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Research article

Biderivations of the extended Schrödinger-Virasoro Lie algebra

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Abstract: Let $\widetilde{\mathfrak{sv}}$ be the extended Schrödinger-Virasoro Lie algebra. In this paper, we consider the skew-symmetric biderivations of the extended Schrödinger-Virasoro Lie algebra. We prove that all biderivations of $\widetilde{\mathfrak{sv}}$ are inner. Based on this result, we show that all linear commuting maps on $\widetilde{\mathfrak{sv}}$, which have the form $\psi(x) = \lambda x$, are standard.

Keywords: extended Schrödinger-Virasoro Lie algebra; biderivation; skew-symmetric; inner biderivation; linear commuting map
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1. Introduction

Throughout the paper, we denote by $\mathbb{C}$ and $\mathbb{Z}$ the sets of complex numbers and integers, respectively. All vector spaces and algebras are over $\mathbb{C}$. The extended Schrödinger-Virasoro Lie algebra $\widetilde{\mathfrak{sv}}$ is an infinite-dimensional algebra that was introduced in [1] in the context of the two-dimensional conformal field theory and statistical physics, and $\widetilde{\mathfrak{sv}}$ can be viewed as an extension of the Schrödinger-Virasoro Lie algebra by a conformal current with conformal weight 1. The Schrödinger-Virasoro Lie algebra $\mathfrak{sv}$, originally introduced by Henkel [2], has been widely studied in [3–8] in recent years.

The extended Schrödinger-Virasoro Lie algebra $\widetilde{\mathfrak{sv}}$ is a vector space spanned by a basis $\{L_n, M_n, N_n, Y_{n+\frac{1}{2}} | n \in \mathbb{Z}\}$ with the following brackets

$$[L_m, L_n] = (n-m)L_{m+n},\ [M_m, M_n] = 0,\ [N_m, N_n] = 0,$$

$$[Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}] = (n-m)M_{m+n+1},\ [L_m, M_n] = nM_{m+n},$$

$$[L_m, N_n] = nN_{m+n},\ [L_m, Y_{n+\frac{1}{2}}] = (n + \frac{1-m}{2})Y_{m+n+\frac{1}{2}},\ [N_m, M_n] = 2M_{m+n},$$

$$[N_m, Y_{n+\frac{1}{2}}] = Y_{m+n+\frac{1}{2}},\ [M_m, Y_{n+\frac{1}{2}}] = 0,$$

where $n,\ m \in \mathbb{Z}$. It is clear that $\widetilde{\mathfrak{sv}}$ is finitely generated with a set of generators $\{L_{-2}, L_{-1}, L_1, L_2, N_1, Y_{\frac{1}{2}}\}$ and is a perfect Lie algebra, i.e., $[\widetilde{\mathfrak{sv}}, \widetilde{\mathfrak{sv}}] = \widetilde{\mathfrak{sv}}$ [9]. Clearly, the center of $\widetilde{\mathfrak{sv}}$ is zero, i.e., $Z(\widetilde{\mathfrak{sv}})$=0.

In [9], authors studied the derivations, the central extensions and the automorphism group of the extended Schrödinger-Virasoro Lie algebra. In [10], Lie bialgebra structures on the extended Schrödinger-Virasoro Lie algebra were classified. n-derivations of the extended Schrödinger-Virasoro Lie algebra were investigated in [11], and the main result when $n = 2$ was applied to characterize the linear commuting maps and the commutative post-Lie algebra structures on $\widetilde{\mathfrak{sv}}$. In this article, we will study the skew-symmetric biderivations and the linear commuting maps of the extended Schrödinger-Virasoro Lie algebra.

It is well know that the derivation algebra of an algebra A plays an important role in the study of the structure of A. As generalizations of derivations, the study of biderivations was initiated by Brešar [12]. In [13], Wang et al. introduced the notion of biderivation of a Lie algebra and showed that the skew-symmetric biderivation of finite-dimensional complex simple Lie algebra is inner. There has been a lot of interest in studying biderivations on the Schrödinger-Virasoro Lie algebra, the conformal Galilei algebra, Kac-Moody algebras and the deformative Schrödinger-Virasoro Lie algebra in [6,14–16], respectively. The case of the deformative Schrodinger-Virasoro Lie algebras yield examples for skew biderivations that are not inner [16].

