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Research article

A novel numerical scheme for reproducing kernel space of 2D fractional diffusion equations

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Abstract: A novel method is presented for reproducing kernel of a 2D fractional diffusion equation. The exact solution is expressed as a series, which is then truncated to get an approximate solution. In addition, some techniques to improve existing methods are also proposed. The proposed approach is easy to implement. It is proved that both the approximate solution and its partial derivatives converge to their exact solutions. Numerical results demonstrate that the proposed approach is effective and can provide a high precision global approximate solution.

Keywords: exact solution; numerical method; 2D fractional diffusion equations; reproducing kernel
Mathematics Subject Classification: 35A40, 65N35, 65R20

1. Introduction

The fractional diffusion equation (FDE) is widely used in engineering and science fields, such as bioengineering, electrochemistry, medicine and signal processing [1–5]. The references [6–10] are extensive and still growing rapidly on numerical approximations to solutions of FDE. Youssri and Atta [11] applied the Petrov-Galerkin Lucas polynomials procedure to time FDEs. Gupta [12] applied simplified differential transformation and homotopy perturbation to the fractional Benney-Lin equation. Attar et al. [13] used Akbar-Ganji's approach to investigate FDEs. Bota and Caruntu [14] used polynomial least squares approach to study the fractional quadratic Riccati differential equation. Moustafa, Youssri and Atta [15] applied the explicit Chebyshev-Galerkin scheme to time FDE. Djennadi, Shawagfeh and Abu Arqub [16] used a reproducing kernel (RK) approach to study the inverse source problem of FDEs. Jiang and Lin [17] studied the fractional advection-dispersion equation using the RK approach.

In this article, 2D FDEs are considered:

$$\partial_t^\alpha u(x,y,t)=f(x,y,t)+\Delta u(x,y,t),\quad (x,y,t)\in G\equiv D\times[0,T],\tag{1.1}$$

$$u(x,y,0)=r(x,y),\quad (x,y)\in D,\tag{1.2}$$

$$u(x,y,t)=0,\quad t\in[0,T],\quad (x,y)\in\partial D.\tag{1.3}$$

Here, $D\subset\mathbb{R}^2$ is a closed bounded field, $T>0$ is fixed and ∂_t^α [18] is defined as

$$\partial_t^\alpha u(x,y,t)=\frac{\int_0^t (t-\tau)^{-\alpha}\,\partial_\tau u(x,y,\tau)\,d\tau}{\Gamma(1-a)},\tag{1.4}$$

where Γ is the gamma function [19, 20], and $r(x,y)$ and $f(x,y,t)$ are given.

The objective for this article is to obtain the numerical format of the 2D FDE by the RK method. Also, superiorities for the scheme are as follows:
(i) A high-precision global approximate solution can be obtained.
(ii) The numerical program is simple and the calculation speed is fast.

The RK function in the Hilbert space and its related theories have important applications in the fields of stochastic processes, machine learning, pattern recognition and neural networks [21–25]. The RK method can obtain the high-precision global approximate solution, and has been widely used in differential and integral equations, linear and nonlinear problems, etc. [26–30]. In recent years, researchers have become more and more interested in using the RK method to solve various FDEs [31–34]. These papers show that the RK method has a number of superior advantages.

The n-term approximate solution for problems (1.1)–(1.3) is given in the RK space in this article. Also, the existing approaches are improved as follows. First, inspired by references [35, 36], a technique is presented to improve an existing method, which avoids the Gram-Schmidt orthogonal (GSO) procedure [17]. This method not only improves the precision, but also greatly reduces the runtime. Second, inspired by [37], a simpler RK is used than that in [38]. This improves accuracy and greatly reduces runtime [35, 36]. Finally, inspired by [35], this paper extends the RK method in [17] from one-dimensional to two-dimensional, and expands the application range of the RK method to solve 2D FDEs.

For simplicity, take $D=[0,1]\times[0,1]\subset\mathbb{R}^2$, $T=1$ and put $u(x,y,t)=r(x,y)+w(x,y,t)$. Thus, problems (1.1)–(1.3) are turned into

$$Lw(x,y,t)=g(x,y,t),\quad (x,y,t)\in G,\tag{1.5}$$

$$w(x,y,0)=0,\quad (x,y)\in D,\tag{1.6}$$

$$w(x,y,t)=0,\quad t\in[0,1],\quad (x,y)\in\partial D.\tag{1.7}$$

Here, set

$$Lw = \partial_t^\alpha w - \Delta w, \quad g\left(x,y,t\right) = \Delta r\left(x,y\right) + f\left(x,y,t\right), \tag{1.8}$$

for $w \in W_{(4,4,3)}(G)$, and $W_{(4,4,3)}(G)$ is defined in the next section.

2. RK space $W_{(4,4,3)}(G)$

In this section, $W_{(4,4,3)}(G)$ is constructed according to [37] as an original book on numerical methods for RK space, which provides a simpler RK than [38].

