AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(12): 28818–28832.
DOI: 10.3934/math.20231477
Received: 26 June 2023
Revised: 12 October 2023
Accepted: 15 October 2023
Published: 24 October 2023

Research article

New facts related to dilation factorizations of Kronecker products of matrices

Yongge Tian and Ruixia Yuan*

College of Business and Economics, Shanghai Business School, Shanghai 201400, China

* **Correspondence:** Email: yuanrx0@163.com.

Abstract: The Kronecker product of two matrices is known as a special algebraic operation of two arbitrary matrices in the computational aspect of matrix theory. This kind of matrix operation has some interesting and striking operation properties, one of which is given by $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$ and is often called the mixed-product equality. In view of this equality, the Kronecker product $A_1 \otimes A_2$ of any two matrices can be rewritten as the dilation factorization $A_1 \otimes A_2 = (A_1 \otimes I_{m_2})(I_{n_1} \otimes A_2)$, and the Kronecker product $A_1 \otimes A_2 \otimes A_3$ can be rewritten as the dilation factorization $A_1 \otimes A_2 \otimes A_3 = (A_1 \otimes I_{m_2} \otimes I_{m_3})(I_{n_1} \otimes A_2 \otimes I_{m_3})(I_{n_1} \otimes I_{n_2} \otimes A_3)$. In this article, we proposed a series of concrete problems regarding the dilation factorizations of the Kronecker products of two or three matrices, and established a collection of novel and pleasing equalities, inequalities, and formulas for calculating the ranks, dimensions, orthogonal projectors, and ranges related to the dilation factorizations. We also present a diverse range of interesting results on the relationships among the Kronecker products $I_{m_1} \otimes A_2 \otimes A_3$, $A_1 \otimes I_{m_2} \otimes A_3$ and $A_1 \otimes A_2 \otimes I_{m_3}$.

Keywords: dilation factorization; dimension; Kronecker product; orthogonal projector; range; rank
Mathematics Subject Classifications: 15A09, 15A10, 15A24

1. Introduction

Throughout this article, $\mathbb{C}^{m\times n}$ denotes the collection of all $m \times n$ matrices over the field of complex numbers, A^* denotes the conjugate transpose of $A \in \mathbb{C}^{m\times n}$, $r(A)$ denotes the rank of $A \in \mathbb{C}^{m\times n}$, $\mathscr{R}(A)$ denotes the range of $A \in \mathbb{C}^{m\times n}$, I_m denotes the identity matrix of order m, and $[\,A,\ B\,]$ denotes a row block matrix consisting of $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{m\times p}$. The Moore-Penrose generalized inverse of $A \in \mathbb{C}^{m\times n}$, denoted by $A^\dagger$, is the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying the four Penrose equations

$$(1)\ AXA = A,\ \ (2)\ XAX = X,\ \ (3)\ (AX)^* = AX,\ \ (4)\ (XA)^* = XA.$$

Further, we denote by

$$P_A = AA^\dagger, \quad E_A = I_m - AA^\dagger \tag{1.1}$$

the two orthogonal projectors induced from $A \in \mathbb{C}^{m\times n}$. For more detailed information regarding generalized inverses of matrices, we refer the reader to [2–4].

Recall that the well-known Kronecker product of any two matrices $A = (a_{ij}) \in \mathbb{C}^{m\times n}$ and $B = (b_{ij}) \in \mathbb{C}^{p\times q}$ is defined to be

$$A \otimes B = (a_{ij}B) = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix} \in \mathbb{C}^{mp\times nq}.$$

The Kronecker product, named after German mathematician Leopold Kronecker, was classified to be a special kind of matrix operation, which has been regarded as an important matrix operation and mathematical technique. This product has wide applications in system theory, matrix calculus, matrix equations, system identification and more (cf. [1, 5–14, 16, 18–21, 23–25, 27, 28, 33, 34]). It has been known that the matrices operations based on Kronecker products have a series of rich and good structures and properties, and thus they have many significant applications in the research areas of both theoretical and applied mathematics. In fact, mathematicians established a variety of useful formulas and facts related to the products and used them to deal with various concrete scientific computing problems. Specifically, the basic facts on Kronecker products of matrices in the following lemma were highly appraised and recognized (cf. [15, 17, 21, 32, 34]).

Fact 1.1. Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times q}$, $C \in \mathbb{C}^{n\times s}$, and $D \in \mathbb{C}^{q\times t}$. Then, the following equalities hold:

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD), \tag{1.2}$$

$$(A \otimes B)^* = A^* \otimes B^*, \quad (A \otimes B)^\dagger = A^\dagger \otimes B^\dagger, \tag{1.3}$$

$$P_{A\otimes B} = P_A \otimes P_B, \qquad r(A \otimes B) = r(A)r(B). \tag{1.4}$$

In addition, the Kronecker product of matrices has a rich variety of algebraic operation properties. For example, one of the most important features is that the product $A_1 \otimes A_2$ can be factorized as certain ordinary products of matrices:

$$A_1 \otimes A_2 = (A_1 \otimes I_{m_2})(I_{n_1} \otimes A_2) = (I_{m_1} \otimes A_2)(A_1 \otimes I_{n_2}) \tag{1.5}$$

for any $A_1 \in \mathbb{C}^{m_1\times n_1}$ and $A_2 \in \mathbb{C}^{m_2\times n_2}$, and the triple Kronecker product $A_1 \otimes A_2 \otimes A_3$ can be written as

$$A_1 \otimes A_2 \otimes A_3 = (A_1 \otimes I_{m_2} \otimes I_{m_3})(I_{n_1} \otimes A_2 \otimes I_{m_3})(I_{n_1} \otimes I_{n_2} \otimes A_3), \tag{1.6}$$

$$A_1 \otimes A_2 \otimes A_3 = (I_{n_1} \otimes A_2 \otimes I_{m_3})(I_{n_1}(I_{n_1} \otimes I_{n_2} \otimes A_3)(A_1 \otimes I_{m_2} \otimes I_{m_3}), \tag{1.7}$$

$$A_1 \otimes A_2 \otimes A_3 = (I_{m_1} \otimes I_{m_2} \otimes A_3)(A_1 \otimes I_{n_2} \otimes I_{n_3})(I_{m_1} \otimes A_2 \otimes I_{n_3}) \tag{1.8}$$

for any $A_1 \in \mathbb{C}^{m_1\times n_1}$, $A_2 \in \mathbb{C}^{m_2\times n_2}$ and $A_3 \in \mathbb{C}^{m_3\times n_3}$, where the five matrices in the parentheses on the right hand sides of (1.5)–(1.8) are usually called the dilation expressions of the given three matrices A_1, A_2 and A_3, and the four equalities in (1.5)–(1.8) are called the dilation factorizations of the Kronecker products $A_1 \otimes A_2$ and $A_1 \otimes A_2 \otimes A_3$, respectively. A common feature of the four matrix equalities in (1.5)–(1.8) is that they factorize Kronecker products of any two or three matrices into certain ordinary products of the dilation expressions of A_1, A_2 and A_3. Particularly, a noticeable fact we have to point out is that the nine dilation expressions of matrices in (1.5)–(1.8) commute each other by the well-known mixed-product property in (1.2) when A_1, A_2 and A_3 are all square matrices. It can further be imagined that there exists proper extension of the dilation factorizations to Kronecker products of multiple matrices. Although the dilation factorizations in (1.5)–(1.8) seem to be technically trivial in form, they can help deal with theoretical and computational issues regarding Kronecker products of two or three matrices through the ordinary addition and multiplication operations of matrices.