Let $\mathfrak{L}$ be a Lie algebra. We call a bilinear map $\varphi : \mathfrak{L} \times \mathfrak{L} \longrightarrow \mathfrak{L}$ a biderivation if it's a derivation with respect to both components:

$$\varphi([x, y], z) = [x, \varphi(y, z)] + [\varphi(x, z), y], \tag{1.1}$$

$$\varphi(x, [y, z]) = [\varphi(x, y), z] + [y, \varphi(x, z)], \forall x, y, z \in \mathfrak{L}.$$

Moreover, a biderivation φ is called skew-symmetric if $\varphi(x, y) = -\varphi(y, x), \forall x, y \in \mathfrak{L}$. The biderivation $\varphi_\lambda : \mathfrak{L} \times \mathfrak{L} \longrightarrow \mathfrak{L}$ for $\lambda \in F$, satisfying $\varphi_\lambda(x, y) = \lambda[x, y]$, is called inner.

A map $\psi : \mathfrak{L} \longrightarrow \mathfrak{L}$ is called a linear commuting map if $[\psi(x), x] = 0$ for all $x \in \mathfrak{L}$. Linear commuting maps on the Schrödinger-Virasoro Lie algebra, the conformal Galilei algebra, Kac-Moody algebras and the deformative Schrödinger-Virasoro Lie algebra were extensively studied in [6,14–16], respectively.

The paper is organized as follows. We will introduce some basic conclusions on biderivations of $\widetilde{\mathfrak{sv}}$ in Section 2. In Section 3, we will prove that every biderivation of $\widetilde{\mathfrak{sv}}$ is inner, and then we will give the form of each linear commuting map on $\widetilde{\mathfrak{sv}}$ in Section 4. In Section 5, in order to find some differences, we compare our main results with those for the Schrödinger-Virasoro Lie algebra in [6] and the deformative Schrödinger-Virasoro Lie algebras in [16].

Throughout this paper, we work over the field $\mathbb{C}$.

2. General results on biderivations of Lie algebra $\widetilde{\mathfrak{sv}}$

In this section, we give some general results on skew-symmetric biderivations of $\widetilde{\mathfrak{sv}}$.

Lemma 2.1. *Let φ be a skew-symmetric biderivation on $\widetilde{\mathfrak{sv}}$, then*

$$[\varphi(x, y), [u, v]] = [[x, y], \varphi(u, v)], \forall x, y, u, v \in \widetilde{\mathfrak{sv}}.$$

In particular, $[\varphi(x,y),[x,y]]=0$.

The proof is similar to that of Corollary 2.2 in [7].

Lemma 2.2. *Suppose that φ is a skew-symmetric biderivation on $\widetilde{\mathfrak{sv}}$. If $[x,y]=0$ for $x,y\in\widetilde{\mathfrak{sv}}$, then $\varphi(x,y)=0$.*

Proof. By Lemma 2.1, we have

$$[\varphi(x,y),[u,v]]=[[x,y],\varphi(u,v)],$$

for all $u,v,x,y\in\widetilde{\mathfrak{sv}}$. Since $[x,y]=0$, then $[\varphi(x,y),[u,v]]=0$. Thus, $\varphi(x,y)$ commutes with $[\widetilde{\mathfrak{sv}},\widetilde{\mathfrak{sv}}]$. Since $[\widetilde{\mathfrak{sv}},\widetilde{\mathfrak{sv}}]$ coincides with the Lie algebra $\widetilde{\mathfrak{sv}}$ [9], we get that $\varphi(x,y)\in Z(\widetilde{\mathfrak{sv}})=0$. This concludes the proof. □

3. Skew-symmetric biderivations of $\widetilde{\mathfrak{sv}}$

Theorem 3.1. *Let φ be a skew-symmetric biderivation of $\widetilde{\mathfrak{sv}}$. We have*

$$\varphi(x,y)=\lambda[x,y],\ \forall x,y\in\widetilde{\mathfrak{sv}},$$

where $\lambda\in\mathbb{C}$.

Proof. This will be completed by verifying the following arguments.

