Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(12): 28753–28765.
DOI: 10.3934/math.20231472
Received: 21 August 2023
Revised: 10 October 2023
Accepted: 17 October 2023
Published: 23 October 2023

Research article

Local Hölder continuity of inverse variation-inequality problem constructed by non-Newtonian polytropic operators in finance

Jia Li and Zhipeng Tong*

School of Management, Shenyang Urban Construction University, Shenyang 110167, China

* **Correspondence:** Email: tongzhipeng_1995@163.com.

Abstract: This paper aims to explore the inverse variation-inequality problems of a specific type of degenerate parabolic operators in a non-divergence form. These problems have significant implications in financial derivative pricing. The study focuses on analyzing the Hölder continuity of weak solutions by employing cut-off factors.

Keywords: inverse variation-inequality problem; non-divergence degenerate parabolic operator; existence; local Hölder continuity
Mathematics Subject Classification: 35K99, 97M30

1. Introduction

In recent years, the study of the following variational inequality has attracted the interest of scholars:

$$\begin{cases} \min\{Lu, u-u_0\} \geq 0, & (x,t) \in \Omega_T, \\ u(0,x) = u_0(x), & x \in \Omega, \\ u(t,x) = \frac{\partial u}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0,T), \end{cases} \tag{1}$$

where Lu is a linear parabolic operator or a degenerate parabolic operator. Due to the satisfaction of the condition $Lu \geq 0$ in Ω_T, researchers have found it convenient to use the comparison principle to obtain upper bounds for the solutions. This approach has been combined with limit methods [1,2], the Leray-Schauder fixed point theorem [3,4], or semi-discrete methods [5,6] to prove the existence of solutions. Additionally, some scholars have started from weak solutions and obtained integral inequalities for the difference between two weak solutions, analyzing the stability and uniqueness of weak solutions with respect to initial values [7–9]. The authors from [10–12] have demonstrated the explosive nature of weak solutions under certain special conditions through energy estimates. The authors have obtained Caccioppoli inequalities that match the variational inequality by analyzing integral inequalities of weak solutions in locally cylindrical regions, and subsequently studied the Schauder estimates for weak solutions [13, 14].

In recent years, research on the pricing of financial derivative products with embedded early exercise provisions has found that inverse variation-inequalities, such as the one shown below, are more suitable for

$$\begin{cases} \min\{-Lu, u - u_0\} \geq 0, & (x,t) \in \Omega_T, \\ u(0,x) = u_0(x), & x \in \Omega, \\ u(t,x) = \frac{\partial u}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0,T). \end{cases} \tag{2}$$

For example, researches from [15, 16] analyzed the pricing problem of American options under the Black-Scholes model, and the value was reduced to the free boundary problem of the variation inequality (2). Therefore, the parabolic operator Lu (denoted as $L_{BS}u$) satisfies

$$L_{BS} = \partial_t u - \frac{1}{2}\sigma^2 \partial_{xx} u + r\partial_x u - ru, \tag{3}$$

where σ represents the volatility of the underlying stock of the option, and r is the risk-free interest rate in the financial market. Based on the aforementioned financial background considerations, the author of this study investigates the inverse variation-inequality problems with the degenerate parabolic operator in non-divergence form

$$Lu = \partial_t u - u^\sigma \Delta_p u - \gamma u^{\sigma-1} |\nabla u|^p, \quad p > 2, \sigma > \gamma > 0. \tag{4}$$

Additionally, we impose the condition that the initial value u_0 satisfies $u_0 \in W_0^{1,p}(\Omega)$. For a recent study on inverse variational inequalities in a different context, please refer to [17]. In that study, an inverse quasi-variational inequality is solved using a dynamical system.

In this paper, we provide the weak solution to the variational inequality (1) and prove the existence of weak solutions. We also prove that the weak solution satisfies an energy inequality in a local cylindrical region, and based on this, we establish the Holder continuity and Harnack inequality of the weak solution. The study of such conclusions is usually focused on degenerate parabolic equation initial-boundary value problems, and research on variational inequalities is still rare.

Due to the fact that the inverse variational inequality (2) implies $Lu \leq 0$ in Ω_T, it no longer allows us to determine the upper bound of the solution u and establish the existence of weak solutions through the comparison principle, as in the traditional variational inequality (1). First, this difficulty is overcome by analyzing the energy upper bound of $(u - M_0)_+$. We can select a suitable M_0 as an upper bound for u, which is also an innovative aspect of this paper. Second, this paper also explores the Harnack inequality and Hölder continuity of weak solutions by analyzing the weak solutions of the reverse variational inequality (2) and combining it with the integral inequality of $(u - k)_\pm$. This analysis is further enhanced by the use of the cut-off factor and the selection of an appropriate k, which adds another innovative aspect to this study.

2. Existence of weak solution

This section is dedicated to addressing our specific problem: We begin by providing a clear definition of a nonnegative weak solution to Eq (1). To begin, it can be inferred from Eq (1) that

$$u \geq u_0 \geq 0 \text{ in } \Omega_T. \tag{5}$$

In fact, by utilizing inequality (1) once again, we have $Lu \leq 0$ in Ω_T. Furthermore, since $u(t,x) = u_0 = 0$ in $\partial\Omega \times (0,T)$, when, using the comparison principle, we can conclude that (5) still holds.

