Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(12): 29017–29041.
DOI: 10.3934/math.20231486
Received: 05 August 2023
Revised: 08 October 2023
Accepted: 11 October 2023
Published: 26 October 2023

Research article

The a posteriori error estimate in fractional differential equations using generalized Jacobi functions

Bo Tang[1,*] **and Huasheng Wang**[2]

[1] School of Mathematics and Information Science, Guangzhou University, Guangzhou 510006, Guangdong, China

[2] School of Mathematics and Computational Science, Wuyi University, Jiangmen, 529020, Guangdong, China

* **Correspondence:** Email: tang08163@163.com.

Abstract: In this work, we study a posteriori error analysis of a general class of fractional initial value problems and fractional boundary value problems. A Petrov-Galerkin spectral method is adopted as the discretization technique in which the generalized Jacobi functions are utilized as basis functions for constructing efficient spectral approximations. The unique solvability of the weak problems is established by verifying the Babuška-Brezzi inf-sup condition. Then, we introduce some residual-type a posteriori error estimators, and deduce their efficiency and reliability in properly weighted Sobolev space. Numerical examples are given to illustrate the performance of the obtained error estimators.

Keywords: fractional initial value problems; fractional boundary value problems; generalized Jacobi functions; Petrov-Galerkin spectral methods; a posteriori error estimators
Mathematics Subject Classification: 34A08, 42C10, 65L05, 65L10, 65L70

1. Introduction

Fractional differential equations (FDEs) appear as tractable mathematical models to depict anomalously diffusive transport, long-range spatial interactions and memory effect [3, 8–10, 13, 19, 30] and have attracted extensive research on the theoretical aspects of the existence and uniqueness of solutions [2, 7, 12, 21, 22, 24]. Due to the non-locality of the fractional operators, the FDEs can rarely be solved explicitly. For this reason, a large number of literature has a growing interest in the development of analytic and numerical analysis of numerical approximations, and spectral method is one of the most widely used numerical methods. Compared with the finite difference method and the finite element method for FDEs, which obtain lower convergence accuracy, the spectral method is a numerical calculation method with high accuracy.

Up to now, there are very promising efforts have been devoted to developing spectral methods for solving FDEs. In pioneer work [11], by constructing intermediate functional spaces in terms of fractional derivatives that are essentially equivalent to the fractional Sobolev spaces, Li and Xu established the well-posedness of the weak problems of fractional diffusion equations and proposed a Galerkin spectral method based on Jacobi polynomials for temporal discretisation and Lagrangian polynomials for space discretization. Subsequently, an efficient space-time Galerkin spectral method based on Jacobi polynomials for temporal discretisation and on Fourier-like basis functions for spatial discretisation was investigated in [33, 34]. An alternating direction implicit Galerkin-Legendre spectral method for the two-dimensional Riesz space fractional nonlinear reaction-diffusion equation is studied in [31]. Mao and Shen [18] then developed a spectral-Galerkin algorithm to solve multi-dimensional fractional elliptic equations with variable coefficients, and they derived rigorous weighted error estimates which improved convergence rate than the usual non-weighted estimates.

On the one hand, the solutions of FDEs are singular near the boundaries because of fractional operators bearing singular kernel/weight functions. On the other hand, the spectral methods based on the traditional polynomial cannot obtain exponential convergence for non-smooth solutions. In order to improve the spectral convergence accuracy and better account for the singularity of solutions of fractional order problems, some scholars have constructed new spectral methods by improving the basis functions. Most notably, Zayernouri and Karniadakis [29] introduce a family of eigenfunctions of a fractional Sturm-Liouville problem in bounded domains, called Jacobi poly-fractonomials, as basis functions, achieving spectral accuracy for some simple fractional model problems. Furthermore, Chen, Shen and Wang [6] extended the range of definition of Jacobi poly-fractonomials and defined a new class of generalised Jacobi functions (GJFs). The optimal approximation results for GJFs in weighted spaces are established in [6], and the a priori error estimates of Petrov-Galerkin method for a class of prototypical FDEs utilizing the GJFs are studied. It turns out that Petrov-Galerkin spectral methods using weighted polynomial bases are particularly well suited for the accurate approximation of FDEs. More recent works in this area can be found in [14, 17].

Besides the a priori error estimation mentioned above, the a posterior error estimation has become an important part of modern scientific computations utilizing various adaptive algorithms since the work of Babuška and Rheinboldt [1]. With a growing number of successful applications of spectral methods to FDEs, the a posterior error analysis of spectral methods for FDEs has been given more and more attention. Mao et al. [15] studied Petrov-Galerkin spectral methods for fractional initial value problems, and a recovery based a posteriori error estimator with postprocessing solutions was obtained. In addition, it is worth noting that Wang et al. studied a posteriori error analysis of the Galerkin spectral methods for space-time fractional diffusion equations [26] and the authors in [25] presented a posteriori error analysis of the Galerkin spectral methods for Multi-term time fractional diffusion equations. In addition, the a posteriori error estimates of the Galerkin spectral method for the fractional optimal control problems is discussed in [5, 27, 28]. To the best of our knowledge, there exists no work currently on residual-type a posteriori error analysis of Petrov-Galerkin spectral methods using GJFs for FDEs, which motivated this work.

In this paper, we investigate the a posteriori error estimates of a class of typical fractional initial value problems and fractional boundary value problems, which pave the way for the research of a posteriori error estimation for spectral element methods. The Petrov-Galerkin spectral method is used as the discretization technique, and some variable involving fractional derivative are discretized by

GJFs with various parameters. A rigorous proof of unique solvability of the spectral discrete problem is presented under some essential assumptions. Then, the a posteriori error estimates without any postprocessing solutions are established, and we investigate numerically the efficiency and reliability of the a posteriori error estimators.

The paper is organized as follows: In Section 2, we introduce some notations and definition of fractional integrals and derivatives, and give some preliminaries on GJFs. First, we state in the first part of Section 3 the Petrov-Galerkin spectral schemes using GJFs for solving a class of fractional initial problems, and we introduce the corresponding a posteriori error estimators, where their efficiency and reliability are proved. Then, we extend the results for a class of fractional boundary problems in the second part of Section 3. Some numerical examples, presented in Section 4, are given to confirm the theoretical findings in above sections, and conclude with some remarks in the final section.

2. Preliminaries

In this section, we collect some basic relations and properties of fractional derivatives and GJFs. Throughout this part, we set $\Lambda := (-1, 1)$. There are some notations that we have to introduce here. Let $\mathbb{N}^+$, $\mathbb{R}$ and $\mathbb{R}^+$ be the set of positive integers, real numbers and positive real numbers, respectively, and denote $\mathbb{N}_0 := \{0\} \cup \mathbb{N}^+$. Set $p \in \mathbb{R}^+$. We denote by $L^p_\omega(\Lambda)$ the class of all measurable functions u with the weight function $\omega(x)$ defined on Λ for which

$$\int_\Lambda |u|^p \omega < \infty,$$

and the functional $\|u\|_{L^p_\omega}$, defined

$$\|u\|_{L^p_\omega} = \Big(\int_\Lambda |u|^p \omega \Big)^{\frac{1}{p}},$$

is a norm on $L^p_\omega(\Lambda)$ provided $1 \le p < \infty$. In general, for $u, v \in L^2_\omega(\Lambda)$, we define

$$(u, v)_{L^2_\omega} = \int_\Lambda uv\omega, \quad \|u\|_{L^2_\omega} = (u, u)^{\frac{1}{2}}_{L^2_\omega}$$

to stand for the inner product and norm of the weighted space $L^2_\omega(\Lambda)$. For convenience, we denote $(u, v)_{L^2_\omega}$ by $(u, v)_\omega$ and $\|\cdot\|_{L^2_\omega}$ by $\|\cdot\|_\omega$. Notice that if $\omega \equiv 1$, then ω will be omitted from the notations, and the weighted space $L^2_\omega(\Lambda)$ is to be $L^2(\Lambda)$.

2.1. Fractional integrals and derivatives

Let us recall the general definitions of fractional integrals and derivatives [20].

Definition 2.1. *(Fractional integrals) For any $u \in L^1(\Lambda)$, the left- and right-sided Riemann-Liouville fractional integral of order $s \in \mathbb{R}^+$ are respectively defined as*

$${}_{-1}I^s_x u(x) = \frac{1}{\Gamma(s)} \int_{-1}^x (x - \tau)^{s-1} u(\tau) d\tau, \quad {}_x I^s_1 u(x) = \frac{1}{\Gamma(s)} \int_x^1 (\tau - x)^{s-1} u(\tau) d\tau,$$

where $\Gamma(\cdot)$ denote the Gamma function.

Definition 2.2. *(Fractional derivatives) Let number $s \geq 0$. For a function u given in Λ, the expression*

$$ {}^{RL}_{-1}D^s_x u(x) = D^n_x({}_{-1}I^{n-s}_x u(x)), \quad {}^{RL}_{x}D^s_1 u(x) = (-1)^n D^n_x({}_xI^{n-s}_1 u(x)), $$

where $n = [s] + 1$, $[s]$ denotes the integer part of s, is called the left- and right-handed Riemann-Liouville fractional derivative of order s, respectively. In addition, the left-handed Caputo fractional derivative of order s is defined as

$$ {}^{C}_{-1}D^s_x u(x) = {}^{RL}_{-1}D^s_x\Big(u(x) - \sum_{k=0}^{n-1} \frac{u^{(k)}(-1)}{k!}(x+1)^k\Big), \tag{2.1} $$

and the right-handed Caputo fractional derivative of order s is defined as

$$ {}^{C}_{x}D^s_1 u(x) = {}^{RL}_{x}D^s_1\Big(u(x) - \sum_{k=0}^{n-1} \frac{u^{(k)}(1)}{k!}(1-x)^k\Big). \tag{2.2} $$

In particular, for any $n \in \mathbb{N}_0$, ${}^{RL}_{-1}D^n_x = D^n_x$, ${}^{RL}_{x}D^n_1 = (-1)^n D^n_x$, where D^n_x is the usual derivative of order n in x. Clearly, it observe from (2.1)-(2.2) that if $u^{(i)}(-1) = 0$, $i = 0, 1, \ldots, n-1$, then ${}^{C}_{-1}D^s_x u(x) = {}^{RL}_{-1}D^s_x u(x)$, and if $u^{(i)}(1) = 0$, $i = 0, 1, \ldots, n-1$, then ${}^{C}_{x}D^s_1 u(x) = {}^{RL}_{x}D^s_1 u(x)$. At the same time, for $u^{(i)}(-1) = 0$, $i = 0, 1, \cdots, n-1$, the Riemann-Liouville fractional derivative operator commutes with integer-order derivative, i.e., that

$$ {}^{RL}_{-1}D^s_x(D^n_x u(x)) = D^n_x({}^{RL}_{-1}D^s_x u(x)) = {}^{RL}_{-1}D^{s+n}_x u(x). \tag{2.3} $$

2.2. GJFs for fractional derivatives

In this subsection, we will introduce the modified GJFs defined in [6], and investigate their properties.

Definition 2.3. *(GJFs) Let $x \in \Lambda$ and $m \in \mathbb{N}_0$.*

- *For $\beta > -1$, $\alpha \in \mathbb{R}$,*

$$ {}^{-}J^{(\alpha,-\beta)}_m(x) := (1+x)^\beta P^{(\alpha,\beta)}_m(x). \tag{2.4} $$

- *For $\alpha > -1$, $\beta \in \mathbb{R}$,*

$$ {}^{+}J^{(-\alpha,\beta)}_m(x) := (1-x)^\alpha P^{(\alpha,\beta)}_m(x), \tag{2.5} $$

where $P^{(\alpha,\beta)}_m(x)$ is the Jacobi polynomials with real parameters $\alpha, \beta \in \mathbb{R}$ on finite interval Λ.

