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Research article

New lower bounds of the minimum eigenvalue for the Fan product of several *M*-matrices

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Abstract: In this study, we generalize the definition of the Fan product of two M-matrices to any k M-matrices $A_1, A_2, \cdots, A_k$ of order n . We introduce two new inequalities for the lower bound of the minimum eigenvalue $\tau(A_1 \star A_2 \star \cdots \star A_k)$. These new lower bounds generalize the existing results. To validate the accuracy of our findings, we present examples in which our results outperform previous ones in certain cases.

Keywords: M-matrix; Fan product; minimum eigenvalue; irreducible
Mathematics Subject Classification: 15A47

1. Introduction

For convenience, this study adopts the following notations. We employ $R^{n\times n}$ ($C^{n\times n}$) to represent the space of real (complex) matrices with dimension n.

Let $A \in R^{n\times n}$ $(n \geq 2)$. Then, A is defined as a reducible matrix if there exists a permutation matrix P such that

$$PAP^{\mathrm{T}} = \begin{pmatrix} A_{11} & A_{12} \\ O & A_{22} \end{pmatrix},$$

where $A_{11} \in R^{l\times l}$, $A_{12} \in R^{l\times(n-l)}$, $A_{22} \in R^{(n-l)\times(n-l)}$ and O is an $(n-l)\times l$ zero matrix with $1 \leq l \leq n-1$. Otherwise, A is termed irreducible.

Let Z_n denote the collection of real matrices of order n, whose non-diagonal elements are nonpositive. If $A \in Z_n$, A is referred to a Z-matrix. It is evident that a sufficient condition for $A \in Z_n$ is that A can be written as:

$$A = sI - P, \tag{1.1}$$

where s is a real number and the elements of the matrix P are nonnegative. For $A \in Z_n$, let us denote

$$\tau(A) \equiv \min\{\mathrm{Re}\,\lambda\},$$

where λ is the characteristic root of the matrix A, with $\tau(A)$ being the minimum eigenvalue of A.

If we restrict $s > \rho(P)$ in (1.1), where $\rho(P)$ is the greatest module of the characteristic root of the matrix P, we will obtain a special class of Z-matrices, namely M-matrices (see Lemma 2.5.2.1 in [1]). The set of nonsingular M-matrices is denoted by M_n.

Notably, if $A \in M_n$, then $\tau(A)$ is a characteristic root of the matrix A (see Problem 19 in Section 2.5 in [1]). M-matrices possess several attractive properties and have been extensively studied [2, 3]. For an M-matrix, research on the minimum eigenvalue holds particular significance and has led to the emergence of numerous new results. In practice, the minimum eigenvalues of the M-matrices can be used to evaluate the stability of a power system. If the absolute value of the minimum eigenvalue is close to zero, it indicates the presence of stability issues in the system. By monitoring and analyzing the minimum eigenvalues of the M-matrices, potential problems in the power system can be detected on time, facilitating the implementation of appropriate measures to improve the stability and reliability of the system.

Unlike the traditional matrix multiplication calculation, the Fan product is a binary operation that takes two matrices of the same dimension and creates a new matrix of the same order. The Fan product of $A_1 = \left(a_{ij}\right) \in R^{n\times n}$ and $A_2 = \left(b_{ij}\right) \in R^{n\times n}$ is denoted by $A_1 \star A_2 = M = (m_{ij})$, where

$$m_{ij} = \begin{cases} a_{ii}b_{ii}, \ i = j, \\ -a_{ij}b_{ij}, \ i \neq j. \end{cases}$$

Notably, the two multiplied matrices must have the same structure. For example, let

$$A_1 = \begin{pmatrix} 4 & -1 & 0 \\ 0 & 2 & -5 \\ -3 & -1 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & -2 & -2 \\ -4 & 2 & -0 \\ 0 & -3 & 2 \end{pmatrix}.$$

Then, we have

$$A_1 \star A_2 = \begin{pmatrix} 4 & -2 & 0 \\ 0 & 4 & 0 \\ 0 & -3 & 2 \end{pmatrix}.$$

The Fan product is a fundamental operation in the study of M-matrices. It plays a crucial role in understanding the properties and characteristics of M-matrices. It is used to analyze the interplay between the elements of two M-matrices and study the properties of the resulting matrix, such as eigenvalues, spectral radius and invertibility. Among these studies, the computation and estimation of the minimum eigenvalue of Fan product has become a popular research topic.

Noticeably, if A_1, A_2 are M-matrices, then $A_1 \star A_2$ and the minimum eigenvalue $\tau(A_1 \star A_2)$ is not greater than any other characteristic roots of the Fan product $A_1 \star A_2$ in absolute value. Based on the Brauer theorem, Gerschgorin theorem and Brualdi theorem, multiple studies involving the bounds of $\tau(A_1 \star A_2)$ were conducted by the authors of [4–6].

Assuming A_1, A_2 as M-matrices, Horn and Johnson [1] established the following classical result describing the relationship between $\tau(A_1 \star A_2)$ and the product of $\tau(A_1), \tau(A_2)$, that is

$$\tau(A_1 \star A_2) \geq \tau(A_1)\,\tau(A_2). \tag{1.2}$$

Inspired by the definition of the Fan product of two M-matrices, we present the concept of the Fan product of k M-matrices as follows.

