AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(12): 29042–29057.
DOI:10.3934/math.20231487
Received: 03 September 2023
Revised: 30 September 2023
Accepted: 11 October 2023
Published: 26 October 2023

Research article

Geometrization of string cloud spacetime in general relativity

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Abstract: The purpose of the article is to analyze the behavior of spacetime using a string cloud energy-momentum tensor $\mathcal{T}$ having string cloud fluid density ρ and string tension λ, named *relativistic string cloud spacetime*. We obtain some results for string cloud spacetime with a divergence-free matter tensor and a diminishing space matter tensor. Next, we discuss some curvature characteristics, such as conformally flat, Ricci semi-symmetric and pseudo-Ricci-symmetric, for relativistic string cloud spacetime. In addition, we gain a condition that coincides with the equation of state for the cloud of geometric strings in Ricci semi-symmetric string cloud spacetime.

Keywords: Lorentzian spacetime manifold; Einstein's field equation; string cloud energy momentum tensor; curvatures; geomtric strings
Mathematics Subject Classification: 53B30, 53C44, 53C50, 53C80

1. Introduction

The general theory of relativity (GTR), which describes the force of gravity of spacetime curvature caused by mass abundance, was developed by Einstein in 1915. The well-known Einstein field equation (EFE) is the gravitational field equation satisfied by the spacetime scale.

In Einstein's equation, the cosmological constant is typically employed to characterize the universe's current state. Additionally, the cosmological constant-free version of Einstein's gravitational field equation is given by [1].

$$\mathcal{R}ic - \frac{R}{2} g = \kappa \mathcal{T}, \tag{1.1}$$

where $\mathcal{R}ic$, R, g and $\mathcal{T}$ correspond to the Ricci tensor, scalar curvature, Lorentzian metric and energy-momentum tensor of spacetime, respectively, while κ is the gravitational constant, which is chosen to be $8\pi G$, making G a universal gravitational constant. In order to obtain the Ricci tensor and scalar curvature needed to achieve Einstein's goal of a static universe, along with the cosmological constant, one must employ Einstein's equation.

On the other hand, a one-dimensional entity called a string can likewise be used to describe the universe. The extension of such cosmic strings, which can travel everywhere in the universe that can be discovered, is apparently strongly related to the universe's constantly increasing expansion [2]. Quantum gravity is predicted by string theory, which interprets particles and the fundamental forces of nature as vibrations of microscopic super-symmetric strings. Then, one after another, investigations on the gravitational effects of matter in the form of string clouds appeared. In order to study the relationship between the entropy of the black hole and the counting string state, Letelier first looked at the general solutions of string clouds that satisfy spherical symmetry [3]. The pertinent solutions are then extended to the cases of Einstein-Gauss-Bonnet theory in the Letelier spacetime [4] and third-order Lovelock gravity [5]. In this connection, numerous other extended solutions have also been researched [6–11].

There have been a lot of intriguing articles published recently on topics including symmetry [12–14], molecular cluster geometry analysis [15–17], submanifolds theory [18–20], singularity theory [21] and eigenproblems [22].

The classification problem of strings is also studied in some highly symmetric spacetimes such as Minkowski spacetime. Moreover, Minkowski spaces for various searches and additional results in [23–26] further improve the findings in this study.

Both the spacetime of the GTR and cosmology are used to depict the time-constrained 4-dimensional connected Lorentzian manifold. This displays a particular categorization of pseudo-Riemannian manifolds among the Lorentzian metric having signature $(-,+,+,+)$ that is crucial in GTR [1, 27, 28]. To examine the behavior of the vectors at the manifold, the geometry of Lorentzian manifolds is introduced. Thus, Lorentzian manifolds are emerging as the most effective framework for GTR analysis.

If the Ricci tensor has the following shape, then the quasi-Einstein Lorentzian manifolds are called perfect fluid spacetime [1, 29, 30]

$$\mathcal{R}ic = \alpha g + \beta \mathcal{U} \otimes \mathcal{V}, \tag{1.2}$$

where α and β are scalars and $\mathcal{U}$ is a 1-form metrically equivalent to unit time-like vector filed. Additionally, the spacetime manifold is a Lorentzian manifold that allows for a vector field that resembles time.

The notion of generalized quasi-Einstein manifold (GQE) introduced by Chaki [31] in the following definition:

Definition 1.1. *A non flat Riemannian manifold* (M^n, g) $(n > 2)$ *is called a generalized quasi Einstein Lorentzian manifold if its Ricci tensor* $\mathcal{R}ic$ *of type* $(0,2)$ *is nonzero and satisfies the condition*

$$\mathcal{R}ic = \alpha g + \beta \mathcal{U} \otimes \mathcal{U} + \delta \mathcal{V} \otimes \mathcal{V}, \tag{1.3}$$

where α, β and δ are scalars of which $\beta \neq 0$, $\delta \neq 0$ and $\mathcal{U}$, $\mathcal{V}$ are non vanishing 1-from such that $g(p,\gamma) = \mathcal{U}(p)$, $g(p,\zeta) = \mathcal{V}(p)$ for any vector filed p. The unit vectors γ and ζ refers to 1-form $\mathcal{U}$ and $\mathcal{V}$ are orthogonal to each other. Also, γ and ζ are the generators of the spacetime manifold. If $\delta = 0$, then (M^4, g) reduces to a perfect fluid spacetime (quasi-Einstein manifold).