Here, readers may refer to [23] for a summary of the messages associated with Jacobi polynomials. Indeed, for $k \in \mathbb{N}^+$, $\alpha \in \mathbb{R}$, there holds the transformation formula for Jacobi polynomials [23]:

$$ P^{(-k,\alpha)}_m(x) = d^{(k,\alpha)}_m\Big(\frac{x-1}{2}\Big)^k P^{(k,\alpha)}_{m-k}(x), \quad P^{(\alpha,-k)}_m(x) = d^{(k,\alpha)}_m\Big(\frac{x+1}{2}\Big)^k P^{(\alpha,k)}_{m-k}(x), \quad (m \geq k \geq 1), \tag{2.6} $$

where

$$ d^{(k,\alpha)}_m = \frac{(m-k)!\,\Gamma(m+\alpha+1)}{m!\,\Gamma(m+\alpha-k+1)}. $$

Combining (2.6) with (2.4) and (2.5), it's clear that for any $\alpha \geq -1,\ k \in \mathbb{N}^+$,

$$ {}^{-}J_m^{(-k,-\alpha)}(x) := (-1)^k 2^{-k} d_m^{(k,\alpha)} (1-x)^k (1+x)^\alpha P_{m-k}^{(k,\alpha)}(x), \quad (m \geq k \geq 1), $$

and

$$ {}^{+}J_m^{(-\alpha,-k)}(x) := 2^{-k} d_m^{(k,\alpha)} (1-x)^\alpha (1+x)^k P_{m-k}^{(\alpha,k)}(x), \quad (m \geq k \geq 1). $$

Notice that for $\alpha, \beta > -1$, the Jacobi polynomials $P_m^{(\alpha,\beta)}(x)$ naturally turn to the classical Jacobi polynomials. They are orthogonal with respect to the weight function $\omega^{(\alpha,\beta)}(x) = (1-x)^\alpha (1+x)^\beta$, namely,

$$ \int_\Lambda P_m^{(\alpha,\beta)} P_{m'}^{(\alpha,\beta)} \omega^{(\alpha,\beta)} = \gamma_m^{(\alpha,\beta)} \delta_{mm'}, \quad (m, m' \geq 0), \tag{2.7} $$

where $\delta_{mm'}$ denotes the dirac Delta symbol, and the constant $\gamma_m^{(\alpha,\beta)}$ is given by

$$ \gamma_m^{(\alpha,\beta)} = \|P_m^{(\alpha,\beta)}\|^2_{\omega^{(\alpha,\beta)}} = \frac{2^{\alpha+\beta+1}\Gamma(m+\alpha+1)\Gamma(m+\beta+1)}{m!(2m+\alpha+\beta+1)\Gamma(m+\alpha+\beta+1)}. \tag{2.8} $$

Accordingly, the GJFs are orthogonal. It straightforwardly from (2.7) and Definition 2.3 that:

- for $\alpha, \beta > -1$,

$$ \int_\Lambda {}^{+}J_m^{(-\alpha,\beta)} {}^{+}J_{m'}^{(-\alpha,\beta)} \omega^{(-\alpha,\beta)} = \gamma_m^{(\alpha,\beta)} \delta_{mm'} = \int_\Lambda {}^{-}J_m^{(\alpha,-\beta)} {}^{-}J_{m'}^{(\alpha,-\beta)} \omega^{(\alpha,-\beta)}, \quad (m, m' \geq 0), \tag{2.9} $$

  where $\gamma_m^{(\alpha,\beta)}$ is defined in (2.8),
- for $\alpha > -1,\ k \in \mathbb{N}^+$,

$$ \int_\Lambda {}^{-}J_m^{(-k,-\alpha)} {}^{-}J_{m'}^{(-k,-\alpha)} \omega^{(-k,-\alpha)} = \gamma_m^{(\alpha,-k)} \delta_{mm'} = \int_\Lambda {}^{+}J_m^{(-\alpha,-k)} {}^{+}J_{m'}^{(-\alpha,-k)} \omega^{(-\alpha,-k)}, \quad (m, m' \geq k), \tag{2.10} $$

  where $\gamma_m^{(\alpha,-k)}$ is defined in (2.8).

Thanks to the above orthogonality, the completeness of GJFs is proved in following lemma.

Lemma 2.1. *(Completeness of GJFs)*

- *For $\alpha > 0, \beta > -1$, $\{{}^{+}J_m^{(-\alpha,\beta)}\}$ is complete in $L^2_{\omega^{(-\alpha,\beta)}}(\Lambda)$.*
- *For $\beta > 0, \alpha > -1$, $\{{}^{-}J_m^{(\alpha,-\beta)}\}$ is complete in $L^2_{\omega^{(\alpha,-\beta)}}(\Lambda)$.*
- *For $\alpha > 0$, $k \in \mathbb{N}^+$, $\{{}^{+}J_m^{(-\alpha,-k)}\}$ and $\{{}^{+}J_m^{(-k,-\alpha)}\}$ are complete in $L^2_{\omega^{(-\alpha,-k)}}(\Lambda)$ and $L^2_{\omega^{(-k,-\alpha)}}(\Lambda)$, respectively.*

Proof. This lemma can be proved by the same process in [6] using the orthogonality of Jacobi polynomials. Hence we omit the proof here. □

We review the fractional calculus properties of GJFs below.

Lemma 2.2. *[6] Let $s \in \mathbb{R}^+$, $m \in \mathbb{N}_0$ and $x \in \Lambda$.*

- *For $\alpha \in \mathbb{R}$ and $\beta - s > -1$,*

$$ {}^{RL}_{-1}D^{s}_{x}\,{}^{-}J^{(\alpha,-\beta)}_{m}(x) = \frac{\Gamma(m+\beta+1)}{\Gamma(m+\beta-s+1)}\,{}^{-}J^{(\alpha+s,-\beta+s)}_{m}(x). \tag{2.11} $$

- *For $\beta \in \mathbb{R}$ and $\alpha - s > -1$,*

$$ {}^{RL}_{x}D^{s}_{1}\,{}^{+}J^{(-\alpha,\beta)}_{m}(x) = \frac{\Gamma(m+\alpha+1)}{\Gamma(m+\alpha-s+1)}\,{}^{+}J^{(-\alpha+s,\beta+s)}_{m}(x). \tag{2.12} $$

Remark 2.1. *Note that it isn't different to derive from* (2.9)–(2.11) *the orthogonality of* $\{{}^{RL}_{-1}D^{s}_{x}\,{}^{-}J^{(\alpha,-\beta)}_{m}\}$ *as $\beta - s > -1$, and $\alpha + s > -1$ or $\alpha + s \in -\mathbb{N}^{+}$. If $\alpha - s > -1$, and $\beta + s > -1$ or $\beta + s \in -\mathbb{N}^{+}$,* $\{{}^{RL}_{x}D^{s}_{1}\,{}^{+}J^{(-\alpha,\beta)}_{m}\}$ *are orthogonal by* (2.9)–(2.12).

Significantly, there are the orthogonality of fractional derivatives of ${}^{-}J^{(\alpha,-\beta)}_{m}(x)$.

Lemma 2.3. *[6] For $\alpha + \beta > -1$, $\beta > 0$ and $m, m' \geq l \geq 0$ with $m, m', l \in \mathbb{N}_0$,*

$$ \int_{\Lambda} {}^{RL}_{-1}D^{\beta+l}_{x}\,{}^{-}J^{(\alpha,-\beta)}_{m}\;{}^{RL}_{-1}D^{\beta+l}_{x}\,{}^{-}J^{(\alpha,-\beta)}_{m'}\,\omega^{(\alpha+\beta+l,l)} = h^{(\alpha,\beta)}_{m,l}\,\delta_{mm'}, \tag{2.13} $$

where

$$ h^{(\alpha,\beta)}_{m,l} := \frac{2^{\alpha+\beta+1}\,\Gamma^{2}(m+\beta+1)\,\Gamma(m+\alpha+\beta+l+1)}{(2m+\alpha+\beta+1)\,m!\,(m-l)!\,\Gamma(m+\alpha+\beta+1)}. $$

Meanwhile, from (2.10) and (2.12) we obtain the orthogonality of integer derivatives of ${}^{+}J^{(-\alpha,-k)}_{m}$ that for $\alpha - n > -1$, $k - n \in \mathbb{N}^{+}$,

$$ \int_{\Lambda} D^{n}_{x}\,{}^{+}J^{(-\alpha,-k)}_{m}\;D^{n}_{x}\,{}^{+}J^{(-\alpha,-k)}_{m'}\,\omega^{(n-\alpha,n-k)} = h^{(\alpha,-k)}_{m,n}\,\delta_{mm'}, \quad (m, m' \geq n \geq 0). \tag{2.14} $$

where

$$ h^{(\alpha,-k)}_{m,n} := \frac{\Gamma^{2}(m+\alpha+1)}{\Gamma^{2}(m+\alpha-n+1)}\,\gamma^{(\alpha-n,n-k)}_{m}. \tag{2.15} $$

3. Spectral methods for fractional differential equations

In this section, we investigate the Petrov-Galerkin spectral methods employing GJFs as basis function for some prototypical fractional differential equations. The unique solvability of the variation formulation is presented by verifying the Babuška-Brezzi inf-sup condition of the involved bilinear form, and then a posteriori error estimates for the spectral approximation is derived.

3.1. Fractional initial value problems

Let $I = (0, T)$, we consider the Caputo fractional differential equation of order $\alpha \in (0, 1)$ with nonzero initial condition

$$ \begin{cases} {}^{C}_{0}D^{\alpha}_{t}u(t) + \lambda u(t) = g(t), & \forall t \in I, \\ u(0) = u_0, \end{cases} \tag{3.1} $$

where $\lambda \in \mathbb{R}$, and g is a given function with regularity to be specified later.

Let $x = 2t/T - 1, t \in I$. We define $\bar{u}(x)$ in the interval Λ as follow:

$$\bar{u}(x) = u[T(1+x)/2] = u(t).$$

Through above substitution, the original problem (3.1) becomes

$$\begin{cases} \rho\, {}_{-1}^{C}D_x^{\alpha}\bar{u}(x) + \lambda\bar{u}(x) = \bar{g}(x), \quad \forall x \in \Lambda, \\ \bar{u}(-1) = u_0, \end{cases} \tag{3.2}$$

where $\rho = (2/T)^{\alpha}$ and $\bar{g}(x) = g[T(1+x)/2] = g(t)$. For the non-homogeneous initial conditions $\bar{u}(-1) = u_0$, we consider decompose the solution $\bar{u}(x)$ into two parts as

$$\bar{u}(x) = u^h(x) + u_0,$$

with $u^h(-1) = 0$. For $\alpha \in (0,1)$, we then derive from (2.1) that the Eq (3.2) is equivalent to the following Riemann-Liouville type fractional differential equation:

$$\begin{cases} \rho\, {}_{-1}^{RL}D_x^{\alpha}u^h(x) + \lambda u^h(x) = f(x), \quad \forall x \in \Lambda, \\ u^h(-1) = 0, \end{cases} \tag{3.3}$$

in which $f(x) = \bar{g}(x) - \lambda u_0$.