Let $A_1 = \left(a_{ij}\right)$, $A_2 = \left(b_{ij}\right)$, $\cdots$, $A_k = \left(k_{ij}\right)$ be n by n M-matrices. Define

$$A_1 \star A_2 \star \cdots \star A_k = H = (h_{ij})$$

where

$$h_{ij} = \begin{cases} a_{ii}b_{ii}\cdots k_{ii}, \ i = j, \\ -\left|a_{ij}b_{ij}\cdots k_{ij}\right|, \ i \neq j. \end{cases}$$

Note that the class of M-matrices is closed under the Fan product and $A_1 \star A_2 \star \cdots \star A_k$ is an M-matrix. Therefore, we can generalize inequality (1.2) to any k M-matrices as follows:

$$\tau (A_1 \star A_2 \star \cdots \star A_k) \geq \tau (A_1) \tau (A_2) \cdots \tau (A_k). \tag{1.3}$$

Inspired by the research in [4–13], we continue to study the lower bound of $\tau (A_1 \star A_2 \star \cdots \star A_k)$. The remainder of this study is organized as follows. First, we present two new types of lower bounds for the minimum eigenvalue involving the Fan product of any k M-matrices $A_1, A_2, \cdots, A_k$ in Section 2. The obtained new bounds generalize some of the previous results. In Section 3, numerical tests are presented to certify our findings and comparisons among these lower bounds are considered.

2. Two new lower bounds for $\tau (A_1 \star A_2 \star \cdots \star A_k)$

To demonstrate our findings, we first introduce some fundamental lemmas. These will be useful in the subsequent proof.

Lemma 1. [1] If $A \in R^{n\times n}$ is an irreducible M-matrix, then
(1) there exists a positive real eigenvalue that is equal to its minimum eigenvalue $\tau (A)$,
(2) there is an eigenvector $u > 0$ such that $Au = \tau (A) u$.

Lemma 2. [14] Given an irreducible M-matrix $A \in R^{n\times n}$ and a nonnegative nonzero vector $z \in R^n$, if $Az \geq kz$, then $\tau (A) \geq k$.

Lemma 3. [15] Let $A = \left(a_{ij}\right) \in C^{n\times n}$ $(n \geq 2)$. If λ is the characteristic root of the matrix A, then there must exist two unequal positive integers i, j that satisfy the following inequality:

$$|\lambda - a_{ii}| \left|\lambda - a_{jj}\right| \leq R_i(A)R_j(A),$$

where $R_i(A) = \sum\limits_{k\neq i}^{n} |a_{ik}|$, $R_j(A) = \sum\limits_{k\neq j}^{n} \left|a_{jk}\right|$.

Lemma 4. [16] Let $x_i \geq y_i \geq 0$, $i = 1, 2, \cdots, n$. If $\sum\limits_{i=1}^{n} \dfrac{1}{p_i} \geq 1$ with $p_i > 0$, then we have

$$\prod_{i=1}^{n} x_i - \prod_{i=1}^{n} y_i \geq \prod_{i=1}^{n} \left({x_i}^{p_i} - {y_i}^{p_i}\right)^{\frac{1}{p_i}}. \tag{2.1}$$

In the following, we present the first lower bound for $\tau (A_1 \star A_2 \star \cdots \star A_k)$.

Theorem 1. Let $A_1 = \left(a_{ij}\right), A_2 = \left(b_{ij}\right), \cdots, A_k = \left(k_{ij}\right)$ be n by n nonsingular M-matrices. Then,

$$\tau (A_1 \star A_2 \star \cdots \star A_k) \geq \min_{1\leq i\leq n} \left\{a_{ii}b_{ii}\cdots k_{ii} - [a_{ii} - \tau (A_1)] [b_{ii} - \tau (A_2)] \cdots [k_{ii} - \tau (A_k)]\right\}. \tag{2.2}$$

Proof. Obviously, inequality (2.2) becomes an equality when $n = 1$. We next assume that $n \geq 2$. To demonstrate this problem, let us distinguish two aspects.
Case 1. First, we assume that $A_1 \star A_2 \star \cdots \star A_k$ is irreducible. Clearly, $A_1, A_2, \cdots, A_k$ are all irreducible. In addition, we have

$$a_{ii} - \tau(A_1) > 0,\ b_{ii} - \tau(A_2) > 0, \ldots, k_{ii} - \tau(A_k) > 0,\ i = 1, 2, \cdots, n.$$

Since $A_1, A_2, \cdots, A_k$ are irreducible M-matrices, according to Lemma 1, there exist

$$u = (u_1, u_2, \cdots, u_n)^{\mathrm{T}} > 0,\ v = (v_1, v_2, \cdots, v_n)^{\mathrm{T}} > 0, \ldots, w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}} > 0$$

satisfying

$$A_1 u = \tau(A_1) u,\ {A_2}^{\mathrm{T}} v = \tau(A_2) v, \ldots, {A_k}^{\mathrm{T}} w = \tau(A_k) w.$$