$W_4[0,1] = \{ w \,|\, w, w', w''$ and w''' are all real valued absolutely continuous functions, $w(1)=0$, $w(0)=0$, $w^{(4)} \in L^2[0,1] \}$. Its inner product is defined in $W_4[0,1]$ as

$$\left\langle w,u \right\rangle_{W_4} = \int_0^1 w^{(4)}(z)u^{(4)}(z)dz + \sum_{i=1}^{2} w^{(i)}(0)u^{(i)}(0). \tag{2.1}$$

$W_3[0,1] = \{ w \,|\, w, w'$ and w'' are all real valued absolutely continuous functions, $w(0)=0$, $w''' \in L^2[0,1] \}$. Its inner product is defined in $W_3[0,1]$ as

$$\left\langle w,u \right\rangle_{W_3} = \int_0^1 w'''(z)u'''(z)dz + \sum_{i=1}^{2} w^{(i)}(0)u^{(i)}(0). \tag{2.2}$$

The norms are defined as $\|w\|_{W_i} = \sqrt{\left\langle w,w \right\rangle_{W_i}}, i = 3,4.$ It can be shown that both $W_4[0,1]$ and $W_3[0,1]$ are RK spaces, and RKs $r1$ and $r2$ are given by (2.1) and (2.2) in [36], respectively.

Furthermore, $\{p_k(x)\}_{k=1}^{\infty}$ and $\{r_k(t)\}_{k=1}^{\infty}$ are assumed to be the orthonormal bases of $W_4[0,1]$ and $W_3[0,1]$, respectively. $W_{(4,4,3)}(G)$ is defined as

$$W_{(4,4,3)}(G) = \Big\{ w \,\Big|\, w(x,y,t) = \sum_{j,k,l=1}^{\infty} c_{jkl}\, p_j(x)\, p_k(y) r_l(t),\ \sum_{j,k,l=1}^{\infty} \left|c_{jkl}\right|^2 < \infty, c_{jkl} \in R \Big\}. \tag{2.3}$$

Its inner product and norm are respectively defined respectively as

$$\left\langle w_1, w_2 \right\rangle_{W_{(4,4,3)}} = \sum_{j,k,l=1}^{\infty} c_{jkl} d_{jkl}, \quad \|w\|_{W_{(4,4,3)}}^2 = \left\langle w,w \right\rangle_{W_{(4,4,3)}}, \tag{2.4}$$

where $w_1 = \sum_{j,k,l=1}^{\infty} c_{ijk}\, p_j(x)\, p_k(y) r_l(t)$ and $w_2 = \sum_{j,k,l=1}^{\infty} d_{jkl}\, p_j(x)\, p_k(y) r_l(t).$

With reference to [37], the following theorem can be obtained.

Lemma 2.1. $W_{(4,4,3)}(G)$ is an RK space whose RK is

$$R(x,\zeta,y,\varsigma,t,\tau)=h(t,\tau)b(x,\zeta)b(y,\varsigma). \tag{2.5}$$

Here, $b(\cdot,\cdot)$ and $h(\cdot,\cdot)$ are given by (2.1)–(2.2) in [36], respectively.

Similarly, $W_{(2,2,2)}(G)$ is also an RK space whose RK is

$$\overline{R}(x,\zeta,y,\varsigma,t,\tau)=s(t,\tau)s(x,\zeta)s(y,\varsigma), \tag{2.6}$$

where $s(\cdot,\cdot)$ is given by (2.3) in [36].

3. Solution to problems (1.5)–(1.8)

A countable dense subset $\left\{\left(x_j,y_j,t_j\right)\right\}_{j\in\mathbb{N}}\subset G$ is chosen. Put $\psi_j(x,y,t)=\overline{R}(x,x_j,y,y_j,t,t_j)$, $\phi_j(x,y,t)=L^*\psi_j(x,y,t)$, where L^* is the formally adjoint operator of L.

Theorem 3.1. Let $\left\{\left(x_j,y_j,t_j\right)\right\}_{j\in\mathbb{N}}\subset G$. Then,

$$\begin{aligned}\phi_j(x,y,t)&=L_{(\zeta,\varsigma,\tau)}R(x,\zeta,y,\varsigma,t,\tau)\big|_{(\zeta,\varsigma,\tau)=(x_j,y_j,t_j)}\\&=\left(\partial_\tau^\alpha R(x,\zeta,y,\varsigma,t,\tau)-\frac{\partial^2R(x,\zeta,y,\varsigma,t,\tau)}{\partial\zeta^2}\right.\\&\left.\quad-\frac{\partial^2R(x,\zeta,y,\varsigma,t,\tau)}{\partial\varsigma^2}\right)\Bigg|_{(\zeta,\varsigma,\tau)=(x_j,y_j,t_j)},\quad j\in\mathbb{N}.\end{aligned} \tag{3.1}$$

Here, $R(x,\zeta,y,\varsigma,t,\tau)$ is shown by (2.5). The proof for this theorem is similar to that in reference [35], where a 2D parabolic inverse source problem is discussed.

