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Research article

Some results on frames by pre-frame operators in Q-Hilbert spaces

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Abstract: Quaternionic Hilbert (Q-Hilbert) spaces are frequently used in applied physical sciences and especially in quantum physics. In order to solve some problems of many nonlinear physical systems, the frame theory of Q-Hilbert spaces was studied. Frames in Q-Hilbert spaces not only retain the frame properties, but also have some advantages, such as a simple structure for approximation. In this paper, we first characterized Hilbert (orthonormal) bases, frames, dual frames and Riesz bases, and obtained the accurate expressions of all dual frames of a given frame by taking advantage of pre-frame operators. Second, we discussed the constructions of frames with the help of the pre-frame operators and gained some more general methods to construct new frames. Moreover, we obtained a necessary and sufficient condition for the finite sum of frames to be a (tight) frame, and the obtained results further enriched and improved the frame theory of the Q-Hilbert space.

Keywords: frame; pre-frame operator; dual frame; sum of frames
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1. Introduction

The concept of frames, which generalizes that of bases, was first introduced in the 1950s when Duffin and Schaeffer [1] studied some ongoing problems in the nonharmonic Fourier series. Looking back upon the a sequence $\{e_j : j \in J\} \subseteq \mathcal{H}$ (Hilbert space), we call $\{e_j : j \in J\}$ is a frame for $\mathcal{H}$ if the following inequality holds,

$$A\|x\|^2 \le \sum_{j \in J} |\langle x, e_j \rangle|^2 \le B\|x\|^2, \quad \forall x \in \mathcal{H},$$

where positive constants A, B are called the frame bounds. Frames have turned into a hot issue even since 1986 when Daubechies, Crossman and Meyer published their pioneering work [2]. Nowadays, great achievements have been made in the research of frame theory [3, 4], and frames have been heavily used in numerous fields, such as coding and wireless communication [5], image and signal processing [6], sampling theory [7], quantum measurements [8], and so on ([9, 10]).

Hilbert space can be defined not only in real field and complex field, but also in quaternion field [11, 12]. In 1936, Birkhoff and von Neumann [13] in their famous pioneering work on quantum logic commented that quantum mechanics can also be formulated in Hilbert space where the ground field of complex numbers is replaced by divisible algebras of quaternions [14]. By now, this opinion has been confirmed in the reference [15]. However, it is worth noting that most existing works on the frame theory only focus on real or complex Hilbert spaces instead of quaternionic Hilbert (Q-Hilbert) spaces. Note that both the real field and the complex field are associative and commutative, while the quaternion field only constitutes noncommutative associative algebra. This key characteristic greatly limited mathematicians to establish a complete theory of functional analysis in Q-Hilbert spaces [16], which affected the development of quantum physics in Q-Hilbert space. Luckily, the study on quaternion field has been developed from the mathematical point of view, and achievements in the frames in Q-Hilbert space especially have been obtained in recent. For example, Khokulan, Thirulogasanthar, Srisatkunarajah [17] and Sharma, Virender [18] introduced and studied frames for finite dimensional Q-Hilbert spaces, Sharma, Goel [19] and Sharma, Singh, Sahu [20] studied frames and dual frames for separable Q-Hilbert spaces, and Ellouz [21] introduced K-Frames and Zhang, Li [22] characterized Riesz bases in Q-Hilbert spaces.

When characterizing dual frames of a frame, constructing new frames is a big issues in frame theory. Finding suitable frames are of great significance in applications, and plenty of achievements have been acquired with regard to such issues. For instance, in [7] Li proved that for a given frame, one could obtain all its dual frames by seeking the left inverse of the invertible operator and then giving the accurate expression of all dual frames of the given frame. In [23], Guo looked to ways of constructing (Ω, μ)-frames, consisting of the structures of new (Ω, μ)-frames and the dual (Ω, μ)-frames in some conditions. In [24], Obeidat, Samarah, Casazza and Tremain went into the sums of frames in Hilbert spaces, and gave simple necessary and sufficient conditions on Bessel sequences $\{x_i\}_{i\in I}$ and $\{y_i\}_{i\in I}$ as well as the operators Q_1, Q_2 on $\mathcal{H}$ so that $\{Q_1x_i + Q_2y_i\}_{i\in I}$ formed a new frame for $\mathcal{H}$. In [25, 26]], the authors discussed the sums of g-frames in Hilbert spaces, it was a simple and effective method to construct new frames by using the sums of known frames. Inspired by these works on frames, and aided by the pre-frame operators, we discuss analogous problems on frames in Q-Hilbert spaces. Especially, we obtain some more general construction methods by means of pre-frame operators (see Theorems 4.1 and 4.2), and other current methods. Usually, a new frame can be constructed by using the frame operator and the synthesis (analysis) operator to satisfy certain conditions, such as in [24] where $\{x_j\}_{j\in J}$ and $\{y_j\}_{j\in J}$ are frames for $\mathcal{H}$ with analysis operators T_1, T_2 and frame operators S_1, S_2, respectively. $\{x_j + y_j\}_{j\in J}$ is a new frame for $\mathcal{H}$ if and only if $S_1 + S_2 + T_1^*T_2 + T_2^*T_1 > 0$. Comparatively, the methods we use are more convenient and direct.

In section two, we give some essential notions and existing results for later use. In section there, we first introduce the notion of the pre-frame operator, which is an important class of operators in frame theory, and characterize orthonormal bases, frames, dual frames and Riesz bases in terms of pre-frame operators. We then we obtain the accurate expressions of all dual frames of a given frame by taking advantage of pre-frame operators. With the help of an operator equation, we also give the characterization of the dual frames in Q-Hilbert spaces. In section four, we discuss the sum of frames and Bessel sequences. By means of pre-frame operators, we gain some more general methods to construct new frames. Moreover, we obtain a necessary and sufficient condition for the finite sum of frames to be a (tight) frame.

2. Preliminaries

In this section we arrange some notions and results of frames in Q-Hilbert spaces (see [19, 20] for details), which are necessary for below. $\mathfrak{Q}$ denotes a noncommutative quaternion field, and J is an index set. Let $\mathcal{H}_R(\mathfrak{Q})$, $\mathcal{K}_R(\mathfrak{Q})$ be right Q-Hilbert spaces (or simply R-Q-Hilbert spaces) and $\mathfrak{B}(\mathcal{H}_R(\mathfrak{Q}), \mathcal{K}_R(\mathfrak{Q}))$ denote the collection of all bounded right $\mathfrak{Q}$-linear operators from $\mathcal{H}_R(\mathfrak{Q})$ to $\mathcal{K}_R(\mathfrak{Q})$, as a special case, $\mathcal{H}_R(\mathfrak{Q}) = \mathcal{K}_R(\mathfrak{Q})$, $\mathfrak{B}(\mathcal{H}_R(\mathfrak{Q}), \mathcal{K}_R(\mathfrak{Q})) = \mathfrak{B}(\mathcal{H}_R(\mathfrak{Q}))$, and $I_{\mathcal{H}_R}$ be the identity operator in $\mathcal{H}_R(\mathfrak{Q})$. For $K \in \mathfrak{B}(\mathcal{H}_R(\mathfrak{Q}))$, the range of K is represented by $R(K)$, and the pseudo-inverse of K is represented by $K^\dagger$ if $R(K)$ is closed.

The noncommutative field of quaternions $\mathfrak{Q}$ is a four-dimensional real algebra with unity. In $\mathfrak{Q}$, zero denotes the null element and one denotes the identity with respect to multiplication. It also includes three so-called imaginary units, denoted by $\vec{i}$, $\vec{j}$, $\vec{k}$, i.e.,

$$\mathfrak{Q} = \{x_0 + x_1\vec{i} + x_2\vec{j} + x_3\vec{k} : x_0,\ x_1,\ x_2,\ x_3 \in \mathbb{R}\},$$

where $\vec{i}^2 = \vec{j}^2 = \vec{k}^2 = -1, \vec{i}\cdot\vec{j} = -\vec{j}\cdot\vec{i} = \vec{k}, \vec{j}\cdot\vec{k} = -\vec{k}\cdot\vec{j} = \vec{i}$ and $\vec{i}\cdot\vec{k} = -\vec{k}\cdot\vec{i} = \vec{j}$. For more information about the properties of quaternions, we refer the readers to [11–16].

Let $\mathbb{H}_R(\mathfrak{Q})$ be a linear vector space under right scalar multiplication over the field of quaternions $\mathfrak{Q}$. $\mathbb{H}_R(\mathfrak{Q})$ is called a right quaternionic pre-Hilbert space or right quaternionic inner product space if it is equipped with a Hermitian quaternionic inner product (or simply the inner product)

$$\langle \cdot \mid \cdot \rangle : \mathbb{H}_R(\mathfrak{Q}) \times \mathbb{H}_R(\mathfrak{Q}) \to \mathfrak{Q}$$

satisfying the following conditions:

(a) $\overline{\langle \phi \mid \psi \rangle} = \langle \psi \mid \phi \rangle$ for all $\phi,\ \psi \in \mathbb{H}_R(\mathfrak{Q})$;

(b) $\langle \phi \mid \phi \rangle > 0$ unless $\phi = 0$;

(c) $\langle \phi \mid \psi + \omega \rangle = \langle \phi \mid \psi \rangle + \langle \phi \mid \omega \rangle$ for all $\phi,\ \psi,\ \omega \in \mathbb{H}_R(\mathfrak{Q})$;

(d) $\langle \phi \mid \psi q \rangle = \langle \phi \mid \psi \rangle q, \langle \phi q \mid \psi \rangle = \bar{q}\langle \phi \mid \psi \rangle$ for all $\phi,\ \psi \in \mathbb{H}_R(\mathfrak{Q})$ and $q \in \mathfrak{Q}$.