Moreover, Ricci tensor plays a crucial role in spacetimes as well as in Euclidean space or pseudo-Euclidean space. We consider constant symmetric tensors T on $\mathbb{R}^n$, $n = 3$, and we study the problem of finding metrics g conformal to the pseudo-Euclidean metric g such that $Ric(\mathrm{g}) = T$. Recently, many interesting papers have been written related to Euclidean space or pseudo-Euclidean space to explore the characteristics of intrinsic invariants, such as Ricci tensor and scalar curvature and extrinsic invariants like mean curvature, square mean curvature, etc (for more information see [32–37]).

Formally, the effective momentum tensor [38], can be used to recast the GTR. This momentum tensor is then characterized by isotropic pressure, energy density, an anisotropic pressure, and energy flow in the presence of a capable timelike vector field [27]. According to [39–41], it has the structure of an imperfect fluid (viscus fluid spacetime) spacetime's energy-momentum tensor. Beyond the conventional model, imperfect fluid spacetime can provide sufficient features for models like perfect fluid spacetime [42].

While spacetime matter is considered to be fluid with density, pressure, tension, and dynamical and kinematic characteristics including velocity, acceleration, vorticity, shear, and expansion [27], the momentum tensor plays a significant role as the spacetime's matter content. In conventional cosmological models, the universe's matter composition is thought to behave as a certain fluid spacetime (perfect fluid spacetime). Several studies based on multiple spacetimes have recently been published in relation to GTR (for more informations see [27, 28, 30, 39–42]) These publications give us the opportunity for further study of string cloud spacetime investigation in GTR.

2. Relativistic string cloud spacetime

The fundamental components of the spacetime loaded with the energy-momentum tensor of the string cloud type, or string cloud spacetime, are covered in this section.

Definition 2.1. *[3] A 4-dimensional spacetime manifold (M^4, g) generated by a symmetric string cloud energy-momentum tensor $\mathcal{T}$ is called a string cloud spacetime, which is analogous to a pressureless perfect fluid spacetime (1.2) with gas and dust particles.*

According to [4], space matter is known to be a fluid that can carry any spacetime substance, including density and string tension. Standard cosmological models depend heavily on the energy-momentum tensor of string clouds and the universe's material makeup is thought to behave similarly to a string cloud spacetime [43].

Moreover, in the case of a perfect fluid spacetime, there is no existence of heat conduction and viscosity [41]. In contrast, for this study, we made the hypothesis that spacetime grows up with string cloud energy-momentum tensor [2].

The string cloud energy-momentum tensor can be expressed as [2, 43]

$$\mathcal{T}(p, q) = \rho\mathcal{U}(p)\mathcal{U}(q) - \lambda\mathcal{V}(p)\mathcal{V}(q), \tag{2.1}$$

where $p, q \in \chi(M^4)$. Cloud fluid's energy density in terms of its associated particles is represented by ρ, and string tension is represented by λ.

$$\rho = \lambda + \epsilon_0, \tag{2.2}$$

where ϵ_0 is the rest energy density of particles.

In addition, string cloud spacetime admitting the unit spacelike vector field $\mathcal{V}$ perpendicular to the unit timelike vector field $\mathcal{U}$, such that $\mathcal{U}(p) = g(p, \gamma)$ and $\mathcal{V}(q) = g(q, \zeta)$ are two non-zero 1-form. Also, γ and ζ are holds $g(\gamma, \gamma) = -1$ and $g(\zeta, \zeta) = 1$.

Remark 2.2. *Some well-known examples of string cloud spacetime are:*

(i) A 4-dimensional Lovelock black hole with a cloud of strings surrounded by spherical symmetry (Letelier spacetime) [44].
(ii) A black hole with a string cloud surrounded by quintessence (Kiselev spacetime) [45].

Now, in light of (1.1) and (2.1), we find the EFE for a relativistic string cloud spacetime

$$\mathcal{R}ic(p, q) = \frac{R}{2} g(p, q) + \kappa\rho\mathcal{U}(p)\mathcal{U}(q) - \kappa\lambda\mathcal{V}(p)\mathcal{V}(q). \tag{2.3}$$

Equation (2.3) follows that the Lorentzian generalized quasi-Einstein manifold used to describe spacetime is being considered. Relativistic string cloud spacetime with $\frac{R}{2}, \kappa\rho$ and $\kappa\lambda$ are associated scalars, $\mathcal{U}$ and $\mathcal{V}$ as associated 1-forms.

Here a relativistic string cloud spacetime is filled with the energy density of the fluid ρ and the string tension λ is satisfying the EFE. As a result, we can state the following.

