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Research article

Rigidity results for closed vacuum static spaces

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Abstract: In this paper we studied rigidity results for closed vacuum static spaces. By using the maximum principle, we achieved rigidity theorems under some pointwise inequalities and showed that the squared norm of the Ricci curvature tensor was discrete.

Keywords: Einstein; vacuum static space; zero radial Weyl curvature
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1. Introduction

Let (M^n, g) be an n-dimensional Riemannian manifold with the metric g and the dimension $n \geq 3$. If there exists a non-constant smooth function f such that

$$f_{ij} = f\Big(R_{ij} - \frac{1}{n-1}Rg_{ij}\Big), \tag{1.1}$$

then (M^n, g, f) is called a vacuum static space (for more backgrounds, see [8, 10, 19, 23]). Here f_{ij}, R_{ij} and R denote components of the Hessian of f, the Ricci curvature tensor and the scalar curvature, respectively. In [8], Fischer-Marsden proposed the following conjecture: *The standard spheres are the only n-dimensional compact vacuum static spaces.* In [18], Kobayashi gave a classification for n-dimensional complete vacuum static spaces that are locally conformally flat. On the other hand, he and Lafontaine [20] also provided some counterexamples for the above conjecture.

In fact, according to the second Bianchi identity, any vacuum static space has constant scalar curvature. Moreover, Bourguignon [2] and Fischer-Marsden [8] have proved that the set $f^{-1}(0)$ has the measure zero and the set $f^{-1}(0)$ is a totally geodesic regular hypersurface.

Let $\mathring{R}_{ij} = R_{ij} - \frac{R}{n}g_{ij}$ be the trace-free Ricci curvature, then (1.1) can be written as

$$f_{ij} = f\mathring{R}_{ij} - \frac{R}{n(n-1)}fg_{ij}, \tag{1.2}$$

which gives

$$\Delta f = -\frac{R}{n-1} f.$$

It is well known that the Weyl curvature tensor W and the Riemannian curvature tensor is related by

$$R_{ijkl} = W_{ijkl} + \frac{1}{n-2}(R_{ik}g_{jl} - R_{il}g_{jk} + R_{jl}g_{ik} - R_{jk}g_{il}) - \frac{R}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{il}g_{jk}).$$

In this paper, we consider rigidity results for closed vacuum static spaces. By using the maximum principle, some rigidity theorems are obtained under some pointwise inequalities and show that the squared norm of the Ricci curvature tensor is discrete.

Theorem 1.1. *Let* (M^n, g, f) *be a closed vacuum static space with the positive scalar curvature and* $f_l W_{lijk} = 0$ *(that is, zero radial Weyl curvature), where* $n \geq 4$*. If*

$$\frac{(n-1)(n-2)}{2}|W|^2 + n(n-1)|\mathring{R}_{ij}|^2 \leq R^2, \tag{1.3}$$

then it must be of Einstein as long as there exists a point such that the inequality in (1.3) *is strict.*

Next, by substituting (1.3) with a stronger condition, we can obtain the following characterizations:

Theorem 1.2. *Let* (M^n, g, f) *be a closed vacuum static space with the positive scalar curvature and* $f_l W_{lijk} = 0$ *(that is, zero radial Weyl curvature), where* $n \geq 4$*. If*

$$\sqrt{\frac{(n-1)(n-2)}{2}}|W| + \sqrt{n(n-1)}|\mathring{R}_{ij}| \leq R, \tag{1.4}$$

then it must be of Einstein or a Riemannian product $\mathbb{S}^1 \times \mathbb{S}^{n-1}$*. In particular, it must be of Einstein as long as there exists a point such that the inequality in* (1.4) *is strict.*

When $W = 0$, the formula (2.1) shows that the Einstein metric with the positive scalar curvature must be of positive constant sectional curvature. Hence, Theorem 1.2 gives the following:

Corollary 1.3. *Let* (M^n, g, f) *be a closed vacuum static space with the positive scalar curvature and* $W = 0$*. If*

$$|\mathring{R}_{ij}| \leq \frac{R}{\sqrt{n(n-1)}}, \tag{1.5}$$

then it must be of either $\mathbb{S}^n$ *with positive constant sectional curvature or a Riemannian product* $\mathbb{S}^1 \times \mathbb{S}^{n-1}$*.*