In this article, we provide a new analysis of performances and properties of Kronecker products of matrices, as well as present a wide range of novel and explicit facts and results through the dilation factorizations described in (1.5)–(1.8) for the purpose of obtaining a deeper understanding and grasping of Kronecker products of matrices, including a number of analytical formulas for calculating ranks, dimensions, orthogonal projectors, and ranges of the dilation expressions of matrices and their algebraic operations.

The remainder of article is organized as follows. In section two, we introduce some preliminary facts and results concerning ranks, ranges, and generalized inverses of matrices. In section three, we propose and prove a collection of useful equalities, inequalities, and formulas for calculating the ranks, dimensions, orthogonal projectors, and ranges associated with the Kronecker products $A_1 \otimes A_2$ and $A_1 \otimes A_2 \otimes A_3$ through the dilation expressions of A_1, A_2 and A_3 and their operations. Conclusions and remarks are given in section four.

2. Some preliminaries

One of the remarkable applications of generalized inverses of matrices is to establish various exact and analytical expansion formulas for calculating the ranks of partitioned matrices. As convenient and skillful tools, these matrix rank formulas can be used to deal with a wide variety of theoretical and computational issues in matrix theory and its applications (cf. [22]). In this section, we present a mixture of commonly used formulas and facts in relation to ranks of matrices and their consequences about the commutativity of two orthogonal projectors, which we shall use as analytical tools to approach miscellaneous formulas related to Kronecker products of matrices.

Lemma 2.1. *[22, 29] Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{m\times k}$. Then, the following rank equalities*

$$r[A,\ B] = r(A) + r(E_A B) = r(B) + r(E_B A), \tag{2.1}$$

$$r[A,\ B] = r(A) + r(B) - 2r(A^*B) + r[P_A P_B,\ P_B P_A], \tag{2.2}$$

$$r[A,\ B] = r(A) + r(B) - r(A^*B) + 2^{-1} r(P_A P_B - P_B P_A), \tag{2.3}$$

$$r[A,\ B] = r(A) + r(B) - r(A^*B) + r(P_{[A,\,B]} - P_A - P_B + P_A P_B) \tag{2.4}$$

hold. Therefore,

$$\begin{aligned} P_A P_B = P_B P_A &\Leftrightarrow P_{[A,\,B]} = P_A + P_B - P_A P_B \\ &\Leftrightarrow r(E_A B) = r(B) - r(A^*B) \\ &\Leftrightarrow r[A,\ B] = r(A) + r(B) - r(A^*B) \end{aligned}$$

$$\Leftrightarrow \mathscr{R}(P_A P_B) = \mathscr{R}(P_B P_A). \tag{2.5}$$

If $P_A P_B = P_B P_A$, $P_A P_C = P_C P_A$ *and* $P_B P_C = P_C P_B$, *then*

$$P_{[A,\,B,\,C]} = P_A + P_B + P_C - P_A P_B - P_A P_C - P_B P_C + P_A P_B P_C. \tag{2.6}$$

Lemma 2.2. *[30] Let* A, B *and* $C \in \mathbb{C}^{m\times m}$ *be three idempotent matrices. Then, the following rank equality*

$$\begin{aligned} r[A,\ B,\ C] = {} & r(A) + r(B) + r(C) - r[AB,\ AC] - r[BA,\ BC] - r[CA,\ CB] \\ & + r[AB,\ AC,\ BA,\ BC,\ CA,\ CB] \end{aligned} \tag{2.7}$$

holds. As a special instance, if $AB = BA$, $AC = CA$ *and* $BC = CB$, *then*

$$r[A,\ B,\ C] = r(A) + r(B) + r(C) - r[AB,\ AC] - r[BA,\ BC] - r[CA,\ CB] + r[AB,\ AC,\ BC]. \tag{2.8}$$

The formulas and facts in the above two lemmas belong to mathematical competencies and conceptions in ordinary linear algebra. Thus they can easily be understood and technically be utilized to establish and simplify matrix expressions and equalities consisting of matrices and their generalized inverses.

3. Main results

We first establish a group of formulas and facts associated with the orthogonal projectors, ranks, dimensions, and ranges of the matrix product in (1.5).

Theorem 3.1. *Let* $A_1 \in \mathbb{C}^{m_1\times n_1}$ *and* $A_2 \in \mathbb{C}^{m_2\times n_2}$, *and denote by*

$$M_1 = A_1 \otimes I_{m_2}, \quad M_2 = I_{m_1} \otimes A_2$$

the two dilation expressions of A_1 *and* A_2, *respectively. Then, we have the following results.*

(a) *The following orthogonal projector equalities hold*:

$$P_{A_1\otimes A_2} = P_{M_1}P_{M_2} = P_{M_2}P_{M_1} = P_{A_1} \otimes P_{A_2}. \tag{3.1}$$

(b) *The following four rank equalities hold*:

$$r[A_1 \otimes I_{m_2},\ I_{m_1} \otimes A_2] = m_1m_2 - (m_1 - r(A_1))(m_2 - r(A_2)), \tag{3.2}$$
$$r[A_1 \otimes I_{m_2},\ I_{m_1} \otimes E_{A_2}] = m_1m_2 - (m_1 - r(A_1))r(A_2), \tag{3.3}$$
$$r[E_{A_1} \otimes I_{m_2},\ I_{m_1} \otimes A_2] = m_1m_2 - r(A_1)(m_2 - r(A_2)), \tag{3.4}$$
$$r[E_{A_1} \otimes I_{m_2},\ I_{m_1} \otimes E_{A_2}] = m_1m_2 - r(A_1)r(A_2), \tag{3.5}$$

and the following five dimension equalities hold:

$$\dim(\mathscr{R}(M_1) \cap \mathscr{R}(M_2)) = r(M_1M_2) = r(A_1)r(A_2), \tag{3.6}$$
$$\dim(\mathscr{R}(M_1) \cap \mathscr{R}^{\perp}(M_2)) = r(M_1E_{M_2}) = r(A_1)(m_2 - r(A_2)), \tag{3.7}$$

$$\dim(\mathscr{R}^{\perp}(M_1) \cap \mathscr{R}(M_2)) = r(E_{M_1} M_2) = (m_1 - r(A_1))r(A_2), \tag{3.8}$$

$$\dim(\mathscr{R}^{\perp}(M_1) \cap \mathscr{R}^{\perp}(M_2)) = r(E_{M_1} E_{M_2}) = (m_1 - r(A_1))(m_2 - r(A_2)), \tag{3.9}$$

$$\begin{aligned}&\dim(\mathscr{R}(M_1) \cap \mathscr{R}(M_2)) + \dim(\mathscr{R}(M_1) \cap \mathscr{R}^{\perp}(M_2)) + \dim(\mathscr{R}^{\perp}(M_1) \cap \mathscr{R}^{\perp}(M_2))\\ &+ \dim(\mathscr{R}^{\perp}(M_1) \cap \mathscr{R}^{\perp}(M_2)) = m_1 m_2.\end{aligned} \tag{3.10}$$

(c) *The following range equalities hold*:

$$\mathscr{R}(M_1) \cap \mathscr{R}(M_2) = \mathscr{R}(M_1 M_2) = \mathscr{R}(M_2 M_1) = \mathscr{R}(A_1 \otimes A_2), \tag{3.11}$$

$$\mathscr{R}(M_1) \cap \mathscr{R}^{\perp}(M_2) = \mathscr{R}(M_1 E_{M_2}) = \mathscr{R}(E_{M_2} M_1) = \mathscr{R}(A_1 \otimes E_{A_2}), \tag{3.12}$$

$$\mathscr{R}^{\perp}(M_1) \cap \mathscr{R}(M_2) = \mathscr{R}(E_{M_1} M_2) = \mathscr{R}(M_2 E_{M_1}) = \mathscr{R}(E_{A_1} \otimes A_2), \tag{3.13}$$

$$\mathscr{R}^{\perp}(M_1) \cap \mathscr{R}^{\perp}(M_2) = \mathscr{R}(E_{M_1} E_{M_2}) = \mathscr{R}(E_{M_2} E_{M_1}) = \mathscr{R}(E_{A_1} \otimes E_{A_2}), \tag{3.14}$$

$$\begin{aligned}&(\mathscr{R}(M_1) \cap \mathscr{R}(M_2)) \oplus (\mathscr{R}^{\perp}(M_1) \cap \mathscr{R}(M_2)) \oplus (\mathscr{R}(M_1) \cap \mathscr{R}^{\perp}(M_2)) \oplus (\mathscr{R}^{\perp}(M_1) \cap \mathscr{R}^{\perp}(M_2))\\ &= \mathbb{C}^{m_1 m_2}.\end{aligned} \tag{3.15}$$