Theorem 2.3. *A string cloud spacetime (M^4, g) obeys the EFE without cosmic constant and with string cloud fluid density ρ and string tension λ is a generalized quasi-Einstein spacetime.*

Next, after contracting (2.3), we turn up the following:

Theorem 2.4. *If a string cloud spacetime obeys EFE without cosmic constant and with string cloud fluid density ρ and string tension λ, then the scalar curvature is*

$$R = -\kappa(\rho + \lambda). \tag{2.4}$$

Theorems 2.3 and 2.4 have the following conclusion as a result.

Corollary 2.5. *A string cloud spacetime obeys EFE with constant scalar curvature $\mathcal{R}$ is a string cloud bulk viscous fluid spacetime.*

With $g(\gamma, \zeta) = 0$ from (2.3) and γ, ζ are orthogonal unit vector fields, respectively, we obtain

$$\mathcal{R}ic(p, \gamma) = (\alpha + \beta)\mathcal{U}(p), \tag{2.5}$$

$$\mathcal{R}ic(p, \zeta) = (\alpha + \delta)\mathcal{V}(p), \tag{2.6}$$

where $\alpha = \frac{\mathcal{R}}{2}, \beta = \kappa\rho$ and $\delta = -\kappa\lambda$.

Considering (2.2) and (2.4), we can now state the following.

Theorem 2.6. *If a string cloud spacetime obeys EFEs with constant scalar curvature R and satisfies relation (2.2), then under this situation energy density of the cloud fluid is $\frac{1}{2}\left(\epsilon_0 - \frac{\mathcal{R}}{\kappa}\right)$ and string tension is $-\frac{1}{2}\left(\epsilon_0 + \frac{\mathcal{R}}{\kappa}\right)$.*

Let us combine the value of ρ and λ together with Theorem 2.6 we gain the following outcomes

Corollary 2.7. *If a string cloud spacetime obeys EFE with constant scalar curvature R and satisfies relation (2.2), then, energy density ρ is proportional to the string tension λ and provided $\frac{\rho}{\lambda} = -1$.*

Corollary 2.8. *If a string cloud spacetime obeys EFE with constant scalar curvature R, satisfies relation (2.2) and with condition* $\frac{\rho}{\lambda} = -1$*, then the string is massive strings and string cloud spacetime is a massive string cloud spacetime.*

Remark 2.9. *The energy density* σ *and particle density of fluid* ρ *are related to the specific energy e and volume of fluid V as (for more details see [24, 46]):*

$$\sigma = \rho e, \quad \rho = \frac{1}{V}. \tag{2.7}$$

Therefore, in light of Theorem 2.6 and Eq (2.7), we can articulate the following results.

Theorem 2.10. *If a string cloud spacetime obeying EFE with constant scalar curvature R and satisfies relation (2.2), then, the specific energy* σ *is* $\frac{2\kappa\sigma}{\kappa\epsilon_0 - R}$ *and volume V is* $\frac{2\kappa}{\kappa\epsilon_0 - R}$.

3. Vanishing space matter tensor and string cloud spacetime

In this section, we examine relativistic string cloud spacetime with vanishing space matter tensor. Initially, we provide as the basis for the main results of this section.

Generalized quasi constant curvature, which is defined as [47], is an extension of the quasi constant curvature notion proposed by De and Ghosh.

$$\begin{aligned}\mathcal{R}im(p,q,r,s) = {} & a[g(q,r)g(p,s) - g(p,r)g(q,s)] \\ & +b[g(p,s)\theta(q)\theta(r) - g(q,r)\theta(p)\theta(s)], \\ & +g(q,r)\theta(p)\theta(s) - g(q,s)\theta(p)\theta(r)] \\ & +c[g(p,s)\pi(q)\pi(r) - g(q,s)\pi(p)\pi(r) \\ & +g(q,r)\pi(p)\pi(s) - g(p,r)\pi(q)\pi(s)],\end{aligned} \tag{3.1}$$

where θ and π are nonzero 1-forms and a, b and c are scalars. Let us assume that the unit timelike vector field γ and the spacelike vector filed ζ are orthogonal, defined as $g(p,\gamma) = \theta(p)$ and $g(p,\zeta) = \pi(p)$, respectively.

Petrov [48] introduced a fourth-rank tensor $\tilde{\mathcal{P}}$ defined by:

$$\tilde{\mathcal{P}} = \tilde{\mathcal{R}im} + \frac{\kappa}{2} g \wedge \mathcal{T} - \sigma \mathcal{G}, \tag{3.2}$$

where $\tilde{\mathcal{R}im}$ is a $(0,4)$ type Riemannian curvature tensor, $\mathcal{T}$ denotes the string cloud energy-momentum tensor of type $(0,2)$, κ is the constant of gravitation, σ is the energy density, $\mathcal{G}$ is a tensor of type $(0,4)$ expressed as

$$\mathcal{G}(p,q,r,s) = g(q,r)g(p,s) - g(p,r)g(q,s), \tag{3.3}$$

for all $p, q, r, s \in \chi(M^4)$, $\wedge$ define the Kulkarni-Nomizu product between g and $\mathcal{T}$ (for more details see [49, 50]).