In particular, when $n = 3$, we have $W = 0$ automatically and Corollary 1.3 yields the following result (which has been proved by Ambrozio in [1, Theorem A]) immediately:

Corollary 1.4. *Let* (M^3, g, f) *be a closed vacuum static space with the positive scalar curvature. If*

$$|\mathring{R}_{ij}| \leq \frac{1}{\sqrt{6}} R, \tag{1.6}$$

then it must be of either $\mathbb{S}^3$ *with positive constant sectional curvature or a Riemannian product* $\mathbb{S}^1 \times \mathbb{S}^2$*.*

Remark 1.1. It is easy to see that the condition (1.4) is stronger than (1.3). On the other hand, one can check that when $M^n = \mathbb{S}^1 \times \mathbb{S}^{n-1}$, we have $|\mathring{R}_{ij}| = \frac{R}{\sqrt{n(n-1)}}$, and when $M^n = \mathbb{S}^n$, we have $|\mathring{R}_{ij}| = 0$. Hence, for closed vacuum static spaces with $W = 0$, Corollary 1.3 gives the following *pinching results*: If $0 \leq |\mathring{R}_{ij}| \leq \frac{R}{\sqrt{n(n-1)}}$, then $|\mathring{R}_{ij}| = 0$ or $|\mathring{R}_{ij}| = \frac{R}{\sqrt{n(n-1)}}$. That is, the value of $|\mathring{R}_{ij}|$ is discrete.

Remark 1.2. Recently, by a generalized maximum principle, Cheng and Wei [6] considered the classifications for three-dimensional complete vacuum static spaces with constant squared norm of Ricci curvature tensor. For the classifications for closed cases, see [17, 24–26] and the references therein.

2. Some necessary lemmas

It is well known that the Weyl curvature tensor and the Cotton tensor are defined respectively as follows:

$$\begin{aligned}R_{ijkl} &= W_{ijkl} + \frac{1}{n-2}(R_{ik}g_{jl} - R_{il}g_{jk} + R_{jl}g_{ik} - R_{jk}g_{il}) \\ &\quad - \frac{R}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{il}g_{jk}) \\ &= W_{ijkl} + \frac{1}{n-2}(\mathring{R}_{ik}g_{jl} - \mathring{R}_{il}g_{jk} + \mathring{R}_{jl}g_{ik} - \mathring{R}_{jk}g_{il}) \\ &\quad + \frac{R}{n(n-1)}(g_{ik}g_{jl} - g_{il}g_{jk}) \end{aligned} \tag{2.1}$$

and

$$C_{ijk} = \mathring{R}_{ij,k} - \mathring{R}_{ik,j} + \frac{n-2}{2n(n-1)}(R_{,k}g_{ij} - R_{,j}g_{ki}). \tag{2.2}$$

From (2.2), it is easy to see that C_{ijk} is skew-symmetric with respect to the last two indices; that is, $C_{ijk} = -C_{ikj}$ and is trace-free in any two indices:

$$C_{iik} = 0 = C_{iji}. \tag{2.3}$$

In addition,

$$C_{ijk} + C_{jki} + C_{kij} = 0, \tag{2.4}$$

and in using the Ricci identity, one has

$$C_{ilk,l} = C_{kli,l}, \quad C_{ijl,l} = C_{jil,l}, \quad C_{lij,l} = 0. \tag{2.5}$$

Associated to (1.1), there is a (0.3)-tensor T_{ijk}, which can be written as

$$T_{ijk} = \frac{n-1}{n-2}(\mathring{R}_{ik}f_j - \mathring{R}_{ij}f_k) + \frac{1}{n-2}(g_{ik}\mathring{R}_{jl} - g_{ij}\mathring{R}_{kl})f_l. \tag{2.6}$$