Next, we analyze the upper bound of u. Let us choose a constant $M_0 > 0$ as a parameter. Multiplying both sides of $Lu \leq 0$ by $(u - M_0)_+$ and integrating over Ω yields (note that $(u - M_0)_+ \geq 0$),

$$\int_\Omega \partial_t u(u - M_0)_+ - u^\sigma \Delta_p u(u - M_0)_+ - \gamma u^{\sigma-1}|\nabla u|^p (u - M_0)_+ \mathrm{d}x \leq 0. \tag{6}$$

On one hand, when $u \geq M_0$ occurs, $\partial_t(u - M_0)_+ = \partial_t u$, thereby resulting in

$$\int_\Omega \partial_t u(u - M_0)_+ \mathrm{d}x = \int_\Omega \partial_t(u - M_0)(u - M_0)_+ \mathrm{d}x = \frac{1}{2}\partial_t \int_\Omega (u - M_0)_+^2 \mathrm{d}x. \tag{7}$$

On the other hand, when $u < M_0$ occurs, $(u - M_0)_+ = 0$ and $\partial_t(u - M_0)_+ = 0$, leading to

$$\int_\Omega \partial_t u(u - M_0)_+ \mathrm{d}x = 0. \tag{8}$$

Combining (6)–(8), we obtain

$$\frac{1}{2}\partial_t \int_\Omega (u - M_0)_+^2 \mathrm{d}x - \int_\Omega u^\sigma \Delta_p u(u - M_0)_+ + \gamma u^{\sigma-1}|\nabla u|^p (u - M_0)_+ \mathrm{d}x \leq 0. \tag{9}$$

Note that

$$\int_\Omega u^\sigma \Delta_p u(u - M_0)_+ + \gamma u^{\sigma-1}|\nabla u|^p (u - M_0)_+ \mathrm{d}x = 0,$$

when $u < M_0$, while

$$\begin{aligned} &- \int_\Omega u^\sigma \Delta_p u(u - M_0)_+ + \gamma u^{\sigma-1}|\nabla u|^p (u - M_0)_+ \mathrm{d}x \\ &= \int_\Omega u^\sigma |\nabla(u - M_0)_+|^p + (\sigma - \gamma) u^{\sigma-1}|\nabla u|^p (u - M_0)_+ \mathrm{d}x, \end{aligned}$$

when u is greater than or equal to M_0. Therefore, (9) implies the significance of

$$\frac{1}{2}\partial_t \int_\Omega (u - M_0)_+^2 \mathrm{d}x + \int_\Omega u^\sigma |\nabla(u - M_0)_+|^p + (\sigma - \gamma) u^{\sigma-1}|\nabla u|^p (u - M_0)_+ \mathrm{d}x \leq 0. \tag{10}$$

Due to $u \geq u_0 \geq 0$ and $\sigma - \gamma \geq 0$,

$$\int_\Omega u^\sigma |\nabla(u - M_0)_+|^p + (\sigma - \gamma) u^{\sigma-1}|\nabla u|^p (u - M_0)_+ \mathrm{d}x$$

is nonnegative. Combining (10), we have

$$\int_\Omega (u - M_0)_+^2 \mathrm{d}x \leq \int_\Omega (u_0 - M_0)_+^2 \mathrm{d}x. \tag{11}$$

Furthermore, due to $u_0 \in W_0^{1,p}$, when M_0 is sufficiently large,

$$\int_\Omega (u_0 - M_0)_+^2 \mathrm{d}x = 0.$$

In this case,

$$u \leq M_0 \text{ in } \Omega_T. \tag{12}$$

By combining (5) and (12), we can demonstrate that the inverse variational inequality (2) satisfies

$$0 \leq u \leq M_0 \text{ in } \Omega_T. \tag{13}$$

Therefore, in [12], before providing a weak solution to the inverse variational inequality (2), we first present a set of maximal monotone maps

$$G(\lambda) = \{\xi | \xi = 0, \lambda > 0; \xi \geq 0, \lambda = 0\}. \tag{14}$$

If $\xi \in G(u - u_0)$, it is easy to see that when $u > u_0$, $\xi = 0$; and in this case $Lu = 0$. When $u = u_0$, $\xi \geq 0$, and in this case we also have $Lu \geq 0$. This inspires us to use $Lu = \xi$ to construct a weak solution for the variational inequality (2).

Definition 2.1. A pair (u, ξ) is considered a generalized solution of the inverse variation-inequality (2) if it satisfies the following conditions:

(a) $u \in L^\infty(0, T, H^1(\Omega))$, $\partial_t u \in L^\infty(0, T, L^2(\Omega))$.

(b) $\xi \in G$ for any $(x, t) \in \Omega_T$.

(c) For fixed $\nu = \frac{\sigma-1}{p} + 1$ and for every test-function $\varphi \in C^1(\bar{\Omega}_T)$, there exists an equality

$$\int\int_{\Omega_T} \partial_t u\varphi + \frac{1}{\nu^{p-1}} u^\nu |\nabla u^\nu|^{p-2} \nabla u^\nu \nabla\varphi + \frac{\sigma - \gamma}{\nu^p} |\nabla u^\nu|^p \varphi \mathrm{d}x\mathrm{d}t = \int\int_{\Omega_T} \xi\varphi \mathrm{d}x\mathrm{d}t.$$

By utilizing (13) and (14), combined with a standard energy method from [2, 12], we can establish the existence of a weak solution for the inverse variational inequality (2).

Theorem 2.1. Assuming that $u_0 \in W_0^{1,p}(\Omega)$ in Ω_T, the inverse variational inequality (2) has a solution (u, ξ) within the class defined in Definition 2.1.

The final part of this section is dedicated to introducing some notation and presenting several previously established results, which will be used in the subsequent proof of the Hölder continuity. The detailed proof can be found in [17].

Lemma 2.1. Assume that $\{Y_n\}, n = 1, 2, 3, \cdots$ is a nonnegative sequence satisfying

$$Y_{n+1} \leq C b^n Y_n^{1+\alpha}, \quad C, b > 1, \alpha > 0.$$

If $Y_0 \leq C^{-1/\alpha} b^{-1/\alpha^2}$, then $Y_n \to 0, n \to \infty$.

Lemma 2.2. Assuming that $p \geq 2$, there exists a positive constant C such that

$$\int\int_{\Omega_T} |u|^p \mathrm{d}x\mathrm{d}t \leq C|\{u > 0\}|^{p/(N+p)} \|u\|^p_{L^p(\Omega_T)},$$

where C depends only on N and p.