That is

$$a_{ii} u_i - \sum_{j \neq i}^{n} \left|a_{ij}\right| u_j = \tau(A_1) u_i,\ i = 1, 2, \cdots, n,$$

$$b_{jj} v_j - \sum_{i \neq j}^{n} \left|b_{ij}\right| v_i = \tau(A_2) v_j,\ j = 1, 2, \cdots, n,$$

......

$$k_{jj} w_j - \sum_{i \neq j}^{n} \left|k_{ij}\right| w_i = \tau(A_k) w_j,\ j = 1, 2, \cdots, n.$$

From the above equations, we have

$$\left|b_{ij}\right| \leq \frac{\left[b_{jj} - \tau(A_2)\right] v_j}{v_i}, \ \ldots, \ \left|k_{ij}\right| \leq \frac{\left[k_{jj} - \tau(A_k)\right] w_j}{w_i}$$

for all $i \neq j$. Let $z = (z_1, z_2, \cdots, z_n) \in R^n$, in which

$$z_i = \frac{u_i}{\left[b_{ii} - \tau(A_2)\right] v_i \cdots \left[k_{ii} - \tau(A_k)\right] w_i} > 0,\ i = 1, 2, \cdots, n.$$

We define $A = A_1 \star A_2 \star \cdots \star A_k$. For any $i = 1, 2, \cdots, n$, we have

$$\begin{aligned}(Az)_i &= a_{ii} b_{ii} \cdots k_{ii} z_i - \sum_{j \neq i}^{n} \left|a_{ij} b_{ij} \cdots k_{ij}\right| z_j \\ &\geq a_{ii} b_{ii} \cdots k_{ii} z_i - \sum_{j \neq i}^{n} \left|a_{ij}\right| \frac{\left[b_{jj} - \tau(A_2)\right] v_j}{v_i} \cdots \frac{\left[k_{jj} - \tau(A_k)\right] w_j}{w_i} z_j.\end{aligned}$$

Noticing that

$$z_j = \frac{u_j}{\left[b_{jj} - \tau(A_2)\right] v_j \cdots \left[k_{jj} - \tau(A_k)\right] w_j} > 0,$$

we get

$$
\begin{aligned}
(Az)_i &\geq a_{ii}b_{ii}\cdots k_{ii}z_i - \frac{1}{v_i\cdots w_i}\sum_{j\neq i}^{n}\left|a_{ij}\right|u_j \\
&= a_{ii}b_{ii}\cdots k_{ii}z_i - \frac{1}{v_i\cdots w_i}\left[a_{ii}-\tau(A_1)\right]u_i \\
&= a_{ii}b_{ii}\cdots k_{ii}z_i - \left[a_{ii}-\tau(A_1)\right]\left[b_{ii}-\tau(A_2)\right]\cdots\left[k_{ii}-\tau(A_k)\right]z_i \\
&= \left\{a_{ii}b_{ii}\cdots k_{ii} - \left[a_{ii}-\tau(A_1)\right]\left[b_{ii}-\tau(A_2)\right]\cdots\left[k_{ii}-\tau(A_k)\right]\right\}z_i.
\end{aligned}
$$

According to Lemma 2, this means that

$$
\tau(A_1\star A_2\star\cdots\star A_k)\geq \min_{1\leq i\leq n}\left\{a_{ii}b_{ii}\cdots k_{ii} - \left[a_{ii}-\tau(A_1)\right]\left[b_{ii}-\tau(A_2)\right]\cdots\left[k_{ii}-\tau(A_k)\right]\right\}.
$$

Case 2. Next, we consider the matrix $A_1\star A_2\star\cdots\star A_k$ as reducible. As is known, a Z-matrix is a nonsingular M-matrix if and only if all of its leading principal minors are positive (see condition (E17) of Theorem 6.2.3 in [14]). At this point, there exists a real number $\varepsilon > 0$ such that $A_1-\varepsilon P$, $A_2-\varepsilon P$, $\ldots$, $A_k-\varepsilon P$ are irreducible nonsingular M-matrices, where $P=\left(p_{ij}\right)$ is a matrix of size n with

$$
p_{12}=p_{23}=\cdots=p_{n-1,n}=p_{n1}=1,
$$

the rest of the elements being zero. If ε is sufficiently small such that all the leading principal minors of $A_1-\varepsilon P$, $A_2-\varepsilon P$, $\ldots$, $A_k-\varepsilon P$ are positive, then we replace $A_1-\varepsilon P$, $A_2-\varepsilon P$, $\ldots$, $A_k-\varepsilon P$ with $A_1, A_2,\cdots,A_k$ in Case 1. Let $\varepsilon\to 0$, we can achieve our desired result by continuity theory. Thus, we have completed the proof of Theorem 1.