Claim 1. *There exists $\lambda\in\mathbb{C}$ such that*

$$\varphi(L_m,L_n)=\lambda(n-m)L_{m+n}=\lambda[L_m,L_n],\forall m,n\in\mathbb{Z}.$$

We write $\varphi(L_m,L_n)$ in terms of the basis as follows

$$\varphi(L_m,L_n)=\sum_{i\in\mathbb{Z}}a^{(1)}_{i,m,n}L_i+\sum_{j\in\mathbb{Z}}b^{(1)}_{j,m,n}M_j+\sum_{k\in\mathbb{Z}}c^{(1)}_{k,m,n}N_k+\sum_{l\in\mathbb{Z}}d^{(1)}_{l,m,n}Y_{l+\frac{1}{2}},$$

where $a^{(1)}_{i,m,n},b^{(1)}_{j,m,n},c^{(1)}_{k,m,n},d^{(1)}_{l,m,n}\in\mathbb{C}$, $i,j,k,l,m,n\in\mathbb{Z}$.

If $n=m$, then $[L_m,L_n]=0$, and based on Lemma 2.2 we have $\varphi(L_m,L_n)=0$, so this claim holds.

Next, we assume $n\neq m$. By Lemma 2.1, we have

$$\frac{1}{n-m}[[L_m,L_n],\varphi(L_m,L_n)]=0,$$

then we get

$$[L_{m+n},\sum_{i\in\mathbb{Z}}a^{(1)}_{i,m,n}L_i+\sum_{j\in\mathbb{Z}}b^{(1)}_{j,m,n}M_j+\sum_{k\in\mathbb{Z}}c^{(1)}_{k,m,n}N_k+\sum_{l\in\mathbb{Z}}d^{(1)}_{l,m,n}Y_{l+\frac{1}{2}}]=0,$$

then

$$\sum_{i\in\mathbb{Z}}a^{(1)}_{i,m,n}(i-m-n)L_{m+n+i}+\sum_{j\in\mathbb{Z}}b^{(1)}_{j,m,n}jM_{m+n+j}+\sum_{k\in\mathbb{Z}}c^{(1)}_{k,m,n}kN_{m+n+k}+\sum_{l\in\mathbb{Z}}d^{(1)}_{l,m,n}(l+\frac{1-m-n}{2})Y_{m+n+l+\frac{1}{2}}=0.$$

Considering the coefficients of terms 'L', 'M', 'N' and 'Y', we conclude that

$$a^{(1)}_{i,m,n}(i-m-n)=0,\ b^{(1)}_{j,m,n}j=0,\ c^{(1)}_{k,m,n}k=0,\ d^{(1)}_{l,m,n}(l+\frac{1-m-n}{2})=0.$$

Thus, $a^{(1)}_{i,m,n}=0$ if $m+n\neq i$, $b^{(1)}_{j,m,n}=0$ if $j\neq 0$, $c^{(1)}_{k,m,n}=0$ if $k\neq 0$, $d^{(1)}_{l,m,n}=0$ if $l\neq \frac{m+n-1}{2}$. Then,

$$\varphi(L_m,L_n)=a^{(1)}_{m+n,m,n}L_{m+n}+b^{(1)}_{0,m,n}M_0+c^{(1)}_{0,m,n}N_0+d^{(1)}_{\frac{m+n-1}{2},m,n}Y_{\frac{m+n-1}{2}+\frac{1}{2}}.$$

By Lemma 2.1, we obtain that

$$[\varphi(L_m,L_n),[L_0,L_1]]=[[L_m,L_n],\varphi(L_0,L_1)],$$

so

$$[a^{(1)}_{m+n,m,n}L_{m+n}+b^{(1)}_{0,m,n}M_0+c^{(1)}_{0,m,n}N_0+d^{(1)}_{\frac{m+n-1}{2},m,n}Y_{\frac{m+n-1}{2}+\frac{1}{2}},L_1]=[(n-m)L_{m+n},a^{(1)}_{1,0,1}L_1+b^{(1)}_{0,0,1}M_0+c^{(1)}_{0,0,1}N_0+d^{(1)}_{0,0,1}Y_{\frac{1}{2}}],$$

and we get

$$\begin{aligned}&a^{(1)}_{m+n,m,n}(1-m-n)L_{m+n+1}+d^{(1)}_{\frac{m+n-1}{2},m,n,}\frac{m+n-1}{2}Y_{\frac{m+n}{2}+1}\\&=a^{(1)}_{1,0,1}(n-m)(1-m-n)L_{m+n+1}+d^{(1)}_{0,0,1}(n-m)\frac{1-m-n}{2}Y_{m+n+\frac{1}{2}}.\end{aligned}$$