3. Integral inequality

Along this section, we assume that u is a nonnegative weak solution to Eq (1) with $p \geq 2$. Our objective is to establish an integral inequality, which will be used to determine the Hölder continuity of the weak solution on the domain

$$Q = Q(\rho, \theta) = B_\rho(x_0) \times (t_0 - \theta, t_0),$$

where ρ and θ are positive undetermined constants. Of course, ρ and θ should be sufficiently small to ensure $Q \subset \Omega_T$. Let us define

$$\mu^+ = \underset{Q(2R,R^p)}{ess\sup} u, \quad \mu^- = \underset{Q(2R,R^p)}{ess\inf} u, \quad \omega = \underset{Q(2R,R^p)}{osc} u = \mu^+ - \mu^-,$$

$$R_n = \frac{1}{2}R + \frac{1}{2^{n+1}}R, \quad Q_n = Q(R_n, dR_n^p), \;\; d \in (0, 1],$$

and introduce the symbol

$$k_n^- = \mu^- + \frac{1}{2^{s_*+1}}\omega + \frac{1}{2^{s_*+n+1}}\omega, \quad k_n^+ = \mu^+ - \frac{1}{2^{s_*+1}}\omega - \frac{1}{2^{s_*+n+1}}\omega,$$

where s_* is a nonnegative undetermined constant. We obtain the Hölder estimate for the weak solution of inequality (2) by using the upper bound estimate of $\underset{Q(\frac{1}{2}R,d(\frac{1}{2}R)^p)}{ess\sup} u$ that includes ω. In order to estimate $\underset{Q(\frac{1}{2}R,d(\frac{1}{2}R)^p)}{ess\inf} u$, we construct $k_n)_-$ and simultaneously construct $k_n)_+$ to estimate $\underset{Q(\frac{1}{2}R,d(\frac{1}{2}R)^p)}{ess\sup} u$. In order to prove the Hölder continuity, s_* must satisfy the condition $s_* > 1$.

Lemma 3.1. Assuming $p \geq 2$ and $\nu = \frac{\sigma-1}{p} + 1$, one can infer

$$(u - k_n^-)_-^{\nu+1} \geq \left(\frac{2^{s_*}}{\omega}\right)^{p\nu-\nu-1} (u - k_n^-)_-^{p\nu}. \tag{15}$$

Proof. According to the definition of k_n^-, it is easy to obtain

$$(u - k_n^-)_- \leq \mu^+ - k_n^- = \frac{1}{2^{s_*+1}}\omega + \frac{1}{2^{s_*+n+1}}\omega \leq \frac{1}{2^{s_*}}\omega.$$

Since $(u - k_n^-)_-^{p\nu}$ reaches its maximum, when u takes the value μ^+,

$$\left(\frac{2^{s_*}}{\omega}\right)^{p\nu-\nu-1} (u - k_n^-)_-^{p\nu} \leq \left(\frac{2^{s_*}}{\omega}\right)^{p\nu-\nu-1} \left(\frac{\omega}{2^{s_*}}\right)^{p\nu} = \left(\frac{\omega}{2^{s_*}}\right)^{\nu+1} \tag{16}$$

holds. At this point, $(u - k_n^-)_-^{\nu+1}$ satisfies

$$(u - k_n^-)_-^{\nu+1} = \left(\frac{1}{2^{s_*+1}}\omega + \frac{1}{2^{s_*+n+1}}\omega\right)^{\nu+1} = \left(\frac{\omega}{2^{s_*}}\right)^{\nu+1}. \tag{17}$$

By combining Eqs (16) and (17), the result is proven to hold.

In order to achieve the desired outcome, a test function $w = \phi^p \times (u-k)_\pm^\nu$ is selected, resulting in

$$\begin{aligned}&\int\int_{\Omega_T} \tfrac{1}{\nu^{p-1}} u^\nu |\nabla u^\nu|^{p-2}\nabla u^\nu \nabla[\phi^p \times (u-k)_\pm^\mu] + \tfrac{\sigma-\gamma}{\nu^p}|\nabla u^\nu|^p \times \phi^p \times (u-k)_\pm^\nu \mathrm{d}x\mathrm{d}t \\ &= \int\int_{\Omega_T} \xi\varphi \mathrm{d}x\mathrm{d}t - \int\int_{\Omega_T} \partial_t u \times \phi^p \times (u-k)_\pm^\nu \mathrm{d}x\mathrm{d}t.\end{aligned} \tag{18}$$

Considering that $\int\int_{\Omega_T} \partial_t u \times \phi^p \times (u-k)_\pm^\nu \mathrm{d}x\mathrm{d}t$ is not suitable for integration calculations, the following transformation is performed:

$$\int_\Omega \partial_t(\phi^p \times (u-k)_\pm^{\nu+1})\mathrm{d}x = (\nu+1)\int_\Omega \phi^p \times (u-k)_\pm^\nu u_t \mathrm{d}x\mathrm{d}t + p\int_\Omega \phi^{p-1} \times \partial_t\phi \times (u-k)_\pm^{\nu+1}\mathrm{d}x. \tag{19}$$

In $\int_\Omega u^\nu |\nabla u^\nu|^{p-2}\nabla u^\nu \nabla[\phi^p \times (u-k)_\pm^\nu]\mathrm{d}x$, a differential transformation is applied to $\nabla[\phi^p \times (u-k)_\pm^\nu]$, resulting in

$$\begin{aligned}&\int_\Omega u^\nu |\nabla u^\nu|^{p-2}\nabla u^\nu \nabla[\phi^p \times (u-k)_\pm^\nu]\mathrm{d}x \\ &= \int_\Omega u^\nu \times |\nabla(u-k)_\pm^\nu|^p \times \phi^p \mathrm{d}x + \int_\Omega |\nabla u^\nu|^{p-2}\nabla u^\nu \times (u-k)_\pm^\nu \times u^\nu \times \nabla\phi^p \mathrm{d}x.\end{aligned} \tag{20}$$

Further utilizing the Hölder and Young inequalities, we can obtain the expression

$$\begin{aligned}&\left|\int_\Omega |\nabla u^\nu|^{p-2}\nabla u^\nu \times (u-k)_\pm^\nu \times u^\nu \nabla\phi^p \mathrm{d}x\right| \\ &\leq \tfrac{p-1}{p}\int_\Omega u^\nu \times |\nabla(u-k)_\pm|^p \times \phi^p \mathrm{d}x + \tfrac{1}{p}\int_\Omega (u-k)_\pm^{p\nu} \times u^\nu \times |\nabla\phi|^p \mathrm{d}x.\end{aligned} \tag{21}$$