(d) *The following orthogonal projector equalities hold*:

$$P_{\mathscr{R}(M_1) \cap \mathscr{R}(M_2)} = P_{M_1} P_{M_2} = P_{A_1} \otimes P_{A_2}, \tag{3.16}$$

$$P_{\mathscr{R}(M_1) \cap \mathscr{R}^{\perp}(M_2)} = P_{M_1} E_{M_2} = P_{A_1} \otimes E_{A_2}, \tag{3.17}$$

$$P_{\mathscr{R}^{\perp}(M_1) \cap \mathscr{R}(M_2)} = E_{M_1} P_{M_2} = E_{A_1} \otimes P_{A_2}, \tag{3.18}$$

$$P_{\mathscr{R}^{\perp}(M_1) \cap \mathscr{R}^{\perp}(M_2)} = E_{M_1} E_{M_2} = E_{A_1} \otimes E_{A_2}, \tag{3.19}$$

$$P_{\mathscr{R}(M_1) \cap \mathscr{R}(M_2)} + P_{\mathscr{R}^{\perp}(M_1) \cap \mathscr{R}(M_2)} + P_{\mathscr{R}(M_1) \cap \mathscr{R}^{\perp}(M_2)} + P_{\mathscr{R}^{\perp}(M_1) \cap \mathscr{R}^{\perp}(M_2)} = I_{m_1 m_2}. \tag{3.20}$$

(e) *The following orthogonal projector equalities hold*:

$$P_{[A_1 \otimes I_{m_2},\, I_{m_1} \otimes A_2]} = P_{A_1} \otimes I_{m_2} + I_{m_1} \otimes P_{A_2} - P_{A_1} \otimes P_{A_2} = I_{m_1 m_2} - E_{A_1} \otimes E_{A_2}, \tag{3.21}$$

$$P_{[A_1 \otimes I_{m_2},\, I_{m_1} \otimes E_{A_2}]} = P_{A_1} \otimes I_{m_2} + I_{m_1} \otimes E_{A_2} - P_{A_1} \otimes E_{A_2} = I_{m_1 m_2} - E_{A_1} \otimes P_{A_2}, \tag{3.22}$$

$$P_{[E_{A_1} \otimes I_{m_2},\, I_{m_1} \otimes A_2]} = E_{A_1} \otimes I_{m_2} + I_{m_1} \otimes P_{A_2} - E_{A_1} \otimes P_{A_2} = I_{m_1 m_2} - P_{A_1} \otimes E_{A_2}, \tag{3.23}$$

$$P_{[E_{A_1} \otimes I_{m_2},\, I_{m_1} \otimes E_{A_2}]} = E_{A_1} \otimes I_{m_2} + I_{m_1} \otimes E_{A_2} - E_{A_1} \otimes E_{A_2} = I_{m_1 m_2} - P_{A_1} \otimes P_{A_2}. \tag{3.24}$$

Proof. It can be seen from (1.2) and (1.4) that

$$\begin{aligned}P_{M_1} P_{M_2} &= (A_1 \otimes I_{m_2})(A_1 \otimes I_{m_2})^{\dagger}(I_{m_1} \otimes A_2)(I_{m_1} \otimes A_2)^{\dagger}\\ &= (A_1 \otimes I_{m_2})(A_1^{\dagger} \otimes I_{m_2})(I_{m_1} \otimes A_2)(I_{m_1} \otimes A_2^{\dagger})\\ &= (P_{A_1} \otimes I_{m_2})(I_{m_1} \otimes P_{A_2})\\ &= P_{A_1} \otimes P_{A_2},\\ P_{M_2} P_{M_1} &= (I_{m_1} \otimes A_2)(I_{m_1} \otimes A_2)^{\dagger}(A_1 \otimes I_{m_2})(A_1 \otimes I_{m_2})^{\dagger}\\ &= (I_{m_1} \otimes A_2)(I_{m_1} \otimes A_2^{\dagger})(A_1 \otimes I_{m_2})(A_1^{\dagger} \otimes I_{m_2})\\ &= (I_{m_1} \otimes P_{A_2})(P_{A_1} \otimes I_{m_2})\end{aligned}$$

$$= P_{A_1} \otimes P_{A_2},$$

thus establishing (3.1).

Applying (2.1) to $[A_1 \otimes I_{m_2},\ I_{m_1} \otimes A_2]$ and then simplifying by (1.2)–(1.4) yields

$$\begin{aligned} r[A_1 \otimes I_{m_2},\ I_{m_1} \otimes A_2] &= r(A_1 \otimes I_{m_2}) + r((I_{m_1m_2} - (A_1 \otimes I_{m_2})(A_1 \otimes I_{m_2})^{\dagger})(I_{m_1} \otimes A_2)) \\ &= r(A_1 \otimes I_{m_2}) + r((I_{m_1m_2} - (A_1 \otimes I_{m_2})(A_1^{\dagger} \otimes I_{m_2}))(I_{m_1} \otimes A_2)) \\ &= r(A_1 \otimes I_{m_2}) + r((I_{m_1m_2} - (A_1A_1^{\dagger}) \otimes I_{m_2})(I_{m_1} \otimes A_2)) \\ &= m_2r(A_1) + r((I_{m_1} - A_1A_1^{\dagger}) \otimes I_{m_2})(I_{m_1} \otimes A_2)) \\ &= m_2r(A_1) + r((I_{m_1} - A_1A_1^{\dagger}) \otimes A_2)) \\ &= m_2r(A_1) + r(I_{m_1} - A_1A_1^{\dagger})r(A_2) \\ &= m_2r(A_1) + (m_1 - r(A_1))r(A_2) \\ &= m_1m_2 - (m_1 - r(A_1))(m_2 - r(A_2)), \end{aligned}$$

as required for (3.2). In addition, (3.2) can be directly established by applying (2.5) to the left hand side of (3.2). Equations (3.3)–(3.5) can be obtained by a similar approach. Subsequently by (3.2),

$$\dim(\mathscr{R}(M_1) \cap \mathscr{R}(M_2)) = r(M_1) + r(M_2) - r[M_1,\ M_2] = r(A_1)r(A_2),$$

as required for (3.6). Equations (3.7)–(3.9) can be established by a similar approach. Adding (3.7)–(3.9) leads to (3.10).

The first two equalities in (3.11) follow from (3.6), and the last two range equalities follow from (3.1).

Equations (3.12)–(3.14) can be established by a similar approach. Adding (3.11)–(3.14) and combining with (3.10) leads to (3.15).

Equations (3.16)–(3.19) follow from (3.11)–(3.14). Adding (3.16)–(3.19) leads to (3.20).

