1. $e^*(f) \neq 0$. Since $L_0 \subseteq {}^{\perp}L_p$, $M_0 \subseteq {}^{\perp}M_p$, $K_0 \subseteq {}^{\perp}K_p$ by Lemma 1.1, we have

$$e^*(e) = f^*(f) = 0$$

and

$$e^*(f) + f^*(e) = (e^* + f^*)(e + f) = 0.$$

It follows that

$$R^2 = e^*(f)e^* \otimes f + f^*(e)f^* \otimes e = e^*(f)R.$$

So,

$$\left(\frac{1}{e^*(f)}R\right)^2 = \frac{1}{e^*(f)}R.$$

Hence, by the linearity of δ and Lemma 3.1, we obtain (1).
Case 2. $e^*(f) = 0$. Note that $K + L = X$. Then,

$$M_0 = M \cap (K + L) = M.$$

Take $f_1 \in M_0$ such that $e^*(f_1) \neq 0$. Actually, if $e^*(f_1) = 0$ for all $f_1 \in M_0$, we have

$$e^* \in M_0^{\perp} = M^{\perp}.$$

Applying the fact that

$$e^* \in L_p = L^{\perp} \cap (M^{\perp} + K^{\perp}),$$

since

$$L^{\perp} \cap M^{\perp} = (0),$$

it follows that

$$e^* \in L^{\perp} \cap M^{\perp} = (0),$$

a contradiction. By Lemma 1.1, there exist unique elements $e_1 \in L_0$ and $g_1 \in K_0$ such that $e_1 + f_1 = g_1$. Denote

$$R_1 = e^* \otimes f_1 - f^* \otimes e_1$$

and

$$R_2 = e^* \otimes (f_1 + f) - f^* \otimes (e_1 + e).$$

Then $R_1, R_2 \in \mathrm{Alg}\mathcal{D}$ are of rank two by Lemma 1.2. Since

$$e^*(f + f_1) = e^*(f_1) \neq 0,$$

it follows from Case 1 that

$$\begin{aligned}\delta(AR)+AR\tau(I) &= \delta(A(R_2-R_1))+A(R_2-R_1)\tau(I)\\ &= \delta(AR_2)+AR_2\tau(I)-\delta(AR_1)-AR_1\tau(I)\\ &= \delta(A)R_2+A\tau(R_2)-\delta(A)R_1-A\tau(R_1)\\ &= \delta(A)R+A\tau(R).\end{aligned}$$

Claim 2. *Let* $A, B \in \mathrm{Alg}\mathcal{D}$*. Then,*

$$\delta(AB)=A\delta(B)+\delta(A)B-A\delta(I)B.$$

Let $R \in \mathrm{Alg}\mathcal{D}$ be any rank-two operator. By Lemma 1.2, we can write

$$R=e^*\otimes f-f^*\otimes e$$

for some nonzero vectors $e \in L_0$, $f \in M_0$, $e+f \in K_0$ and $e^* \in L_p$, $f^* \in M_p$, $e^*+f^* \in K_p$. Since $Ae \in L_0$, $Af \in M_0$ and

$$Ae+Af=A(e+f)\in K_0,$$

we have

$$AR=e^*\otimes Af-f^*\otimes Ae\in \mathrm{Alg}\mathcal{D}$$

by Lemma 1.2. Take $A=I$ in Claim 1 (1), then,

$$\delta(R)+R\tau(I)=\delta(I)R+\tau(R) \tag{3.1}$$

for every rank-two operator $R \in \mathrm{Alg}\mathcal{D}$. Since $AR \in \mathrm{Alg}\mathcal{D}$ has rank at most two, and $\mathrm{Alg}\mathcal{D}$ contains no rank-one operators, we have either $AR=0$ or $\mathrm{rank}(AR)=2$. By Eq (3.1), we can obtain that

$$\delta(AR)+AR\tau(I)=\delta(I)AR+\tau(AR).$$

This together with Claim 1 (1) leads to

$$\delta(I)AR+\tau(AR)=\delta(A)R+A\tau(R) \tag{3.2}$$

for all $A \in \mathrm{Alg}\mathcal{D}$. Replace A by AB in Eq (3.2), we have

$$\delta(I)ABR+\tau(ABR)=\delta(AB)R+AB\tau(R) \tag{3.3}$$

for all $A, B \in \mathrm{Alg}\mathcal{D}$. On the other hand, it follows from Eq (3.2) that