Next, we are about to explore the variational formulation of problem (3.3). First of all, we introduce the solution function space: for $\alpha \in (0,1)$,

$$H_{\omega}^{L}(\Lambda) := \{u^h \in L^2_{\omega^{(-\alpha,-\alpha)}}(\Lambda) : {}_{-1}^{RL}D_x^{\alpha}u^h \in L^2(\Lambda) \text{ such that } u^h(-1) = 0\}, \tag{3.4}$$

endowed with the norms

$$\|u^h\|_{H_{\omega}^{L}} = (\|u^h\|^2_{\omega^{(-\alpha,-\alpha)}} + |u^h|^2_{\alpha})^{\frac{1}{2}}, \tag{3.5}$$

in which $|u^h|_{\alpha} = \|{}_{-1}^{RL}D_x^{\alpha}u^h\|$. By the orthogonality (2.9), we can expand $u^h \in H_{\omega}^{L}(\Lambda)$ as

$$u^h(x) = \sum_{m=0}^{\infty} \hat{u}_m^h\, {}^{-}J_n^{(-\alpha,-\alpha)}(x), \quad \text{where} \quad \hat{u}_m^h = \frac{1}{\gamma_m^{(-\alpha,\alpha)}} \int_{\Lambda} u^h\, {}^{-}J_m^{(-\alpha,-\alpha)} \omega^{(-\alpha,-\alpha)}, \tag{3.6}$$

and there holds the Parseval identity

$$\|u^h\|^2_{\omega^{(-\alpha,-\alpha)}} = \sum_{m=0}^{\infty} \gamma_m^{(-\alpha,\alpha)} |\hat{u}_m^h|^2.$$

Remark 3.1. *The above setup depends on the completeness of* $\{{}^{-}J_m^{(-\alpha,-\alpha)}\}_{m\geq 0}$ *in* $L^2_{\omega^{(-\alpha,-\alpha)}}(\Lambda)$.

Now for $f \in L^2(\Lambda)$, the variational formulation of problem (3.3) is: Find $u^h \in H_{\omega}^{L}(\Lambda)$ such that

$$\mathcal{A}(u^h, v) = (f, v), \quad \forall v \in L^2(\Lambda), \tag{3.7}$$

where the bilinear form $\mathcal{A}(\cdot,\cdot)$ is defined by

$$\mathcal{A}(u^h,v):=\rho({}_{-1}^{RL}D_x^{\alpha}u^h,v)+\lambda(u^h,v).$$

Let $\mathbb{P}_M(\Lambda)$ be the set of algebraic polynomials of degree at most M. In order to discretize problem (3.7), we define the finite-dimensional fractional-polynomial space

$${}^{-}\mathcal{F}_M^{(-\alpha,-\alpha)}(\Lambda)=\{\phi=(1+x)^{\alpha}\psi:\psi\in\mathbb{P}_M(\Lambda)\}=\text{span}\{{}^{-}J_m^{(-\alpha,-\alpha)}:0\le m\le M\},$$

which satisfying the zero initial conditions at $x=-1$. Therefore, we establish the Petrov-Galerkin spectral approximation for (3.7): Find $u_M\in{}^{-}\mathcal{F}_M^{(-\alpha,-\alpha)}(\Lambda)$ such that

$$\mathcal{A}(u_M^h,v_M)=\rho({}_{-1}^{RL}D_x^{\alpha}u_M^h,v_M)+\lambda(u_M^h,v_M)=(I_Mf,v_M),\quad\forall v_M\in\mathbb{P}_M(\Lambda),\tag{3.8}$$

where I_Mf is the Legendre-Gauss-Lobatto interpolation of f relative to $(M+1)$ Legendre-Gauss-Lobatto points, namely,

$$(I_Mf)(x)=\sum_{m=0}^{M}\hat{f}_mL_m(x).$$

Here, $\{\tilde{f}_m\}$ are the 'pseudo-spectral' coefficients computed by the discrete Legendre transform, and $L_m(x)=P_m^{(0,0)}(x)$ denotes the Legendre polynomial in Λ.

We next consider the numerical implementation of Petrov-Galerkin spectral method as follows: Setting

$$u_M^h(x)=\sum_{m=0}^{M}\hat{u}_m^h\,{}^{-}J_m^{(-\alpha,-\alpha)}(x),$$

and let v_M go through all basis functions in $\mathbb{P}_M(\Lambda)=\text{span}\{L_{m'}(x):m'=0,1,\ldots,M\}$. Let $\mathbf{u^h}=[\hat{u}_0^h,\hat{u}_1^h,\ldots,\hat{u}_M^h]$ be the unknown coefficient matrix, we arrive at the linear system

$$\rho\mathbf{u^h}\mathbf{A}+\lambda\mathbf{u^h}\mathbf{B}=\mathbf{f},\tag{3.9}$$

where $\mathbf{A}=[a_{mm'}]_{(M+1)^2}$ is a diagonal matrix with diagonal elements

$$a_{mm}=({}_{-1}^{RL}D_x^{\alpha}\,{}^{-}J_m^{(-\alpha,-\alpha)},L_m)_{\Lambda}=\frac{2\Gamma(m+\alpha+1)}{(2m+1)\Gamma(m+1)},$$

and the matrix $\mathbf{B}$ is defined by element $b_{mm'}$, that is,

$$b_{mm'}=({}^{-}J_m^{(-\alpha,-\alpha)},L_{m'})_{\Lambda}=\int_{\Lambda}P_m^{(-\alpha,\alpha)}L_{m'}\omega^{(0,\alpha)}.$$

Here, $b_{mm'}$ can be exactly calculated by the $(M+2)$-nodes Jacobi-Gauss-Lobatto quadrature with respect to weight $\omega^{(0,\alpha)}$. For the right vector $\mathbf{f}$ in (3.9), it is defined as $\mathbf{f}=[f_0,\cdots,f_{m'},\cdots,f_M]$, and $f_{m'}=(I_Mf,L_{m'})_{\Lambda}$.

3.1.1. Well-posedness

We show the well-posedness of weak formulation (3.7) and Petrov-Galerkin spectral scheme (3.8) using the well-known Babuška-Brezzi theorem. For this purpose, we have to prove the following Lemma, which provide the equivalence relation of norm.

Lemma 3.1. *Let $\alpha \in (0,1)$, and let $H^L_\omega(\Lambda)$ be the spaces defined in* (3.4). *There holds*

$$C_\alpha \|u^h\|_{H^L_\omega} \le \|{}^{RL}_{-1}D^\alpha_x u^h\| \le \|u^h\|_{H^L_\omega}, \quad \forall u^h \in H^L_\omega(\Lambda), \tag{3.10}$$

where

$$C_\alpha = \left(1 + \frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)}\right)^{-\frac{1}{2}}. \tag{3.11}$$

Proof. By the orthogonality (2.9) and (2.13), we obtain that

$$\|u^h\|^2_{\omega^{(-\alpha,-\alpha)}} = \sum_{m=0}^{\infty} \gamma_m^{(-\alpha,\alpha)} |\hat{u}^h_m|^2 \quad \textit{and} \quad \|{}^{RL}_{-1}D^\alpha_x u^h\|^2 = \sum_{m=0}^{\infty} h^{(\alpha,-\alpha)}_{m,\alpha} |\hat{u}^h_m|^2, \tag{3.12}$$

in which

$$h^{(\alpha,-\alpha)}_{m,\alpha} = \frac{\Gamma^2(m+\alpha+1)}{(m!)^2} \gamma_m^{(0,0)}. \tag{3.13}$$

Therefore,

$$\|u^h\|^2_{\omega^{(-\alpha,-\alpha)}} = \sum_{m=0}^{\infty} \frac{\gamma_m^{(-\alpha,\alpha)}}{h^{(\alpha,-\alpha)}_{m,\alpha}} h^{(\alpha,-\alpha)}_{m,\alpha} |\hat{u}^h_m|^2 \le \frac{\gamma_0^{(-\alpha,\alpha)}}{h^{(\alpha,-\alpha)}_{0,\alpha}} \|{}^{RL}_{-1}D^\alpha_x u^h\|^2,$$

and by (3.5), one has

$$\|u^h\|^2_{H^L_\omega} \le \left(1 + \frac{\gamma_0^{(-\alpha,\alpha)}}{h^{(\alpha,-\alpha)}_{0,\alpha}}\right) \|{}^{RL}_{-1}D^\alpha_x u^h\|^2 \le \left(1 + \frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)}\right) \|{}^{RL}_{-1}D^\alpha_x u^h\|^2.$$

This immediately yields the equivalence (3.10). □

Thanks to the above Lemma, the well-posedness of (3.7) can be proved.

Theorem 3.1. *Let $f \in L^2(\Lambda)$. Then, the problem (3.7) exists a unique solution $u^h \in H^L_\omega(\Lambda)$, and it holds*

$$\|u^h\|_{H^L_\omega} \le \frac{1}{\gamma} \|f\|. \tag{3.14}$$

where $\gamma = \rho C_\alpha - |\lambda| > 0$.

Proof. We can verify the continuity of the bilinear form $\mathcal{A}(\cdot,\cdot)$ by the Cauchy-Schwarz inequality, that is, for $\forall u^h \in H^L_\omega(\Lambda)$ and $v \in L^2(\Lambda)$, we have

$$|\mathcal{A}(u^h, v)| \le |\rho({}_{-1}D^\alpha_x u^h, v) + \lambda(u^h, v)| \le \rho\|{}^{RL}_{-1}D^\alpha_x u^h\| \cdot \|v\| + |\lambda|\|u^h\|_{\omega^{(-\alpha,-\alpha)}} \cdot \|v\|.$$

Therefore,

$$|\mathcal{A}(u^h, v)| \le C_{\rho,\lambda}\|u^h\|_{H^L_\omega} \cdot \|v\|.$$

in which $C_{\rho,\lambda}$ is positive constant dependant of ρ and λ.

We are now led to verify the inf-sup condition of $\mathcal{A}(\cdot,\cdot)$, that is, for any $0 \ne u^h \in H^L_\omega(\Lambda)$,

$$\sup_{0\ne v\in L^2(\Lambda)} \frac{|\mathcal{A}(u^h, v)|}{\|v\|} \ge (\rho C_\alpha - |\lambda|)\|u^h\|_{H^L_\omega}. \tag{3.15}$$

where C_α is given in (3.11). For this purpose, we construct $v_* \in L^2(\Lambda)$ from the expansion of $u^h \in H^L_\omega(\Lambda)$ in (3.6) as follows:

$$v_*(x) = \sum_{m=0}^{\infty} \hat{v}^*_m L_m(x) \text{ with } \hat{v}^*_m = \frac{\Gamma(m+\alpha+1)}{m!}\hat{u}^h_m.$$

By the orthogonality of the Legendre Polynomials, we have

$$\|v_*\|^2 = \sum_{m=0}^{\infty} \gamma^{(0,0)}_m |\hat{v}^*_m|^2 = \sum_{m=0}^{\infty} \frac{\Gamma^2(m+\alpha+1)}{(m!)^2}\gamma^{(0,0)}_m |\hat{u}^h_m|^2 = \sum_{m=0}^{\infty} h^{(\alpha,-\alpha)}_{m,0}|\hat{u}^h_m|^2 = \|{}^{RL}_{-1}D^\alpha_x u^h\|^2. \tag{3.16}$$

Then, by a direct calculation, one has

$$\begin{aligned}
|\mathcal{A}(u^h, v_*)| &= |\rho({}_{-1}D^\alpha_x u^h, v_*)_\Lambda + \lambda(u^h, v_*)_\Lambda| \\
&\ge \rho\Big|\int_\Lambda {}_{-1}D^\alpha_x u^h \cdot v_*\Big| - |\lambda|\Big|\int_\Lambda u^h \cdot v_*\Big| \\
&= \rho\Big|\sum_{m=0}^{\infty}\hat{u}^h_m \sum_{m'=0}^{\infty}\hat{v}^*_{m'}\int_\Lambda {}_{-1}D^\alpha_x {}^-J^{(-\alpha,-\alpha)}_m \cdot L_{m'}\Big| - |\lambda|\Big|\int_\Lambda \omega^{(\frac{\alpha}{2},\frac{\alpha}{2})}u^h\omega^{(-\frac{\alpha}{2},-\frac{\alpha}{2})}\cdot v_*\Big| \\
&\ge \rho\Big|\sum_{m=0}^{\infty}\hat{u}^h_m\frac{\Gamma(m+\alpha+1)}{\Gamma(m+1)}\sum_{m'=0}^{\infty}\hat{v}^*_{m'}\int_\Lambda P^{(0,0)}_m L_{m'}\Big| - |\lambda|\Big(\int_\Lambda (u^h)^2\omega^{(-\alpha,-\alpha)}\Big)^{\frac{1}{2}}\Big(\int_\Lambda v_*^2\Big)^{\frac{1}{2}}.
\end{aligned}$$

Thus, using Lemma 3.1 and (3.12), we infer that for any $0 \ne u \in H^L_\omega(\Lambda)$, there exists $0 \ne v_* \in L^2(\Lambda)$ such that

$$|\mathcal{A}(u^h, v_*)| \ge \rho\|{}^{RL}_{-1}D^\alpha_x u^h\| \cdot \|v_*\| - |\lambda|\|u^h\|_{H^L_\omega} \cdot \|v_*\| \ge (\rho C_\alpha - |\lambda|)\|u^h\|_{H^L_\omega} \cdot \|v_*\|. \tag{3.17}$$

This yields (3.15).