Remark 1. We now provide a comparison between the lower bounds in Theorem 1 and the inequality (1.3). In fact, for the nonsingular M-matrices $A_1=\left(a_{ij}\right), A_2=\left(b_{ij}\right),\cdots,A_k=\left(k_{ij}\right)$, we have

$$
a_{ii} > a_{ii}-\tau(A_1)\geq 0,\ b_{ii} > b_{ii}-\tau(A_2)\geq 0,\cdots,\ k_{ii} > k_{ii}-\tau(A_k)\geq 0.
$$

It follows from Lemma 4 that

$$
\begin{aligned}
&a_{ii}b_{ii}\cdots k_{ii} - \left[a_{ii}-\tau(A_1)\right]\left[b_{ii}-\tau(A_2)\right]\cdots\left[k_{ii}-\tau(A_k)\right] \\
\geq&\ \left[a_{ii}-(a_{ii}-\tau(A_1))\right]\left[b_{ii}-(b_{ii}-\tau(A_2))\right]\cdots\left[k_{ii}-(k_{ii}-\tau(A_k))\right] \\
=&\ \tau(A_1)\tau(A_2)\cdots\tau(A_k).
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\tau(A_1\star A_2\star\cdots\star A_k) &\geq \min_{1\leq i\leq n}\left\{a_{ii}b_{ii}\cdots k_{ii} - \left[a_{ii}-\tau(A_1)\right]\left[b_{ii}-\tau(A_2)\right]\cdots\left[k_{ii}-\tau(A_k)\right]\right\} \\
&\geq \tau(A_1)\tau(A_2)\cdots\tau(A_k).
\end{aligned}
$$

This implies that the bound in Theorem 1 is sharper than that in inequality (1.3).

Here, we consider a special case. Let $k=2$ in Theorem 1, we will obtain the following conclusion.

Corollary 1. Let $A_1=\left(a_{ij}\right)$, $A_2=\left(b_{ij}\right)$ be n by n nonsingular M-matrices. Then,

$$
\tau(A_1\star A_2)\geq \min_{1\leq i\leq n}\left\{a_{ii}b_{ii} - \left[a_{ii}-\tau(A_1)\right]\left[b_{ii}-\tau(A_2)\right]\right\}. \tag{2.3}
$$

This happens to be the conclusion of Theorem 9 of Fang [4]. Therefore, the result of Fang [4] is included in Theorem 1 of this paper.

The second inequality regarding $\tau(A_1 \star A_2 \star \cdots \star A_k)$ will be established next.

Theorem 2. Let $A_1 = \left(a_{ij}\right)$, $A_2 = \left(b_{ij}\right), \cdots, A_k = \left(k_{ij}\right)$ be n by n nonsingular M-matrices. Then,

$$\tau(A_1 \star A_2 \star \cdots \star A_k) \geq \min_{i \neq j} \frac{1}{2} \Big\{ a_{ii}b_{ii}\cdots k_{ii} + a_{jj}b_{jj}\cdots k_{jj} - \Big[\left(a_{ii}b_{ii}\cdots k_{ii} - a_{jj}b_{jj}\cdots k_{jj}\right)^2 + 4\left(a_{ii} - \tau(A_1)\right)\cdots\left(k_{ii} - \tau(A_k)\right)\left(a_{jj} - \tau(A_1)\right)\cdots\left(k_{jj} - \tau(A_k)\right)\Big]^{\frac{1}{2}} \Big\}. \quad (2.4)$$

Proof. Obviously, the conclusion is true when $n = 1$. Next, we assume that $n \geq 2$. To demonstrate this problem, let us distinguish two aspects.
Case 1. $A_1 \star A_2 \star \cdots \star A_k$ is irreducible. It is known that $A_1, A_2, \cdots, A_k$ are all irreducible. According to Lemma 1, there exist

$$u = (u_1, u_2, \cdots, u_n)^{\mathrm{T}} > 0,\ v = (v_1, v_2, \cdots, v_n)^{\mathrm{T}} > 0, \cdots,\ w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}} > 0$$

satisfying

$$A_1 u = \tau(A_1)\, u,\ A_2 v = \tau(A_2)\, v,\ \cdots,\ A_k w = \tau(A_k)\, w.$$

Therefore, we have

$$a_{ii} - \sum_{p \neq i}^{n} \frac{\left|a_{ip}\right| u_p}{u_i} = \tau(A_1),$$

$$b_{ii} - \sum_{p \neq i}^{n} \frac{\left|b_{ip}\right| v_p}{v_i} = \tau(A_2),$$

......

$$k_{ii} - \sum_{p \neq i}^{n} \frac{\left|k_{ip}\right| w_p}{w_i} = \tau(A_k).$$

Now, we define k nonsingular positive diagonal matrices as follows:

$$D_1 = \operatorname{diag}(u_1, u_2, \cdots, u_n),\ D_2 = \operatorname{diag}(v_1, v_2, \cdots, v_n),\ \cdots,\ D_k = \operatorname{diag}(w_1, w_2, \cdots, w_n).$$

Let

$$\tilde{A}_1 = {D_1}^{-1} A_1 D_1 = \left(\frac{a_{ij}u_j}{u_i}\right),\ \tilde{A}_2 = {D_2}^{-1} A_2 D_2 = \left(\frac{b_{ij}v_j}{v_i}\right),\ \cdots,\ \tilde{A}_k = {D_k}^{-1} A_k D_k = \left(\frac{k_{ij}w_j}{w_i}\right)$$

and denote $\tilde{A}_1 \star \tilde{A}_2 \star \cdots \star \tilde{A}_k = H = \left(h_{ij}\right)$. By the definition of the matrix $\tilde{A}_1 \star \tilde{A}_2 \star \cdots \star \tilde{A}_k$, we have

$$h_{ij} = \begin{cases} a_{ii}b_{ii}\cdots k_{ii},\ i = j, \\ -\left|\dfrac{a_{ij}u_j}{u_i}\dfrac{b_{ij}v_j}{v_i}\cdots\dfrac{k_{ij}w_j}{w_i}\right|,\ i \neq j. \end{cases}$$