Let $\mathbb{H}_R(\mathfrak{Q})$ be a right quaternionic pre-Hilbert space with the inner product $\langle \cdot \mid \cdot \rangle$. We define the quaternionic norm $\|\cdot\| : \mathbb{H}_R(\mathfrak{Q}) \to \mathbb{R}^+$ on $\mathbb{H}_R(\mathfrak{Q})$ by $\|\phi\| = \sqrt{\langle \phi \mid \phi \rangle}$, $\phi \in \mathbb{H}_R(\mathfrak{Q})$. If $\mathbb{H}_R(\mathfrak{Q})$ is complete with respect to the norm $\|\cdot\|$, it is called an R-Q-Hilbert space and is denoted by $\mathcal{H}_R(\mathfrak{Q})$.

Proposition 2.1. ([12]) Let $\mathcal{H}_R(\mathfrak{Q})$ be an R-Q-Hilbert space and $\mathcal{N} \subseteq \mathcal{H}_R(\mathfrak{Q})$ meet that for $z,\ z' \in \mathcal{N}$ $\langle z|z' \rangle = \begin{cases} 1 & \text{if } z = z'; \\ 0 & \text{if } z \neq z'. \end{cases}$ Then the following assertions are equivalent:

(i) $\forall x,\ y \in \mathcal{H}_R(\mathfrak{Q})$, the progression $\sum_{z\in\mathcal{N}} \langle x|z\rangle\langle z|y\rangle$ is absolute convergence in $\mathcal{H}_R(\mathfrak{Q})$ and it possess:

$$\langle x|y\rangle = \sum_{z\in\mathcal{N}} \langle x|z\rangle\langle z|y\rangle.$$

(ii) $\|x\| = \sum_{z\in\mathcal{N}} |\langle z|x\rangle|^2$, $\forall x \in \mathcal{H}_R(\mathfrak{Q})$.

(iii) $\mathcal{N}^\perp = \{u \in \mathcal{H}_R(\mathfrak{Q}) : \langle u|z\rangle = 0, \forall z \in \mathcal{N}\} = \{0\}$.

(iv) $span\mathcal{N}$ is dense in $\mathcal{H}_R(\mathfrak{Q})$.

Definition 2.1. ([12]) Let $\mathcal{H}_R(\mathfrak{Q})$ be an R-Q-Hilbert space, and $\mathcal{N} \subseteq \mathcal{H}_R(\mathfrak{Q})$ is called a Hilbert basis or orthonormal basis of $\mathcal{H}_R(\mathfrak{Q})$ if it satisfies $\langle z|z'\rangle = \begin{cases} 1 & \text{if } z = z'; \\ 0 & \text{if } z \neq z'. \end{cases}$ for $z, z' \in \mathcal{N}$ and all the conditions in Proposition 2.1. What is more, if $\mathcal{N}$ is a Hilbert basis of $\mathcal{H}_R(\mathfrak{Q})$, then arbitrary $x \in \mathcal{H}_R(\mathfrak{Q})$, the decomposition $x = \sum\limits_{z\in\mathcal{N}} z\langle z|x\rangle$ is unique and the progression $\sum_{z\in\mathcal{N}} z\langle z|x\rangle$ is an absolute convergence in $\mathcal{H}_R(\mathfrak{Q})$.

Compared with complex Hilbert spaces, Q-Hilbert spaces inherit a great deal of standard properties (see [12, 16]).

Definition 2.2. ([11]) An operator $T : \mathcal{H}_R(\mathfrak{Q}) \to \mathcal{H}_R(\mathfrak{Q})$, for arbitrary $\phi, \psi \in \mathcal{H}_R(\mathfrak{Q})$ and $\alpha, \beta \in \mathfrak{Q}$, if $T(\phi\alpha + \psi\beta) = T(\phi)\alpha + T(\psi)\beta$, then T is called right $\mathfrak{Q}$-linear; if there is a constant $M > 0$ such that $\|T\phi\| \leq M\|\phi\|$, then T is bounded.

Proposition 2.2. ([12]) Let $T \in \mathfrak{B}(\mathcal{H}_R(\mathfrak{Q}))$, and satisfy $T = T^*$, then the norm of T is defined as follows

$$\|T\|_{op} = \sup_{f\in\mathcal{H}_R(\mathfrak{Q}),\|x\|=1} |\langle Tx|x\rangle|.$$

Proposition 2.3. ([12]) Let $\mathcal{H}_R(\mathfrak{Q})$ be an R-Q-Hilbert space, $U, V \in \mathfrak{B}(\mathcal{H}_R(\mathfrak{Q}))$. Then

(i) $U + V$ and $UV \in \mathfrak{B}(\mathcal{H}_R(\mathfrak{Q}))$. In addition,

$$\|U + V\|_{op} \leq \|U\|_{op} + \|V\|_{op} \text{ and } \|UV\|_{op} \leq \|U\|_{op}\|V\|_{op};$$

(ii) $(U + V)^* = U^* + V^*$;
(iii) $(UV)^* = V^*U^*, (U^*)^* = U$;
(iv) if the operator U is invertible, then $(U^{-1})^* = (U^*)^{-1}$;
(v) $I^*_{\mathcal{H}_R} = I_{\mathcal{H}_R}$, where $I_{\mathcal{H}_R}$ is the identity operator in $\mathcal{H}_R(\mathfrak{Q})$.

For more background information on Q-Hilbert spaces, see [11, 12].

In [19], Sharma and Goel extended the concept of frame in Hilbert space to the Q-Hilbert space, as described next.

Definition 2.3. ([19]) Let $\mathcal{H}_R(\mathfrak{Q})$ be an R-Q-Hilbert space. A sequence $\{x_j\}_{j\in J} \subset \mathcal{H}_R(\mathfrak{Q})$ is called a frame for $\mathcal{H}_R(\mathfrak{Q})$ if there are two finite constants with $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{j\in J} |\langle x_j \,|\, f\rangle|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H}_R(\mathfrak{Q}). \tag{2.1}$$

The numbers A, B are called frame bounds of $\{x_j\}_{j\in J}$. We call $\{x_j\}_{j\in J}$ a Bessel sequence for $\mathcal{H}_R(\mathfrak{Q})$ if only the righthand inequality of (2.1) is established in these circumstances, B is called Bessel bound. We call $\{x_j\}_{j\in J}$ a λ-tight frame for $\mathcal{H}_R(\mathfrak{Q})$ if $A = B = \lambda$. What is more, we call $\{x_j\}_{j\in J}$ a Parseval frame for $\mathcal{H}_R(\mathfrak{Q})$ if $\lambda = 1$.

Now define the space $l^2(\mathfrak{Q})$ by

$$l^2(\mathfrak{Q}) := \left\{\{q_j\}_{j\in J} : \{q_j\}_{j\in J} \subset \mathfrak{Q} \text{ such that } \sum_{j\in J} |q_j|^2 < +\infty\right\},$$

and endow $l^2(\mathfrak{Q})$ with the inner product

$$\langle p \,|\, q\rangle = \sum_{j\in J} \overline{p_j} q_j, \quad p = \{p_j\}_{j\in J} \text{ and } q = \{q_j\}_{j\in J} \in l^2(\mathfrak{Q}).$$

Then $l^2(\mathfrak{Q})$ is an R-Q-Hilbert space.

If $\{x_j\}_{j\in J}$ is a frame for $\mathcal{H}_R(\mathfrak{Q})$, then the operator $S : \mathcal{H}_R(\mathfrak{Q}) \to \mathcal{H}_R(\mathfrak{Q})$ defined by

$$S f = \sum_{j\in J} x_j \langle x_j \,|\, f\rangle, \quad \forall f \in \mathcal{H}_R(\mathfrak{Q}),$$

and S is called the (right) frame operator related to $\{f_j\}_{j\in J}$. It is understood that S is a right linear bounded invertible operator (see [19]).

Definition 2.4. ([20]) Let $\{x_j\}_{j\in J}$ be a frame for $\mathcal{H}_R(\mathfrak{Q})$. A sequence $\{y_j\}_{j\in J} \subset \mathcal{H}_R(\mathfrak{Q})$ fulfills

$$f = \sum_{j\in J} x_j \langle y_j | f\rangle = \sum_{j\in J} y_j \langle x_j | f\rangle, \quad \forall f \in \mathcal{H}_R(\mathfrak{Q}).$$

Then $\{y_j\}_{j\in J}$ is commonly known as the alternate dual for $\{x_j\}_{j\in J}$ in $\mathcal{H}_R(\mathfrak{Q})$.

Definition 2.5. ([22]) Let $\mathcal{H}_R(\mathfrak{Q})$ be an R-Q-Hilbert space and $\{x_j\}_{j\in J} \subset \mathcal{H}_R(\mathfrak{Q})$. $\{x_j\}_{j\in J}$ has been described as a Riesz basis for $\mathcal{H}_R(\mathfrak{Q})$ if the following conditions are met

(i) $\{x_j\}_{j\in J}$ is complete, that is, for $f \in \mathcal{H}_R(\mathfrak{Q})$, if $\langle x_j \,|\, f\rangle = 0, \forall j \in J$, then $f = 0$.

(ii) There are two positive finite constants A and B such that

$$A \sum_{j\in J_1} |q_j|^2 \le \left\| \sum_{j\in J_1} x_j q_j \right\|^2_{l^2(\mathfrak{Q})} \le B \sum_{j\in J_1} |q_j|^2, \tag{2.2}$$

where $q_j \in \mathfrak{Q}$, $j \in J_1$, J_1 is any finite subset of J. A and B are called Riesz bounds of $\{x_j\}_{j\in J}$.