Under (3.1), we find from (2.5) that

$$P_{[M_1,\,M_2]} = P_{M_1} + P_{M_2} - P_{M_1}P_{M_2} = P_{A_1} \otimes I_{m_2} + I_{m_1} \otimes P_{A_2} - P_{A_1} \otimes P_{A_2} = I_{m_1m_2} - E_{A_1} \otimes E_{A_2},$$

as required for (3.21). Equations (3.22)–(3.24) can be established by a similar approach. □

Equation (3.2) was first shown in [7]; see also [27] for some extended forms of (3.2). Obviously, Theorem 3.1 reveals many performances and properties of Kronecker products of matrices, and it is no doubt that they can be used as analysis tools to deal with various matrix equalities composed of algebraic operations of Kronecker products of matrices. For example, applying the preceding results to the Kronecker sum and difference $A_1 \otimes I_{m_2} \pm I_{m_1} \otimes A_2$ for two idempotent matrices A_1 and A_2, we obtain the following interesting consequences.

Theorem 3.2. *Let $A_1 \in \mathbb{C}^{m_1 \times m_1}$ and $A_2 \in \mathbb{C}^{m_2 \times m_2}$. Then, the following rank inequality*

$$r(A_1 \otimes I_{m_2} + I_{m_1} \otimes A_2) \geq m_1r(A_2) + m_2r(A_1) - 2r(A_1)r(A_2) \tag{3.25}$$

holds. If $A_1 = A_1^2$ and $A_2 = A_2^2$, then the following two rank equalities hold:

$$r(A_1 \otimes I_{m_2} + I_{m_1} \otimes A_2) = m_1r(A_2) + m_2r(A_1) - r(A_1)r(A_2), \tag{3.26}$$

$$r(A_1 \otimes I_{m_2} - I_{m_1} \otimes A_2) = m_1r(A_2) + m_2r(A_1) - 2r(A_1)r(A_2). \tag{3.27}$$

Proof. Equation (3.25) follows from applying the following well-known rank inequality (cf. [22])

$$r(A+B) \geq r\begin{bmatrix} A \\ B \end{bmatrix} + r[A,\ B] - r(A) - r(B)$$

and (2.1) to $A_1 \otimes I_{m_2} + I_{m_1} \otimes A_2$. Specifically, if $A_1 = A_1^2$ and $A_2 = A_2^2$, then it is easy to verify that $(A_1 \otimes I_{m_2})^2 = A_1^2 \otimes I_{m_2} = A_1 \otimes I_{m_2}$ and $(I_{m_1} \otimes A_2)^2 = I_{m_1} \otimes A_2^2 = I_{m_1} \otimes A_2$ under $A_1^2 = A_1$ and $A_2^2 = A_2$. In this case, applying the following two known rank formulas

$$r(A+B) = r\begin{bmatrix} A & B \\ B & 0 \end{bmatrix} - r(B) = r\begin{bmatrix} B & A \\ A & 0 \end{bmatrix} - r(A),$$
$$r(A-B) = r\begin{bmatrix} A \\ B \end{bmatrix} + r[A,\ B] - r(A) - r(B),$$

where A and B are two idempotent matrices of the same size (cf. [29, 31]), to $A_1 \otimes I_{m_2} \pm I_{m_1} \otimes A_2$ and then simplifying by (2.1) and (3.2) yields (3.26) and (3.27), respectively. □

Undoubtedly, the above two theorems reveal some essential relations among the dilation forms of two matrices by Kronecker products, which demonstrate that there still exist various concrete research topics on the Kronecker product of two matrices with analytical solutions that can be proposed and obtained. As a natural and useful generalization of the preceding formulas, we next give a diverse range of results related to the three-term Kronecker products of matrices in (1.6).

Theorem 3.3. *Let* $A_1 \in \mathbb{C}^{m_1 \times n_1}$, $A_2 \in \mathbb{C}^{m_2 \times n_2}$ *and* $A_3 \in \mathbb{C}^{m_3 \times n_3}$, *and let*

$$X_1 = A_1 \otimes I_{m_2} \otimes I_{m_3}, \ \ X_2 = I_{m_1} \otimes A_2 \otimes I_{m_3}, \ \ X_3 = I_{m_1} \otimes I_{m_2} \otimes A_3 \tag{3.28}$$

denote the three dilation expressions of A_1, A_2 *and* A_3, *respectively. Then, we have the following results.*

(a) *The following three orthogonal projector equalities hold*:

$$P_{X_1} = P_{A_1} \otimes I_{m_2} \otimes I_{m_3}, \ \ P_{X_2} = I_{m_1} \otimes P_{A_2} \otimes I_{m_3}, \ \ P_{X_3} = I_{m_1} \otimes I_{m_2} \otimes P_{A_3}, \tag{3.29}$$

the following equalities hold:

$$P_{X_1}P_{X_2} = P_{X_2}P_{X_1} = P_{A_1} \otimes P_{A_2} \otimes I_{m_3}, \tag{3.30}$$
$$P_{X_1}P_{X_3} = P_{X_3}P_{X_1} = P_{A_1} \otimes I_{m_2} \otimes P_{A_3}, \tag{3.31}$$
$$P_{X_2}P_{X_3} = P_{X_3}P_{X_2} = I_{m_1} \otimes P_{A_2} \otimes P_{A_3}, \tag{3.32}$$

and the equalities hold:

$$P_{A_1 \otimes A_2 \otimes A_3} = P_{X_1}P_{X_2}P_{X_3} = P_{A_1} \otimes P_{A_2} \otimes P_{A_3}. \tag{3.33}$$

(b) *The following eight rank equalities hold*:

$$\begin{aligned} & r[A_1 \otimes I_{m_2} \otimes I_{m_3},\ I_{m_1} \otimes A_2 \otimes I_{m_3},\ I_{m_1} \otimes I_{m_2} \otimes A_3] \\ & \quad = m_1m_2m_3 - (m_1 - r(A_1))(m_2 - r(A_2))(m_3 - r(A_3)), \end{aligned} \tag{3.34}$$

$$
\begin{aligned}
&r[A_1\otimes I_{m_2}\otimes I_{m_3},\ I_{m_1}\otimes A_2\otimes I_{m_3},\ I_{m_1}\otimes I_{m_2}\otimes E_{A_3}]\\
&\quad = m_1m_2m_3-(m_1-r(A_1))(m_2-r(A_2))r(A_3), && (3.35)\\
&r[A_1\otimes I_{m_2}\otimes I_{m_3},\ I_{m_1}\otimes E_{A_2}\otimes I_{m_3},\ I_{m_1}\otimes I_{m_2}\otimes A_3]\\
&\quad = m_1m_2m_3-(m_1-r(A_1))r(A_2)(m_3-r(A_3)), && (3.36)\\
&r[E_{A_1}\otimes I_{m_2}\otimes I_{m_3},\ I_{m_1}\otimes A_2\otimes I_{m_3},\ I_{m_1}\otimes I_{m_2}\otimes A_3]\\
&\quad = m_1m_2m_3-r(A_1)(m_2-r(A_2))(m_3-r(A_3)), && (3.37)\\
&r[A_1\otimes I_{m_2}\otimes I_{m_3},\ I_{m_1}\otimes E_{A_2}\otimes I_{m_3},\ I_{m_1}\otimes I_{m_2}\otimes E_{A_3}]\\
&\quad = m_1m_2m_3-(m_1-r(A_1))r(A_2)r(A_3), && (3.38)\\
&r[E_{A_1}\otimes I_{m_2}\otimes I_{m_3},\ I_{m_1}\otimes A_2\otimes I_{m_3},\ I_{m_1}\otimes I_{m_2}\otimes E_{A_3}]\\
&\quad = m_1m_2m_3-r(A_1)(m_2-r(A_2))r(A_3), && (3.39)\\
&r[E_{A_1}\otimes I_{m_2}\otimes I_{m_3},\ I_{m_1}\otimes E_{A_2}\otimes I_{m_3},\ I_{m_1}\otimes I_{m_2}\otimes A_3]\\
&\quad = m_1m_2m_3-r(A_1)r(A_2)(m_3-r(A_3)), && (3.40)\\
&r[E_{A_1}\otimes I_{m_2}\otimes I_{m_3},\ I_{m_1}\otimes E_{A_2}\otimes I_{m_3},\ I_{m_1}\otimes I_{m_2}\otimes E_{A_3}]\\
&= m_1m_2m_3-r(A_1)r(A_2)r(A_3), && (3.41)
\end{aligned}
$$