Considering the two sides of the equation, we have that $a^{(1)}_{m+n,m,n}=(n-m)a^{(1)}_{1,0,1}$ if $m+n\neq 1$, $d^{(1)}_{\frac{m+n-1}{2},m,n}=0$ if $m+n\neq 1$ and $d^{(1)}_{0,0,1}=0$. Then,

$$\varphi(L_m,L_n)=a^{(1)}_{m+n,m,n}L_{m+n}+b^{(1)}_{0,m,n}M_0+c^{(1)}_{0,m,n}N_0.$$

By Lemma 2.1, we obtain that

$$[\varphi(L_m,L_n),[L_0,L_2]]=[[L_m,L_n],\varphi(L_0,L_2)];$$

that is,

$$[a^{(1)}_{m+n,m,n}L_{m+n}+b^{(1)}_{0,m,n}M_0+c^{(1)}_{0,m,n}N_0,2L_2]=[(n-m)L_{m+n},a^{(1)}_{2,0,2}L_2+b^{(1)}_{0,0,2}M_0+c^{(1)}_{0,0,2}N_0],$$

thus, we get

$$2a^{(1)}_{m+n,m,n}(2-m-n)L_{m+n+2}=a^{(1)}_{2,0,2}(n-m)(2-m-n)L_{m+n+2}.$$

We have that $a^{(1)}_{m+n,m,n}=\frac{1}{2}(n-m)a^{(1)}_{2,0,2}$ if $m+n\neq 2$, so $a^{(1)}_{m+n,m,n}=(n-m)a^{(1)}_{1,0,1}$,

$$\varphi(L_m,L_n)=(n-m)a^{(1)}_{1,0,1}L_{m+n}+b^{(1)}_{0,m,n}M_0+c^{(1)}_{0,m,n}N_0.$$

Taking $\lambda=a^{(1)}_{1,0,1}$, thus

$$\varphi(L_m,L_n)=(n-m)\lambda L_{m+n}+b^{(1)}_{0,m,n}M_0+c^{(1)}_{0,m,n}N_0. \tag{3.1}$$

Next, we will prove that $b^{(1)}_{0,m,n} = 0$ and $c^{(1)}_{0,m,n} = 0$. We write $\varphi(L_m, M_n)$ in terms of the basis as follows

$$\varphi(L_m, M_n) = \sum_{i\in\mathbb{Z}} a^{(2)}_{i,m,n} L_i + \sum_{j\in\mathbb{Z}} b^{(2)}_{j,m,n} M_j + \sum_{k\in\mathbb{Z}} c^{(2)}_{k,m,n} N_k + \sum_{l\in\mathbb{Z}} d^{(2)}_{l,m,n} Y_{l+\frac{1}{2}},$$

where $a^{(2)}_{i,m,n}, b^{(2)}_{j,m,n}, c^{(2)}_{k,m,n}, d^{(2)}_{l,m,n} \in \mathbb{C}$, $i, j, k, l, m, n \in \mathbb{Z}$. If $n = 0$, then $[L_m, M_0] = 0$, and based on Lemma 2.2 we have $\varphi(L_m, M_0) = 0$. We let $n \neq 0$. By Lemma 2.1, we also have

$$[\varphi(L_m, M_n), [L_{-1}, L_1]] = [[L_m, M_n], \varphi(L_{-1}, L_1)];$$

that is,

$$[\sum_{i\in\mathbb{Z}} a^{(2)}_{i,m,n} L_i + \sum_{j\in\mathbb{Z}} b^{(2)}_{j,m,n} M_j + \sum_{k\in\mathbb{Z}} c^{(2)}_{k,m,n} N_k + \sum_{l\in\mathbb{Z}} d^{(2)}_{l,m,n} Y_{l+\frac{1}{2}}, 2L_0] = [nM_{m+n}, 2\lambda L_0 + b^{(1)}_{0,-1,1} M_0 + c^{(1)}_{0,-1,1} N_0],$$

then,

$$\sum_{i\in\mathbb{Z}} a^{(2)}_{i,m,n} i L_{i+1} + \sum_{j\in\mathbb{Z}} b^{(2)}_{j,m,n} j M_{j+1} + \sum_{k\in\mathbb{Z}} c^{(2)}_{k,m,n} k N_{k+1} + \sum_{l\in\mathbb{Z}} d^{(2)}_{l,m,n} (l + \frac{1}{2}) Y_{l+\frac{1}{2}} = n\lambda(m+n) M_{m+n+1} + nc^{(1)}_{0,-1,1} M_{m+n}.$$