A direct calculation enables us to observe that T satisfies the following properties:

$$T_{ijk} = -T_{ikj}, \quad T_{iik} = 0 = T_{iji},$$

$$T_{ijk} + T_{jki} + T_{kij} = 0.$$

Moreover, the tensor C_{ijk} is related to T by (see [3, 4, 11, 15, 25]):

$$fC_{ijk} = T_{ijk} + f_l W_{lijk}. \tag{2.7}$$

Lemma 2.1. *Let* (M^n, g, f) *be a vacuum static space with* f *satisfying* (1.2). *We have*

$$\begin{aligned}\Delta f_{ij} = &2f\mathring{R}_{mk}W_{mijk} + \frac{2n}{n-2}f\mathring{R}_{im}\mathring{R}_{mj} + \frac{R^2}{n(n-1)^2}fg_{ij}\\ &- \frac{2}{n-2}f|\mathring{R}_{kl}|^2 g_{ij} + \frac{1}{n-1}Rf\mathring{R}_{ij} + f_m C_{jmi} + f_m\mathring{R}_{mi,j}\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}f\Delta\mathring{R}_{ij} = &2f\mathring{R}_{mk}W_{mijk} + \frac{2n}{n-2}f\mathring{R}_{im}\mathring{R}_{mj} - \frac{2}{n-2}f|\mathring{R}_{kl}|^2 g_{ij}\\ &+ f_m(C_{jmi} + C_{imj}) + \frac{2R}{n-1}f\mathring{R}_{ij} - f_k\mathring{R}_{ij,k}.\end{aligned} \tag{2.9}$$

Proof. By the Ricci identity, we have

$$\begin{aligned}f_{ij,kl} =&f_{ik,jl} + (f_m R_{mijk})_{,l}\\ =&f_{ik,jl} + f_{ml}R_{mijk} + f_m R_{mijk,l}\\ =&f_{ik,lj} + f_{mk}R_{mijl} + f_{im}R_{mkjl} + f_{ml}R_{mijk} + f_m R_{mijk,l}\\ =&f_{kl,ij} + (f_m R_{mkil})_{,j} + f_{mk}R_{mijl}\\ &+f_{im}R_{mkjl} + f_{ml}R_{mijk} + f_m R_{mijk,l}\\ =&f_{kl,ij} + f_{mj}R_{mkil} + f_{mk}R_{mijl} + f_{im}R_{mkjl}\\ &+f_{ml}R_{mijk} + f_m R_{mijk,l} + f_m R_{mkil,j},\end{aligned}$$

which gives

$$\Delta f_{ij} = f_{ij,kk} = (\Delta f)_{,ij} + f_{mj}R_{mi} + 2f_{mk}R_{mijk} + f_{im}R_{mj} + f_m R_{mijk,k} + f_m R_{mi,j}. \tag{2.10}$$

Since the scalar curvature R is constant, then

$$(\Delta f)_{,ij} = -\frac{1}{n-1}Rf\Big[\mathring{R}_{ij} - \frac{R}{n(n-1)}g_{ij}\Big],$$

$$\begin{aligned}f_{mj}R_{mi} =&\Big[f\mathring{R}_{mj} - \frac{R}{n(n-1)}fg_{mj}\Big]\Big(\mathring{R}_{mi} + \frac{R}{n}g_{mi}\Big)\\ =&f\mathring{R}_{im}\mathring{R}_{mj} + \frac{n-2}{n(n-1)}Rf\mathring{R}_{ij} - \frac{R^2}{n^2(n-1)}fg_{ij},\end{aligned}$$

which is equivalent to

$$f_{mj}\mathring{R}_{mi} = f\mathring{R}_{im}\mathring{R}_{mj} - \frac{R}{n(n-1)}f\mathring{R}_{ij},$$

$$\begin{aligned}
f_{mk}R_{mijk} =& f_{mk}\Big[W_{mijk} + \frac{1}{n-2}(\mathring{R}_{mj}g_{ik} - \mathring{R}_{mk}g_{ij} + \mathring{R}_{ik}g_{mj} - \mathring{R}_{ij}g_{mk}) \\
&+ \frac{R}{n(n-1)}(g_{mj}g_{ik} - g_{mk}g_{ij})\Big] \\
=& f\mathring{R}_{mk}W_{mijk} + \frac{1}{n-2}\Big[f_{ik}\mathring{R}_{kj} + f_{jk}\mathring{R}_{ki} - f_{mk}\mathring{R}_{mk}g_{ij} \\
&-(\Delta f)\mathring{R}_{ij}\Big] + \frac{R}{n(n-1)}[f_{ij} - (\Delta f)g_{ij}] \\
=& f\mathring{R}_{mk}W_{mijk} + \frac{1}{n-2}\Big[2f\mathring{R}_{im}\mathring{R}_{mj} - \frac{2R}{n(n-1)}f\mathring{R}_{ij} \\
&-f|\mathring{R}_{kl}|^2g_{ij} + \frac{R}{n-1}f\mathring{R}_{ij}\Big] + \frac{R}{n(n-1)}\Big[f\mathring{R}_{ij} + \frac{R}{n}fg_{ij}\Big].
\end{aligned}$$