the following eight dimension equalities hold:

$$
\begin{aligned}
&\dim(\mathscr{R}(X_1)\cap\mathscr{R}(X_2)\cap\mathscr{R}(X_3)) = r(A_1)r(A_2)r(A_3), && (3.42)\\
&\dim(\mathscr{R}(X_1)\cap\mathscr{R}(X_2)\cap\mathscr{R}^{\perp}(X_3)) = r(A_1)r(A_2)(m_3-r(A_3)), && (3.43)\\
&\dim(\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}(X_3)) = r(A_1)(m_2-r(A_2))r(A_3), && (3.44)\\
&\dim(\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}(X_3)) = (m_1-r(A_1))r(A_2)r(A_3), && (3.45)\\
&\dim(\mathscr{R}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}^{\perp}(X_3)) = r(A_1)(m_2-r(A_2))(m_3-r(A_3)), && (3.46)\\
&\dim(\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}(X_2)\cap\mathscr{R}^{\perp}(X_3)) = (m_1-r(A_1))r(A_2)(m_3-r(A_3)), && (3.47)\\
&\dim(\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}(X_3)) = (m_1-r(A_1))(m_2-r(A_2))r(A_3), && (3.48)\\
&\dim(\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}^{\perp}(X_3)) = (m_1-r(A_1))(m_2-r(A_2))(m_3-r(A_3)), && (3.49)
\end{aligned}
$$

and the following dimension equality holds:

$$
\begin{aligned}
&\dim(\mathscr{R}(X_1)\cap\mathscr{R}(X_2)\cap\mathscr{R}(X_3)) + \dim(\mathscr{R}(X_1)\cap\mathscr{R}(X_2)\cap\mathscr{R}^{\perp}(X_3))\\
&\quad + \dim(\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}(X_3)) + \dim(\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}(X_3))\\
&\quad + \dim(\mathscr{R}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}^{\perp}(X_3)) + \dim(\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}(X_2)\cap\mathscr{R}^{\perp}(X_3))\\
&\quad + \dim(\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}(X_3)) + \dim(\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}^{\perp}(X_3)\\
&= m_1m_2m_3. && (3.50)
\end{aligned}
$$

(c) *The following eight groups of range equalities hold*:

$$
\begin{aligned}
&\mathscr{R}(X_1)\cap\mathscr{R}(X_2)\cap\mathscr{R}(X_3) = \mathscr{R}(X_1X_2X_3) = \mathscr{R}(A_1\otimes A_2\otimes A_3), && (3.51)\\
&\mathscr{R}(X_1)\cap\mathscr{R}(X_2)\cap\mathscr{R}^{\perp}(X_3) = \mathscr{R}(X_1X_2E_{X_3}) = \mathscr{R}(A_1\otimes A_2\otimes E_{A_3}), && (3.52)\\
&\mathscr{R}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}(X_3) = \mathscr{R}(X_1E_{X_2}X_3) = \mathscr{R}(A_1\otimes E_{A_2}\otimes A_3), && (3.53)
\end{aligned}
$$

$$\mathscr{R}^{\perp}(X_1) \cap \mathscr{R}(X_2) \cap \mathscr{R}(X_3) = \mathscr{R}(E_{X_1}X_2X_3) = \mathscr{R}(E_{A_1} \otimes A_2 \otimes A_3), \tag{3.54}$$
$$\mathscr{R}(X_1) \cap \mathscr{R}^{\perp}(X_2) \cap \mathscr{R}^{\perp}(X_3) = \mathscr{R}(X_1E_{X_2}E_{X_3}) = \mathscr{R}(A_1 \otimes E_{A_2} \otimes E_{A_3}), \tag{3.55}$$
$$\mathscr{R}^{\perp}(X_1) \cap \mathscr{R}(X_2) \cap \mathscr{R}^{\perp}(X_3) = \mathscr{R}(E_{X_1}X_2E_{X_3}) = \mathscr{R}(E_{A_1} \otimes A_2 \otimes E_{A_3}), \tag{3.56}$$
$$\mathscr{R}^{\perp}(X_1) \cap \mathscr{R}^{\perp}(X_2) \cap \mathscr{R}(X_3) = \mathscr{R}(X_1E_{X_2}X_3) = \mathscr{R}(E_{A_1} \otimes E_{A_2} \otimes A_3), \tag{3.57}$$
$$\mathscr{R}^{\perp}(X_1) \cap \mathscr{R}^{\perp}(X_2) \cap \mathscr{R}^{\perp}(X_3) = \mathscr{R}(E_{X_1}E_{X_2}E_{X_3}) = \mathscr{R}(E_{A_1} \otimes E_{A_2} \otimes E_{A_3}), \tag{3.58}$$

and the following direct sum equality holds:

$$\begin{aligned}&(\mathscr{R}(X_1) \cap \mathscr{R}(X_2) \cap \mathscr{R}(X_3)) \oplus (\mathscr{R}^{\perp}(X_1) \cap \mathscr{R}(X_2) \cap \mathscr{R}(X_3))\\ &\oplus (\mathscr{R}(X_1) \cap \mathscr{R}^{\perp}(X_2) \cap \mathscr{R}(X_3)) \oplus (\mathscr{R}(X_1) \cap \mathscr{R}(X_2) \cap \mathscr{R}^{\perp}(X_3))\\ &\oplus (\mathscr{R}^{\perp}(X_1) \cap \mathscr{R}^{\perp}(X_2) \cap \mathscr{R}(X_3)) \oplus (\mathscr{R}^{\perp}(X_1) \cap \mathscr{R}(X_2) \cap \mathscr{R}^{\perp}(X_3))\\ &\oplus (\mathscr{R}(X_1) \cap \mathscr{R}^{\perp}(X_2) \cap \mathscr{R}^{\perp}(X_3)) \oplus (\mathscr{R}^{\perp}(X_1) \cap \mathscr{R}^{\perp}(X_2) \cap \mathscr{R}^{\perp}(X_3))\\ &= \mathbb{C}^{m_1m_2m_3}.\end{aligned} \tag{3.59}$$

(d) *The following eight orthogonal projector equalities hold*:

$$P_{\mathscr{R}(X_1)\cap\mathscr{R}(X_2)\cap\mathscr{R}(X_3)} = P_{A_1} \otimes P_{A_2} \otimes P_{A_3}, \tag{3.60}$$
$$P_{\mathscr{R}(X_1)\cap\mathscr{R}(X_2)\cap\mathscr{R}^{\perp}(X_3)} = P_{A_1} \otimes P_{A_2} \otimes E_{A_3}, \tag{3.61}$$
$$P_{\mathscr{R}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}(X_3)} = P_{A_1} \otimes E_{A_2} \otimes P_{A_3}, \tag{3.62}$$
$$P_{\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}(X_2)\cap\mathscr{R}(X_3)} = E_{A_1} \otimes P_{A_2} \otimes P_{A_3}, \tag{3.63}$$
$$P_{\mathscr{R}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}^{\perp}(X_3)} = P_{A_1} \otimes E_{A_2} \otimes E_{A_3}, \tag{3.64}$$
$$P_{\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}(X_2)\cap\mathscr{R}^{\perp}(X_3)} = E_{A_1} \otimes P_{A_2} \otimes E_{A_3}, \tag{3.65}$$
$$P_{\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}(X_3)} = E_{A_1} \otimes E_{A_2} \otimes P_{A_3}, \tag{3.66}$$
$$P_{\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}^{\perp}(X_3)} = E_{A_1} \otimes E_{A_2} \otimes E_{A_3}, \tag{3.67}$$