$$\begin{aligned}\delta(I)ABR+\tau(ABR) &= \delta(A)BR+A\tau(BR)\\ &= \delta(A)BR+A(\delta(B)R+B\tau(R)-\delta(I)BR)\\ &= \delta(A)BR+A\delta(B)R+AB\tau(R)-A\delta(I)BR\end{aligned}$$

for all $A, B \in \mathrm{Alg}\mathcal{D}$. This together with Eq (3.3) implies that

$$\delta(AB)R=A\delta(B)R+\delta(A)BR-A\delta(I)BR$$

for all $A, B \in \mathrm{Alg}\mathcal{D}$ and all rank-two operators $R \in \mathrm{Alg}\mathcal{D}$. By Lemma 1.4 (1),

$$\delta(AB)x = A\delta(B)x + \delta(A)Bx - A\delta(I)Bx$$

for all $A, B \in \mathrm{Alg}\mathcal{D}$ and all $x \in K_0 + L_0 + M_0$. Since $K_0 + L_0 + M_0$ is dense in X by Lemma 1.3 (2), we have

$$\delta(AB) = A\delta(B) + \delta(A)B - A\delta(I)B$$

for all $A, B \in \mathrm{Alg}\mathcal{D}$.

Claim 3. *Let $A, B \in \mathrm{Alg}\mathcal{D}$. Then,*

$$\tau(AB) = A\tau(B) + \tau(A)B - A\tau(I)B.$$

With the similar argument as in the proof of Claim 2, one can get that

$$R\tau(AB) = RA\tau(B) + R\tau(A)B - RA\tau(I)B$$

for all $A, B \in \mathrm{Alg}\mathcal{D}$ and all rank-two operators $R \in \mathrm{Alg}\mathcal{D}$. It follows that

$$(\tau(AB) - A\tau(B) - \tau(A)B + A\tau(I)B)x \in \ker(R)$$

for all $x \in X$. Note that

$$\cap\{\ker(R) : R \in \mathrm{Alg}\mathcal{D},\ \mathrm{rank}(R) = 2\} = {}^{\perp}(K_p + L_p + M_p)$$

by Lemma 1.4. Since R is arbitrary, by Lemma 1.3, we have

$$(\tau(AB) - A\tau(B) - \tau(A)B + A\tau(I)B)x \in {}^{\perp}X^* = (0)$$

for all $x \in X$. This implies that

$$\tau(AB) = A\tau(B) + \tau(A)B - A\tau(I)B$$

for all $A, B \in \mathrm{Alg}\mathcal{D}$.

Claim 4. *Let $A \in \mathrm{Alg}\mathcal{D}$. Then,*

$$\delta(A) - \delta(I)A = \tau(A) - A\tau(I).$$

By Eq (3.1) and Claim 3, we have

$$\begin{aligned}
\delta(AR) &= \delta(I)AR + \tau(AR) - AR\tau(I) \\
&= \delta(I)AR + A\tau(R) + \tau(A)R - A\tau(I)R - AR\tau(I) \\
&= \delta(I)AR + A(\tau(R) - R\tau(I)) + \tau(A)R - A\tau(I)R \\
&= \delta(I)AR + A(\delta(R) - \delta(I)R) + \tau(A)R - A\tau(I)R \\
&= \delta(I)AR + A\delta(R) - A\delta(I)R + \tau(A)R - A\tau(I)R.
\end{aligned}$$

This together with Claim 2 gives us

$$(\delta(A) - \delta(I)A)R = (\tau(A) - A\tau(I))R$$

for all rank-two operators $R \in \mathrm{Alg}\mathcal{D}$. By a similar argument as in Claim 2, we get

$$\delta(A) - \delta(I)A = \tau(A) - A\tau(I).$$

Claim 5. *There exists a linear map γ on* Alg$\mathcal{D}$ *such that (γ, δ, τ) is a ternary derivation.*