Theorem 3.2. $\phi_j\in W_{(4,4,3)}(G)$, $j\in\mathbb{N}$.

Proof. From (3.1),

$$\begin{aligned}\phi_j(x,y,t)=&\frac{b(y,y_j)b(x,x_j)}{\Gamma(1-\alpha)}\int_0^{t_j}(t_j-\tau)^{-\alpha}\partial_\tau h(t,\tau)d\tau\\&-b(y,y_j)h(t,t_j)\partial^2_{\zeta^2}b(x,\zeta)\big|_{\zeta=x_j}\\&-b(x,x_j)h(t,t_j)\partial^2_{\varsigma^2}b(y,\varsigma)\big|_{\varsigma=y_j}.\end{aligned} \tag{3.2}$$

Then,

$$
\begin{aligned}
\left|\partial^{11}_{x^4y^4t^3}\phi_j(x,y,t)\right| = \Bigg| & \frac{\partial^4_{x^4}b(x,x_j)\partial^4_{y^4}b(y,y_j)}{\Gamma(1-\alpha)}\int_0^{t_j}(t_j-\tau)^{-\alpha}\partial^4_{\tau t^3}h(t,\tau)d\tau \\
& -\partial^4_{y^4}b(y,y_j)\partial^3_{t^3}h(t,t_j)\partial^6_{\zeta^2x^4}b(x,\zeta)\Big|_{\zeta=x_j} \\
& -\partial^4_{x^4}b(x,x_j)\partial^3_{t^3}h(t,t_j)\partial^6_{\varsigma^2y^4}b(y,\varsigma)\Big|_{\varsigma=y_j}\Bigg|.
\end{aligned} \tag{3.3}
$$

There are positive constants m_1, m_2, m_3, such that

$$\left|\partial^4_{x^4}b(x,x_j)\partial^4_{y^4}b(y,y_j)\partial^4_{\tau t^3}h(t,\tau)\right| \le m_1, \tag{3.4}$$

$$\left|\partial^4_{y^4}b(y,y_j)\partial^3_{t^3}h(t,t_j)\partial^6_{\zeta^2x^4}b(x,\zeta)\Big|_{\zeta=x_j}\right| \le m_2, \tag{3.5}$$

and

$$\left|\partial^4_{x^4}b(x,x_j)\partial^3_{t^3}h(t,t_j)\partial^6_{\varsigma^2y^4}b(y,\varsigma)\Big|_{\varsigma=y_j}\right| \le m_3, \tag{3.6}$$

for $(x,y,t)\in G$ and $\tau\in[0,1]$. Thus,

$$
\begin{aligned}
\left|\partial^{11}_{x^4y^4t^3}\phi_j(x,y,t)\right| &\le m_1\left|\frac{\int_0^{t_j}(t_j-\tau)^{-\alpha}d\tau}{\Gamma(1-\alpha)}\right| + m_2 + m_3 \\
&\le m_4\left|\int_0^{t_j}(t_j-\tau)^{-\alpha}d\tau\right| + m_2 + m_3 \\
&\le \frac{m_5}{1-\alpha} + m_2 + m_3,
\end{aligned} \tag{3.7}
$$

where m_4, m_5 are positive constants. Hence, $\partial^{11}_{x^4y^4t^3}\phi_j \in L^2(G)$. Since G is closed, $\partial^8_{x^3y^3t^2}\phi_j$ is absolutely continuous in G.

In addition, $b(x,\zeta), b(y,\varsigma)\in W_4[0,1]$, $h(t,\tau)\in W_3[0,1]$ with respect to $\zeta,\varsigma,\tau\in[0,1]$ and $b(1,\zeta)=0$, $b(1,\varsigma)=0$, $b(0,\zeta)=0$, $b(0,\varsigma)=0$, $h(0,\tau)=0$, $\partial^2_{\varsigma^2}b(1,\varsigma)=0$, $\partial^2_{\varsigma^2}b(0,\varsigma)=0$, $\partial^2_{\zeta^2}b(1,\zeta)=0$, $\partial^2_{\zeta^2}b(0,\zeta)=0$, $\partial_\tau h(0,\tau)=0$.

By (3.2), $\phi_j(x,y,0)=0$, $\phi_j(x,1,t)=0$, $\phi_j(x,0,t)=0$, $\phi_j(1,y,t)=0$, $\phi_j(0,y,t)=0$, $j\in\mathbb{N}$. Therefore, $\phi_j\in W_{(4,4,3)}(G)$, $j\in\mathbb{N}$ by definition.

Theorem 3.3. Suppose that the solution to problems (1.1)–(1.3) is unique. Then, in $W_{(4,4,3)}(G)$, $\{\phi_j\}_{j\in\mathbb{N}}$ is complete.