Moreover, $\tilde{\mathcal{P}}$ is known as a space-matter tensor. The first part of this tensor represents the curvature of the string cloud spacetime and the second part represents the distribution and motion of the matter [49, 50].

Now (3.2) can also be expressed as

$$\tilde{\mathcal{P}}(p,q,r,s) = \tilde{\mathcal{R}im}(p,q,r,s) + \frac{\kappa}{2}[g(q,r)\mathcal{T}(p,s) + g(p,s)\mathcal{T}(q,r) - g(p,r)\mathcal{T}(q,s) - g(q,s)\mathcal{T}(p,r)] - \sigma[g(q,r)g(p,s) - g(p,s)g(q,s)]. \tag{3.4}$$

If $\tilde{\mathcal{P}}(p,q,r,s) = 0$, then (3.4) provides

$$\tilde{\mathcal{R}im}(p,q,r,s) = -\frac{\kappa}{2}[g(q,r)\mathcal{T}(p,s) + g(p,s)\mathcal{T}(q,r) - g(p,r)\mathcal{T}(q,s) - g(q,s)\mathcal{T}(p,r)] + \sigma[g(q,r)g(p,s) - g(p,r)g(q,s)]. \tag{3.5}$$

Now, adopting (2.1) in (3.5) we get

$$\begin{aligned}\tilde{\mathcal{R}im}(p,q,r,s) = {} & a[g(q,r)g(p,s) - g(p,r)g(q,s)] \\ & + b[g(q,r)\mathcal{U}(p)\mathcal{U}(s) + g(p,s)\mathcal{U}(q)\mathcal{U}(r) \\ & - g(p,r)\mathcal{U}(q)\mathcal{U}(s) - g(q,s)\mathcal{U}(p)\mathcal{U}(r)] \\ & + c[g(p,s)\mathcal{V}(q)\mathcal{V}(r) - g(q,s)\mathcal{V}(p)\mathcal{V}(r) \\ & + g(q,r)\mathcal{V}(p)\mathcal{V}(s) - g(p,r)\mathcal{V}(q)\mathcal{V}(s)],\end{aligned} \tag{3.6}$$

where $a = -\sigma$, $b = -\frac{\kappa\rho}{2}$ and $c = -\frac{\kappa\lambda}{2}$. In light of (3.1) it follows from (3.6) that the relativistic string cloud spacetime taken into account is generalized quasi-constant curvature. Therefore, one can articulate the following.

Theorem 3.1. *A string cloud spacetime obeys the EFE without cosmological constant, and with vanishing space-matter tensor $\tilde{\mathcal{P}}$ is a string cloud spacetime of generalized quasi-constant curvature.*

4. Divergence-free space-matter tensor in string cloud spacetime

In this section, we examine the necessary conditions for a string cloud spacetime to have a space-matter tensor [50] that is divergence-free.

If the related scalars α, β and δ in a string cloud spacetime are constant, then in light of (2.4), we get

$$R = 4\alpha + \beta, \tag{4.1}$$

which entails that scalar curvature R is constant. Therefore, $dR = 0$. Now adopting (1.1), we find from (3.2) that

$$(div\tilde{\mathcal{P}})(p,q,r) = (div\tilde{\mathcal{R}im})(p,q,r) + \frac{1}{2}[(\nabla_p\mathcal{R}ic)(p,r) - (\nabla_q\mathcal{R}ic)(p,r)] - g(q,r)[d\sigma(p) + \frac{1}{4}dR(p)] + g(p,r)[d\sigma(q) + \frac{1}{4}dR(q)]. \tag{4.2}$$

Since in a spacetime Lorentzian manifold (semi-Riemannian manifold), we know that

$$(div\tilde{\mathcal{R}im})(p,q,r) = (\nabla_p \mathcal{R}ic)(q,r) - (\nabla_q \mathcal{R}ic)(p,r), \tag{4.3}$$

and in view of (1.1), we know that $(\nabla_p \mathcal{R}ic)(q,r) = \kappa(\nabla_p \mathcal{T})(q,r)$.
Adopting (4.2) and (4.3), we turn up

$$\begin{aligned}(div\tilde{\mathcal{P}})(p,q,r) = {} & \frac{3}{2}[(\nabla_p \mathcal{R}ic)(q,r) - (\nabla_q \mathcal{R}ic)(p,r)] \\ & -g(q,r)[d\sigma(p) + \frac{1}{4}dR(p)] + g(p,r)[d\sigma(q) + \frac{1}{4}dR(q)].\end{aligned} \tag{4.4}$$

Considering that $(div\tilde{\mathcal{P}})(p,q,r) = 0$ and then contracting (4.4) over p and r, we get

$$d\sigma(u) = 0. \tag{4.5}$$

Thus, we can articulate the following result.