Consequently, by combining Eqs (18)–(21), we can obtain the equation

$$\begin{aligned}&ess \sup_{t\in(t_0-\theta,t_0)} \int_\Omega (\phi^p \times (u-k)_\pm^{\nu+1})\mathrm{d}x + \tfrac{1}{\nu^{p-1}p}\int_{t_0-\theta}^{t_0}\int_\Omega u^\nu |\nabla(u-k)_\pm^\nu|^p \times \phi^p \mathrm{d}x\mathrm{d}t \\ &\quad + \tfrac{\sigma-\gamma}{\nu^p}\int\int_{\Omega_T} |\nabla u^\nu|^p \times \phi^p \times (u-k)_\pm^\nu \mathrm{d}x\mathrm{d}t \\ &\leq p\int_\Omega \phi^{p-1} \times |\partial_t\phi| \times (u-k)_\pm^{\nu+1}\mathrm{d}x + \int_\Omega (\phi^p(x,t_0-\theta) \times (u(x,t_0-\theta)-k)_\pm^{\nu+1})\mathrm{d}x \\ &\quad + \tfrac{1}{p\nu^{p-1}}\int_{t_0-\theta}^{t_0}\int_\Omega |(u-k)_\pm^{p\nu} \times u^\nu |\nabla\phi|^p \mathrm{d}x\mathrm{d}t.\end{aligned} \tag{22}$$

Theorem 3.1 Let u be a weak solution of the inverse variational inequality (2) with $p \geq 2$, then it follows that

$$\begin{aligned}&ess \sup_{t\in(t_0-\theta,t_0)} \int_\Omega (\phi^p \times (u-k)_\pm^{\nu+1})\mathrm{d}x + \tfrac{1}{\nu^{p-1}p}\int_{t_0-\theta}^{t_0}\int_\Omega u^\nu |\nabla(u-k)_\pm^\nu|^p \times \phi^p \mathrm{d}x\mathrm{d}t \\ &\quad + \tfrac{\sigma-\gamma}{\nu^p}\int\int_{\Omega_T} |\nabla u^\nu|^p \times \phi^p \times (u-k)_\pm^\nu \mathrm{d}x\mathrm{d}t \\ &\leq p\int_\Omega \phi^{p-1} \times |\partial_t\phi| \times (u-k)_\pm^{\nu+1}\mathrm{d}x + \tfrac{1}{p\nu^{p-1}}\int_{t_0-\theta}^{t_0}\int_\Omega |(u-k)_\pm^{p\nu} \times u^\nu |\nabla\phi|^p \mathrm{d}x\mathrm{d}t.\end{aligned} \tag{23}$$

In (23), we utilize the condition $\phi(x, t_0-\theta) = 0$, which readily yields

$$\int_\Omega (\phi^p(x,t_0-\theta) \times (u(x,t_0-\theta)-k)_\pm^{\nu+1})\mathrm{d}x = 0.$$

Additionally, it is worth noting that by selecting suitable ϕ and $(u-k)_\pm$ in (22), we can obtain local estimates for the weak solution u, thereby establishing the Harnack inequality and Hölder continuity.

4. Results towards local continuity

This section is devoted to analyzing the Hölder continuity of weak solutions of the inverse variational inequality (2). We first examine the local lower bound estimate of weak solutions u of the inverse variational inequality (2), and define a cut-off function $\phi_n(x,t)$ on Q_n as described in

$$\phi_n(x,t) = \begin{cases} 0, & (x,t) \in \partial Q_n, \\ 1, & (x,t) \in Q_{n+1}. \end{cases} \tag{24}$$

Additionally, we assume that $\phi_n(x,t)$ satisfies the condition

$$|\nabla \phi_n(x,t)| \leq \frac{2^n}{R_n},\ |\partial_t \phi_n(x,t)| \leq \frac{2^{pn}}{R^p}. \tag{25}$$

In (23), $(u-k)_\pm$ is set as $(u-k)_-$, while k is chosen as k_n^-, resulting in

$$\begin{aligned} &\underset{t\in(t_0-dR/2,t_0)}{ess\sup} \int_\Omega (\phi^p \times (u-k_n^-)_-^{\nu+1})\mathrm{d}x + \tfrac{1}{p\nu^{p-1}} \int_{t_0-dR/2}^{t_0} \int_\Omega u^\nu |\nabla (u-k_n^-)_-^\nu|^p \times \phi^p \mathrm{d}x\mathrm{d}t \\ &+ \tfrac{\sigma-\gamma}{\nu^p} \int \int_{\Omega_T} |\nabla u^\nu|^p \times \phi^p \times (u-k_n^-)_-^\nu \mathrm{d}x\mathrm{d}t \\ &\leq p \int_\Omega \phi^{p-1} \times |\partial_t \phi| \times (u-k_n^-)_-^{\nu+1}\mathrm{d}x + \tfrac{1}{p\nu^{p-1}} \int_{t_0-dR/2}^{t_0} \int_\Omega |(u-k_n^-)_-^{p\nu} \times u^\nu |\nabla \phi|^p \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{26}$$

Due to the presence of $\sigma-\gamma>0$ and $\nu = \frac{\sigma-1}{p}+1>1$,

$$\frac{\sigma-\gamma}{\nu^p} \int \int_{\Omega_T} |\nabla u^\nu|^p \times \phi^p \times (u-k_n^-)_-^\nu \mathrm{d}x\mathrm{d}t \geq 0.$$

After removing them, we have

$$\begin{aligned} &\underset{t\in(t_0-dR/2,t_0)}{ess\sup} \int_\Omega (\phi^p \times (u-k_n^-)_-^{\nu+1})\mathrm{d}x + \tfrac{1}{p\nu^{p-1}} \int_{t_0-dR/2}^{t_0} \int_\Omega u^\nu |\nabla (u-k_n^-)_-^\nu|^p \times \phi^p \mathrm{d}x\mathrm{d}t \\ &\leq p\tfrac{2^{pn}}{R^p} \left(\int_{t_0-dR/2}^{t_0} \int_{B_n} \phi^{p-1} \times (u-k_n^-)_-^{\nu+1}\mathrm{d}x\mathrm{d}t + \tfrac{d}{p^2\nu^{p-1}} \int_{t_0-dR/2}^{t_0} \int_{B_n} |(u-k_n^-)_-^{p\nu}|\mathrm{d}x\mathrm{d}t \right). \end{aligned} \tag{27}$$