Define $D = D_1 D_2 \cdots D_k$ and $D^{-1}\left(A_1 \star A_2 \star \cdots \star A_k\right) D = H' = \left(h_{ij}{}'\right)$. Thus, we get

$$h_{ij}{}' = \begin{cases} \dfrac{1}{u_i v_i \cdots w_i}\left(a_{ii} b_{ii} \cdots k_{ii}\right) u_i v_i \cdots w_i = a_{ii} b_{ii} \cdots k_{ii}, \ i = j, \\ \dfrac{1}{u_i v_i \cdots w_i}\left(-\left|a_{ij} b_{ij} \cdots k_{ij}\right|\right) u_j v_j \cdots w_j = -\left|\dfrac{a_{ij} u_j}{u_i} \dfrac{b_{ij} v_j}{v_i} \cdots \dfrac{k_{ij} w_j}{w_i}\right|, \ i \neq j. \end{cases}$$

Therefore, we have

$$D^{-1}\left(A_1 \star A_2 \star \cdots \star A_k\right) D = \tilde{A}_1 \star \tilde{A}_2 \star \cdots \star \tilde{A}_k.$$

This shows that

$$\tau\left(\tilde{A}_1 \star \tilde{A}_2 \star \cdots \star \tilde{A}_k\right) = \tau\left(A_1 \star A_2 \star \cdots \star A_k\right).$$

In addition, we have

$$\begin{aligned} R_i\left(\tilde{A}_1 \star \tilde{A}_2 \star \cdots \star \tilde{A}_k\right) &= \sum_{p \neq i}^{n}\left|\frac{a_{ip} u_p}{u_i} \frac{b_{ip} v_p}{v_i} \cdots \frac{k_{ip} w_p}{w_i}\right| \\ &\leq \sum_{p \neq i}^{n} \frac{\left|a_{ip}\right| u_p}{u_i} \sum_{p \neq i}^{n} \frac{\left|b_{ip}\right| v_p}{v_i} \cdots \sum_{p \neq i}^{n} \frac{\left|k_{ip}\right| w_p}{w_i} \\ &= \left[a_{ii} - \tau\left(A_1\right)\right]\left[b_{ii} - \tau\left(A_2\right)\right] \ldots \left[k_{ii} - \tau\left(A_k\right)\right]. \end{aligned} \tag{2.5}$$

Similarly, we have

$$R_j\left(\tilde{A}_1 \star \tilde{A}_2 \star \cdots \star \tilde{A}_k\right) \leq \left[a_{jj} - \tau\left(A_1\right)\right]\left[b_{jj} - \tau\left(A_2\right)\right] \ldots \left[k_{jj} - \tau\left(A_k\right)\right]. \tag{2.6}$$

Since $\tau\left(\tilde{A}_1 \star \tilde{A}_2 \star \cdots \star \tilde{A}_k\right)$ is an eigenvalue of $\tilde{A}_1 \star \tilde{A}_2 \star \cdots \star \tilde{A}_k$, in terms of Lemma 3 and inequalities (2.5) and (2.6), there exist two unequal positive integers i, j such that

$$\begin{aligned} &\left|\tau\left(A_1 \star A_2 \star \cdots \star A_k\right) - a_{ii} b_{ii} \cdots k_{ii}\right|\left|\tau\left(A_1 \star A_2 \star \cdots \star A_k\right) - a_{jj} b_{jj} \cdots k_{jj}\right| \\ \leq & R_i\left(\tilde{A}_1 \star \tilde{A}_2 \star \cdots \star \tilde{A}_k\right) R_j\left(\tilde{A}_1 \star \tilde{A}_2 \star \cdots \star \tilde{A}_k\right) \\ \leq & \left[a_{ii} - \tau\left(A_1\right)\right] \ldots \left[k_{ii} - \tau\left(A_k\right)\right]\left[a_{jj} - \tau\left(A_1\right)\right] \ldots \left[k_{jj} - \tau\left(A_k\right)\right]. \end{aligned} \tag{2.7}$$

From inequality (2.7) and $0 < \tau\left(A_1 \star A_2 \star \cdots \star A_k\right) < a_{ii} b_{ii} \cdots k_{ii}$ for $i = 1, 2, \cdots, n$, we get

$$\begin{aligned} &\left[\tau\left(A_1 \star A_2 \star \cdots \star A_k\right) - a_{ii} b_{ii} \cdots k_{ii}\right]\left[\tau\left(A_1 \star A_2 \star \cdots \star A_k\right) - a_{jj} b_{jj} \cdots k_{jj}\right] \\ \leq & \left[a_{ii} - \tau\left(A_1\right)\right] \ldots \left[k_{ii} - \tau\left(A_k\right)\right]\left[a_{jj} - \tau\left(A_1\right)\right] \ldots \left[k_{jj} - \tau\left(A_k\right)\right]. \end{aligned} \tag{2.8}$$