Define a map γ: Alg$\mathcal{D} \to$ Alg$\mathcal{D}$ by

$$\gamma(A) = \delta(A) + A\tau(I) = \tau(A) + \delta(I)A$$

for every $A \in$ Alg$\mathcal{D}$. Clearly, γ is linear. Let $A, B \in$ Alg$\mathcal{D}$. It follows from Claims 3 and 4 that

$$\begin{aligned}\gamma(AB) &= \tau(AB) + \delta(I)AB \\ &= A\tau(B) + \tau(A)B - A\tau(I)B + \delta(I)AB \\ &= A\tau(B) + (\tau(A) - A\tau(I))B + \delta(I)AB \\ &= A\tau(B) + (\delta(A) - \delta(I)A)B + \delta(I)AB \\ &= \delta(A)B + A\tau(B).\end{aligned}$$

Claim 6. *There exist densely defined, closed linear operators $R, T, S : \mathcal{M} \to X$ such that*

$$\tau(A)x = (TA + AS)x,$$

$$\delta(A)x = (RA - AT)x$$

and

$$\gamma(A)x = (RA + AS)x$$

for all $A \in$ Alg$\mathcal{D}$ *and all $x \in \mathcal{M}$. Here $\mathcal{M}$ is invariant under every element of* Alg$\mathcal{D}$.

Define Δ: Alg$\mathcal{D} \to$ Alg$\mathcal{D}$ by

$$\Delta(A) = \tau(A) - A\tau(I)$$

for every $A \in$ Alg$\mathcal{D}$. Obviously, Δ is a linear map. Let $A, B \in$ Alg$\mathcal{D}$. Then by Claim 3, we have

$$\begin{aligned}\Delta(AB) &= \tau(AB) - AB\tau(I) \\ &= A\tau(B) + \tau(A)B - A\tau(I)B - AB\tau(I) \\ &= A(\tau(B) - B\tau(I)) + (\tau(A) - A\tau(I))B \\ &= A\Delta(B) + \Delta(A)B.\end{aligned}$$

So, Δ is a derivation. By [15, Theorem 2.1], we know that every derivation of Alg$\mathcal{D}$ is quasi-spatial, that is, there exists a densely defined, closed linear operator T: $\mathcal{M} \subseteq X \to X$ with $\mathcal{M}$ invariant under every element of Alg$\mathcal{D}$ such that

$$\Delta(A)x = (TA - AT)x \tag{3.4}$$

for all $A \in$ Alg$\mathcal{D}$ and all $x \in \mathcal{M}$. Set

$$R = T + \delta(I)|_{\mathcal{M}}$$

and

$$S = \tau(I)|_{\mathcal{M}} - T.$$

It follows from Eq (3.4) that

$$\tau(A)x = (\Delta(A) + A\tau(I))x$$

$$
\begin{aligned}
&=(TA - AT + A\tau(I))x \\
&=(TA + A(\tau(I) - T))x \\
&=(TA + AS)x
\end{aligned}
$$

for all $A \in \mathrm{Alg}\mathcal{D}$ and all $x \in \mathcal{M}$. Similarly, by Claim 4 and Eq (3.4), we can obtain that

$$\delta(A)x = (RA - AT)x$$

for all $A \in \mathrm{Alg}\mathcal{D}$ and all $x \in \mathcal{M}$. Hence, by the definition of γ, we can get

$$
\begin{aligned}
\gamma(A)x &= (\tau(A) + \delta(I)A)x \\
&= (TA + AS + \delta(I)A)x \\
&= ((T + \delta(I))A + AS)x \\
&= (RA + AS)x
\end{aligned}
$$

for all $A \in \mathrm{Alg}\mathcal{D}$ and all $x \in \mathcal{M}$. □

4. Conclusions

In this paper, we described the structure of linear maps δ and τ satisfying (1.1) on strongly double triangle subspace lattice algebras $\mathrm{Alg}\mathcal{D}$. We proved that if linear maps δ, τ: $\mathrm{Alg}\mathcal{D} \rightarrow \mathrm{Alg}\mathcal{D}$ satisfy (1.1), then exists a linear map γ: $\mathrm{Alg}\mathcal{D} \rightarrow \mathrm{Alg}\mathcal{D}$ such that (γ, δ, τ) is a ternary derivation of $\mathrm{Alg}\mathcal{D}$. Using the result obtained, we characterized linear maps on $\mathrm{Alg}\mathcal{D}$ derivable (centralized) at zero point. Moreover, we showed that every local centralizer of $\mathrm{Alg}\mathcal{D}$ is a centralizer.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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