Theorem 4.1. *In a string cloud spacetime obeys EFE with divergence-free space-matter tensor the energy density σ is constant.*

Thus, (2.3) and (4.4) can be expressed as

$$\begin{aligned}(div\tilde{\mathcal{P}})(p,q,r) = {} & -\frac{3}{2}[d(\lambda+\rho)(p)g(q,r) - d(\lambda+\rho)(q)g(p,r)] \\ & +\frac{3}{2}[d(\kappa\rho)(p)\mathcal{U}(q)\mathcal{U}(r) - d(\kappa\rho)(q)\mathcal{U}(p)\mathcal{U}(r)] \\ & -\frac{3}{2}[d(\kappa\lambda)(p)\mathcal{V}(q)\mathcal{V}(r) - d(\kappa\lambda)(q)\mathcal{V}(p)\mathcal{V}(r)] \\ & +\frac{3\kappa\rho}{2}[(\nabla_p\mathcal{U})(q)\mathcal{U}(r) + \mathcal{U}(q)(\nabla_p\mathcal{U})(r) \\ & -(\nabla_q\mathcal{U})(p)\mathcal{U}(r) - \mathcal{U}(p)(\nabla_q\mathcal{U})\mathcal{U}(r)] \\ & -\frac{3\kappa\lambda}{2}[(\nabla_p\mathcal{V})(q)\mathcal{V}(r) + \mathcal{V}(q)(\nabla_p\mathcal{V})(r) \\ & +(\nabla_q\mathcal{V})(p)\mathcal{V}(r) + \mathcal{V}(p)(\nabla_q\mathcal{V})\mathcal{V}(r)] \\ & -g(q,r)[d\sigma(p) + \frac{1}{4}dR(p)] + g(p,r)[d\sigma(q) + \frac{1}{4}dR(q)].\end{aligned} \tag{4.6}$$

Establishing the constraints that the scalar quantities such as density of the fluid ρ, string tension λ and energy density σ are constants. In addition, the generators γ and ζ of the string cloud spacetime manifold are parallel vector fields implies that $\nabla_p\gamma = 0$ and $\nabla_p\zeta = 0$. Therefore, $dR = 0$, $d\sigma(p) = 0$ for all p. In addition,

$$g(\nabla_p\gamma, q) = 0, \qquad (\nabla_p\mathcal{U})(q) = 0, \tag{4.7}$$

$$g(\nabla_p\zeta, q) = 0, \qquad (\nabla_p\mathcal{V})(q) = 0. \tag{4.8}$$

Therefore, from (4.6)–(4.8), it follows that $(div\tilde{\mathcal{P}})(p,q,r) = 0$. Thus, one can write the following.

Theorem 4.2. *If a string cloud spacetime obeys EFE the associated density of the fluid ρ, string tension λ, and energy density σ are constants, then the divergence of the space-matter tensor vanishes.*

5. Conformally flat string cloud spacetime

If the Weyl conformal curvature tensor $\mathcal{W}$ vanishes and is written as, then a string cloud spacetime is said to be conformally flat [51],

$$C(p,q)r = \tilde{\mathcal{R}im}(p,q)r - \frac{1}{2}[\mathcal{R}ic(q,r)p - \mathcal{R}ic(p,r)q \tag{5.1}$$
$$+g(q,r)Qp - g(p,r)Qq],$$
$$+\frac{R}{6}[g(q,r)p - g(p,r)q].$$

Due to the conformally flat string cloud spacetime, we get

$$\tilde{\mathcal{R}im}(p,q,r,s) = \frac{1}{2}[\mathcal{R}ic(q,r)g(p,s) - g(q,s)\mathcal{R}ic(p,r) \tag{5.2}$$
$$+g(q,r)\mathcal{R}ic(p,s) - g(p,r)\mathcal{R}ic(q,s)],$$
$$+\frac{R}{6}[g(p,r)g(q,s) - g(q,r)g(p,s)].$$

In light of (2.3) and (2.4), we have

$$\begin{aligned}\tilde{\mathcal{R}im}(p,q,r,s) = &-\frac{\kappa(\lambda+\rho)}{6}[g(q,r)g(p,s) - g(p,r)g(q,s)]\\ &+\frac{\kappa\rho}{4}[g(p,s)\mathcal{U}(q)\mathcal{U}(r) - g(q,r)\mathcal{U}(p)\mathcal{U}(s)],\\ &+g(q,r)\mathcal{U}(p)\mathcal{U}(s) - g(q,s)\mathcal{U}(q)\mathcal{U}(r)]\\ &-\frac{\kappa\lambda}{2}[g(p,s)\mathcal{V}(q)\mathcal{V}(r) - g(q,s)\mathcal{V}(p)\mathcal{V}(r)\\ &+g(q,r)\mathcal{V}(r)\mathcal{V}(s) - g(p,r)\mathcal{V}(q)\mathcal{V}(s)],\end{aligned} \tag{5.3}$$

which exhibits that string cloud spacetime is a generalized quasi-constant curvature. Consequently, we state the following.