and the following orthogonal projector equality holds:

$$\begin{aligned}&P_{\mathscr{R}(X_1)\cap\mathscr{R}(X_2)\cap\mathscr{R}(X_3)} + P_{\mathscr{R}(X_1)\cap\mathscr{R}(X_2)\cap\mathscr{R}^{\perp}(X_3)}\\ &\quad + P_{\mathscr{R}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}(X_3)} + P_{\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}(X_2)\cap\mathscr{R}(X_3)}\\ &\quad + P_{\mathscr{R}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}^{\perp}(X_3)} + P_{\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}(X_2)\cap\mathscr{R}^{\perp}(X_3)}\\ &\quad + P_{\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}(X_3)} + P_{\mathscr{R}^{\perp}(X_1)\cap\mathscr{R}^{\perp}(X_2)\cap\mathscr{R}^{\perp}(X_3)} = I_{m_1m_2m_3}.\end{aligned} \tag{3.68}$$

(e) *The following eight orthogonal projector equalities hold*:

$$P_{[A_1\otimes I_{m_2}\otimes I_{m_3},\, I_{m_1}\otimes A_2\otimes I_{m_3},\, I_{m_1}\otimes I_{m_2}\otimes A_3]} = I_{m_1m_2m_3} - E_{A_1} \otimes E_{A_2} \otimes E_{A_3}, \tag{3.69}$$
$$P_{[A_1\otimes I_{m_2}\otimes I_{m_3},\, I_{m_1}\otimes A_2\otimes I_{m_3},\, I_{m_1}\otimes I_{m_2}\otimes E_{A_3}]} = I_{m_1m_2m_3} - E_{A_1} \otimes E_{A_2} \otimes P_{A_3}, \tag{3.70}$$
$$P_{[A_1\otimes I_{m_2}\otimes I_{m_3},\, I_{m_1}\otimes E_{A_2}\otimes I_{m_3},\, I_{m_1}\otimes I_{m_2}\otimes A_3]} = I_{m_1m_2m_3} - E_{A_1} \otimes P_{A_2} \otimes E_{A_3}, \tag{3.71}$$
$$P_{[E_{A_1}\otimes I_{m_2}\otimes I_{m_3},\, I_{m_1}\otimes A_2\otimes I_{m_3},\, I_{m_1}\otimes I_{m_2}\otimes A_3]} = I_{m_1m_2m_3} - P_{A_1} \otimes E_{A_2} \otimes E_{A_3}, \tag{3.72}$$
$$P_{[A_1\otimes I_{m_2}\otimes I_{m_3},\, I_{m_1}\otimes E_{A_2}\otimes I_{m_3},\, I_{m_1}\otimes I_{m_2}\otimes E_{A_3}]} = I_{m_1m_2m_3} - E_{A_1} \otimes P_{A_2} \otimes P_{A_3}, \tag{3.73}$$
$$P_{[E_{A_1}\otimes I_{m_2}\otimes I_{m_3},\, I_{m_1}\otimes A_2\otimes I_{m_3},\, I_{m_1}\otimes I_{m_2}\otimes E_{A_3}]} = I_{m_1m_2m_3} - P_{A_1} \otimes E_{A_2} \otimes P_{A_3}, \tag{3.74}$$
$$P_{[E_{A_1}\otimes I_{m_2}\otimes I_{m_3},\, I_{m_1}\otimes E_{A_2}\otimes I_{m_3},\, I_{m_1}\otimes I_{m_2}\otimes A_3]} = I_{m_1m_2m_3} - P_{A_1} \otimes P_{A_2} \otimes E_{A_3}, \tag{3.75}$$
$$P_{[E_{A_1}\otimes I_{m_2}\otimes I_{m_3},\, I_{m_1}\otimes E_{A_2}\otimes I_{m_3},\, I_{m_1}\otimes I_{m_2}\otimes E_{A_3}]} = I_{m_1m_2m_3} - P_{A_1} \otimes P_{A_2} \otimes P_{A_3}. \tag{3.76}$$

Proof. By (1.1)–(1.3),

$$\begin{aligned} P_{X_1} &= (A_1 \otimes I_{m_2} \otimes I_{m_3})(A_1 \otimes I_{m_2} \otimes I_{m_3})^{\dagger} \\ &= (A_1 \otimes I_{m_2} \otimes I_{m_3})(A_1^{\dagger} \otimes I_{m_2} \otimes I_{m_3}) \\ &= (A_1 A_1^{\dagger}) \otimes I_{m_2} \otimes I_{m_3} \\ &= P_{A_1} \otimes I_{m_2} \otimes I_{m_3}, \end{aligned}$$

thus establishing the first equality in (3.29). The second and third equalities in (3.29) can be shown in a similar way. Also by (1.1)–(1.3),

$$\begin{aligned} P_{A_1 \otimes A_2 \otimes A_3} &= (A_1 \otimes A_2 \otimes A_3)(A_1 \otimes A_2 \otimes A_3)^{\dagger} \\ &= (A_1 \otimes A_2 \otimes A_3)(A_1^{\dagger} \otimes A_2^{\dagger} \otimes A_3^{\dagger}) \\ &= (A_1 A_1^{\dagger}) \otimes (A_2 A_2^{\dagger}) \otimes (A_3 A_3^{\dagger}) \\ &= P_{A_1} \otimes P_{A_2} \otimes P_{A_3}, \end{aligned} \tag{3.77}$$

and by (1.2) and (3.29),

$$P_{X_1} P_{X_2} P_{X_3} = (P_{A_1} \otimes I_{m_2} \otimes I_{m_3})(I_{m_1} \otimes P_{A_2} \otimes I_{m_3})(I_{m_1} \otimes I_{m_2} \otimes P_{A_3}) = P_{A_1} \otimes P_{A_2} \otimes P_{A_3}. \tag{3.78}$$

Combining (3.77) and (3.78) leads to (3.33).

By (2.1), (1.2)–(1.4) and (3.2),

$$\begin{aligned} & r[A_1 \otimes I_{m_2} \otimes I_{m_3},\ I_{m_1} \otimes A_2 \otimes I_{m_3},\ I_{m_1} \otimes I_{m_2} \otimes A_3] \\ &= r(A_1 \otimes I_{m_2} \otimes I_{m_3}) + r((I_{m_1} - A_1 A_1^{\dagger}) \otimes [A_2 \otimes I_{m_3}, I_{m_2} \otimes A_3]) \\ &= m_2 m_3 r(A_1) + r(I_{m_1} - A_1 A_1^{\dagger}) r[A_2 \otimes I_{m_3}, I_{m_2} \otimes A_3] \\ &= m_2 m_3 r(A_1) + (m_1 - r(A_1))(m_2 m_3 - (m_2 - r(A_2))(m_3 - r(A_3))) \\ &= m_1 m_2 m_3 - (m_1 - r(A_1))(m_2 - r(A_2))(m_3 - r(A_3)), \end{aligned}$$

thus establishing (3.34). Equations (3.35)–(3.41) can be established in a similar way.