Hence,

$$\sum_{j\in\mathbb{Z}} b^{(2)}_{j,m,n} j M_j = n\lambda(m+n) M_{m+n} + nc^{(1)}_{0,-1,1} M_{m+n}.$$

This means $b^{(2)}_{m+n,m,n}(m+n) = n\lambda(m+n) + nc^{(1)}_{0,-1,1}$ if $j = m+n$. Based on the arbitrariness of m, n, we obtain $c^{(1)}_{0,-1,1} = 0$. Thus, $c^{(1)}_{0,m,n} = 0$.

We write $\varphi(L_m, N_n)$ in terms of the basis as follows

$$\varphi(L_m, N_n) = \sum_{i\in\mathbb{Z}} a^{(3)}_{i,m,n} L_i + \sum_{j\in\mathbb{Z}} b^{(3)}_{j,m,n} M_j + \sum_{k\in\mathbb{Z}} c^{(3)}_{k,m,n} N_k + \sum_{l\in\mathbb{Z}} d^{(3)}_{l,m,n} Y_{l+\frac{1}{2}},$$

where $a^{(3)}_{i,m,n}, b^{(3)}_{j,m,n}, c^{(3)}_{k,m,n}, d^{(3)}_{l,m,n} \in \mathbb{C}$, $i, j, k, l, m, n \in \mathbb{Z}$. If $n = 0$, then $[L_m, N_0] = 0$, and based on Lemma 2.2 we have $\varphi(L_m, N_0) = 0$. We let $n \neq 0$. By Lemma 2.1, we have

$$[\varphi(L_m, N_n), [L_{-1}, L_1]] = [[L_m, N_n], \varphi(L_{-1}, L_1)];$$

that is,

$$[\sum_{i\in\mathbb{Z}} a^{(3)}_{i,m,n} L_i + \sum_{j\in\mathbb{Z}} b^{(3)}_{j,m,n} M_j + \sum_{k\in\mathbb{Z}} c^{(3)}_{k,m,n} N_k + \sum_{l\in\mathbb{Z}} d^{(3)}_{l,m,n} Y_{l+\frac{1}{2}}, 2L_0] = [nN_{m+n}, 2\lambda L_0 + b^{(1)}_{0,-1,1} M_0],$$

then

$$-\sum_{i\in\mathbb{Z}} a^{(3)}_{i,m,n} i L_i - \sum_{j\in\mathbb{Z}} b^{(3)}_{j,m,n} j M_j - \sum_{k\in\mathbb{Z}} c^{(3)}_{k,m,n} k N_k - \sum_{l\in\mathbb{Z}} d^{(3)}_{l,m,n} (l + \frac{1}{2}) Y_{l+\frac{1}{2}} = -n\lambda(m+n) N_{m+n} + nb^{(1)}_{0,-1,1} M_{m+n}.$$

It follows that $a^{(3)}_{i,m,n} = 0$ if $i \neq 0$ and $c^{(3)}_{k,m,n} = n\lambda$ if $k = m+n \neq 0$. Since $l + \frac{1}{2} \neq 0$, then $d^{(3)}_{l,m,n} = 0$. We obtain that

$$-\sum_{j\in Z} b^{(3)}_{j,m,n} j M_j = nb^{(1)}_{0,-1,1} M_{m+n}.$$

This means $b^{(3)}_{j,m,n} = 0$ if $j \neq m+n$. Now, we conclude that

$$\varphi(L_m, N_n) = a^{(3)}_{0,m,n} L_0 + b^{(3)}_{m+n,m,n} M_{m+n} + n\lambda N_{m+n} + c^{(3)}_{0,m,n} N_0.$$

By Lemma 2.1, we have

$$[\varphi(L_m, N_n), [L_0, L_1]] = [[L_m, N_n], \varphi(L_0, L_1)];$$

that is,

$$[a^{(3)}_{0,m,n} L_0 + b^{(3)}_{m+n,m,n} M_{m+n} + n\lambda N_{m+n} + c^{(3)}_{0,m,n} N_0, L_1] = [nN_{m+n}, \lambda L_1 + b^{(1)}_{0,0,1} M_0],$$

then,

$$a^{(3)}_{0,m,n} L_1 - (m+n) b^{(3)}_{m+n,m,n} M_{m+n+1} - n(m+n)\lambda N_{m+n+1} = -n\lambda(m+n) N_{m+n+1} + 2nb^{(1)}_{0,0,1} N_{m+n}.$$

It follows that $b^{(1)}_{0,0,1} = 0$. Thus, $b^{(1)}_{0,m,n} = 0$.