Furthermore, we can verify the 'transposed' inf-sup condition, that is, for any $0 \ne v \in L^2(\Lambda)$,

$$\sup_{0\ne u^h\in H^L_\omega(\Lambda)} |\mathcal{A}(u^h, v)| > 0. \tag{3.18}$$

In fact, assuming that $0 \neq v_* \in L^2(\Lambda)$ is an arbitrary function, we construct

$$u^h(x) = \sum_{m=0}^{\infty} \hat{u}_m^h \, {}^-J_m^{(-\alpha,-\alpha)}(x), \text{ with } \hat{u}_m^h = \frac{m!}{\Gamma(m+\alpha+1)} \hat{v}_m^*.$$

Using a similar process, we can derive the inf-sup condition (3.18).

To sum up, we can claim from the Babuška-Brezzi theorem that the weak problem (3.7) is well-posed. That means for problem (3.7), there is a unique solution. Furthermore, for $f \in L^2(\Lambda)$, we have from Cauchy-Schwarz inequality that

$$|(f, v)_\Lambda| \leq \|f\| \cdot \|v\|.$$

By taking $v = v_*$, then using (3.17), we get (3.14) right away, which depict the stability. □

Remark 3.2. *The inf-sup condition of $\mathcal{A}(\cdot, \cdot)$ in Theorem 3.1 is also valid for the discrete problem (3.8), which also admits a unique solution.*

3.1.2. A posteriori error estimation

According to the above results, we follow a standard argument to derive the a posteriori error estimates for the spectral Galerkin method in this subsection. First, let us review the importance projection in $L^2(\Lambda)$. Let Π_M^T be the orthogonal projection operator from $L^2(\Lambda)$ onto $\mathbb{P}_M(\Lambda)$. Equivalently, it means that, for any function $\varphi \in L^2(\Lambda)$, $\Pi_M^T \varphi \in \mathbb{P}_M(\Lambda)$, such that

$$(\varphi - \Pi_M^T \varphi, \psi_M) = 0, \quad \forall \psi_M \in \mathbb{P}_M(\Lambda).$$

Then, for any nonnegative real number s, there exists a positive constant C_σ depending only on σ such that, for any function $\varphi \in H^\sigma(\Lambda)$, the following estimate holds [4]:

$$\|\varphi - \Pi_M^T \varphi\| \leq C_\sigma M^{-\sigma} \|\varphi\|_{\sigma,\Lambda}, \quad (\sigma \geq 0). \tag{3.19}$$

Theorem 3.2. *Let u^h, u_N^h be the solutions of (3.7) and (3.8), respectively. Then, there exists positive constants c and C independent of any function and the degree of polynomials, such that*

$$\|u^h - u_M^h\|_{H_\omega^L} \leq C\{\eta + \|f - I_M f\|\},$$
$$\eta \leq c\{\|u^h - u_M^h\|_{H_\omega^L} + \|f - I_M f\|\},$$

where

$$\eta = \|f - \rho \, {}_{-1}^{RL}D_x^\alpha u_M^h - \lambda u_M^h\|.$$

Proof. For any $v \in L^2(\Lambda)$, one has

$$\begin{aligned}
\mathcal{A}(u^h - u_M^h, v) &= \mathcal{A}(u^h - u_M^h, v - \Pi_M^T v) + (f - I_M f, \Pi_M^T v) \\
&= \mathcal{A}(u^h, v - \Pi_M^T v) - \mathcal{A}(u_M^h, v - \Pi_M^T v) + (f - I_M f, \Pi_M^T v) \\
&= (f, v - \Pi_M^T v) - (\rho \, {}_{-1}^{RL}D_x^\alpha u_M^h + \lambda u_M^h, v - \Pi_M^T v) + (f - I_M f, \Pi_M^T v) \\
&= (f - \rho \, {}_{-1}^{RL}D_x^\alpha u_M^h - \lambda u_M^h, v - \Pi_M^T v) + (f - I_M f, \Pi_M^T v).
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
&\frac{|\mathcal{A}(u^h - u_M^h, v)|}{\|v\|} \\
=&\frac{|(f - \rho\, {}_{-1}^{RL}D_x^\alpha u_M^h - \lambda u_M^h, v - \Pi_M^T v)_\Lambda + (f - I_M f, \Pi_M^T v)|}{\|v\|} \\
\leq&\frac{\|f - \rho\, {}_{-1}^{RL}D_x^\alpha u_M^h - \lambda u_M^h\| \cdot \|v - \Pi_M^T v\| + \|f - I_M f\| \cdot \|\Pi_M^T v\|}{\|v\|}.
\end{aligned} \tag{3.20}$$

Then, in view of formula (3.20) and estimate (3.19), we immediately derive from inf-sup condition (3.15) that

$$\|u^h - u_M^h\|_{H_\omega^L} \leq C\{\|f - \rho\, {}_{-1}^{RL}D_x^\alpha u_M^h - \lambda u_M^h\| + \|f - I_M f\|\}.$$

This means that the a posteriori error estimator η along with the truncation of f is an upper bound for $\|u^h - u_M^h\|_H$, i.e., the reliability holds.

In what follows, we investigate the efficiency of η. For any $v \in L^2(\Lambda)$, and with the help of the continuity of $\mathcal{A}(\cdot,\cdot)$ and estimate (3.19), we have

$$\begin{aligned}
(f - \rho\, {}_{-1}^{RL}D_x^\alpha u_M^h - \lambda u_M^h, v) &= \mathcal{A}(u^h - u_M^h, v - \Pi_M^T v) + (f - I_M f, \Pi_M^T v) \\
&\leq C_{\rho,\lambda}\|u^h - h_M^h\|_{H_\omega^L} \cdot \|v - \Pi_M^T v\| + \|f - I_M f\| \cdot \|\Pi_M^T v\| \\
&\leq c(\|u^h - u_M^h\|_{H_\omega^L} + \|f - I_M f\|)\|v\|.
\end{aligned}$$

Thus, it can be seen that

$$\begin{aligned}
\|f - \rho\, {}_{-1}^{RL}D_x^\alpha u_M^h - \lambda u_M^h\| &= \sup_{v \in L^2(\Lambda)\setminus\{0\}} \frac{(f - \rho\, {}_{-1}^{RL}D_x^\alpha u_M^h - \lambda u_M^h, v)}{\|v\|} \\
&\leq c(\|u^h - u_M^h\|_{H_\omega^L} + \|f - I_M f\|).
\end{aligned}$$

Hence, the proof is completed. □

3.2. Fractional boundary value problems

Now we consider the Riemman-Liouville fractional differential equation with homogeneous Dirichlet boundary conditions

$$\begin{cases} {}_{-1}^{RL}D_x^\nu u(x) + \kappa u'(x) - \lambda u(x) = f(x), & \forall x \in \Lambda, \\ u(\pm 1) = 0, \end{cases} \tag{3.21}$$

where $\nu \in (1, 2)$, $\kappa, \lambda \in \mathbb{R}$ and f is a given function.

Let $s = \nu - 1$, the trial and test spaces are introduced as follows:

$$U := \{u \in L^2_{\omega^{(-1,-s)}}(\Lambda) : {}_{-1}^{RL}D_x^s u \in L^2_{\omega^{(s-1,0)}}(\Lambda), u(\pm 1) = 0\}, \tag{3.22}$$

$$V := \{v \in L^2_{\omega^{(-s,-1)}}(\Lambda) : D_x v \in L^2_{\omega^{(1-s,0)}}(\Lambda), v(\pm 1) = 0\}, \tag{3.23}$$

endowed with the norms

$$\|u\|_U = (\|u\|^2_{\omega^{(-1,-s)}} + \|{}_{-1}^{RL}D_x^s u\|^2_{\omega^{(s-1,0)}})^{\frac{1}{2}}, \tag{3.24}$$

$$\|v\|_V = (\|v\|^2_{\omega^{(-s,-1)}} + \|D_x v\|^2_{\omega^{(1-s,0)}})^{\frac{1}{2}}. \tag{3.25}$$

According to the completeness of $\{{}^-J_m^{(-1,-s)}\}$ in $L^2_{\omega^{(-1,-s)}}$, we can expand $u \in U$ as

$$u(x) = \sum_{m=1}^{\infty} \hat{u}_m {}^-J_m^{(-1,-s)}(x) \text{ with } \hat{u}_m = \frac{1}{\gamma_m^{(s,-1)}} \int_\Lambda u \, {}^-J_m^{(-1,-s)} \omega^{(-1,-s)}, \tag{3.26}$$

where $\gamma_m^{(s,-1)}$ is defined in (2.8). Similarly, we write from the completeness of $\{{}^+J_m^{(-s,-1)}\}$ in $L^2_{\omega^{(-s,-1)}}$ that

$$v(x) = \sum_{m=1}^{\infty} \hat{v}_m {}^+J_m^{(-s,-1)}(x) \text{ with } \hat{v}_m = \frac{1}{\gamma_m^{(s,-1)}} \int_\Lambda v \, {}^-J_m^{(-s,-1)} \omega^{(-s,-1)}. \tag{3.27}$$

There holds the Parseval identity

$$\|u\|^2_{\omega^{(-1,-s)}} = \sum_{m=1}^{\infty} \gamma_m^{(s,-1)} |\hat{u}_m|^2, \quad \|v\|^2_{\omega^{(-s,-1)}} = \sum_{m=1}^{\infty} \gamma_m^{(s,-1)} |\hat{v}_m|^2.$$

With the above setup, we now consider the variational formulation of original problem (3.21). By (2.3) and integration by parts, we derive the following identity: for $f \in L^2_{\omega^{(s,1)}}(\Lambda)$, find $u \in U$ such that

$$\mathcal{A}(u, v) := -({}^{RL}_{-1}D^s_x u, D_x v)_\Lambda - \kappa(u, D_x v)_\Lambda - \lambda(u, v)_\Lambda = (f, v)_\Lambda, \quad \forall v \in V. \tag{3.28}$$

Let ${}^-\mathcal{F}_M^{(-1,-s)}(\Lambda$ and ${}^+\mathcal{F}_M^{(-s,-1)}(\Lambda)$ be the finite-dimensional fractional-polynomial space as follows:

$${}^-\mathcal{F}_M^{(-1,-s)}(\Lambda) = \{\phi = (1+x)^s \psi : \psi \in \mathbb{P}_M(\Lambda),\ \psi(1) = 0\} = \text{span}\{{}^-J_m^{(-1,-s)} : 1 \le m \le M\},$$

and

$${}^+\mathcal{F}_M^{(-s,-1)}(\Lambda) = \{\phi = (1-x)^s \psi : \psi \in \mathbb{P}_M(\Lambda),\ \psi(-1) = 0\} = \text{span}\{{}^+J_m^{(-s,-1)} : 1 \le m \le M\}.$$

Then the Petrov-Galerkin approximation for (3.28) is to find $u_M \in {}^-\mathcal{F}_M^{(-1,-s)}(\Lambda)$ such that

$$\mathcal{A}(u_M, v_M) = (I_{M-1} f, v_M)_\Lambda, \quad \forall v_M \in {}^+\mathcal{F}_M^{(-s,-1)}(\Lambda), \tag{3.29}$$

where $I_M f$ is the Jacobi-Gauss-Lobatto interpolation relative to the Jacobi-Gauss-Lobatto points, namely,

$$(I_{M-1} f)(x) = \sum_{m=0}^{M-1} \widetilde{f}_m P_m^{(s,1)}(x).$$

where $\{\widetilde{f}_m\}$ are determined by the Jacobi transform.