In particular, by virtue of the second Bianchi identity, we have

$$R_{jkim,m} = R_{ij,k} - R_{ik,j} = C_{ijk},$$

where, in the last equality, we used the formula (2.2) since the scalar curvature R is constant. Thus, we obtain

$$\begin{aligned}
\Delta f_{ij} =& -\frac{1}{n-1}R\Big[f\mathring{R}_{ij} - \frac{R}{n(n-1)}fg_{ij}\Big] + 2f\mathring{R}_{im}\mathring{R}_{mj} \\
&+ \frac{2(n-2)}{n(n-1)}Rf\mathring{R}_{ij} - \frac{2R^2}{n^2(n-1)}fg_{ij} + 2f\mathring{R}_{mk}W_{mijk} \\
&+ \frac{2}{n-2}\Big[2f\mathring{R}_{im}\mathring{R}_{mj} - \frac{2}{n(n-1)}Rf\mathring{R}_{ij} - f|\mathring{R}_{kl}|^2g_{ij} + \frac{R}{n-1}f\mathring{R}_{ij}\Big] \\
&+ \frac{2R}{n(n-1)}\Big[f\mathring{R}_{ij} + \frac{R}{n}fg_{ij}\Big] + f_mC_{jmi} + f_m\mathring{R}_{mi,j} \\
=& 2f\mathring{R}_{mk}W_{mijk} + \frac{2n}{n-2}f\mathring{R}_{im}\mathring{R}_{mj} + \frac{R^2}{n(n-1)^2}fg_{ij} - \frac{2}{n-2}f|\mathring{R}_{kl}|^2g_{ij} \\
&+ \frac{1}{n-1}Rf\mathring{R}_{ij} + f_mC_{jmi} + f_m\mathring{R}_{mi,j},
\end{aligned} \tag{2.11}$$

and the formula (2.8) is achieved.

From (1.2), we have

$$f\mathring{R}_{ij,k} = f_{ij,k} - f_k\mathring{R}_{ij} + \frac{R}{n(n-1)}f_kg_{ij}, \tag{2.12}$$

$$f_l\mathring{R}_{ij,k} + f\mathring{R}_{ij,kl} = f_{ij,kl} - f_{kl}\mathring{R}_{ij} - f_k\mathring{R}_{ij,l} + \frac{R}{n(n-1)}f_{kl}g_{ij}. \tag{2.13}$$

Therefore,

$$\begin{aligned}
f\Delta\mathring{R}_{ij} =& f\mathring{R}_{ij,kk} = \Delta f_{ij} - (\Delta f)\mathring{R}_{ij} - 2f_k\mathring{R}_{ij,k} + \frac{R}{n(n-1)}(\Delta f)g_{ij} \\
=& 2f\mathring{R}_{mk}W_{mijk} + \frac{2n}{n-2}f\mathring{R}_{im}\mathring{R}_{mj} - \frac{2}{n-2}f|\mathring{R}_{kl}|^2g_{ij} \\
&+ f_m(C_{jmi} + C_{imj}) + \frac{2R}{n-1}f\mathring{R}_{ij} - f_k\mathring{R}_{ij,k}.
\end{aligned} \tag{2.14}$$

The proof of Lemma 2.1 is completed. □

Lemma 2.2. *Let* (M^n, g, f) *be a vacuum static space with* f *satisfying* (1.2). *If* $f_l W_{lijk} = 0$ *(that is, zero radial Weyl curvature), then*

$$\begin{aligned}\frac{1}{2}f\Delta|\mathring{R}_{ij}|^2 + \frac{1}{2}\nabla f\nabla|\mathring{R}_{ij}|^2 =& f\mathring{R}_{ij,k}^2 + 2fW_{mijk}\mathring{R}_{ij}\mathring{R}_{mk} + \frac{2n}{n-2}f\mathring{R}_{im}\mathring{R}_{mj}\mathring{R}_{ji}\\ &+\frac{n-2}{n-1}f|C_{ijk}|^2 + \frac{2R}{n-1}f|\mathring{R}_{ij}|^2. \end{aligned}\tag{2.15}$$