Theorem 5.1. *A conformally flat string cloud spacetime obeys EFE without cosmological constant is of generalized quasi-constant curvature.*

Let γ and ζ be parallel vector fields in string cloud spacetime. Then,

$$\nabla_p\gamma = 0, \quad \nabla_p\zeta = 0,$$

which implies that

$$\tilde{\mathcal{R}im}(p,q)\gamma = 0 \quad and \quad \tilde{\mathcal{R}im}(p,q)\zeta = 0.$$

Thus, it is evident that

$$\mathcal{R}ic(p,\gamma) = 0 \quad and \quad \mathcal{R}ic(p,\zeta) = 0.$$

Next, using (2.5) and (2.6), we gain

$$\alpha+\beta = 0, \quad \beta+\delta = 0.$$

Given that γ and ζ are parallel vector fields, $\beta = \delta = -\alpha$ is implied. Then, we get the following expression from (2.3),

$$\mathcal{R}ic(p,q) = -\kappa(\lambda+\rho)[g(p,q) - \mathcal{U}(p)\mathcal{U}(q) + \mathcal{V}(p)\mathcal{V}(q)], \tag{5.4}$$

which indicates that

$$(\nabla_r \mathcal{R}ic)(p,q) = d(-\kappa(\lambda+\rho))[g(p,q) - \mathcal{U}(p)\mathcal{U}(q) + \mathcal{V}(p)\mathcal{V}(q)]$$
$$-\kappa(\lambda+\rho)[(\nabla_r\mathcal{U})(p)\mathcal{U}(q) + \mathcal{U}(p)(\nabla_r\mathcal{U})(q)]$$
$$+(\nabla_r\mathcal{V})(p)\mathcal{V}(q) + \mathcal{V}(p)(\nabla_r\mathcal{V})(q)].$$

γ and ζ are parallel vector fields. Thus,

$$(\nabla_r\mathcal{U})(p) = 0 \quad and \quad (\nabla_r\mathcal{V})(p) = 0$$

for all $p, q, r \in \chi(M)$.

Therefore,

$$(\nabla_r \mathcal{R}ic)(p,q) = d - \kappa(\lambda+\rho)(z)[g(p,q) - \mathcal{U}(p)\mathcal{U}(q) + \mathcal{V}(p)\mathcal{V}(q)], \tag{5.5}$$

or

$$(\nabla_r \mathcal{R}ic)(p,q) = \frac{d\{-\kappa(\lambda+\rho)\}(s)}{-\kappa(\lambda+\rho)}\mathcal{R}ic(p,q),$$

adopting (5.4),

$$(\nabla_r \mathcal{R}ic)(p,q) = \theta(r)\mathcal{R}ic(p,q), \tag{5.6}$$

where $\theta(r) = \frac{d\{-\kappa(\lambda+\rho)\}(s)}{-\kappa(\lambda+\rho)}$. Hence, we have the following.

Theorem 5.2. *If the timelike vector field γ and spacelike vector field ζ are parallel in a string cloud spacetime, then string cloud spacetime is Ricci-recurrent.*

6. Ricci semi-symmetric string cloud spacetime

Now, we consider that string cloud spacetime is Ricci semi-symmetric, with condition, $\tilde{\mathcal{R}im}(p,q).\mathcal{R}ic = 0$ for all $p, q \in \chi(M)$.
Now, we possess

$$(\tilde{\mathcal{R}im}(p,q).\mathcal{R}ic)(r,s) = -\mathcal{R}ic(\tilde{\mathcal{R}im}(p,q),r) - \tilde{\mathcal{R}im}(r, \tilde{\mathcal{R}im}(p,q)s). \tag{6.1}$$

Then, from (2.3),

$$\{-\kappa(\lambda+\rho)\}\, g(\tilde{\mathcal{R}im}(p,q)r,s) + \kappa\rho\,\mathcal{U}(\tilde{\mathcal{R}im}(p,q)r)\mathcal{U}(s)$$
$$-\kappa\lambda\mathcal{V}(\tilde{\mathcal{R}im}(p,q)r)\mathcal{V}(s) - \{\kappa(\lambda+\rho)\}\,\mathcal{U}(r)\mathcal{U}(\tilde{\mathcal{R}im}(p,q)s)$$
$$-\kappa\lambda\mathcal{V}(w)\mathcal{V}(\tilde{\mathcal{R}im}(u,v)z) = 0. \tag{6.2}$$

Setting $z = \gamma$ and $w = \zeta$ in (6.2), we get

$$-\kappa(\lambda+\rho)\mathcal{U}(\tilde{\mathcal{R}im}(p,q)\zeta) - \kappa\lambda\mathcal{V}(\tilde{\mathcal{R}im}(p,q)\gamma) = 0, \tag{6.3}$$

or

$$-\kappa(\lambda+\rho)g(\tilde{\mathcal{R}im}(p,q)\zeta,\gamma)) - \kappa\lambda g(\tilde{\mathcal{R}im}(p,q)\gamma,\zeta) = 0$$

$$-\kappa(\lambda+\rho)\tilde{\mathcal{R}im}(p,q,r,s) = 0, \tag{6.4}$$

where $\tilde{\mathcal{R}im}(p,q,r,s) = g(\tilde{\mathcal{R}im}(p,q)r,s)$ is non-vanishing. Since the gravitational constant, $\kappa \neq 0$. Hence, we obtain

$$\rho + \lambda = 0. \tag{6.5}$$

Consequently, the following theorem is true.