In the following, we present Petrov-Galerkin spectral method to efficiently calculate the numerical integration. Indeed, the choice of function ${}^+J_m^{(-s,-1)}$ and ${}^-J_m^{(-1,-s)}$ is motivated by the consideration of computing the integral involving fractional derivative. With this choice, the integrand ${}^-J_m^{(-1,-s)}(x){}^+J_m^{(-s,-1)}(x)$ can be converted into a polynomial multiplied by corresponding weight function

$\omega^{(s+1,s+1)}$, and the integrand ${}^{-}J_m^{(-1,-s)}(x)D_x{}^{+}J_m^{(-s,-1)}(x)$ is converted into a polynomial multiplied by weight $\omega^{(s,s)}$. In addition, the integrand ${}_{-1}^{RL}D_x^{s}{}^{-}J_m^{(-1,-s)}(x)D_x{}^{+}J_m^{(-s,-1)}(x)$ is transformed to a a polynomial multiplied by $\omega^{(s-1,0)}$. As a consequence, by expressing u_M in the space ${}^{-}\mathcal{F}_M^{(-1,-s)}(\Lambda)$

$$u_M(x) = \sum_{m=1}^{M} \hat{u}_m {}^{-}J_m^{(-1,-s)}(x),$$

and let the test function v_M go through all basis functions in ${}^{+}\mathcal{F}_M^{(-s,-1)}(\Lambda)$, we arrive at the matrix statement of (3.29):

$$\mathbf{uA} + \kappa\mathbf{uB} - \lambda\mathbf{uC} = \mathbf{f}, \tag{3.30}$$

where $\mathbf{u} = [\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_M]$, and

$$\begin{aligned}
\mathbf{A} &= [a_{mm'}]_{M^2}, \qquad a_{mm'} = ({}_{-1}^{RL}D_x^{s}{}^{-}J_M^{(-1,-s)}, D_x{}^{+}J_M^{(-s,-1)}) = \frac{2^s\Gamma^2(m+s+1)}{m!(2m+s)\,\Gamma(m+s)}\delta_{mm'};\\
\mathbf{B} &= [b_{mm'}]_{M^2}, \qquad b_{mm'} = ({}^{-}J_M^{(-1,-s)}, D_x{}^{+}J_M^{(-s,-1)}) = -\frac{(m+s)\Gamma(m'+s+1)}{2m\Gamma(m'+s)}(P_{m-1}^{(1,s)}, P_{m'}^{(s-1,0)})_{\omega^{(s,s)}};\\
\mathbf{C} &= [c_{mm'}]_{M^2}, \qquad c_{mm'} = ({}^{-}J_M^{(-1,-s)}, {}^{+}J_M^{(-s,-1)}) = -\frac{(m+s)(m'+s)}{4mm'}(P_{m-1}^{(1,s)}, P_{m'-1}^{(s,1)})_{\omega^{(s+1,s+1)}}.
\end{aligned}$$

Here, $b_{mm'}$ and $c_{mm'}$ can be exactly evaluated by Jacobi-Gauss-Lobatto quadrature with weight function $\omega^{(s,s)}$ and $\omega^{(s+1,s+1)}$, respectively. For the right vector $\mathbf{f}$ in (3.30), it is defined as $\mathbf{f} = [f_{m'}]_{M\times 1}$ with $f_{m'} = \frac{m'+s}{2m'}(I_{M-1}f, P_{m'-1}^{(s,1)})_{\omega^{(s,1)}}$.

3.2.1. Well-posedness

In the following part, we show the well-posedness of variational formulation (3.28) and the spectral scheme (3.29). The equivalence of the norms (see, e.g., [16]) will be used subsequently.

Lemma 3.2. *Let $s \in (0,1)$, and let U and V be the space defined in (3.22) and (3.23), respectively. Then, there holds*

$$\begin{aligned}
C_u\|u\|_U &\le \|{}_{-1}^{RL}D_x^{s}u\|_{\omega^{(s-1,0)}} \le \|u\|_U, \quad \forall u \in U,\\
C_v\|v\|_V &\le \|D_x v\|_{\omega^{(1-s,0)}} \le \|v\|_V, \quad \forall v \in V,
\end{aligned}$$

where

$$C_u = \Big(1 + \frac{1}{\Gamma(s+1)\Gamma(s+2)}\Big)^{-\frac{1}{2}}, \quad C_v = \Big(1 + \frac{\Gamma(s+1)}{\Gamma(s+2)}\Big)^{-\frac{1}{2}}. \tag{3.31}$$

Theorem 3.3. *Let $s \in (0,1)$, and let $f \in L^2_{\omega^{(s,1)}}(\Lambda)$. Assume that $\vartheta = C_uC_v - |\kappa| - |\lambda| > 0$, Then the weak problem (3.28) has a unique solution $u \in U$. Moreover,*

$$\|u\|_U \le \frac{1}{\vartheta}\|f\|_{\omega^{(s,1)}}. \tag{3.32}$$

Proof. The unique solvability (3.28) is guaranteed by the well-known Babuška-Brezzi theorem. It is obvious that the bilinear form $\mathcal{A}(\cdot,\cdot)$ is continuous. Next, we focus on proving the inf-sup condition of $\mathcal{A}(\cdot,\cdot)$, that is, for any $0 \neq u \in U$,

$$\sup_{0\neq v\in V} \frac{|\mathcal{A}(u,v)|}{\|u\|_U \|v\|_V} \geq \eta := C_u C_v - |\kappa| - |\lambda|, \tag{3.33}$$

where C_u, C_v is defined in (3.31). For this purpose, we construct function $v_* \in V$ judging by the expansion of $u \in U$

$$v_*(x) = \sum_{m=1}^{\infty} \hat{v}_m^* \, {}^{+}J_m^{(-s,-1)}(x) \text{ with } \hat{v}_m^* = \frac{\Gamma(m+s)}{m!}\hat{u}_m.$$

With the above setup, we have from Lemma 2.2 that for any $0 \neq u \in U$,

$$\begin{aligned}
|({}^{RL}_{-1}D_x^s u, D_x v_*)_\Lambda| &= \Big| \Big(\sum_{m=1}^{\infty} \tilde{u}_m \, {}^{-}J_m^{(s-1,0)}, \sum_{m'=1}^{\infty} \tilde{v}_{m'}^* \, {}^{+}J_{m'}^{(1-s,0)} \Big)_\Lambda \Big| \\
&= \sum_{m=1}^{\infty} |\tilde{u}_m|^2 \cdot \|P_m^{(s-1,0)}\|^2_{\omega^{(s-1,0)}} = \sum_{m=1}^{\infty} \frac{2^s \Gamma^2(m+s+1)}{(2m+s)(m!)^2} |\hat{u}_m|^2 \\
&= \|{}^{RL}_{-1}D_x^s u\|^2_{\omega^{(s-1,0)}} = \|D_x v_*\|^2_{\omega^{(1-s,0)}} \\
&= \|{}^{RL}_{-1}D_x^s u\|_{\omega^{(s-1,0)}} \|D_x v_*\|_{\omega^{(1-s,0)}} \\
&\geq C_u C_v \cdot \|u\|_U \cdot \|v_*\|_V.
\end{aligned} \tag{3.34}$$

where C_u, C_v are defined in (3.31). By Cauchy-Schwarz inequality, one has

$$\begin{aligned}
|(u, D_x v_*)_\Lambda| =& \Big| \int_\Lambda u D_x v_* \Big| \leq \Big| \int_\Lambda u \omega^{(-\frac{1}{2},-\frac{s}{2})} D_x v_* \omega^{(\frac{1-s}{2},0)} \omega^{(\frac{s}{2},\frac{s}{2})} \Big| \\
&\leq \int_\Lambda |u|^2 \omega^{(-1,-s)} \cdot \int_\Lambda |D_x v_*|^2 \omega^{(1-s,0)} \\
=& \|u\|_{\omega^{(-1,-s)}} \cdot \|D_x v_*\|_{\omega^{(1-s,0)}} \leq \|u\|_U \cdot \|v_*\|_V,
\end{aligned} \tag{3.35}$$

and

$$\begin{aligned}
|(u, v_*)_\Lambda| &= \int_\Lambda u v_* \leq \int_\Lambda u \omega^{(-\frac{1}{2},-\frac{s}{2})} v_* \omega^{(-\frac{s}{2},-\frac{1}{2})} \omega^{(\frac{s+1}{2},\frac{s+1}{2})} \\
&\leq \|u\|_{\omega^{(-1,-s)}} \cdot \|v_*\|_{\omega^{(-s,-1)}} \leq \|u\|_U \cdot \|v_*\|_V.
\end{aligned} \tag{3.36}$$

Combining (3.34)–(3.36), then we obtain

$$\begin{aligned}
|\mathcal{A}(u, v_*)| &= | - ({}^{RL}_{-1}D_x^s u, D_x v_*)_\Lambda - \kappa(u, D_x v_*)_\Lambda - \lambda(u, v_*)_\Lambda | \\
&\geq |({}^{RL}_{-1}D_x^s u, D_x v_*)_\Lambda| - |\kappa| \cdot |(u, D_x v_*)_\Lambda| - |\lambda| \cdot |(u, v_*)_\Lambda|, \\
&\geq (C_u C_v - |\kappa| - |\lambda|) \cdot \|u\|_U \cdot \|v_*\|_V.
\end{aligned} \tag{3.37}$$

This means the inf-sup condition (3.33) holds.

Analogously, we are able to verify from a converse process the "transposed" inf-sup condition

$$\sup_{0\neq u\in U} |\mathcal{A}(u,v)| > 0, \quad (0 \neq v \in V).$$

To this end, the well-posedness of weak problem (3.28) is proved by Babuška-Brezzi theorem, which means (3.28) has a unique solution $u \in U$.

Finally, if $f \in L^2_{\omega^{(s,1)}}(\Lambda)$, we directly have from Cauchy-Schwarz inequality that

$$|(f,v)_\Lambda| \leq \|f\|_{\omega^{(s,1)}} \cdot \|v\|_{\omega^{(-s,-1)}} \leq \|f\|_{\omega^{(s,1)}} \cdot \|v\|_V.$$

Then, we can derive (3.32) using (3.37). The proof is complete. □

Remark 3.3. *As the similar arguments for the continuous problem* (3.28)*, the well-posedness of discrete problem (3.29) can be established by verifying the Babuška-Brezzi inf-sup condition of the bilinear form.*

3.2.2. A posteriori error estimation

In the purpose of carrying a posteriori error estimation to the Petrov-Galerkin spectral method for problem (3.21), we first define the $L^2_{\omega^{(-1,-s)}}$-orthogonal projection ${}^-\Pi_M^{(-1,-s)}$ on ${}^-\mathcal{F}_M^{(-1,-s)}(\Lambda)$, such that

$$({}^-\Pi_M^{(-1,-s)}u - u, \varphi_m)_{\omega^{(-1,-s)}} = 0, \quad \forall \varphi_m \in {}^-\mathcal{F}_M^{(-1,-s)}(\Lambda).$$

By the expansion of $u \in U$, we have

$${}^-\Pi_M^{(-1,-s)}u(x) = \sum_{m=1}^{M} \hat{u}_m \, {}^-J_m^{(-1,-s)}(x).$$

Similarly, we let ${}^+\Pi_M^{(-s,-1)}$ denote the $L^2_{\omega^{(-s,-1)}}$-orthogonal projection operator upon ${}^+\mathcal{F}_M^{(-s,-1)}(\Lambda)$, that is,

$${}^+\Pi_M^{(-s,-1)}v(x) = \sum_{m=1}^{M} \hat{v}_m \, {}^+J_m^{(-s,-1)}(x), \tag{3.38}$$

which satisfies

$$(\phi_m, {}^+\Pi_M^{(-s,-1)}v - v)_{\omega^{(-s,-1)}} = 0, \quad \forall \phi_m \in {}^+\mathcal{F}_M^{(-s,-1)}(\Lambda). \tag{3.39}$$

The approximation result on the projection error ${}^+\Pi_M^{(-s,-1)}v - v$ is state as follows.