Solving inequality (2.8), we obtain

$$\begin{aligned} &\tau\left(A_1 \star A_2 \star \cdots \star A_k\right) \\ \geq & \frac{1}{2}\Big\{a_{ii} b_{ii} \cdots k_{ii} + a_{jj} b_{jj} \cdots k_{jj} - \Big[\left(a_{ii} b_{ii} \cdots k_{ii} - a_{jj} b_{jj} \cdots k_{jj}\right)^2 \\ & + 4\left(a_{ii} - \tau\left(A_1\right)\right) \cdots \left(k_{ii} - \tau\left(A_k\right)\right)\left(a_{jj} - \tau\left(A_1\right)\right) \cdots \left(k_{jj} - \tau\left(A_k\right)\right)\Big]^{\frac{1}{2}}\Big\} \end{aligned}$$

$$\geq \min_{i\neq j} \frac{1}{2}\left\{a_{ii}b_{ii}\cdots k_{ii}+a_{jj}b_{jj}\cdots k_{jj}-\left[\left(a_{ii}b_{ii}\cdots k_{ii}-a_{jj}b_{jj}\cdots k_{jj}\right)^{2}\right.\right.$$
$$\left.\left.+4\left(a_{ii}-\tau\left(A_{1}\right)\right)\cdots\left(k_{ii}-\tau\left(A_{k}\right)\right)\left(a_{jj}-\tau\left(A_{1}\right)\right)\cdots\left(k_{jj}-\tau\left(A_{k}\right)\right)\right]^{\frac{1}{2}}\right\}.$$

Case 2. $A_1 \star A_2 \star \cdots \star A_k$ is reducible. We can use the approach of Theorem 1 similarly prove this. Hence, the proof of Theorem 2 is finished.

Remark 2. Following the demonstration of Theorem 2, we present a new proof of Theorem 1. From Theorem 1.11 in [15] , we obtain

$$\left|\tau\left(A_{1}\star A_{2}\star\cdots\star A_{k}\right)-a_{ii}b_{ii}\cdots k_{ii}\right|\leq R_{i}\left(\tilde{A}_{1}\star\tilde{A}_{2}\star\cdots\star\tilde{A}_{k}\right).$$

According to $0<\tau\left(A_{1}\star A_{2}\star\cdots\star A_{k}\right)\leq a_{ii}b_{ii}\cdots k_{ii}$ and inequality (2.5), we obtain

$$a_{ii}b_{ii}\cdots k_{ii}-\tau\left(A_{1}\star A_{2}\star\cdots\star A_{k}\right)\leq\left[a_{ii}-\tau\left(A_{1}\right)\right]\left[b_{ii}-\tau\left(A_{2}\right)\right]\ldots\left[k_{ii}-\tau\left(A_{k}\right)\right].$$

Thus, we acquire

$$\begin{aligned}\tau\left(A_{1}\star A_{2}\star\cdots\star A_{k}\right)&\geq a_{ii}b_{ii}\cdots k_{ii}-\left[a_{ii}-\tau\left(A_{1}\right)\right]\left[b_{ii}-\tau\left(A_{2}\right)\right]\ldots\left[k_{ii}-\tau\left(A_{k}\right)\right]\\&\geq\min_{1\leq i\leq n}\left\{a_{ii}b_{ii}\cdots k_{ii}-\left[a_{ii}-\tau\left(A_{1}\right)\right]\left[b_{ii}-\tau\left(A_{2}\right)\right]\cdots\left[k_{ii}-\tau\left(A_{k}\right)\right]\right\}.\end{aligned}$$

Now, we consider a special case. We can immediately obtain the following corollary from Theorem 2 by setting $k = 2$.

Corollary 2. Let $A_1 = \left(a_{ij}\right), A_2 = \left(b_{ij}\right)$ be n by n nonsingular M-matrices. Then,

$$\begin{aligned}\tau\left(A_{1}\star A_{2}\right)\geq\min_{i\neq j}\frac{1}{2}&\left\{a_{ii}b_{ii}+a_{jj}b_{jj}-\left[\left(a_{ii}b_{ii}-a_{jj}b_{jj}\right)^{2}\right.\right.\\&\left.\left.+4\left(a_{ii}-\tau\left(A_{1}\right)\right)\left(b_{ii}-\tau\left(A_{2}\right)\right)\left(a_{jj}-\tau\left(A_{1}\right)\right)\left(b_{jj}-\tau\left(A_{2}\right)\right)\right]^{\frac{1}{2}}\right\}.\end{aligned}\tag{2.9}$$

This happens to be the conclusion of Theorem 7 of Liu [5]. Therefore, the result of Liu [5] is included in Theorem 2 of this paper.

The following theorem shows that the bound in (2.4) of Theorem 2 is more precise than the bound in (2.2) of Theorem 1.