3. Characterizations of (dual) frames

In this section, we introduce the definition of the pre-frame operators, and utilize the pre-frame operators for characterizing frames and dual frames in the R-Q-Hilbert space $\mathcal{H}_R(\mathfrak{Q})$. For a given frame in $\mathcal{H}_R(\mathfrak{Q})$, we also obtain the accurate expression formula about the dual frames. For these purposes, we first introduce a lemma, which was given by Sharma and Goel in [19].

Lemma 3.1. ([19]) Let $\mathcal{H}_R(\mathfrak{Q})$ be an R-Q-Hilbert space and $\{x_j\}_{j\in J} \subset \mathcal{H}_R(\mathfrak{Q})$. Then $\{f_j\}_{j\in J}$ is a Bessel sequence for $\mathcal{H}_R(\mathfrak{Q})$ with bound B if and only if the right linear operator $T : l^2(\mathfrak{Q}) \to \mathcal{H}_R(\mathfrak{Q})$ defined by

$$T(\{q_j\}_{j\in J}) = \sum_{j\in J} x_j q_j, \quad \{q_j\}_{j\in J} \in l^2(\mathfrak{Q}),$$

is well defined, and $\|T\|_{op} \le \sqrt{B}$.

Proposition 3.1. *Let $\mathcal{H}_R(\mathfrak{Q})$ be an R-Q-Hilbert space and $\mathcal{N} = \{z_j\}_{j\in J}$ be a Hilbert basis for $\mathcal{H}_R(\mathfrak{Q})$. Then $\{x_j\}_{j\in J} \subset \mathcal{H}_R(\mathfrak{Q})$ is a Bessel sequence for $\mathcal{H}_R(\mathfrak{Q})$ if and only if there exists a unique bounded right linear operator $V : \mathcal{H}_R(\mathfrak{Q}) \to \mathcal{H}_R(\mathfrak{Q})$ such that $x_j = V z_j$ for all $j \in J$.*

Proof. (⇒). Note that $\mathcal{N} = \{z_j\}_{j\in J}$ is a Hilbert basis for $\mathcal{H}_R(\mathfrak{Q})$, and therefore $\{\langle z_j|f\rangle\}_{j\in J} \in l^2(\mathfrak{Q})$ for each $f \in \mathcal{H}_R(\mathfrak{Q})$. If $\{x_j\}_{j\in J} \subset \mathcal{H}_R(\mathfrak{Q})$ is a Bessel sequence, then the operator

$$V: \mathcal{H}_R(\mathfrak{Q}) \to \mathcal{H}_R(\mathfrak{Q}), \quad Vf = \sum_{j\in J} x_j\langle z_j|f\rangle, \ \ \forall f \in \mathcal{H}_R(\mathfrak{Q})$$

is well defined by Lemma 3.1. We obtain for each $f \in \mathcal{H}_R(\mathfrak{Q})$ that

$$\begin{aligned}
\|Vf\| &= \sup_{g\in\mathcal{H}_R(\mathfrak{Q}),\|g\|=1} \left|\left\langle \sum_{j\in J} x_j\langle z_j|f\rangle \middle| g\right\rangle\right| \\
&= \sup_{g\in\mathcal{H}_R(\mathfrak{Q}),\|g\|=1} \left|\sum_{j\in J} \overline{\langle z_j|f\rangle}\left\langle x_j|g\right\rangle\right| \\
&= \sup_{g\in\mathcal{H}_R(\mathfrak{Q}),\|g\|=1} \left|\sum_{j\in J}\langle f|z_j\rangle\left\langle x_j|g\right\rangle\right| \\
&\le \sup_{g\in\mathcal{H}_R(\mathfrak{Q}),\|g\|=1} \left(\sum_{j\in J} |\langle z_j|f\rangle|^2\right)^{\frac{1}{2}} \left(\sum_{j\in J} |\left\langle x_j|g\right\rangle|^2\right)^{\frac{1}{2}} \\
&\le \sqrt{B}\|f\|,
\end{aligned}$$

where B is Bessel bound of $\{x_j\}_{j\in J}$. It follows that V is a bounded right linear operator on $\mathcal{H}_R(\mathfrak{Q})$. By the definition of Hilbert basis, for an arbitrary $f \in \mathcal{H}_R(\mathfrak{Q})$, we obtain that $f = \sum_{j\in J} z_jq_j$, where $\{q_j\}_{j\in J} \in l^2(\mathfrak{Q})$ is unique, and

$$\begin{aligned}
Vf = V\left(\sum_{j\in J} z_jq_j\right) &= \sum_{j\in J} x_j\left\langle z_j \middle| \sum_{i\in J} z_iq_i\right\rangle \\
&= \sum_{j\in J} x_j \sum_{i\in J}\left\langle z_j \middle| z_i\right\rangle q_i = \sum_{j\in J} x_jq_j.
\end{aligned}$$

Hence $\sum_{j\in J} Vz_jq_j = \sum_{j\in J} f_jq_j$, which implies that $x_j = Vz_j$. Suppose that $V_1, V_2 \in \mathfrak{B}(\mathcal{H}_R(\mathfrak{Q}))$ and $V_1z_j = V_2z_j = x_j$ for all $j \in J$. For $f \in \mathcal{H}_R(\mathfrak{Q})$, we have $\left\langle (V_1 - V_2)z_j \middle| f\right\rangle = 0$ for all $j \in J$. It follows that

$$0 = \sum_{j\in J}\left|\left\langle (V_1 - V_2)z_j \middle| f\right\rangle\right|^2 = \sum_{j\in J}\left|\left\langle z_j \middle| (V_1^* - V_2^*)f\right\rangle\right|^2 = \|(V_1^* - V_2^*)f\|^2,$$

and thus $V_1 = V_2$. Hence the operator V is unique.

(⇐). If $V \in \mathfrak{B}(\mathcal{H}_R(\mathfrak{Q}))$ satisfies $x_j = Vz_j$ for arbitrary $j \in J$, then

$$\sum_{j\in J} |\left\langle x_j|f\right\rangle|^2 = \sum_{j\in J} |\left\langle Vz_j|f\right\rangle|^2 = \sum_{j\in J} |\left\langle z_j|V^*f\right\rangle|^2 = \|V^*f\|^2 \le \|V\|_{op}^2\|f\|^2, \ \ \forall f \in \mathcal{H}_R(\mathfrak{Q}).$$

This shows that $\{x_j\}_{j\in J}$ is a Bessel sequence for $\mathcal{H}_R(\mathfrak{Q})$. □

Definition 3.1. *Let $\mathcal{N} = \{z_j\}_{j\in J}$ be a Hilbert basis for an R-Q-Hilbert space $\mathcal{H}_R(\mathfrak{Q})$ and $\{x_j\}_{j\in J}$ be a Bessel sequence in $\mathcal{H}_R(\mathfrak{Q})$. The operator V in Proposition 3.1 is called the (right) pre-frame operator associated with $\{x_j\}_{j\in J}$.*

Lemma 3.2. *Let $\mathcal{N} = \{z_j\}_{j\in J}$ be a Hilbert basis for an R-Q-Hilbert space $\mathcal{H}_R(\mathfrak{Q})$ and $\{x_j\}_{j\in J}$ be a Bessel sequence in $\mathcal{H}_R(\mathfrak{Q})$. If V is the pre-frame operator associated with $\{x_j\}_{j\in J}$ and S is the frame operator associated with $\{x_j\}_{j\in J}$, then $S = VV^*$.*

Proof. By the definition of V, $x_j = Vz_j$ for $j \in J$, then

$$\begin{aligned} Sf &= \sum_{j\in J} f_j\langle x_j|f\rangle = \sum_{j\in J} Vz_j\langle Vz_j|f\rangle \\ &= \sum_{j\in J} Vz_j\langle z_j|V^*f\rangle = V\left(\sum_{j\in J} z_j\langle z_j|V^*f\rangle\right) = VV^*f \end{aligned}$$

for $f \in \mathcal{H}_R(\mathfrak{Q})$. Hence $S = VV^*$. □

Now, we characterize frames and dual frames in terms of pre-frame operators. We begin with a lemma.

Lemma 3.3. ([21]) Let $\mathcal{H}_R(\mathfrak{Q}), \mathcal{K}_R(\mathfrak{Q})$ be two R-Q-Hilbert spaces, and $K : \mathcal{H}_R(\mathfrak{Q}) \to \mathcal{K}_R(\mathfrak{Q})$ be a bounded operator. If $R(K)$ is closed, then, there is a bounded operator $K^\dagger : \mathcal{K}_R(\mathfrak{Q}) \to \mathcal{H}_R(\mathfrak{Q})$ for which

$$KK^\dagger f = f, \ \ \forall f \in R(K).$$

Theorem 3.1. *Let $\mathcal{N} = \{z_j\}_{j\in J}$ be a Hilbert basis for an R-Q-Hilbert space $\mathcal{H}_R(\mathfrak{Q})$ and $\{x_j\}_{j\in J}$ be a Bessel sequence for $\mathcal{H}_R(\mathfrak{Q})$. If V and S denote the pre-frame operator and frame operator of $\{x_j\}_{j\in J}$, respectively, then*

(i) $\{x_j\}_{j\in J}$ is a frame for $\mathcal{H}_R(\mathfrak{Q})$ if and only if V is onto.

(ii) $\{x_j\}_{j\in J}$ is a Parseval frame for $\mathcal{H}_R(\mathfrak{Q})$ if and only if V is coisometry (i.e., V^* is isometry).

(iii) $\{x_j\}_{j\in J}$ is a Riesz basis for $\mathcal{H}_R(\mathfrak{Q})$ if and only if V is invertible.