By (3.11), we are able to obtain

$$\mathscr{R}(X_1) \cap \mathscr{R}(X_2) = \mathscr{R}(X_1 X_2) = \mathscr{R}(X_2 X_1) = \mathscr{R}(A_1 \otimes A_2 \otimes I_{m_3}).$$

Consequently,

$$\mathscr{R}(X_1) \cap \mathscr{R}(X_2) \cap \mathscr{R}(X_3) = \mathscr{R}(X_1 X_2) \cap \mathscr{R}(X_3) = \mathscr{R}(X_1 X_2 X_3) = \mathscr{R}(A_1 \otimes A_2 \otimes A_3),$$

as required for (3.51). Equations (3.52)–(3.58) can be established in a similar way. Adding (3.51)–(3.58) leads to (3.59).

Taking the dimensions of both sides of (3.51)–(3.58) and applying (1.4), we obtain (3.42)–(3.50).

Equations (3.60)–(3.68) follow from (3.51)–(3.59).

Equations (3.69)–(3.77) follow from (2.6) and (3.30)–(3.32). □

In addition to (3.28), we can construct the following three dilation expressions

$$Y_1 = I_{m_1} \otimes A_2 \otimes A_3, \quad Y_2 = A_1 \otimes I_{m_2} \otimes A_3 \quad and \quad Y_3 = A_1 \otimes A_2 \otimes I_{m_3} \tag{3.79}$$

from any three matrices $A_1 \in \mathbb{C}^{m_1 \times n_1}$, $A_2 \in \mathbb{C}^{m_2 \times n_2}$ and $A_3 \in \mathbb{C}^{m_3 \times n_3}$. Some concrete topics on rank equalities for the dilation expressions under vector situations were considered in [26]. Below, we give a sequence of results related to the three dilation expressions.

Theorem 3.4. *Let Y_1, Y_2 and Y_3 be the same as given in* (3.79). *Then, we have the following results.*

(a) *The following three projector equalities hold*:

$$P_{Y_1} = I_{m_1} \otimes P_{A_2} \otimes P_{A_3}, \quad P_{Y_2} = P_{A_1} \otimes I_{m_2} \otimes P_{A_3} \quad and \quad P_{Y_3} = P_{A_1} \otimes P_{A_2} \otimes I_{m_3}. \tag{3.80}$$

(b) *The following twelve matrix equalities hold*:

$$\begin{aligned} P_{Y_1}P_{Y_2} &= P_{Y_2}P_{Y_1} = P_{Y_1}P_{Y_3} = P_{Y_3}P_{Y_1} = P_{Y_2}P_{Y_3} = P_{Y_3}P_{Y_2} \\ &= P_{Y_1}P_{Y_2}P_{Y_3} = P_{Y_1}P_{Y_3}P_{Y_2} = P_{Y_2}P_{Y_1}P_{Y_3} = P_{Y_2}P_{Y_3}P_{Y_1} \\ &= P_{Y_3}P_{Y_1}P_{Y_2} = P_{Y_3}P_{Y_2}P_{Y_1} = P_{A_1} \otimes P_{A_2} \otimes P_{A_3}. \end{aligned} \tag{3.81}$$

(c) *The following rank equality holds*:

$$r[Y_1,\ Y_2,\ Y_3] = m_1 r(A_2)r(A_3) + m_2 r(A_1)r(A_3) + m_3 r(A_1)r(A_2) - 2r(A_1)r(A_2)r(A_3). \tag{3.82}$$

(d) *The following range equality holds*:

$$\mathscr{R}(Y_1) \cap \mathscr{R}(Y_2) \cap \mathscr{R}(Y_3) = \mathscr{R}(A_1 \otimes A_2 \otimes A_3). \tag{3.83}$$

(e) *The following dimension equality holds*:

$$\dim(\mathscr{R}(Y_1) \cap \mathscr{R}(Y_2) \cap \mathscr{R}(Y_3)) = r(A_1)r(A_2)r(A_3). \tag{3.84}$$

(f) *The following projector equality holds*:

$$P_{\mathscr{R}(Y_1)\cap\mathscr{R}(Y_2)\cap\mathscr{R}(Y_3)} = P_{A_1} \otimes P_{A_2} \otimes P_{A_3}. \tag{3.85}$$

(g) *The following projector equality holds*:

$$P_{[Y_1,\,Y_2,\,Y_3]} = I_{m_1} \otimes P_{A_2} \otimes P_{A_3} + P_{A_1} \otimes I_{m_2} \otimes P_{A_3} + P_{A_1} \otimes P_{A_2} \otimes I_{m_3} - 2(P_{A_1} \otimes P_{A_2} \otimes P_{A_3}). \tag{3.86}$$

Proof. Equation (3.80) follows directly from (3.79), and (3.81) follows from (3.80). Since P_{Y_1}, P_{Y_2} and P_{Y_3} are idempotent matrices, we find from (2.8) and (3.80) that

$$\begin{aligned} r[Y_1,\ Y_2,\ Y_3] &= r[P_{Y_1},\ P_{Y_2},\ P_{Y_3}] = r(P_{Y_1}) + r(P_{Y_2}) + r(P_{Y_3}) \\ &\quad - r[P_{Y_1}P_{Y_2},\ P_{Y_1}P_{Y_3}] - r[P_{Y_2}P_{Y_1},\ P_{Y_2}P_{Y_3}] - r[P_{Y_3}P_{Y_1},\ P_{Y_3}P_{Y_1}] \\ &\quad + r[P_{Y_1}P_{Y_2},\ P_{Y_1}P_{Y_3},\ P_{Y_2}P_{Y_2}] \\ &= r(P_{Y_1}) + r(P_{Y_2}) + r(P_{Y_3}) - 2r(P_{A_1} \otimes P_{A_2} \otimes P_{A_3}) \\ &= m_1 r(A_2)r(A_3) + m_2 r(A_1)r(A_3) + m_3 r(A_1)r(A_2) - 2r(A_1)r(A_2)r(A_3), \end{aligned}$$

thus establishing (3.82). Equations (3.83)–(3.86) are left as exercises for the reader. □

There are some interesting consequences to Theorems 3.3 and 3.4. For example, applying the following well-known rank inequality (cf. [22]):

$$r(A+B+C) \geq r\begin{bmatrix} A \\ B \\ C \end{bmatrix} + r[A,\ B,\ C] - r(A) - r(B) - r(C)$$

to the sums of matrices in (3.28) and (3.80) yields the two rank inequalities

$$\begin{aligned} & r(A_1 \otimes I_{m_2} \otimes I_{m_3} + I_{m_1} \otimes A_2 \otimes I_{m_3} + I_{m_1} \otimes I_{m_2} \otimes A_3) \\ & \geq m_1 m_2 r(A_3) + m_1 m_3 r(A_2) + m_2 m_3 r(A_1) - 2m_1 r(A_2) r(A_3) - 2m_2 r(A_1) r(A_3) \\ & \quad - 2m_3 r(A_1) r(A_2) + 2r(A_1) r(A_2) r(A_3) \end{aligned}$$

and

$$\begin{aligned} & r(I_{m_1} \otimes A_2 \otimes A_3 + A_1 \otimes I_{m_2} \otimes A_3 + A_1 \otimes A_2 \otimes I_{m_3}) \\ & \geq m_1 r(A_2) r(A_3) + m_2 r(A_1) r(A_3) + m_3 r(A_1) r(A_2) - 4r(A_1) r(A_2) r(A_3), \end{aligned}$$

respectively, where $A_1 \in \mathbb{C}^{m_1 \times m_1}$, $A_2 \in \mathbb{C}^{m_2 \times m_2}$ and $A_3 \in \mathbb{C}^{m_3 \times m_3}$.