In $W_{(4,4,3)}(G)$, the orthonormal system $\{\overline{\phi}_j\}_{j\in\mathbb{N}}$ is obtained by the GSO process for $\{\phi_j\}_{j\in\mathbb{N}}$,

$$\overline{\phi}_j(x,y,t)=\sum_{i=1}^{j}\phi_i(x,y,t)\gamma_{ji},\quad \gamma_{jj}>0,\ \ j\in\mathbb{N}. \tag{3.8}$$

Theorem 3.4. In $W_{(4,4,3)}(G)$, the unique solution to problems (1.5)–(1.8) is expressed as

$$w(x,y,t)=\sum_{j=1}^{\infty}\sum_{i=1}^{j}\overline{\phi}_j(x,y,t)\,g(x_i,y_i,t_i)\gamma_{ji}. \tag{3.9}$$

Thus, the n-term approximate solution $w_n(x,y,t)$ is acquired by

$$w_n(x,y,t)=\sum_{j=1}^{n}\sum_{i=1}^{j}\overline{\phi}_j(x,y,t)\,g(x_i,y_i,t_i)\gamma_{ji}. \tag{3.10}$$

Theorem 3.5. If w is the exact solution (ES) for Eqs (1.5)–(1.8), $w_n=P_nw$, where P_n is an orthogonal projection of $W_{(4,4,3)}$ to $Span\{\overline{\phi}_j\}_{j\in\mathbb{N}}$, then,

$$Lw_n(x_j,y_j,t_j)=g(x_j,y_j,t_j),\quad j=1,2,\cdots,n. \tag{3.11}$$

The proofs of these theorems are similar to those in [35].
Thus,

$$\begin{aligned} w_n(x,y,t)&=\sum_{j=1}^{n}\sum_{i=1}^{j}\overline{\phi}_j(x,y,t)\,g(x_i,y_i,t_i)\gamma_{ji},\\ &=\sum_{j=1}^{n}\sum_{i=1}^{j}\sum_{k=1}^{j}\phi_k(x,y,t)\gamma_{jk}\,g(x_i,y_i,t_i)\gamma_{ji}\\ &=\sum_{j=1}^{n}\phi_j(x,y,t)C_j. \end{aligned} \tag{3.12}$$

Here, $C_j=\sum_{k=j}^{n}\gamma_{kj}\sum_{i=1}^{k}g(x_i,y_i,t_i)\gamma_{ki}$. The verification is listed below.
From (3.10),

$$w_n(x,y,t)=\sum_{j=1}^{n}\sum_{i=1}^{j}\overline{\phi}_j(x,y,t)\,g(x_i,y_i,t_i)\gamma_{ji}=\sum_{j=1}^{n}\sum_{i=1}^{j}\sum_{k=1}^{j}\phi_k(x,y,t)\gamma_{jk}\,g(x_i,y_i,t_i)\gamma_{ji}. \tag{3.13}$$

Let $\overline{C}_j=\sum_{i=1}^{j}g(x_i,y_i,t_i)\gamma_{ji}$. Then,

$$
\begin{aligned}
w_n &= \sum_{j=1}^{n}\sum_{k=1}^{j}\phi_k\gamma_{jk}\overline{C}_j \\
&= \phi_1\gamma_{11}\overline{C}_1+\phi_1\gamma_{21}\overline{C}_2+\phi_2\gamma_{22}\overline{C}_2+\phi_1\gamma_{31}\overline{C}_3+\phi_2\gamma_{32}\overline{C}_3+\phi_3\gamma_{33}\overline{C}_3 \\
&\quad +\cdots+\phi_1\gamma_{n1}\overline{C}_n+\phi_2\gamma_{n2}\overline{C}_n+\phi_3\gamma_{n3}\overline{C}_n+\cdots+\phi_n\gamma_{nn}\overline{C}_n \\
&= \sum_{j=1}^{n}\phi_j\left(\sum_{k=j}^{n}\gamma_{kj}\overline{C}_k\right) \\
&= \sum_{j=1}^{n}\phi_j\left(\sum_{k=j}^{n}\gamma_{kj}\sum_{i=1}^{k}g(x_i,y_i,t_i)\gamma_{ki}\right) \\
&= \sum_{j=1}^{n}\phi_jC_j. \ \square
\end{aligned} \tag{3.14}
$$

Next, Theorem 3.5 tells us that

$$
Lw_n(x_k,y_k,t_k)=\sum_{j=1}^{n}C_jL\phi_j\left(x_k,y_k,t_k\right)=g\left(x_k,y_k,t_k\right), \quad k=1,2,\cdots,n. \tag{3.15}
$$

Here, L, g and ϕ_j are given by (1.8) and (3.1), respectively.

In summary, the main procedures for this proposed approach are listed below.

1) Put $u(x,y,t)=r(x,y)+w(x,y,t)$, problems (1.1)–(1.3) are turned into problems (1.5)–(1.8).

2) From Eq (3.15), $C_j,\ j=1,2,\cdots,n$ can be obtained.

3) Substitute $C_j,\ j=1,2,\cdots,n$ into Eq (3.12), numerical solution $w_n(x,y,t)$ of problems (1.5)–(1.8) can be obtained.