Taking $c^{(1)}_{0,m,n} = 0$ and $b^{(1)}_{0,m,n} = 0$ into (3.1), we get the following

$$\varphi(L_m, L_n) = \lambda(n-m) L_{m+n} = \lambda[L_m, L_n]. \tag{3.2}$$

Claim 2. $\varphi(x, y) = \lambda[x, y]$ *for all* $x, y \in \widetilde{\mathfrak{sv}}$.

Based on (3.2), we can assume that $\varphi_1(x, y) = \varphi(x, y) - \lambda[x, y]$. Thus, $\varphi_1(x, y)$ is also a skew-symmetric biderivation of $\widetilde{\mathfrak{sv}}$, and $\varphi_1(L_i, L_j) = 0$. The result to be proved now is that this new biderivation φ_1 is zero.

Lemma 2.1 is applied to $[\varphi_1(x, y), [L_m, L_n]]$. Since

$$[\varphi_1(x, y), [L_m, L_n]] = [[x, y], \varphi_1(L_m, L_n)],$$

then

$$(n-m)[\varphi_1(x, y), L_{m+n}] = 0.$$

Hence, we have $[\varphi_1(x, y), L_i] = 0$ for all $x, y \in \widetilde{\mathfrak{sv}}$ *and* $i \in \mathbb{Z}$. This shows that $\varphi_1(x, y)$ belongs to $< M_0, N_0 >$, where $< M_0, N_0 >$ is the subspace generated by M_0, N_0, for any $x, y \in \widetilde{\mathfrak{sv}}$.

Based on (1.1), we have

$$\varphi_1([L_i, u], v) = [L_i, \varphi_1(u, v)] + [\varphi_1(L_i, v), u].$$

Thus,

$$\varphi_1([L_i, u], v) = [\varphi_1(L_i, v), u] \tag{3.3}$$

for any $u, v \in \widetilde{\mathfrak{sv}}$ and $i \in \mathbb{Z}$. The left hand side of (3.3) is in $< M_0, N_0 >$ and the righthand side of (3.3) is in $< [M_0, u], [N_0, u] >$. Thus, there exists complex numbers a, b, c, d such that

$$\varphi_1([L_i, u], v) = aM_0 + bN_0, \tag{3.4}$$

$$[\varphi_1(L_i, v), u] = c[M_0, u] + d[N_0, u], \tag{3.5}$$

for any $u, v \in \widetilde{\mathfrak{sv}}$ and $i \in \mathbb{Z}$. It follows that

$$aM_0 + bN_0 = c[M_0, u] + d[N_0, u]. \tag{3.6}$$

By replacing u by $Y_{\frac{1}{2}}$ in (3.6)

$$aM_0 + bN_0 = c[M_0, Y_{\frac{1}{2}}] + d[N_0, Y_{\frac{1}{2}}] = dY_{\frac{1}{2}},$$

we get that $a = b = d = 0$. Hence, $\varphi_1([L_i, u], v) = 0$ for any $u, v \in \widetilde{\mathfrak{sv}}$ and $i \in \mathbb{Z}$.

On the other hand, based on

$$[L_0, L_n] = nL_n,\ [L_0, M_n] = nM_n,\ [L_0, N_n] = nN_n,\ [L_0, Y_{n+\frac{1}{2}}] = (n + \frac{1}{2})Y_{n+\frac{1}{2}},$$

we have that

$$Y_{n+\frac{1}{2}} = \frac{1}{(n + \frac{1}{2})}[L_0, Y_{n+\frac{1}{2}}],$$

and if $n \neq 0$,

$$L_n = \frac{1}{n}[L_0, L_n],\ M_n = \frac{1}{n}[L_0, M_n],\ N_n = \frac{1}{n}[L_0, N_n].$$

So, for any $x \in \{L_n,\ M_n,\ N_n \mid n \neq 0,\ n \in \mathbb{Z}\} \bigcup \{Y_{n+\frac{1}{2}} \mid n \in \mathbb{Z}\}$, there exists complex numbers $a \neq 0$ such that $x = a[L_0, x]$. Then,

$$\varphi_1(x, y) = a\varphi_1([L_0, x], y) = 0,\ for\ y \in \widetilde{\mathfrak{sv}}.$$