Further analysis of $\int_{t_0-dR/2}^{t_0} \int_{B_n} \phi^{p-1} \times (u-k_n^-)_-^{\nu+1}\mathrm{d}x\mathrm{d}t + \frac{d}{p^2\nu^{p-1}} \int_{t_0-dR/2}^{t_0} \int_{B_n} |(u-k_n^-)_-^{p\nu}|\mathrm{d}x\mathrm{d}t$ is conducted by applying Lemma 3.1,

$$\begin{aligned} &\int_{t_0-dR/2}^{t_0} \int_{B_n} \phi^{p-1} \times (u-k_n^-)_-^{\nu+1}\mathrm{d}x\mathrm{d}t + \tfrac{d}{p^2\nu^{p-1}} \int_{t_0-dR/2}^{t_0} \int_{B_n} |(u-k_n^-)_-^{p\nu}|\mathrm{d}x\mathrm{d}t \\ &\leq \left(\tfrac{\omega}{2^{s_*}}\right)^{\nu+1} \left[p^2 + \tfrac{d}{p^2\nu^{p-1}} \left(\tfrac{\omega}{2^{s_*}}\right)^{p\nu-\nu-1} \right] \int_{t_0-dR/2}^{t_0} \int_{B_n} (u-k_n^-)_-^{\nu+1}\mathrm{d}x\mathrm{d}t. \end{aligned}$$

By substituting the aforementioned results into Eq (27), we can obtain

$$\begin{aligned} &\underset{t\in(t_0-dR/2,t_0)}{ess\sup} \int_\Omega (\phi^p \times (u-k_n^-)_-^{\nu+1})\mathrm{d}x + \tfrac{1}{p\nu^{p-1}} \int_{t_0-dR/2}^{t_0} \int_\Omega u^\nu |\nabla (u-k_n^-)_-^\nu|^p \times \phi^p \mathrm{d}x\mathrm{d}t \\ &\leq p\tfrac{2^{pn}}{R^p} \left(\tfrac{\omega}{2^{s_*}}\right)^{\nu+1} \left[p^2 + \tfrac{1}{p^2\nu^{p-1}} \left(\tfrac{\omega}{2^{s_*}}\right)^{p\nu-\nu-1} \right] \int_{t_0-dR/2}^{t_0} \int_{B_n} \chi_{(u-k_n^-)_->0}\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{28}$$

For the purpose of facilitating the discussion, let us define $A_n = \{x \in B_n | u \le k_n^-\}$. Consequently, it can be derived from Eq (28) that

$$\|(u-k_n^-)_-\phi_n\|_{L^p(Q_n)}^p \le p\frac{2^{pn}}{R^p}\left(\frac{\omega}{2^{s_*}}\right)^{\nu+1}\left[p^2+\frac{1}{p^2\nu^{p-1}}\left(\frac{\omega}{2^{s_*}}\right)^{p\nu-\nu-1}\right]\int_{t_0-dR/2}^{t_0}|A_n|\mathrm{d}t. \tag{29}$$

Applying Lemma 2.2 to $\|(u-k_n^-)_-\phi_n\|_{L^p(Q_n)}^p$, we obtain

$$\|(u-k_n^-)_-\|_{L^p(Q_n)}^p \le \|(u-k_n^-)_-\phi_n\|_{L^p(Q_n)}^p\left(\int_{t_0-dR/2}^{t_0}|A_n|\mathrm{d}t\right)^{\frac{p}{N+p}}. \tag{30}$$

Lemma 4.1. If u is a weak solution of the inverse variational inequality (2) with $p > 2$, then

$$\|(u-k_n^-)_-\|_{L^p(Q_{n+1})}^p \ge \frac{1}{2^{p(n+2)}}\left(\frac{\omega}{2^{s_*}}\right)^p\int_{t_0-dR/2}^{t_0}|A_{n+1}|\mathrm{d}t.$$

Proof. Due to

$$k_n^- = \mu^- + \frac{1}{2^{s_*+1}}\omega + \frac{1}{2^{s_*+n+1}}\omega,$$

it follows that

$$\|(u-k_n^-)_-\|_{L^p(Q_{n+1})}^p = \int\int_{Q_{n+1}}(u-k_n^-)_-^p\mathrm{d}x\mathrm{d}t \ge \sum_{l=n+1}^{\infty}\int\int_{Q_{n+1}}(k_l^- - k_n^-)_-^p\mathrm{d}x\mathrm{d}t \ge \int_{Q_{n+1}}(k_{n+1}^- - k_n^-)_-^p\mathrm{d}x\mathrm{d}t,$$

thereby

$$\|(u-k_n^-)_-\|_{L^p(Q_{n+1})}^p \ge |k_n^- - k_{n+1}^-|^p\int_{t_0-dR/2}^{t_0}|A_{n+1}|\mathrm{d}t. \tag{31}$$

Furthermore, due to

$$|k_n^- - k_{n+1}^-|^p = \frac{1}{2^{s_*+n+1}}\omega - \frac{1}{2^{s_*+n+2}}\omega = \frac{1}{2^{s_*+n+2}}\omega,$$

it follows that

$$|k_n^- - k_{n+1}^-|^p \ge \frac{1}{2^{p(n+2)}}\left(\frac{\omega}{2^{s_*}}\right)^p,$$

thereby

$$|k_n^- - k_{n+1}^-|^p\int_{t_0-dR/2}^{t_0}|A_{n+1}|\mathrm{d}t \ge \frac{1}{2^{p(n+2)}}\left(\frac{\omega}{2^{s_*}}\right)^p\int_{t_0-dR/2}^{t_0}|A_{n+1}|\mathrm{d}t. \tag{32}$$

By combining Eqs (31) and (32), Lemma 4.1 is proven.