Theorem 3. Let $A_1 = \left(a_{ij}\right), A_2 = \left(b_{ij}\right), \cdots, A_k = \left(k_{ij}\right)$ be n by n nonsingular M-matrices. Then,

$$\begin{aligned}\tau\left(A_{1}\star A_{2}\star\cdots\star A_{k}\right)\geq\min_{i\neq j}\frac{1}{2}&\left\{a_{ii}b_{ii}\cdots k_{ii}+a_{jj}b_{jj}\cdots k_{jj}-\left[\left(a_{ii}b_{ii}\cdots k_{ii}-a_{jj}b_{jj}\cdots k_{jj}\right)^{2}\right.\right.\\&\left.\left.+4\left(a_{ii}-\tau\left(A_{1}\right)\right)\cdots\left(k_{ii}-\tau\left(A_{k}\right)\right)\left(a_{jj}-\tau\left(A_{1}\right)\right)\cdots\left(k_{jj}-\tau\left(A_{k}\right)\right)\right]^{\frac{1}{2}}\right\}\\\geq\min_{1\leq i\leq n}&\left\{a_{ii}b_{ii}\cdots k_{ii}-\left[a_{ii}-\tau\left(A_{1}\right)\right]\left[b_{ii}-\tau\left(A_{2}\right)\right]\cdots\left[k_{ii}-\tau\left(A_{k}\right)\right]\right\}.\end{aligned}$$

Proof. We assume, without losing generality, that

$$a_{ii}b_{ii}\cdots k_{ii}-\left[a_{ii}-\tau\left(A_{1}\right)\right]\left[b_{ii}-\tau\left(A_{2}\right)\right]\cdots\left[k_{ii}-\tau\left(A_{k}\right)\right]$$

$$\leq a_{jj}b_{jj}\cdots k_{jj}-\left[a_{jj}-\tau(A_1)\right]\left[b_{jj}-\tau(A_2)\right]\cdots\left[k_{jj}-\tau(A_k)\right].$$

As a result, we can express the inequality above in the following way:

$$\begin{aligned}&\left[a_{jj}-\tau(A_1)\right]\left[b_{jj}-\tau(A_2)\right]\cdots\left[k_{jj}-\tau(A_k)\right]\\ &\leq\left[a_{ii}-\tau(A_1)\right]\left[b_{ii}-\tau(A_2)\right]\cdots\left[k_{ii}-\tau(A_k)\right]+a_{jj}b_{jj}\cdots k_{jj}-a_{ii}b_{ii}\cdots k_{ii}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\left(a_{ii}b_{ii}\cdots k_{ii}-a_{jj}b_{jj}\cdots k_{jj}\right)^2+4\left[a_{ii}-\tau(A_1)\right]\cdots\left[k_{ii}-\tau(A_k)\right]\left[a_{jj}-\tau(A_1)\right]\cdots\left[k_{jj}-\tau(A_k)\right]\\ \leq&\left(a_{ii}b_{ii}\cdots k_{ii}-a_{jj}b_{jj}\cdots k_{jj}\right)^2+4[a_{ii}-\tau(A_1)]^2\cdots[k_{ii}-\tau(A_k)]^2\\ &+4\left[a_{ii}-\tau(A_1)\right]\cdots\left[k_{ii}-\tau(A_k)\right]\left(a_{jj}b_{jj}\cdots k_{jj}-a_{ii}b_{ii}\cdots k_{ii}\right)\\ =&\left\{a_{jj}b_{jj}\cdots k_{jj}-a_{ii}b_{ii}\cdots k_{ii}+2\left[a_{ii}-\tau(A_1)\right]\cdots\left[k_{ii}-\tau(A_k)\right]\right\}^2.\end{aligned}\tag{2.10}$$

From inequalities (2.4) and (2.10), we obtain

$$\begin{aligned}\tau(A_1\star A_2\star\cdots\star A_k)\geq&\min_{i\neq j}\frac{1}{2}\left\{a_{ii}b_{ii}\cdots k_{ii}+a_{jj}b_{jj}\cdots k_{jj}-\left[\left(a_{ii}b_{ii}\cdots k_{ii}-a_{jj}b_{jj}\cdots k_{jj}\right)^2\right.\right.\\ &\left.\left.+4\left(a_{ii}-\tau(A_1)\right)\cdots\left(k_{ii}-\tau(A_k)\right)\left(a_{jj}-\tau(A_1)\right)\cdots\left(k_{jj}-\tau(A_k)\right)\right]^{\frac{1}{2}}\right\}\\ \geq&\min_{i\neq j}\frac{1}{2}\left\{a_{ii}b_{ii}\cdots k_{ii}+a_{jj}b_{jj}\cdots k_{jj}-\left[a_{jj}b_{jj}\cdots k_{jj}-a_{ii}b_{ii}\cdots k_{ii}+2\left(a_{ii}-\tau(A_1)\right)\cdots\left(k_{ii}-\tau(A_k)\right)\right]\right\}\\ =&\min_{1\leq i\leq n}\left\{a_{ii}b_{ii}\cdots k_{ii}-\left[a_{ii}-\tau(A_1)\right]\cdots\left[k_{ii}-\tau(A_k)\right]\right\}.\end{aligned}$$

The proof of Theorem 3 is finished.