Lemma 3.3. *Let* $s > 0$. *For any* $v \in V$*, we have the* $L^2_{\omega^{(-s,-1)}}$*-estimates*

$$\|{}^+\Pi_M^{(-s,-1)}v - v\|_{\omega^{(-s,-1)}} \leq ((M+1)(M+s+1))^{-\frac{1}{2}} \|D_x v\|_{\omega^{(1-s,0)}}. \tag{3.40}$$

For $0 < l \leq n \leq M$, $l, n \in \mathbb{N}$*, we also obtain the estimates*

$$\|D_x^l({}^+\Pi_M^{(-s,-1)}v - v)\|_{\omega^{(l-s,l-1)}} \leq cM^{l-n}\|D_x^n v\|_{\omega^{(n-s,n-1)}}. \tag{3.41}$$

Proof. In view of the expansion (3.27), we find from (2.14) that for any $v \in V$,

$$\|D_x^n v\|^2_{\omega^{(n-s,n-1)}} = \sum_{m=1}^{\infty} h_{m,n}^{(s,-1)} |\hat{v}_m|^2.$$

By (2.14), (3.27) and (3.38), we obtain

$$\begin{aligned}\|D_x^l({}^{+}\Pi_M^{(-s,-1)} v - v)\|^2_{\omega^{(l-s,l-1)}} = \sum_{m=M+1}^{\infty} h_{m,l}^{(s,-1)} |\hat{v}_m|^2 &= \sum_{m=M+1}^{\infty} \frac{h_{m,l}^{(s,-1)}}{h_{m,n}^{(s,-1)}} h_{m,n}^{(s,-1)} |\hat{v}_m|^2 \\ &\leq \frac{h_{M+1,l}^{(s,-1)}}{h_{M+1,n}^{(s,-1)}} \|D_x^n v\|^2_{\omega^{(n-s,n-1)}}.\end{aligned}$$

We now turn to estimate the constant term on the right-hand side of the above equation. By (2.15), and a direct calculation, we have

$$\frac{h_{M+1,l}^{(s,-1)}}{h_{M+1,n}^{(s,-1)}} = \frac{\Gamma(M+s-n+2)}{\Gamma(M+s-l+2)} \frac{\Gamma(M+l+1)}{\Gamma(M+n+1)} \leq cM^{l-n} \frac{\Gamma(M+l+1)}{\Gamma(M+n+1)}.$$

Thanks to the Lemma 2.1 in [32], for $0 < l \leq n \leq M$, we have

$$\frac{\Gamma(M+l+1)}{\Gamma(M+n+1)} \leq cM^{l-n}.$$

Combining the above three inequalities, we can derive (3.41).

The $L^2_{\omega^{(-s,-1)}}$-estimates can be derived by a similar proof. In fact, by (2.10) and

$$\begin{aligned}\|{}^{+}\Pi_M^{(-s,-1)} v - v\|^2_{\omega^{(-s,-1)}} &= \sum_{m=M+1}^{\infty} \gamma_m^{(s,-1)} |\hat{v}_m|^2 \\ &= \sum_{m=M+1}^{\infty} \frac{2^s \Gamma(m+s+1)\Gamma(m)}{(2m+s)\, m!\, \Gamma(m+s)} |\hat{v}_m|^2 \\ &\leq \frac{1}{(M+1)(M+s+1)} \sum_{m=M+1}^{\infty} h_{m,1}^{(s,-1)} |\hat{v}_m|^2 \\ &\leq \frac{1}{(M+1)(M+s+1)} \|D_x v\|^2_{\omega^{(1-s,0)}}.\end{aligned}$$

Hence, the estimate (3.40) holds immediately. □

Actually, the approximation properties of the projection operator ${}^{-}\Pi_M^{(-1,-s)}$ can be directly derived using the similar argument. Only we perform the corresponding results below.

Corollary 3.1. *Let $s > 0$. For any $u \in U$, we have the $L^2_{\omega^{(-1,-s)}}$-estimates*

$$\|{}^{-}\Pi_M^{(-1,-s)} u - u\|_{\omega^{(-1,-s)}} \leq ((M+1)(M+s+1))^{-\frac{1}{2}} \|D_x u\|_{\omega^{(0,1-s)}},$$

For $0 < l \leq n \leq M$, $l, n \in \mathbb{N}$, we also obtain the estimates

$$\|D_x^l({}^{-}\Pi_M^{(-1,-s)} v - v)\|_{\omega^{(l-1,l-s)}} \leq cM^{l-n} \|D_x^n v\|_{\omega^{(n-1,n-s)}}.$$

With the aid of the above results, we can follow a standard argument to carry out the reliability of the a posteriori error estimates.

Theorem 3.4. *Let u^h, u^h_M be the solutions of (3.28) and (3.29), respectively. Then there exists a positive constants C and c independent of any function and the degree of polynomials, such that*

$$\begin{aligned}\|u-u_M\|_U &\le C\{\eta_u+\|f-I_{M-1}f\|_{\omega^{(s,1)}}\},\\ \eta_l &\le c\{\|u-u_M\|_U+\|f-I_{M-1}f\|_{\omega^{(s,1)}}\},\end{aligned}$$

where

$$\begin{aligned}\eta_u &= ((M+1)(M+\nu))^{-\frac{1}{2}}\|f-{}^{RL}_{-1}D^{\nu}_x u_M-\kappa D_x u_M+\lambda u_M\|_{\omega^{(s,1)}},\\ \eta_l &= |(f-{}^{RL}_{-1}D^{\nu}_x u_M-\kappa D_x u_M+\lambda u_M,\omega^{(s,1)})|.\end{aligned}$$

Proof. For any $v\in V$, we have

$$\begin{aligned}\mathcal{A}(u-u_M,v) &= \mathcal{A}(u-u_M,v-{}^{+}\Pi^{(-s,-1)}_M v)+(f-I_{M-1}f,{}^{+}\Pi^{(-s,-1)}_M v)\\ &= \mathcal{A}(u,v-{}^{+}\Pi^{(-s,-1)}_M v)-\mathcal{A}(u_M,v-{}^{+}\Pi^{(-s,-1)}_M v)+(f-I_{M-1}f,{}^{+}\Pi^{(-s,-1)}_M v)\\ &= (f,v-{}^{+}\Pi^{(-s,-1)}_M v)+({}^{RL}_{-1}D^{s}_x u_M,D_x(v-{}^{+}\Pi^{-s,-1}_M v))\\ &\quad+\kappa(u_M,D_x(v-{}^{+}\Pi^{(-s,-1)}_M v))+\lambda(u,v-{}^{+}\Pi^{(-s,-1)}_M v)+(f-I_{M-1}f,{}^{+}\Pi^{(-s,-1)}_M v)\\ &= (f-{}^{RL}_{-1}D^{\nu}_x u_M-\kappa D_x u_M+\lambda u_M,v-{}^{+}\Pi^{(-s,-1)}_M v)+(f-I_{M-1}f,{}^{+}\Pi^{(-s,-1)}_M v).\end{aligned}$$

Furthermore, one has

$$\begin{aligned}\frac{|\mathcal{A}(u-u_M,v)|}{\|v\|_V} &= \frac{|(f-{}^{RL}_{-1}D^{\nu}_x u_M-\kappa D_x u_M+\lambda u_M,v-{}^{+}\Pi^{(-s,-1)}_M v)+(f-I_{M-1}f,{}^{+}\Pi^{(-s,-1)}_M v)|}{\|v\|_V}\\ &\le \frac{\|f-{}^{RL}_{-1}D^{\nu}_x u_M-\kappa D_x u_M+\lambda u_M\|_{\omega^{(s,1)}}\cdot\|v-{}^{+}\Pi^{(-s,-1)}_M v\|_{\omega^{(-s,-1)}}}{\|v\|_V}\\ &\quad+\frac{\|f-I_{M-1}f\|_{\omega^{(s,1)}}\cdot\|{}^{+}\Pi^{(-s,-1)}_M v\|_{\omega^{(-s,-1)}}}{\|v\|_V}.\end{aligned}$$

So by (3.25), (3.40) and (3.33), we obtain from Lemma 3.3 that

$$\|u-u_M\|_U\le C\{((M+1)(M+\nu))^{-\frac{1}{2}}\|f-{}^{RL}_{-1}D^{\nu}_x u_M-\kappa D_x u_M+\lambda u_M\|_{\omega^{(s,1)}}+\|f-I_{M-1}f\|_{\omega^{(s,1)}}\}.$$

Therefore, we claim that the a posteriori error estimator η_u with the truncation error of f is an upper bound for $\|u-u_M\|_U$, i.e., the reliable property holds.

We next investigate the lower bound property of η_l, which means the efficient property. For any $v\in V$, and with the help of the estimates listed in Lemma 3.3, we have

$$\begin{aligned}&(f-{}^{RL}_{-1}D^{\nu}_x u_M-\kappa D_x u_M+\lambda u_M,v)\\ &= \mathcal{A}(u-u_M,v-{}^{+}\Pi^{(-s,-1)}_M v)+(f-I_{M-1}f,{}^{+}\Pi^{(-s,-1)}_M v)\\ &\le C\|u-u_M\|_U\cdot\|v-{}^{+}\Pi^{(-s,-1)}_M v\|_V+\|f-I_{M-1}f\|_{\omega^{(s,1)}}\cdot\|v-{}^{+}\Pi^{(-s,-1)}_M v\|_{\omega^{(-s,-1)}}\\ &\le c\|u-u_M\|_U\cdot\|D_x(v-{}^{+}\Pi^{(-s,-1)}_M v)\|_{\omega^{(1-s,0)}}+\|f-I_{M-1}f\|_{\omega^{(s,1)}}\cdot\|v-{}^{+}\Pi^{(-s,-1)}_M v\|_{\omega^{(-s,-1)}}.\end{aligned}$$

Hence, we obtain from the dual space of $L^2_{\omega^{(-s,-1)}}$ itself that

$$\begin{aligned}
&\|f - {}^{RL}_{-1}D^{\nu}_x u_M - \kappa D_x u_M + \lambda u_M\|_{\omega^{(s,1)}} \\
= &\sup_{v\in V\setminus\{0\}} \frac{(f - {}^{RL}_{-1}D^{\nu}_x u_M - \kappa D_x u_M + \lambda u_M, v)}{\|v\|_V} \\
\leq &\sup_{v\in V\setminus\{0\}} \frac{c\|u - u_M\|_U \cdot \|D_x v\|_{\omega^{(1-s,0)}} + \|f - I_{M-1}f\|_{\omega^{(s,1)}} \cdot \|v\|_{\omega^{(-s,-1)}}}{\|v\|_V}.
\end{aligned}$$

By Lemma 3.2, one immediately goes to

$$\|f - {}^{RL}_{-1}D^{\nu}_x u_M - \kappa D_x u_M + \lambda u_M\|_{\omega^{(s,1)}} \leq c\{\|u - u_M\|_U + \|f - I_M f\|_{\omega^{(s,1)}}\}.$$

The proof is completed. □

4. Numerical experiments

In what follows, we provide numerical results to illustrate the accuracy of the proposed Petrov-Galerkin schemes and to validate the reliability and efficiency of the a posteriori error estimators. The corresponding data reveal that the a posteriori error estimators η_u and η_l can depict the error estimates of the numerical solution.