(iv) $\{x_j\}_{j\in J}$ is a Hilbert basis for $\mathcal{H}_R(\mathfrak{Q})$ if and only if V is unitary.

Proof. (i) If $\{x_j\}_{j\in J}$ is a frame for $\mathcal{H}_R(\mathfrak{Q})$, then S is invertible by (Theorem 3.5 in [19]). By Lemma 3.2, we have $S = VV^*$, so V is onto. On the other hand, if V is onto, then $\{x_j\}_{j\in J}$ is a Bessel sequence for $\mathcal{H}_R(\mathfrak{Q})$ by Lemma 3.1. Next we only need to show the existence of lower frame bound. Note that V is onto, we have $VV^\dagger = I_{\mathcal{H}_R}$ by Lemma 3.3. It follows that $(V^\dagger)^*V^* = I_{\mathcal{H}_R}$. Accordingly,

$$\|f\|^2 = \|(V^\dagger)^*V^*f\|^2 \leq \|(V^\dagger)^*\|_{op}^2\|V^*f\|^2, \ \ \forall f \in \mathcal{H}_R(\mathfrak{Q}).$$

Thus,

$$\begin{aligned} \sum_{j\in J} |\langle x_j|f\rangle|^2 &= \sum_{j\in J} |\langle Vz_j|f\rangle|^2 \\ &= \sum_{j\in J} |\langle z_j|V^*f\rangle|^2 = \|V^*f\|^2 \geq \frac{1}{\|(V^\dagger)^*\|_{op}^2}\|f\|^2. \end{aligned}$$

(ii) It is easy to check that $\{x_j\}_{j\in J}$ is a Parseval frame for $\mathcal{H}_R(\mathfrak{Q})$ iff S is an identity operator on $\mathcal{H}_R(\mathfrak{Q})$. By Lemma 3.2, we have $S = VV^*$, so $S = I_{\mathcal{H}_R}$ if and only if V is a coisometry.

(iii) See Theorem 3.7 in [22].

(iv) If $\{x_j\}_{j\in J}$ is a Hilbert basis for $\mathcal{H}_R(\mathfrak{Q})$, then we have for any $f \in \mathcal{H}_R(\mathfrak{Q})$ that

$$\|f\|^2 = \sum_{j\in J} |\langle x_j|f\rangle|^2 = \sum_{j\in J} |\langle Vz_j|f\rangle|^2 = \sum_{j\in J} |\langle z_j|V^*f\rangle|^2 = \|V^*f\|^2.$$

Therefore, $VV^* = I_{\mathcal{H}_R}$. It follows that V is a unitary operator. On the contrary, if V is a unitary operator, then for any $f \in \mathcal{H}_R(\mathfrak{Q})$, by simple calculation we have

$$\sum_{j\in J} |\langle x_j|f\rangle|^2 = \sum_{j\in J} |\langle Vz_j|f\rangle|^2 = \sum_{j\in J} |\langle z_j|V^*f\rangle|^2 = \|V^*f\|^2 = \|f\|^2.$$

$$\langle x_i|x_j\rangle = \langle Vz_i|Vz_j\rangle = \langle V^*Vz_i|z_j\rangle = \langle z_i|z_j\rangle = \delta_{ij}, \quad \text{for each } i,\ j \in J.$$

So, $\{x_j\}_{j\in J}$ is a Hilbert basis for $\mathcal{H}_R(\mathfrak{Q})$. □

Theorem 3.2. *Let $\mathcal{N} = \{z_j\}_{j\in J}$ be a Hilbert basis for an R-Q-Hilbert space $\mathcal{H}_R(\mathfrak{Q})$. Let $\{x_j\}_{j\in J}$ and $\{y_j\}_{j\in J}$ be Bessel sequences for $\mathcal{H}_R(\mathfrak{Q})$, and let the pre-frame operators related with $\{x_j\}_{j\in J}$ and $\{y_j\}_{j\in J}$ be V and W, respectively. Then, $\{x_j\}_{j\in J}$ and $\{y_j\}_{j\in J}$ are dual frames if and only if $VW^* = I_{\mathcal{H}_R}$ or $WV^* = I_{\mathcal{H}_R}$.*

Proof. Note that V and W are pre-frame operators related with $\{x_j\}_{j\in J}$ and $\{y_j\}_{j\in J}$, respectively, so we have

$$x_j = Vz_j \ \text{ and } \ y_j = Wz_j, \ \ \forall j \in J.$$

Hence, for $f \in \mathcal{H}_R(\mathfrak{Q})$, we have

$$\sum_{j\in J} x_j\langle y_j|f\rangle = \sum_{j\in J} Vz_j\langle Wz_j|f\rangle = V\left(\sum_{j\in J} z_j\langle z_j\middle|W^*f\rangle\right) = VW^*f.$$

Similarly,

$$\sum_{j\in J} y_j\langle x_j|f\rangle = \sum_{j\in J} Wz_j\langle Vz_j|f\rangle = W\left(\sum_{j\in J} z_j\langle z_j\middle|V^*f\rangle\right) = WV^*f.$$

It can be seen from this that $\{x_j\}_{j\in J}$ and $\{y_j\}_{j\in J}$ are dual frames if and only if $VW^* = I_{\mathcal{H}_R}$ or $WV^* = I_{\mathcal{H}_R}$. □

If $U, V \in \mathfrak{B}(\mathcal{H}_R(\mathfrak{Q}))$ and $UV = I_{\mathcal{H}_R}$, then U is called a left inverse operator of V. Our next goal is to characterize dual frames for the existing frame in an R-Q-Hilbert space. The following theorem gave the characterization of the right linear bounded left inverses of the existing pre-frame operator.

Theorem 3.3. *Let $\mathcal{N} = \{z_j\}_{j\in J}$ be a Hilbert basis for an R-Q-Hilbert space $\mathcal{H}_R(\mathfrak{Q})$ and $\{x_j\}_{j\in J}$ be a frame for $\mathcal{H}_R(\mathfrak{Q})$. Suppose that $\{y_j\}_{j\in J} \subset \mathcal{H}_R(\mathfrak{Q})$ and the pre-frame operator of $\{x_j\}_{j\in J}$ is V, then $\{y_j\}_{j\in J}$ is a dual frame of $\{x_j\}_{j\in J}$ if and only if $y_j = Wz_j$ for an arbitrary $j \in J$, where W is a right linear bounded left inverse of V^*.*

Proof. (⇒). Let $\{y_j\}_{j\in J}$ be an arbitrary dual frame of $\{x_j\}_{j\in J}$, and W be the pre-frame operator of $\{y_j\}_{j\in J}$. There is a bounded right linear operator W such that $y_j = Wz_j$ for arbitrary $j \in J$ by Proposition 3.1, and so for $f \in \mathcal{H}_R(\mathfrak{Q})$, we have

$$f = \sum_{j\in J} x_j\langle y_j|f\rangle = \sum_{j\in J} y_j\langle x_j|f\rangle.$$

Note as V as the pre-frame operator of $\{x_j\}_{j\in J}$, we have $x_j = Vz_j$ for arbitrary $j \in J$, so

$$f = \sum_{j\in J} x_j\langle y_j|f\rangle = \sum_{j\in J} Vz_j\langle Wz_j|f\rangle = V\left(\sum_{j\in J} z_j\langle z_j\middle|W^*f\rangle\right) = VW^*f.$$

This implies that $VW^* = I_{\mathcal{H}_R}$. Therefore, $WV^* = I_{\mathcal{H}_R}$, as required.

(⇐). Let $y_j = Wz_j$ for any $j \in J$, where the bounded right linear operator W is a left inverse of V^*. Since $WV^* = I_{\mathcal{H}_R}$, $W \in \mathfrak{B}(\mathcal{H}_R(\mathfrak{Q}))$ is surjective. Hence $\{y_j\}_{j\in J}$ is a frame for $\mathcal{H}_R(\mathfrak{Q})$ by Theorem 3.1 (i). For all $f \in \mathcal{H}_R(\mathfrak{Q})$, we have

$$\sum_{j\in J} x_j\langle y_j|f\rangle = \sum_{j\in J} Vz_j\langle Wz_j|f\rangle = V\left(\sum_{j\in J} z_j\langle z_j\middle|W^*f\rangle\right) = VW^*f = WV^*f = f.$$

Thus, $\{y_j\}_{j\in J}$ is an arbitrary dual frame of $\{x_j\}_{j\in J}$ by Definition 2.4. □

Theorem 3.3 suggests the operator W has great independent interest. To have a better understanding of W, we prove the following lemma.

Lemma 3.4. *Let $\{x_j\}_{j\in J}$ be a frame for an R-Q-Hilbert space $\mathcal{H}_R(\mathfrak{Q})$, and its pre-frame operator and frame operator are V and S, respectively. Then $W \in \mathfrak{B}(\mathcal{H}_R(\mathfrak{Q}))$ is a left invertible operator of V^* if and only if*

$$W = S^{-1}V + U(I_{\mathcal{H}_R} - V^*S^{-1}V),$$

where $U \in \mathfrak{B}(\mathcal{H}_R(\mathfrak{Q}))$.