4. Conclusions

We presented a new analysis of the dilation factorizations of the Kronecker products of two or three matrices, and obtained a rich variety of exact formulas and facts related to ranks, dimensions, orthogonal projectors, and ranges of Kronecker products of matrices. Admittedly, it is easy to understand and utilize these resulting formulas and facts in dealing with Kronecker products of matrices under various concrete situations. Given the formulas and facts in the previous theorems, there is no doubt to say that this study clearly demonstrates significance and usefulness of the dilation factorizations of Kronecker products of matrices. Therefore, we believe that this study can bring deeper insights into performances of Kronecker products of matrices, and thereby can lead to certain advances of enabling methodology in the domain of Kronecker products. We also hope that the findings in this resultful study can be taken as fundamental facts and useful supplementary materials in matrix theory when identifying and approaching various theoretical and computational issues associated with Kronecker products of matrices.

Moreover, the numerous formulas and facts in this article can be extended to the situations for dilation factorizations of multiple Kronecker products of matrices, which can help us a great deal in producing more impressive and useful contributions of researches related to Kronecker products of matrices and developing other relevant mathematical techniques applicable to solving practical topics. Thus, they can be taken as a reference and a source of inspiration for deep understanding and exploration of numerous performances and properties of Kronecker products of matrices.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors would like to express their sincere thanks to anonymous reviewers for their helpful comments and suggestions.

Conflict of interest

The authors declare that they have no conflict of interest.

References

1. A. Barrlund, Efficient solution of constrained least squares problems with Kronecker product structure, *SIAM J. Matrix Anal. Appl.,* **19** (1994), 154–160. https://doi.org/10.1137/S0895479895295027
2. A. Ben-Israel, T. N. E. Greville, *Generalized inverses: Theory and applications*, 2 Eds., Springer, New York, 2003.
3. D. S. Bernstein, *Scalar, vector, and matrix mathematics: Theory, facts, and formulas-revised and expanded edition*, Princeton University Press, Princeton, NJ, 2018.
4. S. L. Campbell, C. D. Meyer, *Generalized inverses of linear transformations*, SIAM, Philadelphia, 2009.
5. C. Canuto, V. Simoncini, M. Verani, On the decay of the inverse of matrices that are sum of Kronecker products, *Linear Algebra Appl.,* **452** (2014), 21–39. https://doi.org/10.1016/j.laa.2014.03.029
6. T. T. Chen, W. Li, On Condition numbers for the weighted Moore-Penrose inverse and the weighted least squares problem involving Kronecker products, *East Asian J. Appl. Math.,* **4** (2014), 1–20. https://doi.org/10.4208/eajam.230313.070913a
7. J. Chuai, Y. Tian, Rank equalities and inequalities for Kronecker products of matrices with applications, *Appl. Math. Comput.,* **150** (2004), 129–137. https://doi.org/10.1016/S0096-3003(03)00203-0
8. S. Czesław, Inverting covariance matrices, *Discuss. Math. Probab. Statist.,* **26** (2006), 163–177.
9. H. Diao, R. Jayaram, Z. Song, W. Sun, D. P. Woodruff, Optimal sketching for Kronecker product regression and low rank approximation, *Adv. Neural Inf. Process. Syst.,* **32** (2019), 4739–4750.
10. H. Diao, W. Wang, Y. Wei, S. Qiao, On condition numbers for Moore-Penrose inverse and linear least squares problem involving Kronecker products, *Numer. Linear Algebra Appl.,* **20** (2013), 44–59. https://doi.org/10.1002/nla.1823
11. M. Fahrbach, G. Fu, M. Ghadiri, Subquadratic Kronecker regression with applications to tensor decomposition, *Adv. Neural Inf. Process. Syst.,* **35** (2022), 28776–28789.
12. D. W. Fausett, C. T. Fulton, Large least squares problems involving Kronecker products, *SIAM J. Matrix Anal. Appl.,* **15** (1994), 219–227. https://doi.org/10.1137/S0895479891222106

13. D. W. Fausett, C. T. Fulton, H. Hashish, Improved parallel QR method for large least squares problems involving Kronecker products, *J. Comput. Appl. Math.,* **78** (1997), 63–78. https://doi.org/10.1016/S0377-0427(96)00109-4

14. C. T. Fulton, L. Wu, Parallel algorithms for large least squares problems involving kronecker products, *Nonlin. Anal. Theor. Meth. Appl.,* **30** (1997), 5033–5040. https://doi.org/10.1016/S0362-546X(97)00189-2

15. A. Graham, *Kronecker products and matrix calculus with applications,* Wiley, New York, 1981.

16. S. J. Haberman, Direct products and linear models for complete factorial tables, *Ann. Statist.,* **3** (1975), 314–333. https://doi.org/10.1214/aos/1176343059

17. Y. Hardy, W. H. Steeb, *Matrix calculus, Kronecker product and tensor product: A practical approach to linear algebra, multilinear algebra and tensor calculus with software implementations,* 3 Eds., World Scientific Pub., 2019.

18. M. Huhtanen, Real linear Kronecker product operations, *Linear Algebra Appl.,* **418** (2006), 347–361. https://doi.org/10.1016/j.laa.2006.02.020

19. H. V. Jemderson, F. Pukelsheim, S. R. Searle, On the history of the Kronecker product, *Linear Multilinear A.,* **14** (1983), 113–120.

20. R. H. Koning, H. Neudecker, T. Wansbeek, Block Kronecker products and the vecb operator, *Linear Algebra Appl.,* **149** (1991), 165–184. https://doi.org/10.1016/0024-3795(91)90332-Q

21. P. Lancaster, M. Tismenetsky, *The theory of matrices: With applications,* 2 Eds., Academic Press, San Diego, 1985.

22. G. Marsaglia, G. P. H. Styan, Equalities and inequalities for ranks of matrices, *Linear Multilinear A.,* **2** (1974), 269–292.

23. L. Meng, L. Li, Condition numbers of the minimum norm least squares solution for the least squares problem involving Kronecker products, *AIMS Math.,* **6** (2021), 9366–9377. https://doi.org/10.3934/math.2021544

24. G. S. Rogers, Kronecker products in ANOVA—a first step, *Amer. Statist.,* **38** (1984), 197–202. https://doi.org/10.1080/00031305.1984.10483199

25. W. H. Steeb, Y. Hardy, *Matrix calculus and Kronecker product: A practical approach to linear and multilinear algebra,* World Scientific, River Edge, NJ, USA, 2011.

26. D. A. Stefonishin, On the generic rank of matrices composed of Kronecker products, *Doklady Math.,* **97** (2018), 125–128. https://doi.org/10.1134/S1064562418020060

27. Y. Tian, Some rank equalities and inequalities for Kronecker products of matrices, *Linear Multilinear A.,* **53** (2005), 445–454. https://doi.org/10.1080/03081080500055072

28. Y. Tian, Problem 815: Comparing ranges of Kronecker products of matrices, solution by E. Herman et al., *Coll. Math. J.,* **37** (2006), 397.

29. Y. Tian, On relationships between two linear subspaces and two orthogonal projectors, *Spec. Matrices,* **7** (2019), 142–212. https://doi.org/10.1515/spma-2019-0013

30. Y. Tian, Miscellaneous equalities for idempotent matrices with applications, *Open Math.,* **18** (2020), 671–714. https://doi.org/10.1515/math-2020-0147

31. Y. Tian, G. P. H. Styan, Rank equalities for idempotent and involutory matrices, *Linear Algebra Appl.,* **335** (2001), 101–117.

32. C. F. Van Loan, The ubiquitous Kronecker product, *J. Comp. Appl. Math.,* **123** (2000), 85–100. https://doi.org/10.1016/S0377-0427(00)00393-9

33. C. F. Van Loan, N. Pitsianis, *Approximation with Kronecker products*, Linear algebra for large scale and real-time applications, Springer, Dordrecht, **232** (1993), 293–314. https://doi.org/10.1007/978-94-015-8196-7_17

34. H. Zhang, F. Ding, On the Kronecker products and their applications, *J. Appl. Math.,* **13** (2013), 296185.

AIMS Press