Example 4.1. *Test problem 1 upon fractional initial value problems.* We consider the problem (3.1) with the case

$$\begin{cases} {}^{C}_{0}D^{\alpha}_t u(t) + \lambda u(t) = g(t), \quad \forall t \in (0,1), \\ u(0) = 1, \end{cases} \tag{4.1}$$

and given the source term $g(t) = \frac{2^{2.5}\Gamma(3.5)}{\Gamma(3.5-\alpha)} t^{2.5-\alpha} + \lambda(2t)^{2.5} + 1$.

Let $x = 2t - 1$. Then (4.1) is written as

$$\begin{cases} 2^{\alpha}\,{}^{RL}_{-1}D^{\alpha}_x u^h(x) + \lambda u^h(x) = f(x), \quad \forall x \in (-1,1), \\ u^h(-1) = 0, \end{cases} \tag{4.2}$$

where $f(x) = g((x+1)/2) - 1 = \frac{2^{\alpha}\Gamma(3.5)}{\Gamma(3.5-\alpha)}(x+1)^{2.5-\alpha} + \lambda(x+1)^{2.5}$. Here, It's not hard to calculate that the solution $u^h(x) = u((x+1)/2) + 1 = (x+1)^{2.5}$.

For the modified problem (4.2) with $\alpha = 0.5$ and $\lambda = 0$, we note that the source term f is polynomial while u^h has singularity at $x = -1$. Table 1 shows H^L_ω-numerical errors reach machine accuracy fast, which illustrates the GJFs basis absolutely matches the singularities of the solution in this case. Furthermore, we observe that η-error indicators have the same rate of convergence as the numerical errors, which conforms to the results in Theorem 3.2.

Table 1. Errors of Example 4.1 with $\alpha = 0.5$, $\lambda = 0$ on $M = 4,\ 6,\ 8$.

M	$\|u^h - u^h_M\|_{H^L_\omega}$	η	$\|f - I_M f\|$
4	1.5203×10^{-15}	1.0694×10^{-15}	1.2186×10^{-15}
6	4.2902×10^{-15}	5.5510×10^{-15}	5.4775×10^{-15}
8	2.8798×10^{-15}	3.8731×10^{-15}	3.9362×10^{-15}

Since $\rho C_\alpha - |\lambda| > 0$ in Theorem 3.1, hence we could set $\lambda = 0.5$. In this case, we list the numerical errors $\|u - u_M\|_{H^L_\omega}$, truncation error $\|f - I_M f\|$ and error indicators η mentioned in Theorem 3.2 against various M and α. We see from the Figure 1 that its approximation indicates the algebraic convergence. It is a matter of fact that the numerical errors at this point are determined not only by the regularity of the source term but also by the exact solution. In view of the numerical results, $\|u^h - u^h_M\|_{H^L_\omega}$ can be depicted with the a posteriori error estimator, i.e., η-indicator. The numerical results suggest that the a posteriori error estimator is valid, which is consistent with our theoretical results of Theorem 3.2.

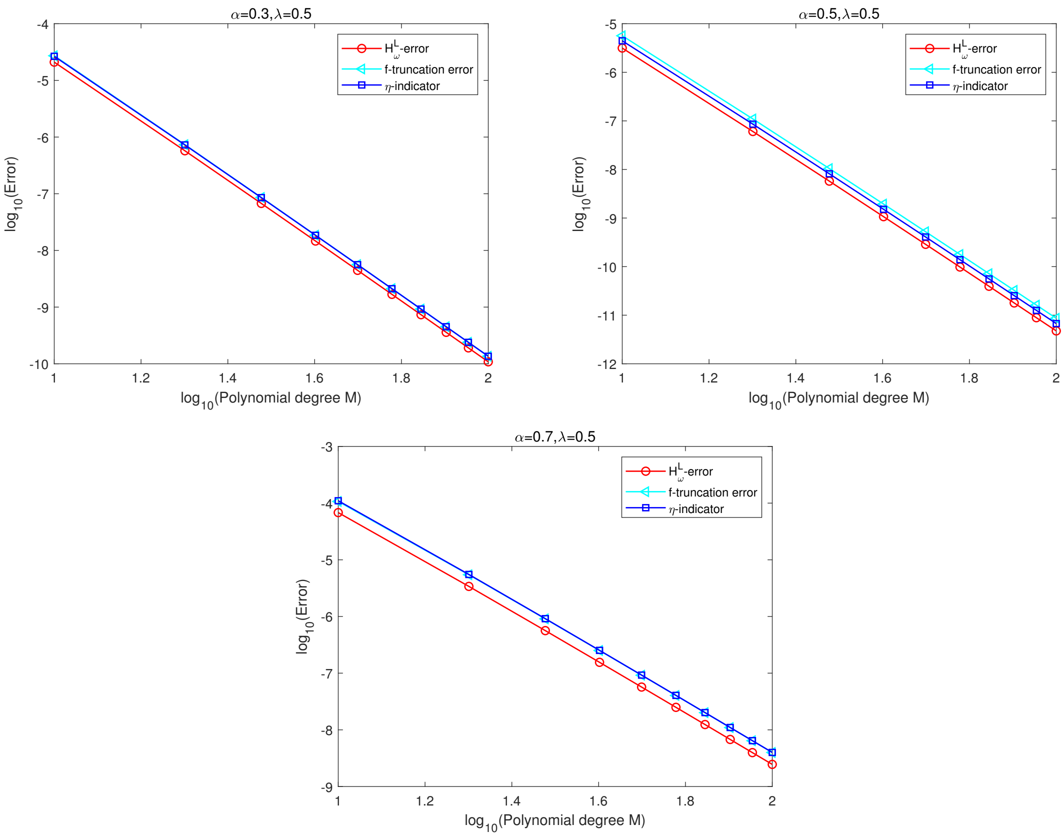

Figure 1. H^L_ω-errors, f-truncation errors and η-indicator in log-log sale against various M and α for Example 4.1 with $\lambda = 0.5$.

Example 4.1'. *Test problem 1 upon fractional initial value problems.* We consider (4.1) with a given source function $g(t) = \sin(2t)$, whose exact solution has singularity at $t = 0$ due to the Caputo fractional derivative. Obviously, the source term $f(x) = \sin(x + 1) - 1$ in (4.2). Here, we determine a numerical solution with $M = 150$ as the reference 'exact' solution. In Figure 2, we list the H^L_ω-errors, f-truncation errors and η-indicator in log-log scale against various M and α for this case with $\lambda = 0.1$. As expected, we observe that the truncation errors of f has spectral accuracy and the numerical errors can be depicted with the η-indicator. Hence, we declare that the a posteriori error estimators stated in Theorem 3.2 are reliable and efficient.

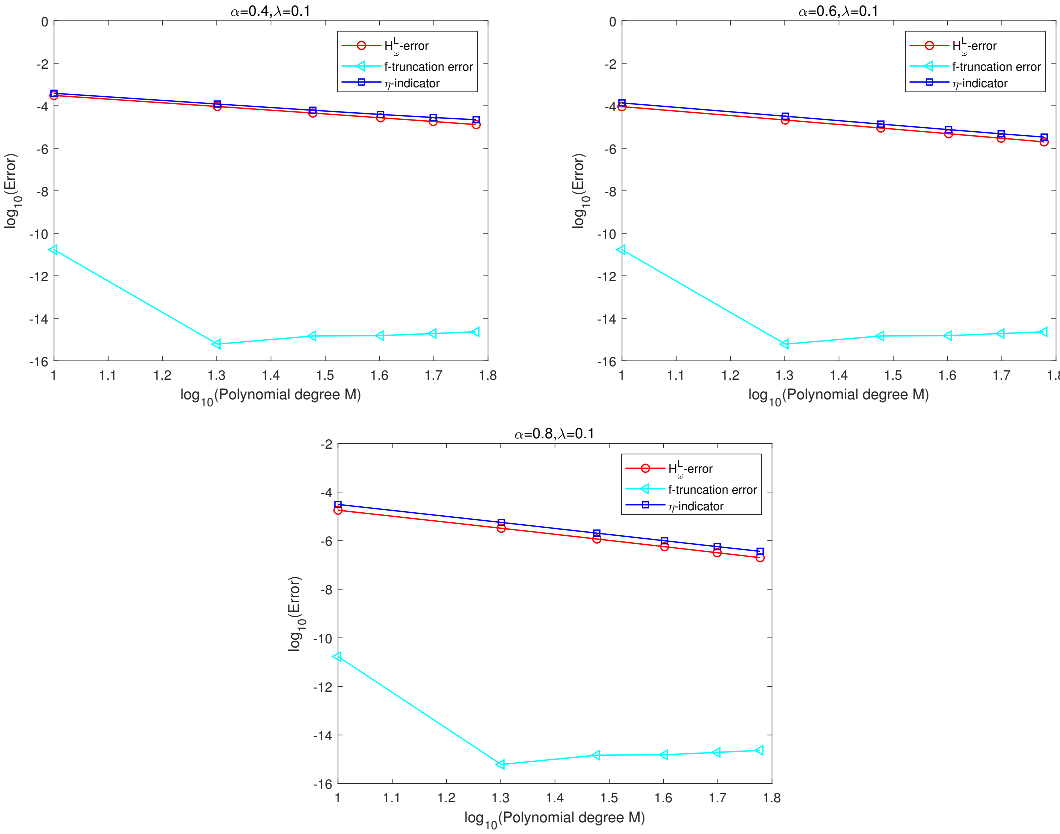

Figure 2. H^L_ω-errors, f-truncation errors and η-indicator in log-log sale against various M and α for Example 4.1' with $\lambda = 0.1$.

Example 4.2. *Test problem 2 upon fractional boundary value problems.* Let $k = \lambda = 1/4$ in (3.21) to fix the hypothesis of Theorem 3.3. We now consider the following fractional boundary value problem:

$$\begin{cases} {}^{RL}_{-1}D^{\nu}_{x}u(x) + \frac{1}{4}D_x u(x) - \frac{1}{4}u(x) = f(x), & x \in (-1, 1), \\ u(\pm 1) = 0, \end{cases} \tag{4.3}$$

with the source function is $u(x) = (1 - x^2)^{2.5}$.

We let $\nu = 1.3,\ 1.7$ in (4.3). Figure 3 shows the data containing the numerical errors $\|u - u_M\|_U$, error indicators η and truncation error $\|f - I_M f\|_{\omega^{(s,1)}}$ against various M and ν. Like the previous cases, we can only observe an algebraic convergence. Furthermore, the errors between the numerical and exact solutions have the same order of accuracy as the a posteriori error indicator η, which agree well with our theoretical analysis of Theorem 3.4.

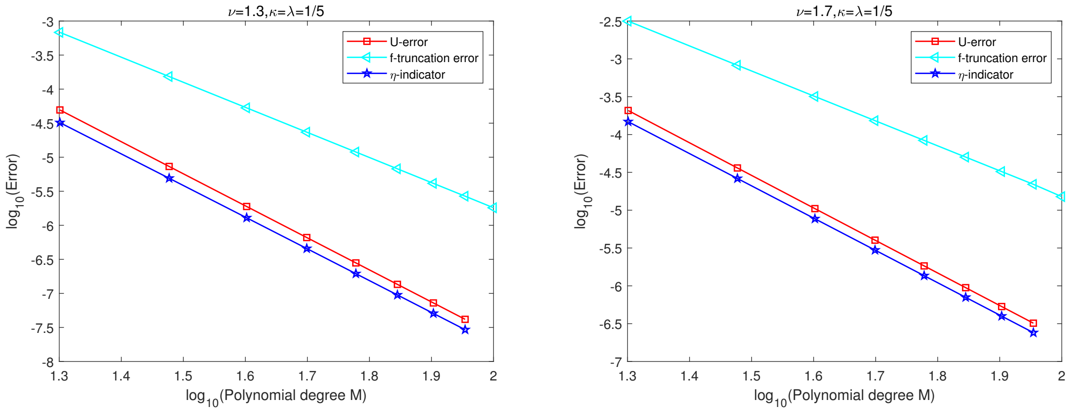

Figure 3. U-errors, f-truncation errors and η-indicator in log-log sale against various M and ν for Example 4.2 with $\kappa = \lambda = 0.25$.