Proof. Using (2.9), we have

$$\begin{aligned}\frac{1}{2}f\Delta|\mathring{R}_{ij}|^2 + \frac{1}{2}\nabla f\nabla|\mathring{R}_{ij}|^2 =& f\mathring{R}_{ij,k}^2 + f\mathring{R}_{ij}\Delta\mathring{R}_{ij} + f_k\mathring{R}_{ij}\mathring{R}_{ij,k}\\ =& f\mathring{R}_{ij,k}^2 + 2fW_{mijk}\mathring{R}_{ij}\mathring{R}_{mk} + \frac{2n}{n-2}f\mathring{R}_{im}\mathring{R}_{mj}\mathring{R}_{ij}\\ &+(C_{jmi} + C_{imj})\mathring{R}_{ij}f_m + \frac{2R}{n-1}f|\mathring{R}_{ij}|^2\\ =& f\mathring{R}_{ij,k}^2 + 2fW_{mijk}\mathring{R}_{ij}\mathring{R}_{mk} + \frac{2n}{n-2}f\mathring{R}_{im}\mathring{R}_{mj}\mathring{R}_{ji}\\ &-2C_{ijk}\mathring{R}_{ij}f_k + \frac{2R}{n-1}f|\mathring{R}_{ij}|^2.\end{aligned}\tag{2.16}$$

Since $f_l W_{lijk} = 0$, then (2.7) gives

$$fC_{ijk} = T_{ijk}$$

and

$$\begin{aligned}fC_{ijk}\mathring{R}_{ij}f_k =& T_{ijk}\mathring{R}_{ij}f_k\\ =& \Big[\frac{n-1}{n-2}(\mathring{R}_{ik}f_j - \mathring{R}_{ij}f_k) + \frac{1}{n-2}(g_{ik}\mathring{R}_{jl} - g_{ij}\mathring{R}_{kl})f_l\Big]\mathring{R}_{ij}f_k\\ =& \frac{n}{n-2}\mathring{R}_{ki}\mathring{R}_{kj}f_if_j - \frac{n-1}{n-2}|\mathring{R}_{ij}|^2|\nabla f|^2.\end{aligned}\tag{2.17}$$

On the other hand,

$$\begin{aligned}f^2|C_{ijk}|^2 =& |T_{ijk}|^2\\ =& \Big|\frac{n-1}{n-2}(\mathring{R}_{ik}f_j - \mathring{R}_{ij}f_k) + \frac{1}{n-2}(g_{ik}\mathring{R}_{jl} - g_{ij}\mathring{R}_{kl})f_l\Big|^2\\ =& -\frac{2n(n-1)}{(n-2)^2}\mathring{R}_{ki}\mathring{R}_{kj}f_if_j + \frac{2(n-1)^2}{(n-2)^2}|\mathring{R}_{ij}|^2|\nabla f|^2.\end{aligned}\tag{2.18}$$

Combining (2.17) and (2.18), we achieve

$$-2(n-1)C_{ijk}\mathring{R}_{ij}f_k = (n-2)f|C_{ijk}|^2.$$

Thus, (2.16) becomes

$$\begin{aligned}\frac{1}{2}f\Delta|\mathring{R}_{ij}|^2 + \frac{1}{2}\nabla f\nabla|\mathring{R}_{ij}|^2 =& f\mathring{R}_{ij,k}^2 + 2fW_{mijk}\mathring{R}_{ij}\mathring{R}_{mk} + \frac{2n}{n-2}f\mathring{R}_{im}\mathring{R}_{mj}\mathring{R}_{ji}\\ &+\frac{n-2}{n-1}f|C_{ijk}|^2 + \frac{2R}{n-1}f|\mathring{R}_{ij}|^2,\end{aligned}\tag{2.19}$$

and the formula (2.15) is attained. □

We also need the following lemma (see [9, 13, 14, 21]):