4) Again by $u(x,y,t)=r(x,y)+w(x,y,t)$, the approximation $u_n(x,y,t)$ for the original problems (1.1)–(1.3) can be obtained.

From the above calculation steps, it can be seen that the GSO process for $\left\{\phi_j\right\}_{j=1}^{\infty}$ in [17] may be avoided. (Here, only the proofs require the GSO step, however numerical computation does not require this step.) Therefore, compared with the method in [17], this method may improve the accuracy and greatly reduce the runtime [35, 36]. This method can solve some model problems effectively and provide a high-precision global approximate solution.

4. Convergence analysis

$w(x,y,t)$ and $w_n(x,y,t)$ are respectively ES and the n-term approximation solution to Eqs (1.5)–(1.8). Set $\|w(x,y,t)\|_C \triangleq \max_{(x,y,t)\in G}|w(x,y,t)|$. Similarly to [35], the following theorem can be obtained.

Theorem 4.1. Suppose $w\in W_{(4,4,3)}(G)$. Then,

1) $\|w-w_n\|_{W_{(4,4,3)}(G)}\to 0, \quad n\to\infty$. Also, $\|w-w_n\|_{W_{(4,4,3)}(G)}$ decreases monotonically with n.

2) $\left\| \frac{\partial^{l+k+j} w}{\partial x^l \partial y^k \partial t^j} - \frac{\partial^{l+k+j} w_n}{\partial x^l \partial y^k \partial t^j} \right\|_C \to 0, n \to \infty; j = 0,1; l,k = 0,1,2; l+k+j = 0,1,2.$

Proof. (1) From (3.9) and (3.10),

$$\left\| w - w_n \right\|_{W_{(4,4,3)}(G)} = \left\| \sum_{j=n+1}^{\infty} \sum_{i=1}^{j} \bar{\phi}_j \left(x, y, t \right) g\left(x_i, y_i, t_i \right) \gamma_{ji} \right\|_{W_{(4,4,3)}(G)} . \tag{4.1}$$

Thus,

$$\left\| w - w_n \right\|_{W_{(4,4,3)}(G)} \to 0, \quad n \to \infty. \tag{4.2}$$

In addition,

$$\begin{aligned} \left\| w - w_n \right\|^2_{W_{(4,4,3)}(G)} &= \left\| \sum_{j=n+1}^{\infty} \sum_{i=1}^{j} \bar{\phi}_j \left(x, y, t \right) g\left(x_i, y_i, t_i \right) \gamma_{ji} \right\|^2_{W_{(4,4,3)}(G)} \\ &= \sum_{j=n+1}^{\infty} \left(\sum_{i=1}^{j} g\left(x_i, y_i, t_i \right) \gamma_{ji} \right)^2 . \end{aligned} \tag{4.3}$$

Clearly, $\left\| w - w_n \right\|_{W_{(4,4,3)}(G)}$ decreases monotonically with n.

Note that

$$\frac{\partial^{l+k+j} w(x,y,t)}{\partial x^l \partial y^k \partial t^j} - \frac{\partial^{l+k+j} w_n(x,y,t)}{\partial x^l \partial y^k \partial t^j} = \left\langle w(\zeta,\varsigma,\tau) - w_n(\zeta,\varsigma,\tau), \frac{\partial^{l+k+j} K(x,\zeta,y,\varsigma,t,\tau)}{\partial x^l \partial y^k \partial t^j} \right\rangle; \quad j = 0,1; l,k = 0,1,2; l+k+j = 0,1,2, \tag{4.4}$$

$$\left\| \frac{\partial^{l+k+j} K(x,\zeta,y,\varsigma,t,\tau)}{\partial x^l \partial y^k \partial t^j} \right\|_{W_{(4,4,3)}(G)} \le C_i, \quad j = 0,1; l,k = 0,1,2; l+k+j = 0,1,2; i = 1,2,\cdots,9. \tag{4.5}$$

Here, $C_i, i = 1,2,\cdots,9$ are normal numbers. For all $(x,y,t) \in G,$

$$\begin{aligned} \left| \frac{\partial^{l+k+j} w(x,y,t)}{\partial x^l \partial y^k \partial t^j} - \frac{\partial^{l+k+j} w_n(x,y,t)}{\partial x^l \partial y^k \partial t^j} \right| &= \left| \left\langle w(\zeta,\varsigma,\tau) - w_n(\zeta,\varsigma,\tau), \frac{\partial^{l+k+j} K(x,\zeta,y,\varsigma,t,\tau)}{\partial x^l \partial y^k \partial t^j} \right\rangle \right| \\ &\le \left\| w - w_n \right\|_{W_{(4,4,3)}(G)} \left\| \frac{\partial^{l+k+j} K(x,\zeta,y,\varsigma,t,\tau)}{\partial x^l \partial y^k \partial t^j} \right\|_{W_{(4,4,3)}(G)} \\ & \quad j = 0,1; l,k = 0,1,2; l+k+j = 0,1,2, \\ &\le C_i \left\| w - w_n \right\|_{W_{(4,4,3)}(G)} . \end{aligned} \tag{4.6}$$

Hence,

$$\left\| \frac{\partial^{l+k+j} w}{\partial x^l \partial y^k \partial t^j} - \frac{\partial^{l+k+j} w_n}{\partial x^l \partial y^k \partial t^j} \right\|_C \to 0, n \to \infty; j = 0,1; l,k = 0,1,2; l+k+j = 0,1,2. \tag{4.7}$$

5. Numerical results

The effectiveness of the presented technique and the high precision of the approximate solution are verified by two examples in this section. Mathematica 5.0 software was used for all numerical calculations on a personal laptop computer.