Proof. Suppose that $W \in \mathfrak{B}(\mathcal{H}_R(\mathfrak{Q}))$ is an arbitrary left invertible of V^*. Let $U = W$, then

$$\begin{aligned} S^{-1}V + U(I_{\mathcal{H}} - V^*S^{-1}V) &= S^{-1}V + W - WV^*S^{-1}V \\ &= S^{-1}V + W - S^{-1}V = W. \end{aligned}$$

Conversely, we suppose that $W = S^{-1}V + U(I_{\mathcal{H}_R} - V^*S^{-1}V)$, by Lemma 3.2, and we have

$$\begin{aligned} WV^* &= S^{-1}VV^* + U(I_{\mathcal{H}_R} - V^*S^{-1}V)V^* \\ &= S^{-1}S + U(V^* - V^*S^{-1}VV^*) \\ &= I_{\mathcal{H}_R} + UV^* - UV^*S^{-1}VV^* = I_{\mathcal{H}_R}. \end{aligned}$$

Hence W is a bounded right linear left inverse of V^*. □

Based on Theorem 3.3 and Lemma 3.4, we characterize all dual frames for an arbitrarily given frame in R-Q-Hilbert spaces, and give the accurate expressions of all dual frames by taking advantage of pre-frame operators.

Theorem 3.4. *Let $\mathcal{N} = \{z_j\}_{j\in J}$ be a Hilbert basis for an R-Q-Hilbert space $\mathcal{H}_R(\mathfrak{Q})$. If $\{x_j\}_{j\in J}$ is a frame for $\mathcal{H}_R(\mathfrak{Q})$, V and S are the pre-frame operator and frame operator of $\{x_j\}_{j\in J}$, respectively, then the sequence $\{y_j\}_{j\in J} \subset \mathcal{H}_R(\mathfrak{Q})$ is a dual frame for $\{x_j\}_{j\in J}$ if and only if*

$$y_j = S^{-1}x_j + Uz_j - UV^*S^{-1}x_j, \quad \forall j \in J,$$

where $U \in \mathfrak{B}(\mathcal{H}_R(\mathfrak{Q}))$ is a right linear operator.

Proof. (⇒). Suppose that the sequence $\{y_j\}_{j\in J} \subset \mathcal{H}_R(\mathfrak{Q})$ is an arbitrary dual frame for $\{x_j\}_{j\in J}$. The results in Theorem 3.3 show that $y_j = Wz_j$ for arbitrary $j \in J$, where W is a left inverse of V^*. By Lemma 3.4, we have

$$W = S^{-1}V + U(I_{\mathcal{H}_R} - V^*S^{-1}V)$$

for some right linear operator $U \in \mathfrak{B}(\mathcal{H}_R(\mathfrak{Q}))$. Hence, for any $j \in J$, we have

$$\begin{aligned} y_j = Wz_j &= (S^{-1}V + U(I_{\mathcal{H}} - V^*S^{-1}V))z_j \\ &= S^{-1}Vz_j + U(I_{\mathcal{H}_R} - V^*S^{-1}V)z_j \\ &= S^{-1}x_j + Uz_j - UV^*S^{-1}Vz_j \\ &= S^{-1}x_j + Uz_j - UV^*S^{-1}x_j. \end{aligned}$$

(⇐). Assume that

$$y_j = S^{-1}x_j + Uz_j - UV^*S^{-1}j_j, \quad \text{for all } j \in J.$$

where $U \in \mathfrak{B}(\mathcal{H}_R(\mathfrak{Q}))$ is a right linear operator. Next to prove that $\{y_j\}_{j\in J}$ is an arbitrary dual frame for $\{x_j\}_{j\in J}$, note that V is the pre-frame operator of $\{x_j\}_{j\in J}$, then

$$\begin{aligned} y_j &= S^{-1}x_j + Uz_j - UV^*S^{-1}x_j \\ &= S^{-1}Vz_j + Uz_j - UV^*S^{-1}Vz_j \\ &= (S^{-1}V + U - UV^*S^{-1}V)z_j. \end{aligned}$$

It is easy to prove that $\{y_j\}_{j\in J}$ is a Bessel sequence for $\mathcal{H}_R(\mathfrak{Q})$. Let W denote the pre-frame operator of $\{y_j\}_{j\in J}$, then

$$W = S^{-1}V + U - UV^*S^{-1}V = S^{-1}V + U(I_{\mathcal{H}} - V^*S^{-1}V).$$

Thus, W is a bounded right linear left inverse of V^* by Lemma 3.4. We conclude that $\{y_j\}_{j\in J}$ is what we are looking for by Theorem 3.3. □

At the end of this section, we give some characterizations of dual frames by taking advantage of operator equations.

Theorem 3.5. *Let $\{x_j\}_{j\in J}$ be a Parseval frame for an R-Q-Hilbert space $\mathcal{H}_R(\mathfrak{Q})$, and $\{y_j\}_{j\in J}$ be a frame for $\mathcal{H}_R(\mathfrak{Q})$. Use T_x, T_y to denote the pre-frame operators of $\{x_j\}_{j\in J}$ and $\{y_j\}_{j\in J}$, respectively. Suppose that P_x is the orthogonal projection: $l^2(\mathfrak{Q}) \to R(T_x^*)$, then $\{y_j\}_{j\in J}$ is a dual frame of $\{x_j\}_{j\in J}$ if and only if $P_xT_y^* = T_x^*$.*

Proof. (⇒). Note that $\{x_j\}_{j\in J}$ is a Parseval frame for $\mathcal{H}_R(\mathfrak{Q})$, $T_xT_x^* = I_{\mathcal{H}_R}$ by Lemma 3.2. Hence $T_x^*T_x = P_x$. If $\{y_j\}_{j\in J}$ is a dual frame of $\{x_j\}_{j\in J}$, then $T_xT_y^* = I_{\mathcal{H}_R}$. It follows that $P_xT_y^* = T_x^*T_xT_y^* = T_x^*$.

(⇐). Since $T_x^* = P_xT_y^* = T_x^*T_xT_y^*$, $T_x^* - T_x^*T_xT_y^* = 0$, i.e., $T_x^*(I_{\mathcal{H}_R} - T_xT_y^*) = 0$. By Theorem 3.12 in [19], we know that T_x^* is onto, so we have $T_xT_y^* = I_{\mathcal{H}_R}$. Therefore, $\{y_j\}_{j\in J}$ is a dual frame of $\{x_j\}_{j\in J}$. □

Theorem 3.6. *Let $\{x_j\}_{j\in J}$ and $\{y_j\}_{j\in J}$ be frames for an R-Q-Hilbert space $\mathcal{H}_R(\mathfrak{Q})$, and their pre-frame operators be T_x and T_y, respectively. If P_x is an orthogonal projection from $l^2(\mathfrak{Q})$ to $R(T_x^*)$, then $\{y_j\}_{j\in J}$ is a dual frame of $\{x_j\}_{j\in J}$ if and only if $P_xT_y^* = T_x^*S_x^{-1}$, where S_x denotes the frame operator of $\{x_j\}_{j\in J}$.*

Proof. In accordance with Theorem 3.9 in [19], we have $\{S_x^{-\frac{1}{2}} f_x\}_{j\in J}$ is a Parseval frame for $\mathcal{H}_R(\mathfrak{Q})$ if $\{x_j\}_{j\in J}$ is a frame for $\mathcal{H}_R(\mathfrak{Q})$. The rest is similar to the proof of Theorem 3.5. □

4. Sums of frames in R-Q-Hilbert spaces

In application, constructing new frames is one of the active research directions. In [24], the authors debated the constructions of frames by means of the sum of frames in Hilbert spaces. Inspired by their work, in this section, we will apply the pre-frame operators to discuss the finite sum of frames in R-Q-Hilbert spaces, which generalize the corresponding results on general frames in Hilbert spaces. At first, we give an example.

Example 4.1. *Let $\mathcal{N} = \{z_j\}_{j\in J}$ be a Hilbert basis for an R-Q-Hilbert space $\mathcal{H}_R(\mathfrak{Q})$. Define two sequences $\{x_j\}_{j\in J}$, $\{y_j\}_{j\in J} \subset \mathcal{H}_R(\mathfrak{Q})$ by*

$$\begin{cases} x_1 = z_1, \\ x_j = z_{j-1}, \quad \text{for } j \geq 2,\ j \in J, \end{cases}$$

and $y_j = -x_j$ for all $j \in J$. Through simple calculation, we know that $\{x_j\}_{j\in J}$ and $\{y_j\}_{j\in J}$ are frames for $\mathcal{H}_R(\mathfrak{Q})$, but $\{x_j + y_j\}_{j\in J}$ is not frame for $\mathcal{H}_R(\mathfrak{Q})$. Define $x_j = z_j$ for every $j \in J$ and $y_j = \frac{1}{j}z_j$ for every $j \in J$, then $\{x_j + y_j\}_{j\in J}$ is frame for $\mathcal{H}_R(\mathfrak{Q})$. However, $\{y_j\}_{j\in J}$ is not a frame but a Bessel sequence for $\mathcal{H}_R(\mathfrak{Q})$.

By Example 4.1, it shows that the sum of frames for $\mathcal{H}_R(\mathfrak{Q})$ is not a new frame. It is natural to ask for some proper conditions, and when the conditions have been established, the sum of frames is a frame in R-Q-Hilbert spaces. The following theorems give some sufficient conditions on the frame $\{x_j\}_{j\in J}$ and Bessel sequence $\{y_j\}_{j\in J}$, which lead to new frames of the form $\{\alpha x_j + \beta y_j\}_{j\in J}$ or $\{\alpha_j x_j + \beta_j y_j\}_{j\in J}$.

Theorem 4.1. *Suppose that $\{x_j\}_{j\in J}$ is a frame for an R-Q-Hilbert space $\mathcal{H}_R(\mathfrak{Q})$, and its frame bounds are A and B; $\{y_j\}_{j\in J}$ is a Bessel sequence for $\mathcal{H}_R(\mathfrak{Q})$ and its Bessel bound is B_1. If A, B and B_1 satisfy*

$$A|\alpha|^2 - 2B_1|\beta|^2 > 0$$

for non-zero constants $\alpha, \beta \in \mathfrak{Q}$, then a new frame of the form $\{\alpha x_j + \beta y_j\}_{j\in J}$ can be constructed for $\mathcal{H}_R(\mathfrak{Q})$.