Lemma 2.3. *For any $\rho \in \mathbb{R}$, the following estimate holds:*

$$\Big| - W_{ijkl}\mathring{R}_{jl}\mathring{R}_{ik} + \frac{\rho}{n-2}\mathring{R}_{ij}\mathring{R}_{jk}\mathring{R}_{ki}\Big| \leq \sqrt{\frac{n-2}{2(n-1)}}\Big(|W|^2 + \frac{2\rho^2}{n(n-2)}|\mathring{R}_{ij}|^2\Big)^{\frac{1}{2}}|\mathring{R}_{ij}|^2. \tag{2.20}$$

3. Proof of results

3.1. Proof of Theorem 1.1

Multiplying both sides of (2.15) with f, we have

$$\begin{aligned}\frac{1}{2}f^2\Delta|\mathring{R}_{ij}|^2 + \frac{1}{2}f\nabla f\nabla|\mathring{R}_{ij}|^2 = f^2\mathring{R}_{ij,k}^2 + 2f^2W_{mijk}\mathring{R}_{ij}\mathring{R}_{mk} + \frac{2n}{n-2}f^2\mathring{R}_{im}\mathring{R}_{mj}\mathring{R}_{ji}\\ + \frac{n-2}{n-1}f^2|C_{ijk}|^2 + \frac{2R}{n-1}f^2|\mathring{R}_{ij}|^2.\end{aligned} \tag{3.1}$$

Since the manifold is closed, then (3.1) together with (2.20) yields

$$\begin{aligned}\frac{1}{2}f^2\Delta|\mathring{R}_{ij}|^2 + \frac{1}{2}f\nabla f\nabla|\mathring{R}_{ij}|^2 \geq f^2\Big(\mathring{R}_{ij,k}^2 + \frac{n-2}{n-1}|C_{ijk}|^2\Big)\\ + 2f^2\Big[\frac{R}{n-1} - \sqrt{\frac{n-2}{2(n-1)}}\Big(|W|^2 + \frac{2n}{n-2}|\mathring{R}_{ij}|^2\Big)^{\frac{1}{2}}\Big]|\mathring{R}_{ij}|^2.\end{aligned} \tag{3.2}$$

Therefore, under the assumption (1.3), it follows from (3.2) that

$$\begin{aligned}\frac{1}{2}f^2\Delta|\mathring{R}_{ij}|^2 + \frac{1}{2}f\nabla f\nabla|\mathring{R}_{ij}|^2 \geq & f^2\Big(\mathring{R}_{ij,k}^2 + \frac{n-2}{n-1}|C_{ijk}|^2\Big)\\ & +2f^2\Big[\frac{R}{n-1} - \sqrt{\frac{n-2}{2(n-1)}}\Big(|W|^2 + \frac{2n}{n-2}|\mathring{R}_{ij}|^2\Big)^{\frac{1}{2}}\Big]|\mathring{R}_{ij}|^2\\ \geq & 0,\end{aligned} \tag{3.3}$$

which shows that $|\mathring{R}_{ij}|^2$ is subharmonic on M^n. Using the maximum principle, we obtain that $|\mathring{R}_{ij}|$ is constant and $\mathring{R}_{ij,k} = 0$. In this case, (3.3) becomes

$$\Big[\frac{R}{n-1} - \sqrt{\frac{n-2}{2(n-1)}}\Big(|W|^2 + \frac{2n}{n-2}|\mathring{R}_{ij}|^2\Big)^{\frac{1}{2}}\Big]|\mathring{R}_{ij}|^2 = 0. \tag{3.4}$$

If there exists a point x_0 such that (1.3) is strict, then from (3.4) we have $|\mathring{R}_{ij}|(x_0) = 0$, which with $|\mathring{R}_{ij}|$ constant shows that $\mathring{R}_{ij} \equiv 0$. That is, the metric is Einstein and the proof of Theorem 1.1 is completed.