The domain G is divided into an $m_1 \times m_2 \times m_3$ grid with steps $1/m_1, 1/m_2$ and $1/m_3$ in the x, y and t directions, where $m_1, m_2, m_3 \in \mathbb{N}$.

Example 5.1. Eqs (1.1)–(1.3) are considered based on the following conditions:

$$\begin{cases} u(x,y,0) = x^2 sin^3(y)(1-x)^2 sin^3(1-y), \quad (x,y) \in D, \\ u(x,y,t) = 0, \quad t \in [0,1], \quad (x,y) \in \partial D, \\ \alpha = 0.6. \end{cases} \tag{5.1}$$

The ES is

$$u(x,y,t) = \exp(t) x^2 sin^3(y)(1-x)^2 sin^3(1-y) . \tag{5.2}$$

According to the main steps (1)–(4) of the proposed method in Section 3, the absolute errors (AE) for $u(x,y,t)$ are given in Table 1 with grid $m_1 \times m_2 \times m_3 = 3\times3\times3$. Root mean square errors (RMSE) for $u(x,y,t)$ and CPU time are shown in Table 2.

It is not difficult to see from Tables 1 and 2 that good results can be achieved when the step size is larger. Further, the CPU time is short. In addition, the accuracy increases as the step size decreases.

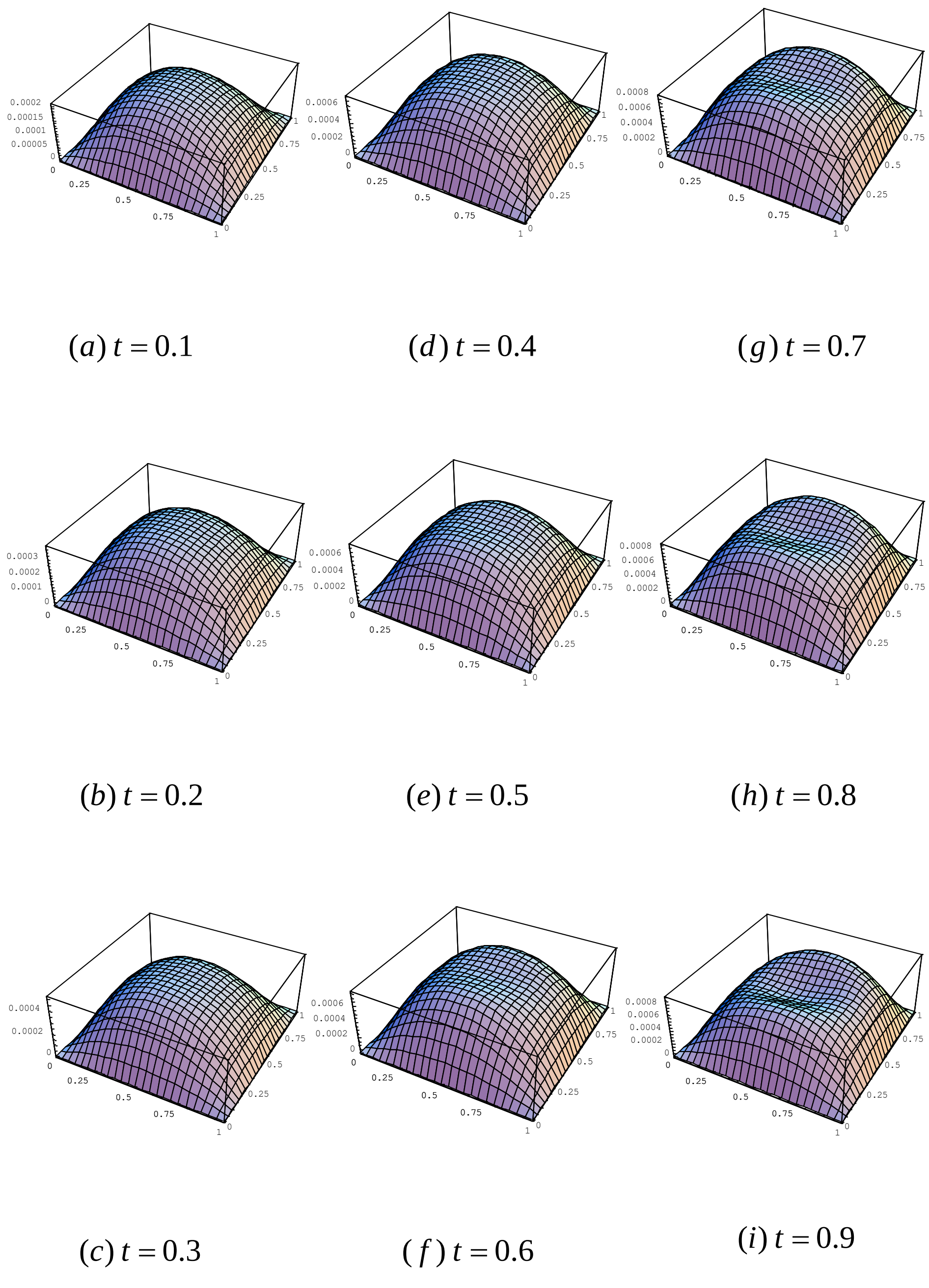
Table 1. AE for $u(x,y,t)$ in Example 5.1.