Proof. To prove that $\{\alpha x_j + \beta y_j\}_{j\in J}$ is a newly constructed frame for $\mathcal{H}_R(\mathfrak{Q})$, we must find the upper and lower bounds of $\{\alpha x_j + \beta y_j\}_{j\in J}$. For $\forall f \in \mathcal{H}_R(\mathfrak{Q})$, we have

$$\begin{aligned} \sum_{j\in J} \left|\langle(\alpha x_j + \beta y_j)|f\rangle\right|^2 &\leq \sum_{j\in J} \left(\left|\langle\alpha x_j|f\rangle\right| + \left|\langle\beta y_j|f\rangle\right|\right)^2 \\ &= \sum_{j\in J} \left|\langle\alpha x_j|f\rangle\right|^2 + \sum_{j\in J} \left|\langle\beta y_j|f\rangle\right|^2 + 2\sum_{j\in J} \left|\langle\alpha x_j|f\rangle\right| \left|\langle\beta y_j|f\rangle\right| \end{aligned}$$

$$
\begin{aligned}
&\le \sum_{j\in J}\left|\langle \alpha x_j|f\rangle\right|^2+\sum_{j\in J}\left|\langle \beta y_j|f\rangle\right|^2+2\left(\sum_{j\in J}\left|\langle \alpha x_j|f\rangle\right|^2\right)^{\frac{1}{2}}\left(\sum_{j\in J}\left|\langle \beta y_j|f\rangle\right|^2\right)^{\frac{1}{2}}\\
&\le 2\sum_{j\in J}\left|\langle \alpha x_j|f\rangle\right|^2+2\sum_{j\in J}\left|\langle \beta y_j|f\rangle\right|^2\\
&= 2|\alpha|^2\sum_{j\in J}\left|\langle x_j|f\rangle\right|^2+2|\beta|^2\sum_{j\in J}\left|\langle y_j|f\rangle\right|^2\\
&\le 2(|\alpha|^2B+|\beta|^2B_1)\|f\|^2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\sum_{j\in J}\left|\langle \alpha x_j|f\rangle\right|^2 &= \sum_{j\in J}\left|\langle(\alpha x_j+\beta y_j)|f\rangle-\langle \beta y_j|f\rangle\right|^2\\
&\le 2\sum_{j\in J}\left|\langle(\alpha x_j+\beta y_j)|f\rangle\right|^2+2\sum_{j\in J}\left|\langle \beta y_j|f\rangle\right|^2,
\end{aligned}
$$

and it follows that

$$
\begin{aligned}
2\sum_{j\in J}\left|\langle(\alpha x_j+\beta y_j)|f\rangle\right|^2 &\ge \sum_{j\in J}\left|\langle \alpha x_j|f\rangle\right|^2-2\sum_{j\in J}\left|\langle \beta y_j|f\rangle\right|^2\\
&= |\alpha|^2\sum_{j\in J}\left|\langle x_j|f\rangle\right|^2-2|\beta|^2\sum_{j\in J}\left|\langle y_j|f\rangle\right|^2\\
&\ge (|\alpha|^2A-2|\beta|^2B_1)\|f\|^2.
\end{aligned}
$$

Thus, we have

$$
\frac{1}{2}(|\alpha|^2A-2|\beta|^2B_1)\|f\|^2\le \sum_{j\in J}\left|\langle(\alpha x_j+\beta y_j)|f\rangle\right|^2\le 2(|\alpha|^2B+|\beta|^2B_1)\|f\|^2.
$$

Observe that $A|\alpha|^2-2B_1|\beta|^2>0$, then $\{\alpha x_j+\beta y_j\}_{j\in J}$ is a newly constructed frame for $\mathcal{H}_R(\mathfrak{Q})$. □

Theorem 4.2. *Let $\mathcal{N}=\{z_j\}_{j\in J}$ be a Hilbert basis for an R-Q-Hilbert space $\mathcal{H}_R(\mathfrak{Q})$. Suppose that $\{x_j\}_{j\in J}$ is a frame for $\mathcal{H}_R(\mathfrak{Q})$, and its frame bounds are A and B; $\{y_j\}_{j\in J}$ is a Bessel sequence for $\mathcal{H}_R(\mathfrak{Q})$ and its pre-frame operator is V. For any two families $\{\alpha_j\}_{j\in J}$ and $\{\beta_j\}_{j\in J}$ ($\alpha_j,\beta_j\in\mathfrak{Q},\ \ j\in J$), if*

$$
\|V\|^2<\frac{A\inf_{j\in J}|\alpha_j|^2}{2\sup_{j\in J}|\beta_j|^2},
$$

then $\{\alpha_jx_j+\beta_jy_j\}_{j\in J}$ is a newly constructed frame for $\mathcal{H}_R(\mathfrak{Q})$.

Proof. For all $f\in\mathcal{H}_R(\mathfrak{Q})$, we have

$$
\sum_{j\in J}\left|\langle(\alpha x_j+\beta y_j)|f\rangle\right|^2\le 2\left(\sum_{j\in J}\left|\langle \alpha x_j|f\rangle\right|^2+\sum_{j\in J}\left|\langle \beta y_j|f\rangle\right|^2\right)
$$

$$
\begin{aligned}
&\le 2\left(\left(\sup_{j\in J}|\alpha_j|^2\right)\sum_{j\in J}\left|\langle x_j\middle|f\rangle\right|^2+\left(\sup_{j\in J}|\beta_j|^2\right)\sum_{j\in J}\left|\langle y_j\middle|f\rangle\right|^2\right)\\
&= 2\left(\left(\sup_{j\in J}|\alpha_j|^2\right)\sum_{j\in J}\left|\langle x_j\middle|f\rangle\right|^2+\left(\sup_{j\in J}|\beta_j|^2\right)\sum_{j\in J}\left|\langle Vz_j\middle|f\rangle\right|^2\right)\\
&\le 2\left(\left(\sup_{j\in J}|\alpha_j|^2\right)B\|f\|^2+\left(\sup_{j\in J}|\beta_j|^2\right)\|V^*f\|^2\right)\\
&\le 2\left(\left(\sup_{j\in J}|\alpha_j|^2\right)B+\left(\sup_{j\in J}|\beta_j|^2\right)\|V\|^2\right)\|f\|^2.
\end{aligned}
$$

On the other hand, since

$$
\begin{aligned}
\sum_{j\in J}\left|\langle \alpha x_j\middle|f\rangle\right|^2 &= \sum_{j\in J}\left|\langle(\alpha x_j+\beta y_j)\middle|f\rangle-\langle\beta y_j\middle|f\rangle\right|^2\\
&\le 2\sum_{j\in J}\left|\langle(\alpha x_j+\beta y_j)\middle|f\rangle\right|^2+2\sum_{j\in J}\left|\langle\beta y_j\middle|f\rangle\right|^2,
\end{aligned}
$$

then,

$$
\begin{aligned}
2\sum_{j\in J}\left|\langle(\alpha x_j+\beta y_j)\middle|f\rangle\right|^2 &\ge \sum_{j\in J}\left|\langle\alpha x_j\middle|f\rangle\right|^2-2\sum_{j\in J}\left|\langle\beta y_j\middle|f\rangle\right|^2\\
&\ge \left(\inf_{j\in J}|\alpha_j|^2\right)\sum_{j\in J}\left|\langle x_j\middle|f\rangle\right|^2-2\left(\sup_{j\in J}|\beta_j|^2\right)\sum_{j\in J}\left|\langle Vz_j\middle|f\rangle\right|^2\\
&= \left(\inf_{j\in J}|\alpha_j|^2\right)\sum_{j\in J}\left|\langle x_j\middle|f\rangle\right|^2-2\left(\sup_{j\in J}|\beta_j|^2\right)\|V^*f\|^2\\
&\ge \left(A\left(\inf_{j\in J}|\alpha_j|^2\right)-2\left(\sup_{j\in J}|\beta_j|^2\right)\|V\|^2\right)\|f\|^2.
\end{aligned}
$$

If $\|V\|^2 < \frac{A\inf_{j\in J}|\alpha_j|^2}{2\sup_{j\in J}|\beta_j|^2}$, then $A\left(\inf_{j\in J}|\alpha_j|^2\right)-2\left(\sup_{j\in J}|\beta_j|^2\right)\|V\|^2 > 0$, so $\{\alpha_j x_j+\beta_j y_j\}_{j\in J}$ is a newly constructed frame for $\mathcal{H}_R(\mathfrak{Q})$. □

From Theorem 4.2, we have the following corollary.