3.2. Proof of Theorem 1.2

We recall the following inequality, which was first proved by Huisken (cf. [16, Lemma 3.4]):

$$|W_{ikjl}\mathring{R}_{ij}\mathring{R}_{kl}| \leq \sqrt{\frac{n-2}{2(n-1)}}|W||\mathring{R}_{ij}|^2 \tag{3.5}$$

and

$$|\mathring{R}_{ij}\mathring{R}_{jk}\mathring{R}_{ki}| \leq \frac{n-2}{\sqrt{n(n-1)}}|\mathring{R}_{ij}|^3, \tag{3.6}$$

with the equality in (3.6) at some point $p \in M$ if, and only if, $\mathring{R}_{ij}$ can be diagonalized at p and the eigenvalue multiplicity of $\mathring{R}_{ij}$ is at least $n-1$ [12,22]. Thus, from (2.15), we obtain

$$\begin{aligned}\frac{1}{2}f^2\Delta|\mathring{R}_{ij}|^2 + \frac{1}{2}f\nabla f\nabla|\mathring{R}_{ij}|^2 \geq & f^2\Big(\mathring{R}_{ij,k}^2 + \frac{n-2}{n-1}|C_{ijk}|^2 - \sqrt{\frac{2(n-2)}{n-1}}|W||\mathring{R}_{ij}|^2 \\ & -2\sqrt{\frac{n}{n-1}}|\mathring{R}_{ij}|^3 + \frac{2R}{n-1}|\mathring{R}_{ij}|^2\Big) \\ = & f^2\Big(\mathring{R}_{ij,k}^2 + \frac{n-2}{n-1}|C_{ijk}|^2\Big) + 2f^2\Big(\frac{R}{n-1} \\ & - \sqrt{\frac{n-2}{2(n-1)}}|W| - \sqrt{\frac{n}{n-1}}|\mathring{R}_{ij}|\Big)|\mathring{R}_{ij}|^2.\end{aligned}$$

Similarly, under the assumption (1.4), we obtain

$$\begin{aligned}\frac{1}{2}f^2\Delta|\mathring{R}_{ij}|^2 + \frac{1}{2}f\nabla f\nabla|\mathring{R}_{ij}|^2 \geq & f^2\Big(\mathring{R}_{ij,k}^2 + \frac{n-2}{n-1}|C_{ijk}|^2\Big) + 2f^2\Big(\frac{R}{n-1} \\ & - \sqrt{\frac{n-2}{2(n-1)}}|W| - \sqrt{\frac{n}{n-1}}|\mathring{R}_{ij}|\Big)|\mathring{R}_{ij}|^2 \\ \geq & 0,\end{aligned} \tag{3.7}$$

which shows that $|\mathring{R}_{ij}|^2$ is subharmonic on M^n. Using the maximum principle again, we obtain that $|\mathring{R}_{ij}|$ is constant and $\mathring{R}_{ij,k} = 0$. In this case, (3.7) becomes

$$\Big(\frac{R}{n-1} - \sqrt{\frac{n-2}{2(n-1)}}|W| - \sqrt{\frac{n}{n-1}}|\mathring{R}_{ij}|\Big)|\mathring{R}_{ij}|^2 = 0 \tag{3.8}$$

and the equalities in (3.5) and (3.6) occur.

In particular, writing $\mathring{R}_{ij} = ag_{ij} + bv_iv_j$ at p with some scalars a, b and a vector v, we see that the left hand side of (3.5) is zero [12] at every point p. As (3.5) is an equality and, according to [7], g is real-analytic, the metric g must be conformally flat or Einstein.

If there exists a point x_0 such that (1.4) is strict, then from (3.8) we have $|\mathring{R}_{ij}|(x_0) = 0$. Which with $|\mathring{R}_{ij}|$ constant shows that $\mathring{R}_{ij} \equiv 0$ and the metric is Einstein. Otherwise, we have that the equality in (1.4) occurs and

$$\sqrt{\frac{(n-1)(n-2)}{2}}|W| + \sqrt{n(n-1)}|\mathring{R}_{ij}| = R. \tag{3.9}$$

In this case, we have $W = 0$ and (3.9) becomes $|\mathring{R}_{ij}| = \frac{R}{\sqrt{n(n-1)}}$, and then $M^n = \mathbb{S}^1 \times \mathbb{S}^{n-1}$ [5].

Therefore, we complete the proof of Theorem 1.2.

4. Conclusions

The aim of this paper is to study rigidity results for closed vacuum static spaces. The main tool is to apply the maximum principle to the function $|\mathring{R}_{ij}|^2$ since the manifolds are closed. More precisely, we obtain rigidity theorems by establishing some pointwise inequalities and applying the maximum principle, which further proves that the squared norm of the Ricci curvature tensor is discrete.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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