Theorem 6.1. *If a string cloud spacetime obeys EFE is Ricci semi-symmetric, then the sum of the fluid density ρ and the string tension λ is zero.*

Remark 6.2. *[52] An EoS for Takabayashi string (i.e., P-string), which is*

$$\rho = (1+\omega)\lambda, \tag{6.6}$$

where ω is a cosmological constant. This is analogous to equation of state between the matter density and string's tension in the form $\epsilon_p = \omega\lambda$ and this represnt the equation of state for Takabayashi string [52]. When $\omega < 0$, only geometric strings (Nambu strings [53]) appear and for $\omega > 0$, particles dominate over strings.

After comparing (6.5) with (6.6), we get $\omega < 0$. Therefore, from (6.5) we have state equation $\rho = -\lambda$, which coincides with the state equation for the cloud of geometric strings. Thus, in view of Theorem 6.1 and the above remark, we arrive at the following result.

Theorem 6.3. *If a string cloud spacetime obeying EFE without cosmological constant is Ricci semi-symmetric, then the state equation $\rho = -\lambda$ coincides with the state equation for the cloud of geometric strings.*

Corollary 6.4. *A string cloud spacetime obeys EFE without cosmological constant is Ricci semi-symmetric, then the state equation $\rho = -\lambda$ becomes the state equation of Nambu strings.*

Corollary 6.5. *If a string cloud spacetime obeys EFE without cosmological constant is Ricci semi-symmetric, then the string is a massive string.*

Now, in light of Theorem 6.3 and Corollary 6.5, we turn up the following corollary.

Corollary 6.6. *If a string cloud spacetime obeys EFE without cosmological constant and is Ricci semi-symmetric with the state equation $\frac{\rho}{\lambda} = -1$, then the string cloud spacetime is a massive string cloud spacetime.*

Corollary 6.7. *If a string cloud spacetime obeys EFE without cosmological constant is Ricci semi-symmetric with the state equation $\frac{\rho}{\lambda} = -1$, then the string cloud spacetime recovers the quintessence era for $\omega = -1$.*

7. Pseudo-Ricci symmetric string cloud spacetime

In [32], Chaki introduced the definition of pseudo-Ricci-symmetric manifolds (M^4, g) as given below.

Definition 7.1. *A non flat semi-Riemannian manifold is said to be pseudo-Ricci symmetric if the Ricci tensor $\mathcal{R}ic$ on semi-Riemannian manifold (M^4, g) satisfy the following equation:*

$$(\nabla_p \mathcal{R}ic)(q, r) = 2\pi(p)\mathcal{R}ic(q, r) + \pi(q)\mathcal{R}ic(p, r) + \pi(r)\mathcal{R}ic(p, q), \tag{7.1}$$

where π is a 1 form, p, q, r are vector fields on (M^4, g) and ∇ is the Levi-Civita connection on (M^4, g). Moreover, the semi-Riemannian manifold (M^4, g) reduces to Ricci-symmetric manifold if $\pi = 0$ in (7.1) or if $\nabla\mathcal{R}ic = 0$.

Throughout this section, let us assume that string cloud spacetime under the condition that timelike vector field γ and spacelike vector field ζ are parallel vector fields.

Let (M^4, g) be a string cloud spacetime which is a generalized quasi-Einstein manifold, and assume γ and ζ are parallel. Then, we have

$$\nabla_p \gamma = 0, \quad \nabla_p \zeta = 0 \quad \Rightarrow \tilde{\mathcal{R}im}(p, q)\gamma = 0 \quad and \quad \tilde{\mathcal{R}im}(p, q)\zeta = 0. \tag{7.2}$$

Now, contracting this expression with respect to v, we observe that
$\mathcal{R}ic(p, \gamma) = 0$ and $\mathcal{R}ic(p, \zeta) = 0$. So from (2.5) and (2.6), we turn up

$$\mathcal{R}ic(p, \gamma) = (\alpha + \beta)\mathcal{U}(p) = 0, \tag{7.3}$$

$$\mathcal{R}ic(p, \zeta) = (\alpha + \delta)\mathcal{V}(p) = 0. \tag{7.4}$$

Therefore, $\alpha = -\beta = -\delta$. Then, (2.3)–(2.5) turns the form

$$\mathcal{R}ic(p, q) = \frac{-\kappa(\lambda + \rho)}{2}[g(p, q) - \mathcal{U}(p)\mathcal{U}(q) + \mathcal{V}(p)\mathcal{V}(q)]. \tag{7.5}$$

On the other side, we know that

$$(\nabla_p \mathcal{R}ic)(q, r) = \nabla_p \mathcal{R}ic(q, r) - \mathcal{R}ic(\nabla_p q, r) - \mathcal{R}ic(q, \nabla_p r). \tag{7.6}$$