Corollary 4.1. *Let $\mathcal{N}=\{z_j\}_{j\in J}$ be a Hilbert basis for an R-Q-Hilbert space $\mathcal{H}_R(\mathfrak{Q})$. Suppose that $\{x_j\}_{j\in J}$ is a frame for $\mathcal{H}_R(\mathfrak{Q})$, and its frame bounds are A and B; $\{y_j\}_{j\in J}$ is a Bessel sequence for $\mathcal{H}_R(\mathfrak{Q})$ with the pre-frame operator V. If $\|V\|^2<\frac{A}{2}$, then $\{x_j+y_j\}_{j\in J}$ is a new frame for $\mathcal{H}_R(\mathfrak{Q})$.*

Theorem 4.3. *Let $\mathcal{N}=\{z_j\}_{j\in J}$ be a Hilbert basis for an R-Q-Hilbert space $\mathcal{H}_R(\mathfrak{Q})$. Assume that $\{x_j\}_{j\in J}$ and $\{y_j\}_{j\in J}$ are both frames for $\mathcal{H}_R(\mathfrak{Q})$, and V_x, V_y are their pre-frame operators, respectively. If the condition $V_yV_x^*=0$ is met, then $\{x_j+y_j\}_{j\in J}$ is a newly constructed frame for $\mathcal{H}_R(\mathfrak{Q})$. Moreover, if $\{x_j\}_{j\in J}$ and $\{y_j\}_{j\in J}$ are both one-tight frames for $\mathcal{H}_R(\mathfrak{Q})$ and $V_yV_x^*=0$, then $\{x_j+y_j\}_{j\in J}$ is a two-tight frame for $\mathcal{H}_R(\mathfrak{Q})$.*

Proof. Note that the pre-frame operators of $\{x_j\}_{j\in J}$ and $\{y_j\}_{j\in J}$ are V_x and V_y, respectively. It can be seen from their definitions

$$x_j = V_f z_j \text{ and } y_j = V_g z_j, \text{ for all } j \in J.$$

Hence $x_j + y_j = V_x z_j + V_y z_j = (V_x + V_y)z_j$ for any $j \in J$. To show $\{x_j + y_j\}_{j\in J}$ is a frame for $\mathcal{H}_R(\mathfrak{Q})$, it is sufficient to show $V_x + V_y$ is onto by Theorem 3.1. Using $V_y V_x^* = 0$, we have

$$(V_x + V_y)V_x^* = V_x V_x^* + V_y V_x^* = V_x V_x^*.$$

Once again, to utilize the invertibility of $V_x V_x^*$, for an arbitrary element g in $\mathcal{H}_R(\mathfrak{Q})$, taking $f = V_x^*(V_x V_x^*)^{-1}g$, undoubtedly, $f \in \mathcal{H}_R(\mathfrak{Q})$ satisfies

$$(V_x + V_y)f = (V_x + V_y)V_x^*(V_x V_x^*)^{-1}g = (V_x V_x^*)(V_x V_x^*)^{-1}g = g.$$

Thus $V_x + V_y$ is onto.

Especially, if $\{x_j\}_{j\in J}$ and $\{y_j\}_{j\in J}$ are both one-tight frames for $\mathcal{H}_R(\mathfrak{Q})$, and their pre-frame satisfies operators $V_y V_x^* = 0$, then $\{x_j + y_j\}_{j\in J}$ is a frame for $\mathcal{H}_R(\mathfrak{Q})$ by the proof of the previous part. Letting $S_{\langle x+y\rangle}$ denote the frame operator of $\{x_j + y_j\}_{j\in J}$, for any $f \in \mathcal{H}_R(\mathfrak{Q})$, we know that

$$\begin{aligned} S_{\langle x+y\rangle} f &= \sum_{j\in J}(x_j + y_j)\langle (x_j + y_j)|f\rangle \\ &= \sum_{j\in J} x_j\langle x_j|f\rangle + \sum_{j\in J} x_j\langle y_j|f\rangle + \sum_{j\in J} y_j\langle x_j|f\rangle + \sum_{j\in J} y_j\langle y_j|f\rangle \\ &= V_x V_x^* f + \sum_{j\in J} V_x z_j\langle V_y z_j|f\rangle + \sum_{j\in J} V_y z_j\langle V_x z_j|f\rangle + V_y V_y^* f \\ &= 2f + V_x V_y^* f + V_y V_x^* f + = 2f. \end{aligned}$$

Thus,

$$\sum_{j\in J} \left|\langle (x_j + y_j)\middle|f\rangle\right|^2 = \langle S_{\langle x+y\rangle} f|f\rangle = \langle 2f|f\rangle = 2\|f\|^2.$$

It follows that $\{x_j + y_j\}_{j\in J}$ is a two-tight frame for $\mathcal{H}_R(\mathfrak{Q})$. □

Extend the number of frames to a finite number and we have the following corollary.

Corollary 4.2. *Let $\mathcal{N} = \{z_j\}_{j\in J}$ be a Hilbert basis for an R-Q-Hilbert space $\mathcal{H}_R(\mathfrak{Q})$. Suppose that $\{x_{1,j}\}_{j\in J}$, $\{x_{2,j}\}_{j\in J}$, $\cdots$, $\{x_{l,j}\}_{j\in J}$ are frames for $\mathcal{H}_R(\mathfrak{Q})$, and V_1, V_2, $\cdots$, V_l are pre-frame operators associated with $\{x_{1,j}\}_{j\in J}$, $\{x_{2,j}\}_{j\in J}$, $\cdots$, $\{x_{l,j}\}_{j\in J}$, respectively. If $V_m V_n^* = 0$, $m, n = 1, 2, \cdots, l$, then $\{x_{1,j} + x_{2,j} + \cdots + x_{l,j}\}_{j\in J}$ is a frame for $\mathcal{H}_R(\mathfrak{Q})$. Moreover, if $\{x_{1,j}\}_{j\in J}$, $\{x_{2,j}\}_{j\in J}$, $\cdots$, $\{x_{l,j}\}_{j\in J}$ are one-tight frames for $\mathcal{H}_R(\mathfrak{Q})$ and $V_m V_n^* = 0$, $m, n = 1, 2, \cdots, l$, then $\{x_{1,j} + x_{2,j} + \cdots + x_{l,j}\}_{j\in J}$ is an l-tight frame for $\mathcal{H}_R(\mathfrak{Q})$.*

More generally, we have the following theorem.

Theorem 4.4. *Let $\mathcal{N} = \{z_j\}_{j\in J}$ be a Hilbert basis for an R-Q-Hilbert space $\mathcal{H}_R(\mathfrak{Q})$. Assume that $\{x_j\}_{j\in J}$ and $\{y_j\}_{j\in J}$ are both frames for $\mathcal{H}_R(\mathfrak{Q})$, and V_x, V_y are their pre-frame operators, respectively, and satisfy $V_y V_x^* = 0$. If $P, Q \in \mathfrak{B}(\mathcal{H}_R(\mathfrak{Q}))$, and P or Q is onto, then $\{Px_j + Qy_j\}_{j\in J}$ is a frame for $\mathcal{H}_R(\mathfrak{Q})$.*

Proof. Note that the pre-frame operators of $\{x_j\}_{j\in J}$ and $\{y_j\}_{j\in J}$ are V_x and V_y, respectively. It can be seen from their definitions

$$x_j = V_x z_j \text{ and } y_j = V_y z_j, \text{ for all } j \in J.$$

After a simple calculation,

$$Px_j + Qy_j = PV_x z_j + QV_y z_j = (PV_x + QV_y)z_j.$$

To show $\{Px_j + Qy_j\}_{j\in J}$ is a frame for $\mathcal{H}_R(\mathfrak{Q})$, it is sufficient to show that the operator $PV_x + QV_y$ is onto by Theorem 3.1. By Lemma 3.2, we know that $V_x V_x^*$ is invertible. Without loss of generality, let us suppose P is onto. For an arbitrary element $g \in \mathcal{H}_R(\mathfrak{Q})$, there is always $f \in \mathcal{H}_R(\mathfrak{Q})$ meets $Pf = g$. Thus, for any $g \in \mathcal{H}_R(\mathfrak{Q})$ and taking $h = V_x^*(V_x V_x^*)^{-1} f$, undoubtedly, $h \in \mathcal{H}_R(\mathfrak{Q})$ satisfies

$$\begin{aligned}(PV_x + QV_y)h &= (PV_x + QV_y)V_x^*(V_x V_x^*)^{-1} f \\ &= PV_x V_x^*(V_x V_x^*)^{-1} f + QV_y V_x^*(V_x V_x^*)^{-1} f \\ &= PV_x V_x^*(V_x V_x^*)^{-1} f = Pf = g.\end{aligned}$$

So $PV_x + QV_y$ is onto. □

In particular, to two one-tight frames in an R-Q-Hilbert space, a necessary and sufficient condition is given, for which the new frame is tight.

Theorem 4.5. *Let $\mathcal{N} = \{z_j\}_{j\in J}$ be a Hilbert basis for an R-Q-Hilbert space $\mathcal{H}_R(\mathfrak{Q})$. Assume that $\{x_j\}_{j\in J}$ and $\{y_j\}_{j\in J}$ are two one-tight frames for $\mathcal{H}_R(\mathfrak{Q})$, V_x, V_y are their pre-frame operators, respectively, and satisfy $V_y V_x^* = 0$. Let $U_1, U_2 \in \mathfrak{B}(\mathcal{H}_R(\mathfrak{Q}))$, then $\{U_1 x_j + U_2 y_j\}_{j\in J}$ is a λ-tight frame for $\mathcal{H}_R(\mathfrak{Q})$ if and only if $U_1 U_1^* + U_2 U_2^* = \lambda I_{\mathcal{H}_R}$.*