Continuing the analysis of the lower bound for weak solutions by combining (30) and Lemma 4.1 and substituting the obtained result into (29), it can be easily deduced that

$$\frac{1}{2^{p(n+2)}}\left(\frac{\omega}{2^{s_*}}\right)^p\int_{t_0-\theta}^{t_0}|A_{n+1}|\mathrm{d}t \le p\frac{2^{pn}}{R^p}\left(\frac{\omega}{2^{s_*}}\right)^{\nu+1}\left[p^2+\frac{1}{p^2}\left(\frac{\omega}{2^{s_*}}\right)^{p\nu-\nu-1}\right]\int_{t_0-dR/2}^{t_0}|A_n|\mathrm{d}t. \tag{33}$$

Consequently, simplifying (33) yields

$$\int_{t_0-\theta}^{t_0}|A_{n+1}|\mathrm{d}t \le \frac{p4^p}{R^p}\left(\frac{\omega}{2^{s_*}}\right)^{\nu-p+1}\left[p^2+\frac{1}{p^2}\left(\frac{\omega}{2^{s_*}}\right)^{p\nu-\nu-1}\right]4^{pn}\int_{t_0-dR/2}^{t_0}|A_n|\mathrm{d}t.$$

This from Lemma 2.2 implies that $\int_{t_0-dR/2}^{t_0}|A_n|\mathrm{d}t \to 0$ as $\to n$, if

$$\int_{t_0-\theta}^{t_0}|\{x\in B_{\frac{1}{2}R}|u\geq\mu^-+\frac{1}{2^{s_*+1}}\omega\}|\mathrm{d}t\leq\frac{p^{\frac{2p}{N+p}-\frac{N+p}{p}}}{R^{-\frac{p}{N+p}p}}\Big(\frac{\omega}{2^{s_*}}\Big)^{-(1+\nu)\frac{p^2}{N+p}}4^{-\frac{p^3}{(N+p)^2}-\frac{p^2}{N+p}}.\tag{34}$$

It is worth noting that $\sigma\geq 1$ when selecting a sufficiently large S_* such that (34) always holds, thus leading to the following result.

Theorem 4.1. If $\sigma\geq 1$, selecting a sufficiently large $s_*>1$, it holds that

$$u\geq\mu^-+\frac{\omega}{2^{s_*+1}}\ a.e.\ (x,t)\in Q(\frac{1}{2}R,d(\frac{1}{2}R)^p).\tag{35}$$

Furthermore, if

$$\int_{t_0-\theta}^{t_0}|\{x\in B_{\frac{1}{2}R}|u\geq\mu^-+\frac{1}{2^{s_*+1}}\omega\}|\mathrm{d}t\leq\frac{p^{\frac{2p}{N+p}-\frac{N+p}{p}}}{R^{-\frac{p}{N+p}p}}\Big(\frac{\omega}{2^{s_*}}\Big)^{-(1+\nu)\frac{p^2}{N+p}}4^{-\frac{p^3}{(N+p)^2}-\frac{p^2}{N+p}},$$

then, (35) still holds.

Next, we analyze the upper bound of the weak solution. By applying Lemma 4.1, it is easy to obtain

$$(u-k_n^+)_+^{\nu+1}\geq\left(\frac{2^{s_*}}{\omega}\right)^{p\nu-\nu-1}(u-k_n^+)_+^{p\nu}.\tag{36}$$

Consequently, in Eq (23) we set $(u-k)_\pm$ as $(u-k_n^+)_+$ and eliminate the nonpositive term $\frac{\sigma-\gamma}{\nu^p}\int\int_{\Omega_T}|\nabla u^\nu|^p\times\phi^p\times(u-k_n^+)_+^\nu\mathrm{d}x\mathrm{d}t$, resulting in

$$\begin{aligned}&\underset{t\in(t_0-dR/2,t_0)}{ess\sup}\int_\Omega(\phi^p\times(u-k_n^+)_+^{\nu+1})\mathrm{d}x+\tfrac{1}{p\nu^{p-1}}\int_{t_0-dR/2}^{t_0}\int_\Omega u^\nu|\nabla(u-k_n^+)_+^\nu|^p\times\phi^p\mathrm{d}x\mathrm{d}t\\&\leq p\tfrac{2^{pn}}{R^p}\Big(\int_{t_0-dR/2}^{t_0}\int_{B_n}\phi^{p-1}\times(u-k_n^+)_+^{\nu+1}\mathrm{d}x\mathrm{d}t+\tfrac{d}{p^2\nu^{p-1}}\int_{t_0-dR/2}^{t_0}\int_{B_n}|(u-k_n^+)_+^{p\nu}|\mathrm{d}x\mathrm{d}t\Big).\end{aligned}\tag{37}$$

Note that

$$k_n^+=\mu^+-\frac{1}{2^{s_*+1}}\omega-\frac{1}{2^{s_*+n+1}}\omega.$$

By applying Lemma 3.1, we can obtain

$$(u-k_n^+)_+^{\nu+1}\geq\left(\frac{2^{s_*}}{\omega}\right)^{p\nu-\nu-1}(u-k_n^+)_+^{p\nu},\tag{38}$$

which implies that

$$\begin{aligned}&\underset{t\in(t_0-dR/2,t_0)}{ess\sup}\int_\Omega(\phi^p\times(u-k_n^+)_+^{\nu+1})\mathrm{d}x+\tfrac{1}{p\nu^{p-1}}\int_{t_0-dR/2}^{t_0}\int_\Omega u^\nu|\nabla(u-k_n^+)_+^\nu|^p\times\phi^p\mathrm{d}x\mathrm{d}t\\&\leq p\tfrac{2^{pn}}{R^p}\Big(\tfrac{\omega}{2^{s_*}}\Big)^{\nu+1}\Big[p^2+\tfrac{1}{p^2\nu^{p-1}}\Big(\tfrac{\omega}{2^{s_*}}\Big)^{p\nu-\nu-1}\Big]\int_{t_0-dR/2}^{t_0}\int_{B_n}\chi_{(u-k_n^+)_+>0}\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{39}$$