(x,y,t)	AE	(x,y,t)	AE
$\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$	6.02456E-4	$\left(\frac{2}{3},\frac{1}{3},\frac{1}{3}\right)$	6.03771E-4
$\left(\frac{1}{3},\frac{1}{3},\frac{2}{3}\right)$	7.22032E-4	$\left(\frac{2}{3},\frac{1}{3},\frac{2}{3}\right)$	7.21850E-4
$\left(\frac{1}{3},\frac{2}{3},\frac{1}{3}\right)$	6.03771E-4	$\left(\frac{2}{3},\frac{2}{3},\frac{1}{3}\right)$	6.05237E-4
$\left(\frac{1}{3},\frac{2}{3},\frac{2}{3}\right)$	7.21850E-4	$\left(\frac{2}{3},\frac{2}{3},\frac{2}{3}\right)$	7.21535E-4

Table2. RMSE for $u(x,y,t)$ and CPU time in Example 5.1.

$m_1 \times m_2 \times m_3$	RMSE for $u(x,y,t)$	CPU time (s)
$2\times2\times2$	2.52686E-3	0.016
$3\times3\times3$	6.65434E-4	0.718

Moreover, errors $|u-u_{27}|: t=0.1, 0.2, \cdots, 0.9$ are given in Figure 1. It is easy to see from the figure that this method can provide a high-precision global approximate solution.



$(a)\ t=0.1$ $(d)\ t=0.4$ $(g)\ t=0.7$

$(b)\ t=0.2$ $(e)\ t=0.5$ $(h)\ t=0.8$

$(c)\ t=0.3$ $(f)\ t=0.6$ $(i)\ t=0.9$

Figure 1. Errors $|u-u_{27}|: t=0.1, 0.2, \cdots, 0.9$ in Example 5.1.

Example 5.2. Questions (1.1)–(1.3) are considered based on the following conditions:

$$\begin{cases} u(x,y,0)=sin^2(x)sin^5(y)sin^2(1-x)sin^5(1-y), \quad (x,y)\in D, \\ u(x,y,t)=0, \quad t\in[0,1], \quad (x,y)\in\partial D, \\ \alpha=0.5. \end{cases} \tag{5.3}$$

The ES is

$$u(x,y,t)=\exp(t)sin^2(x)sin^5(y)sin^2(1-x)sin^5(1-y)\,. \tag{5.4}$$

According to the main steps (1)–(4) of the proposed method in Section 3, the AE for $u(x,y,t)$ are given in Table 3 with grid $m_1\times m_2\times m_3=3\times3\times3$. Also, the RMSE for $u(x,y,t)$ and CPU time are shown in Table 4. It is easy to see from the two tables that good results can be achieved when the step size is larger. Further, the CPU time is short. Besides, as step size decreases, accuracy improves.