Proof. Note that V_x and V_y are pre-frame operators associated with $\{x_j\}_{j\in J}$ and $\{y_j\}_{j\in J}$, respectively. For every $j \in J$, we have

$$x_j = V_x z_j \text{ and } y_j = V_y z_j.$$

For any $f \in \mathcal{H}_R(\mathfrak{Q})$, we have

$$\begin{aligned}\sum_{j\in J} \left|\langle (U_1 x_j + U_2 y_j) | f\rangle\right|^2 &= \sum_{j\in J} \left\langle f \middle| (U_1 x_j + U_2 y_j)\right\rangle \left\langle (U_1 x_j + U_2 y_j) \middle| f\right\rangle \\ &= \sum_{j\in J} \left(\left\langle f \middle| U_1 x_j\right\rangle + \left\langle f \middle| U_2 y_j\right\rangle\right)\left(\left\langle U_1 x_j \middle| f\right\rangle + \left\langle U_2 y_j \middle| f\right\rangle\right) \\ &= \sum_{j\in J} \left|\langle U_1 x_j | f\rangle\right|^2 + \sum_{j\in J} \left\langle f \middle| U_1 x_j\right\rangle \left\langle U_2 y_j \middle| f\right\rangle \\ &\quad + \sum_{j\in J} \left\langle f \middle| U_2 y_j\right\rangle \left\langle U_1 x_j \middle| f\right\rangle + \sum_{j\in J} \left|\langle U_2 y_j | f\rangle\right|^2 \\ &= \sum_{j\in J} \left|\langle x_j | U_1^* f\rangle\right|^2 + \sum_{j\in J} \left\langle U_1^* f \middle| x_j\right\rangle \left\langle y_j \middle| U_2^* f\right\rangle \\ &\quad + \sum_{j\in J} \left\langle U_2^* f \middle| y_j\right\rangle \left\langle x_j \middle| U_1^* f\right\rangle + \sum_{j\in J} \left|\langle y_j | U_2^* f\rangle\right|^2\end{aligned}$$

$$
\begin{aligned}
&= \|U_1^* f\|^2 + \|U_2^* f\|^2 + \sum_{j\in J} \left\langle V_x^* U_1^* f \middle| z_j \right\rangle \left\langle z_j \middle| V_y^* U_2^* f \right\rangle \\
&\quad + \sum_{j\in J} \left\langle V_y^* U_2^* f \middle| z_j \right\rangle \left\langle z_j \middle| V_x^* U_1^* f \right\rangle \\
&= \|U_1^* f\|^2 + \|U_2^* f\|^2 + \langle V_x^* U_1^* f \big| V_y^* U_2^* f\rangle + \langle V_y^* U_2^* f \big| V_x^* U_1^* f\rangle \\
&= \|U_1^* f\|^2 + \|U_2^* f\|^2 + \langle V_y V_f^* U_1^* f \big| U_2^* f\rangle + \langle U_2^* f \big| V_y V_x^* U_1^* f\rangle \\
&= \|U_1^* f\|^2 + \|U_2^* f\|^2 = \langle (U_1 U_1^* + U_2 U_2^*) f \big| f\rangle.
\end{aligned}
$$

It follows that $\{U_1 x_j + U_2 y_j\}_{j\in J}$ is a λ-tight frame for $\mathcal{H}_R(\mathfrak{Q})$ if and only if $U_1 U_1^* + U_2 U_2^* = \lambda I_{\mathcal{H}_R}$. □

In the end, a necessary and sufficient condition is given, for which the finite sum of frames to be a frame in an R-Q-Hilbert space.

Theorem 4.6. *Let $\{x_{1,j}\}_{j\in J}$, $\{x_{2,j}\}_{j\in J}$, $\cdots$, $\{x_{l,j}\}_{j\in J}$ be frames for an R-Q-Hilbert space $\mathcal{H}_R(\mathfrak{Q})$, and A_i and B_i be the lower and upper bounds of the frame $\{x_{i,J}\}_{j\in J}$ for each $i \in \{1, 2, \cdots, l\}$, respectively. Let $\{\alpha_1, \alpha_2, \cdots, \alpha_l\}$ ($\alpha_i \in \mathfrak{Q}$, $i = 1, 2, \cdots, l$) be any given scalars, then $\left\{\sum_{i=1}^{l} \alpha_i x_{i,j}\right\}_{j\in J}$ is a frame for $\mathcal{H}_R(\mathfrak{Q})$ if and only if there exists an $M > 0$ and some $p \in \{1, 2, \cdots, l\}$ such that*

$$
M \sum_{j\in J} \left|\langle x_{p,j} \big| f\rangle\right|^2 \le \sum_{j\in J} \left|\left\langle \sum_{i=1}^{l} \alpha_i x_{i,j} \middle| f \right\rangle\right|^2, \quad f \in \mathcal{H}_R(\mathfrak{Q}).
$$

Proof. ($\Rightarrow$). Note that $\{x_{i,j}\}_{j\in J}$ ($i = 1, 2, \cdots, l$) is a frame for $\mathcal{H}_R(\mathfrak{Q})$ with frame bounds A_i and B_i. We have for some $p \in \{1, 2, \cdots, l\}$ that

$$
A_p \|f\|^2 \le \sum_{j\in J} \left|\langle x_{p,j} \big| f\rangle\right|^2 \le B_p \|f\|^2, \quad \forall f \in \mathcal{H}_R(\mathfrak{Q}).
$$

It follows that

$$
\frac{1}{B_p} \sum_{j\in J} \left|\langle x_{p,j} \big| f\rangle\right|^2 \le \|f\|^2, \ \ \forall f \in \mathcal{H}_R(\mathfrak{Q}).
$$

Assume that $\left\{\sum_{i=1}^{l} \alpha_i x_{i,j}\right\}_{j\in J}$ is a frame for $\mathcal{H}_R(\mathfrak{Q})$ with the lower and upper bounds A and B, respectively. We have for each $f \in \mathcal{H}_R(\mathfrak{Q})$ that

$$
A\|f\|^2 \le \sum_{j\in J} \left|\left\langle \sum_{i=1}^{l} \alpha_i x_{i,j} \middle| f \right\rangle\right|^2 \le B\|f\|^2,
$$

so

$$
\|f\|^2 \le \frac{1}{A} \sum_{j\in J} \left|\left\langle \sum_{i=1}^{l} \alpha_i x_{i,j} \middle| f \right\rangle\right|^2,
$$

for $f \in \mathcal{H}_R(\mathfrak{Q})$. Therefore, we can conclude that

$$
\frac{A}{B_p} \sum_{j\in J} \left|\langle x_{p,j} \big| f\rangle\right|^2 \le \sum_{j\in J} \left|\left\langle \sum_{i=1}^{l} \alpha_i x_{i,j} \middle| f \right\rangle\right|^2,
$$

for $f \in \mathcal{H}_R(\mathfrak{Q})$. Taking $M = \frac{A}{B_p} > 0$, we have for any $f \in \mathcal{H}_R(\mathfrak{Q})$ that

$$M \sum_{j\in J} \left|\langle x_{p,j}|f\rangle\right|^2 \le \sum_{j\in J} \left|\left\langle \sum_{i=1}^{l} \alpha_i x_{i,j} \middle| f \right\rangle\right|^2 .$$

(⇐). For each $i \in \{1, 2, \cdots, l\}$, let $M > 0$ be a constant such that for some $p \in \{1, 2, \cdots, l\}$,

$$M \sum_{j\in J} \left|\langle x_{p,j}|f\rangle\right|^2 \le \sum_{j\in J} \left|\left\langle \sum_{i=1}^{l} \alpha_i x_{i,j} \middle| f \right\rangle\right|^2 , \quad f \in \mathcal{H}_R(\mathfrak{Q}).$$

Since $\{x_{i,j}\}_{j\in J}$ $(i = 1, 2, \cdots, l)$ is a frame, we have

$$A_p\|f\|^2 \le \sum_{j\in J} \left|\langle x_{p,j}|f\rangle\right|^2 \le B_p\|f\|^2, \quad \forall f \in \mathcal{H}_R(\mathfrak{Q}),$$

so

$$MA_p\|f\|^2 \le M \sum_{j\in J} \left|\langle x_{p,j}|f\rangle\right|^2 \le \sum_{j\in J} \left|\left\langle \sum_{i=1}^{l} \alpha_i x_{i,j} \middle| f \right\rangle\right|^2 , \ \forall f \in \mathcal{H}_R(\mathfrak{Q}),$$

and the lower bound of $\left\{\sum_{i=1}^{l} \alpha_i x_{i,j}\right\}_{j\in J}$ exists. Next, we look for the upper bound of $\left\{\sum_{i=1}^{l} \alpha_i x_{i,j}\right\}_{j\in J}$, we will show that $\left\{\sum_{i=1}^{l} a_i x_{i,j}\right\}_{j\in J}$ is a Bessel sequence for $\mathcal{H}_R(\mathfrak{Q})$, we have for all $f \in \mathcal{H}_R(\mathfrak{Q})$ that

$$\begin{aligned}
\sum_{j\in J} \left|\left\langle \sum_{i=1}^{l} \alpha_i x_{i,j} \middle| f \right\rangle\right|^2 &\le \sum_{j\in J} l \left(\sum_{i=1}^{l} \left|\left\langle \alpha_i x_{i,j} \middle| f \right\rangle\right|^2 \right) \\
&= l \sum_{i=1}^{l} \left(|\alpha_i|^2 \sum_{j\in J} \left|\left\langle x_{i,j} \middle| f \right\rangle\right|^2 \right) \\
&= l(\max_{1\le i\le l}\{|\alpha_i|^2\}) \left(\sum_{i=1}^{l} B_i \right) \|f\|^2 \\
&\le l^2 \max_{1\le i\le l}\{|\alpha_i|^2\} \max_{1\le i\le l}\{B_i\} \|f\|^2.
\end{aligned}$$

Therefore, $\left\{\sum_{i=1}^{l} a_i x_{i,j}\right\}_{j\in J}$ is a frame for $\mathcal{H}_R(\mathfrak{Q})$. □

5. Conclusions

Frames in Q-Hilbert spaces both retain the frame properties, and also have some advantages, such as simple structure for approximation. In this paper, the definition of pre-frame operator was introduced. We characterized the Hilbert (orthonormal) bases, frames, dual frames and Riesz bases, and obtained the accurate expressions of all dual frames of a given frame by taking advantage of pre-frame operators. We also discussed the constructions of frames with the help of the pre-frame operators, and gained some more general methods to construct new frames. The obtained results further enriched the frame theory in Q-Hilbert spaces.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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