For the sake of convenience in the discussion, we will continue to use the symbol $A_n=\{x\in B_n|u\leq k_n^+\}$, hence

$$\|(u-k_n^+)_+\phi_n\|_{L^p(Q_n)}^p\leq p\frac{2^{pn}}{R^p}\Big(\frac{\omega}{2^{s_*}}\Big)^{\nu+1}\left[p^2+\frac{1}{p^2\nu^{p-1}}\Big(\frac{\omega}{2^{s_*}}\Big)^{p\nu-\nu-1}\right]\int_{t_0-dR/2}^{t_0}|A_n|\mathrm{d}t.\tag{40}$$

By utilizing Lemma 2.2, we can obtain the following estimation

$$\|(u-k_n^+)_+\|^p_{L^p(Q_n)} \le \|(u-k_n^+)_+\phi_n\|^p_{L^p(Q_n)}\left(\int_{t_0-dR/2}^{t_0}|A_n|\mathrm{d}t\right)^{\frac{p}{N+p}}. \tag{41}$$

Following the same approach as in Lemma 4.1, we can deduce that

$$\|(u-k_n^+)_+\|^p_{L^p(Q_{n+1})} \ge \frac{1}{2^{p(n+2)}}\left(\frac{\omega}{2^{s_*}}\right)^p\int_{t_0-dR/2}^{t_0}|A_{n+1}|\mathrm{d}t. \tag{42}$$

Combining (41) and (42) and substituting the result into (40), we can simplify and deduce that

$$\int_{t_0-dR/2}^{t_0}|A_{n+1}|\mathrm{d}t \le C(p,\nu)\frac{4^{pn}}{R^p}\left(\int_{t_0-dR/2}^{t_0}|A_n|\mathrm{d}t\right)^{1+\frac{p}{N+p}}. \tag{43}$$

Clearly, the equation above and Lemma 2.2 imply that $\int_{t_0-dR/2}^{t_0}|A_n|\mathrm{d}t \to 0$ as $\to n$, if

$$\int_{t_0-\theta}^{t_0}|\{x\in B_{\frac{1}{2}R}|u\le\mu^+-\frac{1}{2^{s_*+1}}\omega\}|\mathrm{d}t \le \frac{p^{\frac{2p}{N+p}-\frac{N+p}{p}}}{R^{-\frac{p}{N+p}p}}\left(\frac{\omega}{2^{s_*}}\right)^{-(1+\nu)\frac{p^2}{N+p}}4^{-\frac{p^3}{(N+p)^2}-\frac{p^2}{N+p}}. \tag{44}$$

Thus we have

$$u \le \mu^+ - \frac{\omega}{2^{s_*+1}} \text{ a.e. } (x,t)\in Q(\frac{1}{2}R, d(\frac{1}{2}R)^p). \tag{45}$$

Due to the fact that

$$\underset{Q(\frac{1}{2}R,d(\frac{1}{2}R)^p)}{osc} u = \underset{Q(\frac{1}{2}R,d(\frac{1}{2}R)^p)}{ess\sup} u - \underset{Q(\frac{1}{2}R,d(\frac{1}{2}R)^p)}{ess\inf} u,$$

combining Eqs (35) and (45), we obtain

$$\underset{Q(\frac{1}{2}R,d(\frac{1}{2}R)^p)}{osc} u \le (1-\frac{1}{2^{s_*}})\omega \ a.e.\ (x,t)\in Q(\frac{1}{2}R, d(\frac{1}{2}R)^p). \tag{46}$$

Theorem 4.2. (Hölder continuity) For any $(x,t)\in Q(\frac{1}{2}R, d(\frac{1}{2}R)^p)$, if $\sigma > 1$, there exists a nonnegative constant C such that

$$\underset{Q(\frac{1}{2}R,d(\frac{1}{2}R)^p)}{osc} u \le C\omega.$$

Furthermore, if (34) and (44) hold, the above inequality still holds.

In fact, by choosing

$$C = (1-\frac{1}{2^{s_*}})$$

in (46), the conclusion of Theorem 3.3 is evident. Furthermore, by selecting

$$C = (2^{s_*}+1)/(2^{s_*}-1) \le 2,$$

we have the following result.

Theorem 4.3. (Harnack's inequality) Assuming $\sigma \ge 1$, there exists a nonnegative constant C such that

$$\underset{Q(\frac{1}{2}R,d(\frac{1}{2}R)^p)}{ess\sup} u \le C \underset{Q(\frac{1}{2}R,d(\frac{1}{2}R)^p)}{ess\inf} u.$$

The Harnack inequality implies the following Hölder modulus estimate, as indicated by the literature in [17].

Theorem 4.4. (Hölder's modulus estimate) Let $u \in L^\infty(0,T;W_0^{1,p}(\Omega))$ be a weak solution of the inverse variation-inequality (2) and $\sigma \geq 1$. Then, there exists a constant C and $\beta \in (0,1)$, for any $\Omega' \subset \Omega$, such that $[u]_{\beta,\frac{1}{2}\beta;\Omega'_T} \leq C$.

5. Conclusions

This paper aimed to explore a specific type of inverse variation inequality problem

$$\begin{cases} \min\{-Lu, u-u_0\} \geq 0, & (x,t) \in \Omega_T, \\ u(0,x) = u_0(x), & x \in \Omega, \\ u(t,x) = \frac{\partial u}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0,T), \end{cases}$$

which was formulated using degenerate parabolic operators in non-divergence form

$$Lu = \partial_t u - u^\sigma \Delta_p u - \gamma u^{\sigma-1}|\nabla u|^p, \quad p > 2, \sigma > \gamma > 0.$$

First, by incorporating $(u - M_0)_+$ into $Lu \leq 0$ in Ω_T, we obtained the following integral inequality:

$$\frac{1}{2}\partial_t \int_\Omega (u - M_0)_+^2 \mathrm{d}x + \int_\Omega u^\sigma |\nabla(u - M_0)_+|^p + (\sigma - \gamma)u^{\sigma-1}|\nabla u|^p (u - M_0)_+ \mathrm{d}x \leq 0.$$

Subsequently, we derived the upper and lower bounds for the inverse variation inequality problem (2) and utilized them to construct a weak solution for the inverse variation-inequality problem (2).