Remark 3. From the previous statements, we observe that

$$\begin{aligned}\tau(A_1\star A_2\star\cdots\star A_k)\geq&\min_{i\neq j}\frac{1}{2}\left\{a_{ii}b_{ii}\cdots k_{ii}+a_{jj}b_{jj}\cdots k_{jj}-\left[\left(a_{ii}b_{ii}\cdots k_{ii}-a_{jj}b_{jj}\cdots k_{jj}\right)^2\right.\right.\\ &\left.\left.+4\left(a_{ii}-\tau(A_1)\right)\cdots\left(k_{ii}-\tau(A_k)\right)\left(a_{jj}-\tau(A_1)\right)\cdots\left(k_{jj}-\tau(A_k)\right)\right]^{\frac{1}{2}}\right\}\\ \geq&\min_{1\leq i\leq n}\left\{a_{ii}b_{ii}\cdots k_{ii}-\left[a_{ii}-\tau(A_1)\right]\left[b_{ii}-\tau(A_2)\right]\cdots\left[k_{ii}-\tau(A_k)\right]\right\}\\ \geq&\tau(A_1)\tau(A_2)\cdots\tau(A_k).\end{aligned}$$

3. Numerical examples

To demonstrate that our new lower bounds are more precise than the previous results, we consider two specific examples in this section.

Example 1. First, we employ two M-matrices from [6].

$$A_1=\begin{pmatrix}2&-1&0\\0&1&-0.5\\-0.5&-1&2\end{pmatrix},\ A_2=\begin{pmatrix}1&-0.25&-0.25\\-0.5&1&-0.25\\-0.25&-0.5&1\end{pmatrix}.$$

We compute the Fan product:

$$A_1 \star A_2 = \begin{pmatrix} 2 & -0.25 & 0 \\ 0 & 1 & -0.125 \\ -0.125 & -0.5 & 2 \end{pmatrix}.$$

It is simple to see that $\tau(A_1) = 0.5402$, $\tau(A_2) = 0.3432$ and $\tau(A_1 \star A_2) = 0.9377$. By inequality (1.2) in [1], we get

$$\tau(A_1 \star A_2) \geq \tau(A_1)\tau(A_2) = 0.1854.$$

According to Corollary 1 (see also Theorem 9 in [4]), we have

$$\tau(A_1 \star A_2) \geq \min_{1\leq i\leq n} \{a_{ii}b_{ii} - [a_{ii} - \tau(A_1)][b_{ii} - \tau(A_2)]\} = 0.6980.$$

In terms of Corollary 2 (see also Theorem 7 in [5]), we obtain

$$\tau(A_1 \star A_2) \geq \min_{i\neq j} \frac{1}{2}\Big\{a_{ii}b_{ii}+a_{jj}b_{jj} - \Big[\big(a_{ii}b_{ii} - a_{jj}b_{jj}\big)^2 \\ +4(a_{ii} - \tau(A_1))(b_{ii} - \tau(A_2))\big(a_{jj} - \tau(A_1)\big)\big(b_{jj} - \tau(A_2)\big)\Big]^{\frac{1}{2}}\Big\} = 0.7655.$$

Example 2. Now, we present the second example and examine the following three M-matrices.

$$A_1 = \begin{pmatrix} 100 & -21 & -30 \\ 0 & 1 & -12 \\ 0 & 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 0 & 0 \\ -19 & 100 & 0 \\ -81 & -17 & 1 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 100 \end{pmatrix}.$$

We compute the Fan product:

$$A_1 \star A_2 \star A_3 = \begin{pmatrix} 100 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 100 \end{pmatrix}.$$

It is easy to observe that $\tau(A_1) = \tau(A_2) = \tau(A_3) = 1$, $\tau(A_1 \star A_2 \star A_3) = 100$. By inequality (1.3), we obtain

$$\tau(A_1 \star A_2 \star A_3) = 100 \geq \tau(A_1)\tau(A_2)\tau(A_3) = 1.$$

We observe that this result is trivial. If we apply Theorem 1 in this study, we acquire

$$\tau(A_1 \star A_2 \star A_3) \geq \min_{1\leq i\leq n} \{a_{ii}b_{ii}c_{ii} - [a_{ii} - \tau(A_1)][b_{ii} - \tau(A_2)][c_{ii} - \tau(A_3)]\} = 100.$$

Surprisingly, the proposed is the actual minimum eigenvalue of $\tau(A_1 \star A_2 \star A_3)$. From the presented examples, we can see that our results are more accurate than the earlier results in some cases.

4. Conclusions

M-matrices are a special class of matrices with important properties. The Fan product is a binary operation defined for M-matrices, which plays an important role in understanding the properties and characteristics of M-matrices. Inspired by the definition of the Fan product of two M-matrices, we introduced the concept of the Fan product of k M-matrices.

Additionally, for M-matrices $A_1, A_2, \cdots, A_k$, we have proposed two new inequalities for the lower bound of the minimum eigenvalue of the Fan product $A_1 \star A_2 \star \cdots \star A_k$. The derived new type lower bounds generalize some of the existing results to a certain extent.

In summary, this study established the relationship between the minimum eigenvalue of the Fan product of k M-matrices and the minimum eigenvalues of the original k M-matrices. The conclusions of this study can be considered as a valuable addition to the theoretical study of M-matrices.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this paper.

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