In addition, we also consider (4.3) with the smooth solution $u(x) = (1 - x)\sin^2(\pi x)$. In this case, we list the errors mentioned in Theorem 3.4 against various M with $\nu = 1.2, 1.8$ in Figure 4. In view of this figure, the case suggests that we only obtain algebraic convergence even for smooth data due to the regularity of the source term. As predicted by Theorem 3.4, the error of the Petrov-Galerkin spectral method between the numerical and exact solutions has the same convergence behaviors as the a posteriori error indicators. Hence, we declare that the a posteriori error estimators η stated in Theorem 3.4 are valid.

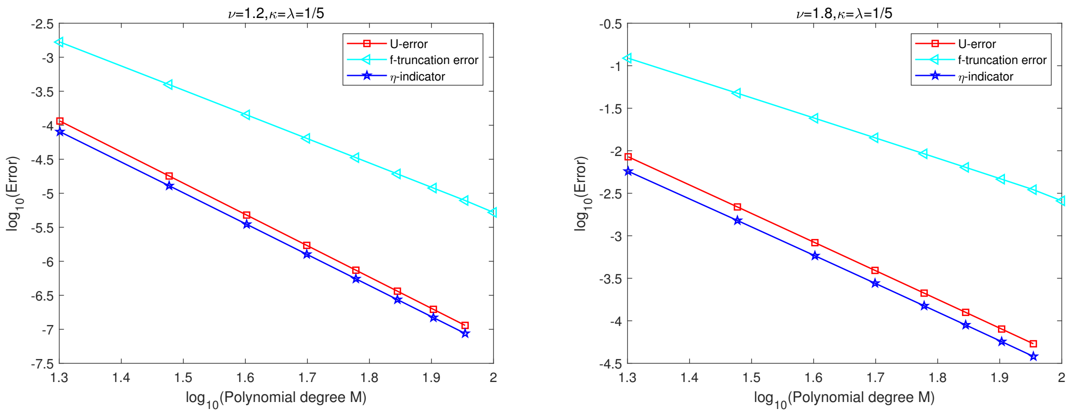

Figure 4. U-errors, f-truncation errors and η-indicator in log-log sale against various M and ν for Example 4.2 with $\kappa = \lambda = 0.25$.

5. Conclusions

We investigate in this paper the a posteriori error estimators of the generalized Jacobi function spectral methods for solving FDEs. In this study, we constructed a global Petrov-Galerkin spectral method to approximate a general class of fractional initial value problems and fractional boundary value problems without discretization, which reduce the computational complexity. With rigorous analyses, the efficiency and reliability of the a posteriori error estimators of proposed methods without any postprocessing solutions is established. Using these error bounds, a suitable degree M can be found without the need to solve the discrete system step by step, which will lead to a reduction in our economic computational cost. The corresponding numerical data reveal that the obtained a posteriori error estimators can capture the error estimates of its approximations between the numerical and exact solutions. This study is just the first step for a posteriori error analysis of generalized Jacobi function spectral methods for FDEs. The problem focusing on adaptive p-version finite element methods with this kind of estimators are included in our ongoing work.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work is supported by the Postdoctoral Research Project of Guangzhou (62306510), the Special Foundation in Key Fields for Universities of Guangdong Province (2022ZDZX1034) and the Joint Research and Development Fund of Wuyi University, Hong Kong and Macao (2021WGALH16). The authors would like to thank Dr. Zhankuan, Zeng of Jiaying University for advice on the numerical program.

Conflict of interest

The authors declare no conflicts of interest in this paper.

References

1. I. Babuška, W. C. Rheinboldt, A-posteriori error estimates for the finite element method, *Int. J. Numer. Meth. Eng.*, **12** (1978), 1597–1615. https://doi.org/10.1002/nme.1620121010
2. M. Benchohra, S. Hamani, S. K. Ntouyas, Boundary value problems for differential equations with fractional order and nonlocal conditions, *Nonlinear Anal.-Theor.*, **71** (2009), 2391–2396. https://doi.org/10.1016/j.na.2009.01.073
3. D. A. Benson, S. W. Wheatcraft, M. M. Meerschaert, The fractional-order governing equation of Lévy motion, *Water Resour. Res.*, **36** (2000), 1413–1423. https://doi.org/10.1029/2000WR900032
4. C. Bernardi, Y. Maday, Spectral methods, In: *Handbook numerical analysis*, **5** (1997), 209–485. https://doi.org/10.1016/S1570-8659(97)80003-8
5. Y. Chen, X. Lin, Y. Huang, Error analysis of spectral approximation for space-time fractional optimal control problems with control and state constraints, *J. Comput. Appl. Math.*, **413** (2022), 114293. https://doi.org/10.1016/j.cam.2022.114293
6. S. Chen, J. Shen, L. Wang, Generalized Jacobi functions and their applications to fractional differential equations, *Math. Comput.*, **85** (2016), 1603–1638. http://doi.org/10.1090/mcom3035
7. R. W. Ibrahim, Global controllability of a set of fractional differential equations, *Miskolc Math. Notes*, **12** (2011), 51–60. https://doi.org/10.18514/MMN.2011.259

8. R. Klages, G. Radons, I. M. Sokolov, *Anomalous transport: Foundations and applications*, Wiley, 2008.

9. D. Kusnezov, A. Bulgac, G. D. Dang, Quantum Lévy processes and fractional kinetics, *Phys. Rev. Lett.*, **82** (1999), 1136–1139. https://doi.org/10.1103/PhysRevLett.82.1136

10. E. Lutz, Fractional transport equations for Lévy stable processes, *Phys. Rev. Lett.*, **86** (2001), 2208–2211. https://doi.org/10.1103/PhysRevLett.86.2208

11. X. Li, C. Xu, A space-time spectral method for the time fractional diffusion equation, *SIAM. J. Numer. Anal.*, **47** (2009), 2108–2131. https://doi.org/10.1137/080718942

12. F. Mainardi, Y. Luchko, G. Pagnini, The fundamental solution of the space-time fractional diffusion equation, 2007, arXiv: cond-mat/0702419v1. https://doi.org/10.48550/arXiv.cond-mat/0702419

13. B. B. Mandelbrot, J. W. Van Ness, Fractional Brownian motions, fractional noises and applications, *SIAM Rev.*, **10** (1968), 422–437. https://doi.org/10.1137/1010093

14. Z. Mao, S. Chen, J. Shen, Efficient and accurate spectral method using generalized Jacobi functions for solving Riesz fractional differential equations, *Appl. Numer. Math.*, **106** (2016), 165–181. https://doi.org/10.1016/j.apnum.2016.04.002

15. W. Mao, Y. Chen, H. Wang, A-posteriori error estimations of the GJF-Petrov-CGalerkin methods for fractional differential equations, *Comput. Math. Appl.*, **90** (2021), 159–170. https://doi.org/10.1016/j.camwa.2021.03.021

16. W. Mao, Y. Chen, H. Wang, A posteriori error estimations of the Petrov-Galerkin methods for fractional Helmholtz equations, *Numer. Algor.*, **89** (2022), 1095–1127. https://doi.org/10.1007/s11075-021-01147-0

17. Z. Mao, G. E. Karniadakis, A spectral method (of exponential convergence) for singular solutions of the diffusion equation with general two-sided fractional derivative, *SIAM J. Numer. Anal.*, **56** (2018), 24–49. https://doi.org/10.1137/16M1103622

18. Z. Mao, J. Shen, Efficient spectral-Galerkin methods for fractional partial differential equations with variable coefficients, *J. Comput. Phys.*, **307** (2016), 243–261. https://doi.org/10.1016/j.jcp.2015.11.047

19. R. Metzler, J. Klafter, The random walk's guide to anomalous diffusion: A fractional dynamics approach, *Phys. Rep.*, **339** (2000), 1–77. https://doi.org/10.1016/S0370-1573(00)00070-3

20. S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives: Theory and applications*, New York: Gordon and Breach Sciences Publishers, 1993.

21. L. Silvestre, Regularity of the obstacle problem for a fractional power of the Laplace operator, *Commun. Pur. Appl. Math.*, **60** (2007), 67–112. https://doi.org/10.1002/cpa.20153

22. G. Su, L. Lu, B. Tang, Z. Liu, Quasilinearization technique for solving nonlinear Riemann-Liouville fractional-order problems, *Appl. Math. Comput.*, **378** (2020), 125199. https://doi.org/10.1016/j.amc.2020.125199

23. G. Szegö, *Orthogonal polynomials*, American Mathematical Society, Providence, 1975.

24. B. Tang, J. Zhao, Z. Liu, Monotone iterative method for two-point fractional boundary value problems, *Adv. Differ. Equ.*, **2018** (2018), 182. https://doi.org/10.1186/s13662-018-1632-9

25. B. Tang, Y, Chen, X. Lin, A posteriori error estimates of spectral Galerkin methods for multi-term time fractional diffusion equations, *Appl. Math. Lett.*, **120** (2021), 107259. https://doi.org/10.1016/j.aml.2021.107259

26. H. Wang, Y. Chen, Y. Huang, W. Mao, A posteriori error estimates of the Galerkin spectral methods for space-time fractional diffusion equations, *Adv. Appl. Math. Mech.*, **12** (2020), 87–100. https://doi.org/10.4208/aamm.OA-2019-0137

27. X. Ye, C. Xu, A posteriori error estimates of spectral method for the fractional optimal control problems with non-homogeneous initial conditions, *AIMS Mathematics*, **6** (2021), 12028–12050. https://doi.org/10.3934/math.2021697

28. X. Ye, C. Xu, A posteriori error estimates for the fractional optimal control problems, *J. Inequal. Appl.*, **2015** (2015), 141. https://doi.org/10.1186/s13660-015-0662-z

29. M. Zayernouri, G. E. Karniadakis, Fractional Sturm-Liouville eigen-problems: Theory and numerical approximation, *J. Comput. Phys.*, **252** (2013), 495–517. https://doi.org/10.1016/j.jcp.2013.06.031

30. G. M. Zaslavsky, Chaos, fractional kinetics, and anomalous transport, *Phys. Rep.*, **371** (2002), 461–580. https://doi.org/10.1016/S0370-1573(02)00331-9

31. F. Zeng, F. Liu, C. Li, K. Burrage, I. Turner, V. Anh, A Crank-Nicolson ADI spectral method for a two-dimensional Riesz space fractional nonlinear reaction-diffusion equation, *SIAM J. Numer. Anal.*, **52** (2014), 2599–2622. https://doi.org/10.1137/130934192

32. X. Zhao, L. Wang, Z. Xie, Sharp error bounds for Jacobi expansions and Gegenbauer-Gauss quadrature of analytic functions, *SIAM J. Numer. Anal.*, **51** (2013), 1443–1469. https://doi.org/10.1137/12089421X

33. M. Zheng, F. Liu, V. Anh, I. Turner, A high-order spectral method for the multi-term time-fractional diffusion equations, *Appl. Math. Model.*, **40** (2016), 4970–4985. https://doi.org/10.1016/j.apm.2015.12.011

34. M. Zheng, F. Liu, I. Turner, V. Anh, A novel high order space-time spectral method for the time fractional Fokker-Planck equation, *SIAM J. Sci. Comput.*, **37** (2015), A701–A724. https://doi.org/10.1137/140980545