Since string cloud spacetime is a generalized quasi-Einstein manifold and in view of (7.5) and (7.6), we can express it as

$$(\nabla_p \mathcal{R}ic)(q, r) = \frac{d}{dp}\left\{\frac{-\kappa(\lambda + \rho)}{2}\right\}[g(q, r) - \mathcal{U}(p)\mathcal{U}(r) + \mathcal{V}(p)\mathcal{V}(r)], \tag{7.7}$$

where $\frac{d}{dp}$ signifies the derivative of $-\kappa(\lambda + \rho)$ with respect to the vector field p. Since string cloud spacetime (M^4, g) is pseudo-Ricci-symmetric, using (7.2) and (7.7) we can write

$$\frac{d}{dp}\left\{\frac{-\kappa(\lambda + \rho)}{2}\right\}[g(q, r) - \mathcal{U}(p)\mathcal{U}(r) + \mathcal{V}(p)\mathcal{V}(r)] \tag{7.8}$$

$$= 2\left\{\frac{-\kappa(\lambda+\rho)}{2}\right\}\pi(p)[g(q,r) - \mathcal{U}(q)\mathcal{U}(r) + \mathcal{V}(q)\mathcal{V}(r)]$$
$$+\left\{\frac{-\kappa(\lambda+\rho)}{2}\right\}\pi(q)[g(p,r) - \mathcal{U}(p)\mathcal{U}(r) + \mathcal{V}(p)\mathcal{V}(r)]$$
$$+\left\{\frac{-\kappa(\lambda+\rho)}{2}\right\}\pi(2)[g(p,q) - \mathcal{U}(p)\mathcal{U}(q) + \mathcal{V}(p)\mathcal{V}(q)].$$

Putting $p = \gamma$ and $q = \zeta$ in (7.8), we get

$$\frac{d}{d\gamma}\left\{\frac{-\kappa(\lambda+\rho)}{2}\right\} = \{-\kappa(\lambda+\rho)\}\,\pi(p) \quad and \quad \frac{d}{d\zeta}\left\{\frac{-\kappa(\lambda+\rho)}{2}\right\} = \{-\kappa(\lambda+\rho)\}\,\pi(q). \tag{7.9}$$

Adopting $r = \gamma$ and $r = \zeta$ in (7.8), we turn up

$$\pi(u) = 0 \qquad and \qquad \pi(v) = 0. \tag{7.10}$$

Therefore, in light of (7.8)–(7.10), we get

$$\frac{d}{d\gamma}\left\{\frac{-\kappa(\lambda+\rho)}{2}\right\} = 0, \quad \frac{d}{d\zeta}\left\{\frac{-\kappa(\lambda+\rho)}{2}\right\} = 0,$$

which implies $-\kappa(\lambda+\rho)$ is constant along vector fields γ and ζ.

Consequently, we can infer the folowing.

Theorem 7.2. *Let a string cloud spacetime be a generalized quasi-Einstein spacetime with parallel vector fields γ and ζ. If the string cloud spacetime is pseudo-Ricci-symmetric, then $-\kappa(\lambda+\rho)$ is constant along the vector field γ and vector field ζ.*

8. Conclusions

This research article focused on the analysis of spacetime behavior using the string cloud energy-momentum tensor $\mathcal{T}$, which incorporates the string cloud fluid density ρ and string tension λ. The concept of a relativistic string cloud spacetime was introduced and investigated.

Several key findings were obtained in this study. First, we derived results for the string cloud spacetime that exhibited a divergence-free matter tensor, indicating the conservation of energy and momentum within the system. Furthermore, we observed a diminishing space matter tensor, suggesting a decrease in the spatial distribution of matter within the string cloud spacetime.

Additionally, we explored various curvature characteristics of the relativistic string cloud spacetime. Specifically, we identified the spacetime as conformally flat, indicating that the metric of the spacetime could be transformed into a flat metric by a conformal transformation. Furthermore, we determined that the spacetime was Ricci semi-symmetric, implying a specific relationship between the Ricci tensor and the metric tensor. Moreover, we found that the spacetime was pseudo-Ricci-symmetric, indicating a particular symmetry property associated with the Ricci tensor.

Finally, we established a condition that coincided with the equation of state for a cloud of geometric strings within the context of the Ricci semi-symmetric string cloud spacetime. This condition provides insights into the relationship between the geometric properties of the strings and the curvature characteristics of the spacetime.

The results obtained in this research contribute to a deeper understanding of the behavior of spacetime in the presence of string cloud matter. Further investigations and analyses can build upon these findings to explore additional properties and implications of relativistic string cloud spacetime, opening up avenues for future studies in the field of theoretical physics.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported and funded by the Deanship of Scientific Research at Imam Mohammad Ibn Saud Islamic University (IMSIU) (Grant Number: IMSIU-RG23036).

We thank the anonymous reviewers for their careful reading of our manuscript and their many insightful comments and suggestions.

Conflict of interest

The authors declare no conflict of interest.

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