Next, in the weak solution, the test function $w = \phi^p \times (u-k)_\pm^\nu$ was chosen and an integral inequality was obtained using the Hölder and Young inequalities, as shown in Theorem 3.1. Finally, the incorporation of a cut-off factor in Theorem 3.1 yields the Hölder continuity of the weak solution to problem (2), the Harnack inequality and the Hölder modulus estimate.

There are still some areas in this paper that can be improved. The current study only considered the case where $\sigma > \gamma$, and the existence of weak solutions cannot be proven if $\sigma < \gamma$. Additionally, in the proof process of the Hölder continuity in Section 4, the condition $\sigma > \gamma$ was also used to ensure that $\frac{\sigma-\gamma}{\nu^p} \int\int_{\Omega_T} |\nabla u^\nu|^p \times \phi^p \times (u-k)_+^\nu \mathrm{d}x\mathrm{d}t$ was nonnegative. In Lemma 2.2, the parameter p was restricted to be greater than two. In future research, we will attempt to analyze the impact of these restrictive conditions on the results.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The author sincerely thanks the editors and anonymous reviewers for their insightful comments and constructive suggestions, which greatly improved the quality of the paper.

Conflict of interest

The authors declare no conflicts of interest.

References

1. C. O. Alves, L. M. Barros, C. E. T. Ledesma, Existence of solution for a class of variational inequality in whole $\mathbb{R}^N$ with critical growth, *J. Math. Anal. Appl.*, **494** (2021), 124672. https://doi.org/10.1016/j.jmaa.2020.124672
2. J. Li, C. Bi, Study of weak solutions of variational inequality systems with degenerate parabolic operators and quasilinear terms arising American option pricing problems, *AIMS Math.*, **7** (2022), 19758–19769. https://doi.org/10.3934/math.20221083
3. T. Ye, W. Liu, T. Shen, Existence of nontrivial rotating periodic solutions for second-order Hamiltonian systems, *Appl. Math. Lett.*, **142** (2023), 108630. https://doi.org/10.1016/j.aml.2023.108630
4. K. K. Saha, N. Sukavanam, S. Pan, Existence and uniqueness of solutions to fractional differential equations with fractional boundary conditions, *Alexandria Eng. J.*, **72** (2023), 147–155. https://doi.org/10.1016/j.aej.2023.03.076
5. G. C. G. dos Santos, N. de A. Lima, R. N. de Lima, Existence and multiple of solutions for a class integro-differential equations with singular term via variational and Galerkin methods, *Nonlinear Anal.*, **69** (2023), 103752. https://doi.org/10.1016/j.nonrwa.2022.103752
6. F. O. Gallego, H. Ouyahya, M. Rhoudaf, Existence of a capacity solution to a nonlinear parabolic Celliptic coupled system in anisotropic Orlicz-Sobolev spaces, *Results Appl. Math.*, **18** (2023), 100376. https://doi.org/10.1016/j.rinam.2023.100376
7. L. Lussardi, E. Mascolo, A uniqueness result for a class of non-strictly convex variational problems, *J. Math. Anal. Appl.*, **446** (2017), 1687–1694. https://doi.org/10.1016/j.jmaa.2016.09.060
8. M. Boukrouche, D. A. Tarzia, Existence, uniqueness, and convergence of optimal control problems associated with parabolic variational inequalities of the second kind, *Nonlinear Anal.*, **12** (2011), 2211–2224. https://doi.org/10.1016/j.nonrwa.2011.01.003
9. A. Dieb, I. Ianni, A. Saldaña, Uniqueness and nondegeneracy for Dirichlet fractional problems in bounded domains via asymptotic methods, *Nonlinear Anal.*, **236** (2023), 113354. https://doi.org/10.1016/j.na.2023.113354
10. M. G. Ghimenti, A. M. Micheletti, Compactness and blow up results for doubly perturbed Yamabe problems on manifolds with umbilic boundary, *Nonlinear Anal.*, **229** (2023), 113206. https://doi.org/10.1016/j.na.2022.113206
11. L. Li, M. Wang, Global existence and blow-up of solutions of nonlocal diffusion problems with free boundaries, *Nonlinear Anal.*, **58** (2021), 103231. https://doi.org/10.1016/j.nonrwa.2020.103231
12. Q. M. Tran, T. T. Vu, H. D. T. Huynh, H. D. Pham, Global existence, blow-up in finite time and vacuum isolating phenomena for a system of semilinear wave equations associated with the helical flows of Maxwell fluid, *Nonlinear Anal.*, **69** (2023), 103734. https://doi.org/10.1016/j.nonrwa.2022.103734

13. L. Marino, Schauder estimates for degenerate Lévy Ornstein-Uhlenbeck operators, *J. Math. Anal. Appl.*, **500** (2021), 125168. https://doi.org/10.1016/j.jmaa.2021.125168

14. Y. Sun, T. Wu, Hölder and Schauder estimates for weak solutions of a certain class of non-divergent variation inequality problems in finance, *AIMS Math.*, **8** (2023), 18995–19003. https://doi.org/10.3934/math.2023968

15. D. Wang, K. Serkh, C. Christara, A high-order deferred correction method for the solution of free boundary problems using penalty iteration, with an application to American option pricing, *J. Comput. Appl. Math.*, **432** (2023), 115272. https://doi.org/10.1016/j.cam.2023.115272

16. S. Hussain, H. Arif, M. Noorullah, A. A. Pantelous, Pricing American options under Azzalini Ito-McKean skew Brownian motions, *Appl. Math. Comput.*, **451** (2023), 128040. https://doi.org/10.1016/j.amc.2023.128040

17. Y. Wang, Local Hölder continuity of nonnegative weak solutions of degenerate parabolic equations, *Nonlinear Anal.*, **72** (2010), 3289–3302. https://doi.org/10.1016/j.na.2009.12.007