Moreover, $L_0 = \frac{1}{2}[L_{-1}, L_1]$, $M_0 = [L_{-1}, M_1]$, $N_0 = [L_{-1}, N_1]$. Hence,

$$\varphi_1(L_0, y) = \frac{1}{2}\varphi_1([L_{-1}, L_1], y) = 0,$$
$$\varphi_1(M_0, y) = \varphi_1([L_{-1}, M_1], y) = 0,$$
$$\varphi_1(N_0, y) = \varphi_1([L_{-1}, N_1], y) = 0,$$

for $y \in \widetilde{\mathfrak{sv}}$. Therefore, we have checked all cases for $\varphi_1(x, y) = 0$. This completes the proof.

□

4. Linear commuting map of $\widetilde{\mathfrak{sv}}$

In this section, we study the linear commuting maps of $\widetilde{\mathfrak{sv}}$ based on Theorem 3.1. Recall the concept of linear commuting map ψ on the Lie algebra $\widetilde{\mathfrak{sv}}$. We have

$$[\psi(x), x] = 0$$

for all $x \in \widetilde{\mathfrak{sv}}$. Undoubtedly, if ψ on $\widetilde{\mathfrak{sv}}$ is such a map, then,

$$[\psi(x), y] = [x, \psi(y)]$$

for all $x, y \in \widetilde{\mathfrak{sv}}$.

A linear commuting map $\psi(x)$ on $\mathfrak{L}$ is said to be standard if it has the following form $\psi(x) = \lambda x + f(x), \forall x \in \mathfrak{L}$, where $\lambda \in \mathbb{C}$, $f : \mathfrak{L} \to Z(\mathfrak{L})$. All commuting maps of other forms are called non-standard. If $Z(\mathfrak{L}) = 0$, then $f(x) = 0$, and thus ψ is standard if and only if $\psi(x) = \lambda x$ for $\forall x \in \mathfrak{L}$.

Theorem 4.1. *Let ψ be a linear commuting map of $\widetilde{\mathfrak{sv}}$. Then, ψ has the following form*

$$\psi(x) = \lambda x$$

for all $x \in \widetilde{\mathfrak{sv}}$, where $\lambda \in \mathbb{C}$. This means all commuting maps of $\widetilde{\mathfrak{sv}}$ are standard.

The proof of this theorem is similar to that of Theorem 3.1 in [6].

5. Discussion

In Tables 1 and 2, we denote $\mathfrak{sv}$ by the Schrödinger-Virasoro Lie algebra and denote $\mathfrak{L}(\lambda, \mu, s)$ by the deformative Schrödinger-Virasoro Lie algebras. Comparing our main result with those for $\mathfrak{sv}$ in [6] and $\mathfrak{L}(\lambda, \mu, s)$ in [16], we find some differences.

Table 1. (Q1) Whether all the skew-symmetric biderivations of $\mathfrak{L}$ are inner.

	Answer for question Q1
$\mathfrak{sv}$	inner (see [6])
$\mathfrak{L}(\lambda, \mu, s)$	exist non-inner for the certain $\mathfrak{L}(\lambda, \mu, s)$ (see [16])
$\widetilde{\mathfrak{sv}}$	inner (this paper)

Table 2. (Q2) Whether all the linear commuting maps on $\mathfrak{L}$ are standard.

	Answer for question Q2
$\mathfrak{sv}$	non-standard (see [6])
$\mathfrak{L}(\lambda, \mu, s)$	non-standard (see [16])
$\widetilde{\mathfrak{sv}}$	standard (this paper)

6. Conclusions

In this work, we mainly determined the skew-symmetric biderivations of the Lie algebra $\widetilde{\mathfrak{sv}}$. The results showed that all the skew-symmetric biderivations of the Lie algebra $\widetilde{\mathfrak{sv}}$ are inner. Furthermore, we proved that every linear commuting map ψ on $\widetilde{\mathfrak{sv}}$ had the form $\psi(x) = \lambda x$, where $\lambda \in \mathbb{C}$, which indicated that all linear commuting maps of $\widetilde{\mathfrak{sv}}$ are standard.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declare no conflict of interest.

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