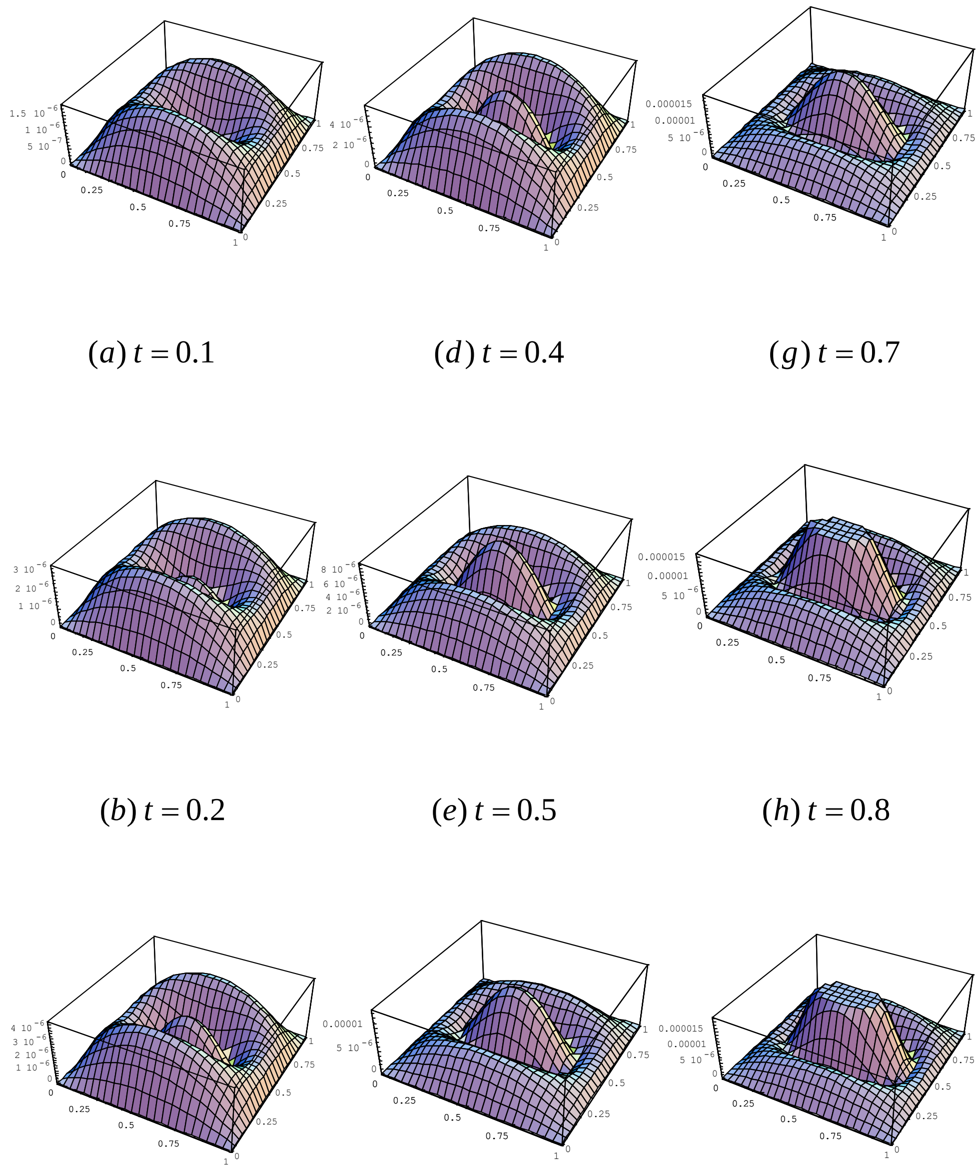
Table 3. AE for $u(x,y,t)$ in Example 5.2.

(x,y,t)	AE	(x,y,t)	AE
$\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$	2.81463E-6	$\left(\frac{2}{3},\frac{1}{3},\frac{1}{3}\right)$	2.82680E-6
$\left(\frac{1}{3},\frac{1}{3},\frac{2}{3}\right)$	9.91324E-7	$\left(\frac{2}{3},\frac{1}{3},\frac{2}{3}\right)$	9.88143E-7
$\left(\frac{1}{3},\frac{2}{3},\frac{1}{3}\right)$	2.82680E-6	$\left(\frac{2}{3},\frac{2}{3},\frac{1}{3}\right)$	2.84024E-6
$\left(\frac{1}{3},\frac{2}{3},\frac{2}{3}\right)$	9.88143E-7	$\left(\frac{2}{3},\frac{2}{3},\frac{2}{3}\right)$	9.85582E-7

Table4. RMSE for $u(x,y,t)$ and CPU time in Example 5.2.

$m_1\times m_2\times m_3$	RMSE for $u(x,y,t)$	CPU time (s)
$2\times2\times2$	1.73194E-4	0.015
$3\times3\times3$	2.11771E-6	0.640

Moreover, Figure 2 shows the errors $|u-u_{27}|:t=0.1,0.2,\cdots,0.9.$ It is not difficult to see from the figure that this method can provide a high-precision global approximate solution.



$(a)\ t=0.1$ $(d)\ t=0.4$ $(g)\ t=0.7$

$(b)\ t=0.2$ $(e)\ t=0.5$ $(h)\ t=0.8$

$(c)\ t=0.3$ $(f)\ t=0.6$ $(i)\ t=0.9$

Figure 2. Errors $|u-u_{27}|: t=0.1, 0.2, \cdots, 0.9$ in Example 5.2.

6. Conclusions and discussion

The presented method has been successfully applied to the 2D FDE in this article. Based on the RK space, the method improves existing methods [17,38], extends the RK method in [17] from one-dimensional to two-dimensional, and obtains a simpler RK than that in [38]. This improves the accuracy and greatly reduces the runtime. Numerical results demonstrate that this approch has high precision, and the error for the approximate solution decreases monotonically in the sense of $\|\cdot\|_{W_{(4,4,3)}}$. In addition, this approch is applicable to more general FDEs, which we will discuss in an upcoming paper.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare that they have no conflicts of interest.

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